Classification of Segre holonomies of torsion-free affine connections

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Abstract

Using twistor methods, a complete classification of irreducible holonomies of torsion-free affine connections which can be represented as a tensor product of non-Abelian representations, is given.

As a by-product, a complete list of all compact complex homogeneous-rational manifolds X and ample line bundles $L \to X$ such that $H^0(X, TX \otimes L^*) \neq 0$ and/or $H^1(X, TX \otimes L^*) \neq 0$ is obtained.

1 Introduction

Holonomy group is one of the most informative characteristics of an affine connection. The problem of classification of holonomy groups has a long history which starts in 1920s with the works of Cartan [16, 17] where he used this notion to classify locally symmetric Riemannian manifolds. In 1955, Berger [7] showed that the list of irreducibly acting matrix Lie groups which can, in principle, occur as the holonomy of a torsion-free affine connection must be very restrictive. This is in a sharp contrast to the result of Hano and Ozeki [20] which says that there is no interesting holonomy classification in the class of arbitrary affine connections — *any* closed subgroup of $GL(n, \mathbb{R})$ can be realized as the holonomy of an affine connection (with torsion, in general).

Berger presented his classification list of all possible candidates to irreducible holonomies[1] in two parts — the first part is claimed to contain all possible groups which preserve a non-degenerate symmetric bilinear form, and the second part is claimed to contain all the rest, *up to a finite number of missing terms* which Bryant [13] suggested to call the *exotic holonomies.* The proof of the second part was omitted; as of this writing, no proof has yet been published.

The classification of all metric holonomies has been recently completed [14]. This is a culmination of efforts of many people to show that most entries of Berger's original metric list do occur as holonomies of Levi-Civita connections and that just a few of them are superious (see, e.g., [3, 9, 12, 13, 14, 33] and the references cited therein).

[1] From now on by a holonomy group we always understand the irreducibly acting holonomy of a *torsion-free* affine connection which is *not* locally symmetric. The second assumption is motivated by the fact that, due to Cartan [17] and Berger [8], the list of locally symmetric affine spaces is completely known.

Berger's original list of non-metric holonomies		
group G	representation V	restrictions
$\mathrm{T}_{\mathbb{R}} \cdot \mathrm{SL}(n, \mathbb{R})$	$\mathbb{R}^n$ $\odot^2\mathbb{R}^n \simeq \mathbb{R}^{n(n+1)/2}$ $\Lambda^2\mathbb{R}^n \simeq \mathbb{R}^{n(n-1)/2}$	$n \geqslant 2$ $n \geqslant 3$ $n \geqslant 5$
$\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SL}(n, \mathbb{C})$	$\mathbb{C}^n \simeq \mathbb{R}^{2n}$ $\odot^2\mathbb{C}^n \simeq \mathbb{R}^{n(n+1)}$ $\Lambda^2\mathbb{C}^n \simeq \mathbb{R}^{n(n-1)}$	$n \geqslant 1$ $n \geqslant 3$ $n \geqslant 5$
$\mathbb{R}^* \cdot \mathrm{SL}(n, \mathbb{C})$	$\{A \in M_n(\mathbb{C}) : \bar{A} = A^t\} \simeq \mathbb{R}^{n^2}$	$n \geqslant 3$
$\mathrm{T}_{\mathbb{R}} \cdot \mathrm{SL}(n, \mathbb{H})$	$\mathbb{H}^n \simeq \mathbb{R}^{4n}$ $\{A \in M_n(\mathbb{H}) : A^* = -A^t\} \simeq \mathbb{R}^{n(2n+1)}$ $\{A \in M_n(\mathbb{H}) : A^* = A^t\} \simeq \mathbb{R}^{n(2n-1)}$	$n \geqslant 1$ $n \geqslant 2$ $n \geqslant 3$
$\mathrm{T}_{\mathbb{R}} \cdot \mathrm{Sp}(n, \mathbb{R})$ $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{Sp}(n, \mathbb{C})$	$\mathbb{R}^{2n}$ $\mathbb{C}^{2n} \simeq \mathbb{R}^{4n}$	$n \geqslant 2$ $n \geqslant 2$
$\mathbb{R}^* \cdot \mathrm{SO}(p, q)$ $\mathrm{T}^*_{\mathbb{C}} \cdot \mathrm{SO}(n, \mathbb{C})$	$\mathbb{R}^{p+q}$ $\mathbb{C}^n \simeq \mathbb{R}^{2n}$	$p+q \geqslant 3$ $n \geqslant 3$
$\mathrm{T}_{\mathbb{R}} \cdot \mathrm{SL}(m, \mathbb{R}) \cdot \mathrm{SL}(n, \mathbb{R})$ $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SL}(m, \mathbb{C}) \cdot \mathrm{SL}(n, \mathbb{C})$ $\mathrm{T}_{\mathbb{R}} \cdot \mathrm{SL}(m, \mathbb{H}) \cdot \mathrm{SL}(n, \mathbb{H})$ $\mathrm{SU}(2) \cdot \mathrm{SO}(n, \mathbb{H})$	$\mathbb{R}^{mn}$ $\mathbb{C}^m \otimes \mathbb{C}^n \simeq \mathbb{R}^{2mn}$ $\mathbb{R}^{16mn}$ $\mathbb{R}^2 \otimes \mathbb{R}^{4n} \simeq \mathbb{R}^{8n}$	$m > n \geqslant 2$ or $m \geqslant n > 2$ $m > n \geqslant 2$ or $m \geqslant n > 2$ $m > n \geqslant 1$ or $m \geqslant n > 1$ $n \geqslant 2$
Notations: $\mathrm{T}_{\mathbb{F}}$ denotes any connected Lie subgroup of $\mathbb{F}^*$, $\mathrm{T}^*_{\mathbb{F}}$ denotes any non-trivial connected Lie subgroup of $\mathbb{F}^*$, $M_n(\mathbb{F})$ denotes the algebra of $n \times n$ matrices with entries in $\mathbb{F}$.		

Table 1: List of non-metric holonomies (Part I)

List of exotic holonomies		
group G	representation V	restrictions
$T_{\mathbb{R}} \cdot \mathrm{Spin}(5,5)$ $T_{\mathbb{R}} \cdot \mathrm{Spin}(1,9)$ $T_{\mathbb{C}} \cdot \mathrm{Spin}(10,\mathbb{C})$	$\mathbb{R}^{16}$ $\mathbb{R}^{16}$ $\mathbb{C}^{16} \simeq \mathbb{R}^{32}$	
$T_{\mathbb{R}} \cdot E_6^1$ $T_{\mathbb{R}} \cdot E_6^4$ $T_{\mathbb{C}} \cdot E_6^{\mathbb{C}}$	$\mathbb{R}^{27}$ $\mathbb{R}^{27}$ $\mathbb{C}^{27} \simeq \mathbb{R}^{54}$	
$T_{\mathbb{R}} \cdot SL(2,\mathbb{R})$ $T_{\mathbb{C}} \cdot SL(2,\mathbb{C})$ $\mathbb{R}^* \cdot SO(2) \cdot SL(2,\mathbb{R})$ $\mathbb{C}^* \cdot SU(2)$	$\odot^3\mathbb{R}^2 \simeq \mathbb{R}^4$ $\odot^3\mathbb{C}^2 \simeq \mathbb{R}^8$ $\mathbb{R}^2 \otimes \mathbb{R}^2 \simeq \mathbb{R}^4$ $\mathbb{C}^2 \simeq \mathbb{R}^4$	
$SL(2,\mathbb{R}) \cdot SO(p,q)$ $SL(2,\mathbb{C}) \cdot SO(n,\mathbb{C})$	$\mathbb{R}^2 \otimes \mathbb{R}^{p+q} \simeq \mathbb{R}^{2p+2q}$ $\mathbb{C}^2 \otimes \mathbb{C}^n \simeq \mathbb{R}^{4n}$	$p+q>2$ $n \geqslant 3$
E_7^5 E_7^7 $E_7^{\mathbb{C}}$	$\mathbb{R}^{56}$ $\mathbb{R}^{56}$ $\mathbb{R}^{112} \simeq \mathbb{C}^{56}$	
$Sp(3,\mathbb{R})$ $Sp(3,\mathbb{C})$	$\mathbb{R}^{14} \subset \Lambda^3\mathbb{R}^6$ $\mathbb{R}^{28} \simeq \mathbb{C}^{14} \subset \Lambda^3\mathbb{C}^6$	
$SL(6,\mathbb{R})$ $SL(6,\mathbb{C})$	$\mathbb{R}^{20} \simeq \Lambda^3\mathbb{R}^6$ $\mathbb{R}^{40} \simeq \Lambda^3\mathbb{C}^6$	
$\mathrm{Spin}(2,10)$ $\mathrm{Spin}(6,6)$ $\mathrm{Spin}(12,\mathbb{C})$	$\mathbb{R}^{32}$ $\mathbb{R}^{32}$ $\mathbb{R}^{64} \simeq \mathbb{C}^{32}$	
Notations: $T_{\mathbb{F}}$ denotes any connected Lie subgroup of $\mathbb{F}^*$		

Table 2: List of non-metric holonomies (Part II)

Berger's second list of non-metric holonomies, refined and extended, is given in Tables 1 and 2. The 4-dimensional representations of $T_{\mathbb{R}} \cdot SL(2,\mathbb{R})$, $T_{\mathbb{C}} \cdot SL(2,\mathbb{C})$, $\mathbb{R}^* \cdot SO(2) \cdot SL(2,\mathbb{R})$ and $\mathbb{C}^* \cdot SU(2)$, and the fundamental representations of various real forms of $T_{\mathbb{C}} \cdot Spin(10,\mathbb{C})$ and $T_{\mathbb{C}} \cdot E_6^{\mathbb{C}}$ have been added to the list of non-metric holonomies by Bryant [13, 14]. The series $SL(2,\mathbb{R}) \cdot SO(p,q)$ and $SL(2,\mathbb{C}) \cdot SO(n,\mathbb{C})$ have been found by Chi et al [18]. Recently [30] it has been shown that representations of E_7^5, E_7^7, $E_7^{\mathbb{C}}$, $Sp(3,\mathbb{R})$, $Sp(3,\mathbb{C})$, $SL(6,\mathbb{R})$, $SL(6,\mathbb{C})$, $Spin(2,10)$, $Spin(6,6)$ and $Spin(12,\mathbb{C})$ listed in Table 2 also occur as holonomies of torsion-free affine connections.

In summary, due to [5, 13, 14, 18, 30, 33] all entries of Tables 1 and 2 are known to occur as holonomies. The completeness status of these tables is not clear at present. However, we can make a definite statement about a part of modified Berger's non-metric list.

Theorem A *Let G be the irreducible holonomy of a torsion-free affine connection which is not locally symmetric and does not preserve any (pseudo-)Riemannian metric. If the semisimple part of G is not simple, then G is one of the groups listed in the following table*

LIST OF NON-METRIC SEGRE HOLONOMIES		
group G	representation V	restrictions
$T_{\mathbb{R}} \cdot SL(m,\mathbb{R}) \cdot SL(n,\mathbb{R})$	$\mathbb{R}^{mn}$	$m > n \geqslant 2$ or $m \geqslant n > 2$
$T_{\mathbb{C}} \cdot SL(m,\mathbb{C}) \cdot SL(n,\mathbb{C})$	$\mathbb{C}^m \otimes \mathbb{C}^n \simeq \mathbb{R}^{2mn}$	$m > n \geqslant 2$ or $m \geqslant n > 2$
$T_{\mathbb{R}} \cdot SL(m,\mathbb{H}) \cdot SL(n,\mathbb{H})$	$\mathbb{R}^{16mn}$	$m > n \geqslant 1$ or $m \geqslant n > 1$
$\mathbb{R}^* \cdot SO(p,q)$	$\mathbb{R}^4$	$p = q = 2$ or $p = 4, q = 0$
$T_{\mathbb{C}}^* \cdot SL(2,\mathbb{C}) \cdot SL(2,\mathbb{C})$	$\mathbb{R}^8 \simeq \mathbb{C}^4$	
$SL(2,\mathbb{R}) \cdot SO(p,q)$	$\mathbb{R}^2 \otimes \mathbb{R}^{p+q} \simeq \mathbb{R}^{2p+2q}$	$p + q > 2$
$SU(2) \cdot SO(n,\mathbb{H})$	$\mathbb{R}^2 \otimes \mathbb{R}^{4n} \simeq \mathbb{R}^{8n}$	$n \geqslant 2$
$SL(2,\mathbb{C}) \cdot SO(n,\mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^n \simeq \mathbb{R}^{4n}$	$n \geqslant 3$
NOTATIONS: $T_{\mathbb{F}}$ denotes any connected Lie subgroup of $\mathbb{F}^*$, $T_{\mathbb{F}}^*$ denotes any non-trivial connected Lie subgroup of $\mathbb{F}^*$.		

The class of holonomy groups (and the associated geometric structures) studied by Theorem A appear in the literature under different names. For example, the authors of [2, 26] call these *almost Grassmanian*, the authors of [5] call these *paraconformal*. In this paper we follow the terminology of Bryant [14] who suggested to call them *Segre* holonomies. Some applications of Segre structures to high energy physics are discussed in [25, 27].

The second classification result of this paper has, at first sight, nothing to do with the holonomy problem.

Theorem B *Let X be a compact complex homogeneous-rational manifold and L an ample line bundle on X. Then*

$$\text{(i) } \mathrm{H}^0(X, TX \otimes L^*) = \begin{cases} \mathbb{C} & \text{for } (X, L) = (\mathbb{CP}_1, \mathcal{O}(2)) \\ \mathbb{C}^n & \text{for } (X, L) = (\mathbb{CP}_n, \mathcal{O}(1)),\ n \geqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

(ii) $\mathrm{H}^1(X, TX \otimes L^*) = 0$ *unless (X, L) is one of the entries in Table 3.*

One may compare this result with the vanishing theorem of Kobayashi and Ochiai [24] which says that if X is a compact complex rational manifold and $L \to X$ a line bundle such that $\det(TX) \otimes L^*$ is ample, then $H^i(X, TX \otimes L^*) = 0$ for all $i \geqslant 2$.

The paper is organized as follows. After recalling a few basic facts about holonomy groups in the beginning of section 2, we show in sections 2 and 3 that Theorem B implies Theorem A (more precisely, it is the part of Table 3 classifying all (X, L) with $H^1(X, TX \otimes L^{2*}) \neq 0$ which implies Theorem A). As a by-product, we get a very simple group-theoretic explanation of the effectiveness of twistor methods in differential geometry. In the second half of the paper (section 4) we prove Theorem B.

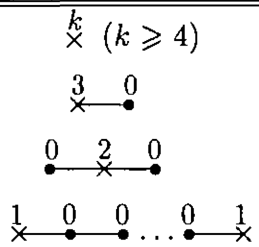
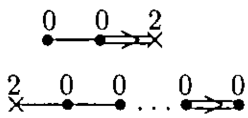
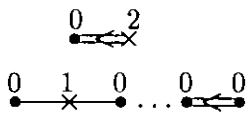
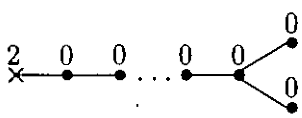
2 Holonomy groups in the Borel-Weil context

1. Definition of holonomy groups. Consider the following data:

- Let M be a smooth connected and simply connected n-manifold.
- Fix a point $x \in M$.
- Let $\mathcal{L}_x = \{\gamma : [0,1] \to M \mid \gamma(0) = \gamma(1)\}$ be the set of piecewise smooth loops based at x.
- Let ∇ be an affine connection on M.
- For $\gamma \in \mathcal{L}_x$, let $P_\gamma : T_xM \longrightarrow T_xM$ be a linear automorphism induced by the ∇-parallel translations along γ.

The *holonomy of* ∇ *at* $x \in M$ is defined as a subset $H_x := \{P_\gamma \mid \gamma \in \mathcal{L}_x\} \subseteq \mathrm{GL}(T_xM)$. Its basic properties are: (i) H_x is a connected Lie subgroup of $\mathrm{GL}(T_xM)$; (ii) if one fixes an isomorphism $i : T_xM \simeq V$, where V is any fixed vector space with $\dim V = \dim M$ (typically, $V = \mathbb{R}^n$), then the conjugacy class of $i(H_x) \subset \mathrm{GL}(V)$ does not depend on the choice of $x \in M$ (see, e.g., [9]). The *holonomy group* of ∇ is defined as any linear subgroup $G \subseteq \mathrm{GL}(V)$ in the conjugacy class of $i(H_x)$ for some $x \in M$. The Lie algebra $\mathfrak{g} \subseteq gl(V)$ of G is called the *holonomy algebra* of ∇.

Let V be a vector space and $\mathfrak{g} \subseteq gl(V)$ a Lie subalgebra. What is a necessary condition for $\mathfrak{g}$ to be the holonomy algebra of a torsion-free affine connection? One of the answers is that at least one of the Spencer $\mathfrak{g}$-modules, $\mathfrak{g}^{(1)}$ or $\mathrm{H}^{1,2}(\mathfrak{g})$, must be non-zero. We shall recall their definition in the next subsection.

$\mathrm{Aut}^0 X$	(X, L)	$\mathrm{H}^1(X, TX \otimes L^*)$
$A_l \quad (l \geqslant 1)$		$\odot^{k-4}\mathbb{C}^2$ $\mathbb{C}$ $\mathbb{C}$ $\mathbb{C}$
$B_l \quad (l \geqslant 3)$		$\mathbb{C}$ $\mathbb{C}$
$C_l \ (l \geqslant 2)$		$\mathbb{C}$ $\mathbb{C}$
$D_l \quad (l \geqslant 4)$		$\mathbb{C}$
F_4		$\mathbb{C}$
$A_1 \times A_1$	 $(k \geqslant 2)$	$\odot^{k-2}\mathbb{C}^2$
$A_1 \times A_l \ (l \geqslant 1)$	 $(k \geqslant 2)$	$\odot^{k-2}\mathbb{C}^2 \otimes \mathbb{C}^{l+1}$

NOTATION: $\mathrm{Aut}^0 X$ denotes the universal covering of the component the identity of the group of all automorphisms of X

Table 3: The list of all (X, L) with $\mathrm{H}^1(X, TX \otimes L^*) \neq 0$

2. Spencer cohomology. Let V be a vector space and $\mathfrak{g}$ a Lie subalgebra of $gl(V) \simeq V \otimes V^*$. Define recursively the $\mathfrak{g}$-modules

$$\begin{aligned} \mathfrak{g}^{(-1)} &= V \\ \mathfrak{g}^{(0)} &= \mathfrak{g} \\ \mathfrak{g}^{(k)} &= [\mathfrak{g}^{(k-1)} \otimes V^*] \cap [V \otimes \odot^{k+1} V^*] \ , \ k = 1, 2, \ldots, \end{aligned}$$

and define the map

$$\partial : \mathfrak{g}^{(k)} \otimes \Lambda^{l-1} V^* \longrightarrow \mathfrak{g}^{(k-1)} \otimes \Lambda^l V^*$$

as the antisymmetrisation over the last l indices. Here and elsewhere the symbols $\odot^k$ and Λ^k stand for k-th order symmetric and antisymmetric powers respectively.

Since $\partial^2 = 0$, there is a complex

$$\mathfrak{g}^{(k)} \otimes \Lambda^{l-1} V^* \xrightarrow{\partial} \mathfrak{g}^{(k-1)} \otimes \Lambda^l V^* \xrightarrow{\partial} \mathfrak{g}^{(k-2)} \otimes \Lambda^{l+1} V^*$$

whose cohomology at the center term is denoted by $\mathrm{H}^{k,l}(\mathfrak{g})$ and is called the (k, l) *Spencer cohomology group*. In particular,

$$\begin{aligned} \mathrm{H}^{k,1}(\mathfrak{g}) &= 0 \\ \mathrm{H}^{k,2}(\mathfrak{g}) &= \frac{\mathrm{Ker} : \mathfrak{g}^{(k-1)} \otimes \Lambda^2 V^* \xrightarrow{\partial} \mathfrak{g}^{(k-2)} \otimes \Lambda^3 V^*}{\mathrm{Image} : \mathfrak{g}^{(k)} \otimes V^* \xrightarrow{\partial} \mathfrak{g}^{(k-1)} \otimes \Lambda^2 V^*} . \end{aligned} \quad (1)$$

The $\mathfrak{g}$-module $\mathrm{H}^{k,2}(\mathfrak{g})$ has the following geometric meaning: if G is a matrix Lie group whose Lie algebra is $\mathfrak{g}$ and $\mathcal{G} \to M$ is a G-structure on a manifold M which is infinitesimally flat to k-th order, then the obstruction for $\mathcal{G}$ to be infinitesimally flat to $(k+1)$-th order is given by a section of the associated vector bundle $\mathcal{G} \times_G \mathrm{H}^{k,2}(\mathfrak{g})$.

Another $\mathfrak{g}$-module $\mathfrak{g}^{(1)}$ has a clear geometric interpretation as well. If a G-structure $\mathcal{G} \to M$ is infinitesimally flat to 1st order (which is equivalent to saying that $\mathcal{G}$ admits a *torsion-free* affine connection), then the set of all torsion-free affine connections in $\mathcal{G}$ is an affine space modelled on the vector space $\mathrm{H}^0(M, \mathcal{G} \times_G \mathfrak{g}^{(1)})$. In particular, if $G \subseteq \mathrm{GL}(V)$ is such that $\mathfrak{g}^{(1)} = 0$, then any G-structure admits at most one torsion-free affine connection. If $K(\mathfrak{g})$ denotes the $\mathfrak{g}$-module of formal curvature tensors of torsion-free affine connections with holonomy in $\mathfrak{g}$, i.e.

$$K(\mathfrak{g}) = [\mathfrak{g} \otimes \Lambda^2 V^*] \cap [\mathrm{Ker} : V \otimes V^* \otimes \Lambda^2 V^* \to V \otimes \Lambda^3 V^*],$$

then

$$\mathrm{H}^{1,2}(\mathfrak{g}) = \frac{K(\mathfrak{g})}{\partial(\mathfrak{g}^{(1)} \otimes V^*)}$$

i.e. the cohomology group $\mathrm{H}^{1,2}(\mathfrak{g})$ represents the part of $K(\mathfrak{g})$ which is invariant under $\mathfrak{g}^{(1)}$-valued shifts in a formal torsion-free affine connection with holonomy in $\mathfrak{g}$. For example, if $(G, V) = (\mathrm{CO}(n, \mathbb{R}), \mathbb{R}^n)$, then $\mathfrak{g}^{(1)} = V^*$ and $\mathrm{H}^{1,2}(\mathfrak{g})$ is the vector space of formal Weyl tensors.

If $\mathfrak{g}^{(1)} = 0$, then $\mathrm{H}^{1,2}(\mathfrak{g})$ is exactly $K(\mathfrak{g})$, the $\mathfrak{g}$-module which plays a key role in the theory of torsion-free affine connections with holonomy in $\mathfrak{g}$. The case $\mathfrak{g}^{(1)} = 0$ is generic — there are very few irreducibly acting Lie subgroups $\mathfrak{g} \subset gl(V)$ which have $\mathfrak{g}^{(1)} \neq 0$. For future reference we list in Table 4 all complex irreducible Lie subgroups $G \subset \mathrm{GL}(V)$ with

The list of all irreducible complex Lie subgroups $G \subseteq \mathrm{GL}(V, \mathbb{C})$ with $\mathfrak{g}^{(1)} \neq 0$		
group G	representation V	$\mathfrak{g}^{(1)}$
$\mathrm{SL}(n, \mathbb{C})$	$V = \mathbb{C}^n$, $n \geqslant 2$	$(V \otimes \odot^2 V^*)_0$
$\mathrm{GL}(n, \mathbb{C})$	$V = \mathbb{C}^n$, $n \geqslant 1$	$V \otimes \odot^2 V^*$
$\mathrm{GL}(n, \mathbb{C})$	$V \simeq \odot^2 \mathbb{C}^n$, $n \geqslant 2$	V^*
$\mathrm{GL}(n, \mathbb{C})$	$V \simeq \Lambda^2 \mathbb{C}^n$, $n \geqslant 5$	V^*
$\mathrm{GL}(m, \mathbb{C}) \cdot \mathrm{GL}(n, \mathbb{C})$	$V \simeq \mathbb{C}^m \otimes \mathbb{C}^n$, $m, n \geqslant 2$	V^*
$\mathrm{Sp}(n, \mathbb{C})$	$V = \mathbb{C}^n$, $n \geqslant 4$	$\odot^3 V^*$
$\mathbb{C}^* \cdot \mathrm{Sp}(n, \mathbb{C})$	$V = \mathbb{C}^n$, $n \geqslant 4$	$\odot^3 V^*$
$\mathrm{CO}(n, \mathbb{C})$	$V = \mathbb{C}^n$, $n \geqslant 5$	V^*
$\mathbb{C}^* \cdot \mathrm{Spin}(10, \mathbb{C})$	$V = \mathbb{C}^{16}$	V^*
$\mathbb{C}^* \cdot \mathrm{E}_6^{\mathbb{C}}$	$V = \mathbb{C}^{27}$	V^*

Table 4: Classification list of Cartan (1909) and Kobayashi & Nagano (1965)

$\mathfrak{g}^{(1)} \neq 0$ which is due to Cartan [15] and Kobayashi & Nagano [23]. As of this writing, the list of all irreducibly acting $\mathfrak{g} \subseteq gl(V)$ which have $\mathrm{H}^{1,2}(\mathfrak{g}) \neq 0$ is not known — otherwise the holonomy classification problem would be solved long ago.

Another $\mathfrak{g}$-module, $K^1(\mathfrak{g})$, which is of interest in the holonomy context can be defined as the kernel of the composition

$$K(\mathfrak{g}) \otimes V^* \longrightarrow \mathfrak{g} \otimes \Lambda^2 V^* \otimes V^* \longrightarrow \mathfrak{g} \otimes \Lambda^3 V^*,$$

where the first map is the natural inclusion and the second map is the antisymmetrization on the last three indices. If there exist a torsion-free affine connection ∇ on a smooth manifold M with the holonomy algebra in $\mathfrak{g}$, then the curvature tensor R of ∇ can be represented locally as a function on M with values in $K(\mathfrak{g})$, while the covariant derivative ∇R can be represented locally as a function on M with values in $K^1(\mathfrak{g})$. Therefore, $K^1(\mathfrak{g}) \neq 0$ (in particular, $K(\mathfrak{g}) \neq 0$) is one of the necessary conditions for ∇ to be the holonomy of a torsion-free affine connection which is not locally symmetric. Note that $\mathfrak{g}' \subseteq \mathfrak{g}$ implies $K(\mathfrak{g}') \subseteq K(\mathfrak{g})$ and $K^1(\mathfrak{g}') \subseteq K^1(\mathfrak{g})$.

It is well-known [22] that the classification of real irreducible representations of real reductive Lie algebras can be accomplished via the classification of complex irreducible representations of complex reductive Lie algebras. The problem of classifying real reductive holonomies can, in principle, be handled in a similar way (see subsection 5 in §3), with the first and the most important step being the classification of all possible candidates to *complex* reductive holonomies. With this motivation, we restrict our attention in the rest of Section 2 and in the most of Section 3 to complex irreducible representations of complex reductive Lie groups G and their Lie algebras $\mathfrak{g}$.

Unless otherwise explicitly stated, $\mathrm{T}_{\mathbb{C}}$ denotes in what follows either a trivial group or the multiplicatitive group $\mathbb{C}^*$ and $t_{\mathbb{C}}$ denotes the Lie algebra of $\mathrm{T}_{\mathbb{C}}$.

3. Twistor formulae for Spencer cohomology. Let V be a finite dimensional complex vector space and $G \subseteq \mathrm{GL}(V)$ an irreducible representation of a reductive complex Lie group in V. Then G also acts irreducibly in V^* via the dual representation. Let $\tilde{X}$ be the G-orbit of a highest weight vector in $V^* \setminus 0$. Then the quotient $X := \tilde{X}/\mathbb{C}^*$ is a compact complex homogeneous-rational manifold canonically embedded into $\mathbb{P}(V^*)$, and there is a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \hookrightarrow & V^* \setminus 0 \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \mathbb{P}(V^*) \end{array}$$

In fact, $X = G_s/P$, where G_s is the semisimple part of G and P is the parabolic subgroup of G_s leaving a highest weight vector in V^* invariant up to a scale. Let L be the restriction of the hyperplane section bundle $\mathcal{O}(1)$ on $\mathbb{P}(V^*)$ to the submanifold X. Clearly, L is an ample homogeneous line bundle on X.

In summary, there is a natural map

$$(G, V) \longrightarrow (X, L)$$

which associates with an irreducibly acting reductive Lie group $G \subseteq \mathrm{GL}(V)$ a pair (X, L) consisting of a compact complex homogeneous-rational manifold X and an ample line bundle L on X. We call (X, L) the *Borel-Weil data* associated with (G, V).

Can this map be reversed? According to Borel-Weil, the representation space V can be reconstructed very easily:

$$V = \mathrm{H}^0(X, L).$$

What about G? According to Onishchik, with a few (but notable) exceptions, G can be reconstructed as well.

Fact 2.1 [1] *Assume that G is simple. The Lie algebra of G is isomorphic to $\mathrm{H}^0(X, TX)$ unless one of the following holds:*

(i) G is the representation of $\mathrm{Sp}(n, \mathbb{C})$ in $\mathbb{C}^{2n}$ in which case $\mathrm{H}^0(X, TX) \simeq sl(n, \mathbb{C})$;

(ii) G is the representation of G_2 in $\mathbb{C}^7$ in which case $\mathrm{H}^0(X, TX) \simeq so(7, \mathbb{C})$;

(iii) G is the fundamental spinor representation of $\mathrm{SO}(2n+1, \mathbb{C})$ in which case $\mathrm{H}^0(X, TX) \simeq so(2n+2, \mathbb{C})$.

Another proof of this fact is given in [35].

Therefore, if $G \subseteq \mathrm{GL}(V)$ is semisimple then, with a few exceptions, G can be reconstructed from (X, L). However, it is often undesirable to restrict oneself to semisimple groups only (especially in the context of the holonomy classification problem). There is a natural central extension of the Lie algebra $\mathrm{H}^0(X, TX)$:

Fact 2.2 *For any (X, L), $\mathfrak{g} := \mathrm{H}^0(X, L \otimes (J^1 L)^*)$ is a reductive Lie algebra canonically represented in $\mathrm{H}^0(X, L)$.*

This fact is easy to explain — $\mathrm{H}^0(X, L \otimes (J^1 L)^*)$ is exactly the Lie algebra of the Lie group G of all global biholomorphisms of the line bundle L which commute with the projection $L \rightarrow X$.

In summary, with a given irreducible representation $G \subseteq \mathrm{GL}(V)$ there is canonically associated a pair (X, L) consisting of a compact complex homogenous-rational manifold X and a very ample line bundle on X such that much of the original information about G can be restored from (X, L). For our purposes the crucial observation is that the $\mathfrak{g}$-modules $\mathfrak{g}^{(k)}$ and $\mathrm{H}^{k,2}(\mathfrak{g})$ also admit a simple description in terms of (X, L).

Theorem 2.3 *For a compact complex manifold X and a very ample line bundle L on X, there is an isomorphism*

$$\mathfrak{g}^{(k)} = \mathrm{H}^0\left(X, L \otimes \odot^{k+1} N^*\right), \quad k = 0, 1, 2, \ldots$$

and an exact sequence of $\mathfrak{g}$-modules,

$$0 \longrightarrow \mathrm{H}^{k,2}(\mathfrak{g}) \longrightarrow \mathrm{H}^1\left(X, L \otimes \odot^{k+2} N^*\right) \longrightarrow \mathrm{H}^1\left(X, L \otimes \odot^{k+1} N^*\right) \otimes V^*, \quad k = 1, 2, \ldots$$

where $\mathfrak{g} := \mathrm{H}^0(X, L \otimes N^*)$, $N := J^1 L$, and $\mathrm{H}^{k,2}(\mathfrak{g})$ are the Spencer cohomology groups associated with the canonical representation of $\mathfrak{g}$ in the vector space $V := \mathrm{H}^0(X, L)$.

Proof. Since L is very ample, there is a natural "evaluation" epimorhism

$$V \otimes \mathcal{O}_X \to J^1 L \to 0$$

whose dualization gives rise to the canonical monomorphism $0 \to N^* \to V^* \otimes \mathcal{O}_X$. Then one may construct the following sequences of locally free sheaves,

$$0 \longrightarrow L \otimes \odot^{k+1} N^* \longrightarrow L \otimes \odot^k N^* \otimes V^* \longrightarrow L \otimes \odot^{k-1} N^* \otimes \Lambda^2 V^* \tag{2}$$

and

$$0 \longrightarrow L \otimes \odot^{k+2} N^* \longrightarrow L \otimes \odot^{k+1} N^* \otimes V^* \longrightarrow L \otimes \odot^k (N^*) \otimes \Lambda^2 V^* \longrightarrow L \otimes \odot^{k-1} N^* \otimes \Lambda^3 V^*, \tag{3}$$

and notice that they both are exact. [Hint: for any vector space W one has $W \otimes \Lambda^2 W$ mod $\Lambda^3 W \simeq W \otimes \odot^2 W$ mod $\odot^3 W$.]

Then computing $\mathrm{H}^0(X, \ldots)$ of (2) and using the inductive definition of $\mathfrak{g}^{(k)}$ one easily obtains the first statement of the Theorem.

The second statement follows from (3) and the definition (1) of $\mathrm{H}^{k,2}(\mathfrak{g})$. Indeed, define E_k by the exact sequence

$$0 \longrightarrow L \otimes \odot^{k+2} N^* \longrightarrow L \otimes \odot^{k+1} N^* \otimes V^* \longrightarrow E_k \longrightarrow 0.$$

The associated long exact sequence implies the following *exact* sequence of vector spaces

$$0 \longrightarrow \mathrm{H}^0(X, E_k)/\partial[\mathfrak{g}^{(k)} \otimes V^*] \longrightarrow \mathrm{H}^1\left(X, L \otimes \odot^{k+2} N^*\right) \longrightarrow \mathrm{H}^1\left(X, L \otimes \odot^{k+1} N^*\right) \otimes V^*.$$

On the other hand, the exact sequence

$$0 \longrightarrow E_k \longrightarrow L \otimes \odot^k N^* \otimes \Lambda^2 V^* \longrightarrow L \otimes \odot^{k-1} N^* \otimes \Lambda^3 V^*$$

implies

$$\mathrm{H}^0(X, E_k) = \ker : \mathfrak{g}^{(k-1)} \otimes \Lambda^2 V^* \xrightarrow{\partial} \mathfrak{g}^{(k-2)} \otimes \Lambda^3 V^*.$$

which in turn implies

$$\mathrm{H}^{k,2}(\mathfrak{g}) = \mathrm{H}^0(X, E_k)/\partial[\mathfrak{g}^{(k)} \otimes V^*].$$

This completes the proof of the second part of the Theorem. □

In 1976 Penrose [32] considered the data $(X \hookrightarrow Z, N)$ consisting of a rational curve $X = \mathbb{CP}^1$ embedded into a complex 3-fold Z with normal bundle $N = \mathcal{O}(1) \oplus \mathcal{O}(1)$; and showed that the Kodaira moduli space of all rational curves obtained by holomorphic deformations of X inside Z is a complex 4-dimensional manifold M which comes equipped with a canonically induced self-dual conformal structure. Moreover, he showed that any local conformal self-dual structure arises in this way. Since this pioneering work, several other manifestations of this strange phenomenon have been observed when a complex analytic data of the form $(X \hookrightarrow Z, N)$ gives rise to a full category of local geometric structures C_{geo}. More precisely, it is asuccessful choice of a pair (X, N) consisting of a complex homogeneous manifold X and a homogeneous vector bundle N on X which uniquely specifies C_{geo}, the choice of a particular ambient manifold Z corresponding to the choice of a particular object in C_{geo}.

The fact that, according to Theorem 2.3, the spaces of formal curvature tensors fit nicely into the Borel-Weil paradigm gives a simple group-theoretic explanation of why a twistorial data (X, N) can, in principle, be used as a building block for basic differential-geometric objects. If $\mathrm{rank} N \geqslant 2$, then, following a common practice in complex analysis, one should replace the pair (X, N) by an equivalent one $(\hat{X} = \mathbb{P}(N^*), L = \mathcal{O}(1))$ and then apply Theorem 2.3 to find out which geometric category C_{geo} may correspond to (X, N). Applying this procedure, e.g., to the pair $(\mathbb{CP}_1, \mathbb{C}^{2k} \otimes \mathcal{O}(1))$, $k > 1$, one immediately concludes that C_{geo} can only be the category of complexified quaternionic manifolds.

Also, this purely group-theoretic result suggests that there should exist a universal twistor construction for *all* torsion-free geometries. Details of this construction are given in [28, 29].

3 Classification of Segre holonomies

1. Cohomology on reducible rational homogeneous manifolds. From now on we assume that $\mathbb{X} = X_1 \times X_2$ is a direct product of two compact complex homogeneous-rational manifolds X_1 and X_2 and that $\mathbb{L}$ is an ample holomorphic line bundle on X. Denoting by $\pi_1 : \mathbb{X} \to X_1$ and $\pi_2 : \mathbb{X} \to X_2$ the natural projections, we may write $\mathbb{L} = \pi_1^*(L_1) \otimes \pi_2^*(L_2)$ for some uniquely specified ample line bundles L_1 and L_2 on X_1 and X_2 respectively. We denote $\mathbb{N} := J^1\mathbb{L}$ and $N_i := J^1 L_i$, $i = 1, 2$.

Since

$$0 \longrightarrow \Omega X_i \otimes L_i \longrightarrow N_i \longrightarrow L_i \longrightarrow 0,$$

one has

$$0 \longrightarrow \pi_1^*(\Omega^1 X_1) \otimes \mathbb{L} + \pi_2^*(\Omega^1 X_2) \otimes \mathbb{L} \longrightarrow \pi_1^*(N_1) \otimes \pi_2^*(L_2) + \pi_2^*(N_2) \otimes \pi_1^*(L_1) \longrightarrow \mathbb{L} + \mathbb{L} \longrightarrow 0.$$

The latter extension combined with

$$0 \longrightarrow \pi_1^*(\Omega^1 X_1) \otimes \mathbb{L} + \pi_2^*(\Omega^1 X_2) \otimes \mathbb{L} \longrightarrow \mathbb{N} \longrightarrow \mathbb{L} \longrightarrow 0.$$

implies

$$0 \longrightarrow \mathbb{N} \longrightarrow \pi_1^*(N_1) \otimes \pi_2^*(L_2) + \pi_1^*(L_1) \otimes \pi_2^*(N_2) \longrightarrow \mathbb{L} \longrightarrow 0,$$

or

$$0 \longrightarrow \mathbb{L}^* \longrightarrow \pi_1^*(N_1^*) \otimes \pi_2^*(L_2^*) + \pi_1^*(N_1^*) \otimes \pi_2^*(L_2^*) \longrightarrow \mathbb{N}^* \longrightarrow 0, \tag{4}$$

which in turn implies the following two exact sequences

$$0 \longrightarrow \begin{array}{c} \pi_1^*(N_1^*) \otimes \pi_2^*(L_2^*) \\ + \\ \pi_1^*(L_1^*) \otimes \pi_2^*(N_2^*) \end{array} \longrightarrow \begin{array}{c} \pi_1^*(L_1 \otimes \odot^2 N_1^*) \otimes \pi_2^*(L_2^*) \\ + \\ \pi_1^*(N_1^*) \otimes \pi_2^*(N_2^*) \\ + \\ \pi_1^*(L_1^*) \otimes \pi_2^*(L_2 \otimes \odot^2 N_2^*) \end{array} \longrightarrow \mathbb{L} \otimes \odot^2 \mathbb{N}^* \longrightarrow 0 \tag{5}$$

and

$$0 \longrightarrow \begin{array}{c} \pi_1^*(\odot^2 N_1^*) \otimes \pi_2^*(L_2^*)^2 \\ + \\ \pi_1^*(L_1 \otimes N_1^*) \otimes \pi_2^*(L_2^* \otimes N_2^*) \\ + \\ \pi_1^*(L_1^*)^2 \otimes \pi_2^*(\odot^2 N_2^*) \end{array} \longrightarrow \begin{array}{c} \pi_1^*(L_1 \otimes \odot^3 N_1^*) \otimes \pi_2^*(L_2^*)^2 \\ + \\ \pi_1^*(\odot^2 N_1^*) \otimes \pi_2^*(L_2^* \otimes N_2^*) \\ + \\ \pi_1^*(L_1^* \otimes N_1^*) \otimes \pi_2^*(\odot N_2^*) \\ + \\ \pi_1^*(L_1^*)^2 \otimes \pi_2^*(L_2 \otimes \odot^3 N_2^*) \end{array} \longrightarrow \mathbb{L} \otimes \odot^3 \mathbb{N}^* \longrightarrow 0. \tag{6}$$

Proposition 3.1 *Let X be a compact complex homogeneous-rational manifold and L an ample line bundle on X. Then*

$$\mathrm{H}^0(X, TX \otimes L^*) = \begin{cases} \mathbb{C} & \text{for } (X,L) = (\mathbb{CP}_1, \mathcal{O}(2)) \\ \mathbb{C}^{n+1} & \text{for } (X,L) = (\mathbb{CP}_n, \mathcal{O}(1)),\ n \geqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. If $\dim X = 1$, i.e. $X = \mathbb{CP}^1$, then the statement follows from the isomorphism $TX \simeq \mathcal{O}(2)$.

Asssume now that $\dim X \geqslant 2$. Then, by the Kodaira vanishing theorem, $\mathrm{H}^1(X, L^*) = 0$ for any ample line bundle L on X. Applying the Künneth formular to the long exact sequence of (5) with $\mathbb{X} = X \times X$ and $\mathbb{L} = \pi_1^*(L) \otimes \pi_2^*(L)$, one easily obtains

$$\mathrm{H}^0(\mathbb{X}, \mathbb{L} \otimes \odot^2 \mathbb{N}^*) = \mathrm{H}^0(X, N^*) \otimes \mathrm{H}^0(X, N^*) = \mathrm{H}^0(X, TX \otimes L^*) \otimes \mathrm{H}^0(X, TX \otimes L^*).$$

On the other hand, by Theorem 2.3,

$$\mathrm{H}^0(\mathbb{X}, \mathbb{L} \otimes \odot^2 \mathbb{N}^*) = \mathfrak{g}^{(1)},$$

where $\mathfrak{g}$ is the irreducible representation of

$$\mathrm{H}^0(\mathbb{X}, \mathbb{L} \otimes \mathbb{N}^*) \simeq \mathbb{C} \oplus \mathrm{H}^0(X, TX) \oplus \mathrm{H}^0(X, TX)$$

in $V \otimes V$ with $V = \mathrm{H}^0(X, L)$. Table 4 implies that such a $\mathfrak{g}^{(1)}$ can be non-zero if and only if $\mathrm{H}^0(X, TX) \simeq sl(n+1, \mathbb{C})$ irreducibly represented in $\mathbb{C}^{n+1}$, i.e. $X = \mathbb{CP}_n$. Then the isomorphism $\mathrm{H}^0(X, L) = \mathbb{C}^{n+1}$ implies $L = \mathcal{O}(1)$. Therefore, $\mathrm{H}^0(\mathbb{X}, \mathbb{L} \otimes \odot^2 \mathbb{N}^*)$ with $\mathbb{X} = X \times X$ and $\mathbb{L} = \pi_1^*(L) \otimes \pi_2^*(L)$ vanishes unless $(X, L) = (\mathbb{CP}^n, \mathcal{O}(1))$ which implies that $\mathrm{H}^0(X, TX \otimes L^*)$ vanishes unless $(X, L) = (\mathbb{CP}_n, \mathcal{O}(1))$. Finally, the extension

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathbb{C}^{n+1} \otimes \mathcal{O}_{\mathbb{CP}_n} \longrightarrow T\mathbb{CP}_n(-1) \longrightarrow 0$$

implies $\mathrm{H}^0(\mathbb{CP}_n, T\mathbb{CP}_n(-1)) = \mathbb{C}^{n+1}$ which completes the proof of Proposition 3.1 in the case $\dim X \geqslant 2$. □

Corollary 3.2 *Let X be a compact complex homogeneous-rational manifold, L an ample line bundle on X and $N = J^1L$. Then, for any $k \geqslant 1$,*

$$\mathrm{H}^0\left(X, \odot^k N^*\right) = \begin{cases} \odot^k \mathbb{C}^{n+1} & \text{for } (X,L) = (\mathbb{CP}^n, \mathcal{O}(1)),\ n \geqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The statement is true for $(X,L) = (\mathbb{CP}^n, \mathcal{O}(1))$ since $J^1\mathcal{O}(1) = \mathbb{C}^{n+1} \otimes \mathcal{O}_X$.

The case $k = 1$ of the required statement follows immediately from Propostion 3.1 and the extension

$$0 \longrightarrow L^* \longrightarrow N^* \longrightarrow TX \otimes L^* \longrightarrow 0 \tag{7}$$

The latter also implies

$$0 \longrightarrow \odot^k N^* \longrightarrow L \otimes \odot^{k+1} N^* \longrightarrow \odot^{k+1} TX \otimes L^{*k} \longrightarrow 0$$

which in turn implies

$$\mathrm{H}^0(X, \odot^k N^*) \subseteq \mathrm{H}^0(X, L \otimes \odot^{k+1} N^*).$$

According to Cartan [15] (see also [34] for another proof), the only irreducible complex Lie subalgebras $\mathfrak{g} \subseteq gl(V)$ which have $\mathfrak{g}^{(k)} \neq 0$ for $k \geqslant 3$ are $gl(m,\mathbb{C})$, $sl(m,\mathbb{C})$, $sp(m/2,\mathbb{C})$ and $sp(m/2,\mathbb{C}) \oplus \mathbb{C}$ standardly represented in $\mathbb{C}^m$, $m \geqslant 2$. The Borel-Weil data (X,L) associated with these four representations are $(\mathbb{CP}_{m-1}, \mathcal{O}(1))$. Therefore, if $(X,L) \neq (\mathbb{CP}_n, \mathcal{O}(1))$, then, by Theorem 2.3, $\mathrm{H}^0(X, L \otimes \odot^{k+1} N^*) = 0$ for all $k \geqslant 3$. Hence $\mathrm{H}^0(X, \odot^k N^*) = 0$ for all $k \geqslant 3$. This proves our Corollary for $k \geqslant 3$.

Asume now that $k = 2$ and denote $\tilde{L} := L^2$ and $\tilde{N} := J^1\tilde{L} \simeq L \otimes N$. Then

$$\mathrm{H}^0(X, \tilde{L} \otimes \odot^2 \tilde{N}^*) = \mathrm{H}^0(X, \odot^2 N^*).$$

Again, using Theorem 2.3 and Table 4 one concludes that the only irreducibly acting reductive Lie subalgebra $\mathfrak{g} \subset gl(V)$ which has $\mathfrak{g}^{(1)} \neq 0$ and whose associated pair $(X, \tilde{L})$ is such that $\tilde{L}$ is a *square* of an ample line bundle on X is $gl(n,\mathbb{C})$ represented in $\odot^2\mathbb{C}^n$ with the associated Borel-Weil data $(\mathbb{CP}_{n-1}, \mathcal{O}(2))$. Therefore, $\mathrm{H}^0(X, \odot^2 N^*) = 0$ for all $(X,L) \neq (\mathbb{CP}_n, \mathcal{O}(1))$. The proof is completed. □

2. The case $\mathbb{X} = X_1 \times X_2$ with $\dim X_i \geqslant 2$. The long exact sequence of (5) implies

$$\mathrm{H}^0\left(\mathbb{X}, \mathbb{L} \otimes \odot^2 \mathbb{N}^*\right) = \mathrm{H}^0(X_1, N_1^*) \otimes \mathrm{H}^0(X_2, N_2^*) \tag{8}$$

while the long exact sequence of (6) contains the following piece

$$0 \longrightarrow \begin{array}{c} \mathrm{H}^0\left(X_1, \odot^2 N_1^*\right) \otimes \mathrm{H}^1\left(X_2, TX_2 \otimes L_2^{*2}\right) \\ + \\ \mathrm{H}^0\left(X_2, \odot^2 N_2^*\right) \otimes \mathrm{H}^1\left(X_1, TX_1 \otimes L_1^{*2}\right) \end{array} \longrightarrow \mathrm{H}^1\left(\mathbb{X}, \mathbb{L} \otimes \odot^3 \mathbb{N}^*\right) \longrightarrow$$

$$\longrightarrow \mathrm{H}^1\left(X_1, TX_1 \otimes L_1^{*2}\right) \otimes \mathrm{H}^1\left(X_2, TX_2 \otimes L_2^{*2}\right) \longrightarrow \ldots . \tag{9}$$

Lemma 3.3 *Let $G_i \subseteq \mathrm{GL}(V_i)$, $i = 1, 2$, be an irreducible complex semisimple matrix Lie group such that the associated Borel-Weil data (X_i, L_i) satisfies $\dim X_i \geqslant 2$. Then $G = \mathrm{T}_{\mathbb{C}} \cdot G_1 \cdot G_2 \subset \mathrm{GL}(V_1 \otimes V_2)$ can have $K(\mathfrak{g}) \neq 0$ only if each G_i is isomorphic to one of the following representations*

Group:	SL$(n,\mathbb{C})$	Sp$(m,\mathbb{C})$	SO$(p,\mathbb{C})$	G_2	Spin$(7,\mathbb{C})$
Representation space:	$\mathbb{C}^n$	$\mathbb{C}^{2m}$	$\mathbb{C}^p$	$\mathbb{C}^7$	$\mathbb{C}^8$

where $n \geqslant 3$, $m \geqslant 2$ *and* $p \geqslant 4$.

Proof. $K(\mathfrak{g}) \neq 0$ if only if $\mathfrak{g}^{(1)} \neq 0$ and/or $\mathrm{H}^{1,2}(\mathfrak{g}) \neq 0$. By Theorem 2.3, Corollary 3.2 for $k=1$ and (8), one has

$$\mathfrak{g}^{(1)} = \mathrm{H}^0\left(\mathbb{X}, \mathbb{L} \otimes \odot^2 \mathbb{N}^*\right) = \begin{cases} \mathbb{C}^{n_1+1} \otimes \mathbb{C}^{n_2+1} & \text{for } (X_i, L_i) = (\mathbb{CP}_{n_i}, \mathcal{O}(1)),\ n_i \geqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, a glance at Table 3 shows that $\mathrm{H}^1(X, TX \otimes L^{*2}) \neq 0$ only if $(X, L) = (Q_n, j^*\mathcal{O}(1))$ where Q_n is the n-dimensional quadric and $j : Q_n \hookrightarrow \mathbb{CP}_{n+1}$ is its standard embedding. Then Theorem 2.3, Corollary 3.2 for $k=2$ and (9)imply that $\mathrm{H}^{1,2}(\mathfrak{g})$ can be non-zero only if each pair (X_i, L_i) is isomorphic either to $(\mathbb{CP}_n, \mathcal{O}(1))$ or to $(Q_n, j^*\mathcal{O}(1))$.

These observations combined with Fact 2.1 imply that $K(\mathfrak{g})$ can be non-zero only for representations listed in Lemma 3.3. □

EXAMPLE 1. Let G be the representation of $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SL}(m,\mathbb{C}) \cdot \mathrm{SL}(n,\mathbb{C})$ in the vector space $V = V_m \otimes V_n$ where V_m and V_n are m- and, respectively, n-dimensional complex vector spaces with $m, n \geqslant 3$. The associated Borel-Weil data $(\mathbb{X}, \mathbb{L})$ is $(\mathbb{CP}_{m-1} \times \mathbb{CP}_{n-1}, \pi_1^*(\mathcal{O}(1)) \otimes \pi_2^*(\mathcal{O}(1)))$ implying

$$\mathfrak{g}^{(1)} = \mathrm{H}^0(\mathbb{X}, \mathbb{L} \otimes \odot^2 \mathbb{N}^*) = V^*, \qquad \mathrm{H}^{1,2}(\mathfrak{g}) = \mathrm{H}^1(\mathbb{X}, \mathbb{L} \otimes \odot^3 \mathbb{N}^*) = 0.$$

Therefore, $K(\mathfrak{g}) = \partial(\mathfrak{g}^{(1)} \otimes V^*) \simeq V^* \otimes V^*$. Denoting typical elements of V, V_m and V_n by v^a, v^A and $v^{\dot{A}}$ respectively[2] and identifying $v^a \in V$ with its image $v^{A\dot{A}}$ under the isomorphism $V = V_m \otimes V_n$, one may write a typical element $R_{abc}{}^d \in K(\mathfrak{g}) \subset \Lambda^2 V^* \otimes V^* \otimes V$ as

$$R_{abc}{}^d \equiv R_{A\dot{A}B\dot{B}C\dot{C}}{}^{D\dot{D}} = \left[\delta_A^D Q_{B\dot{B}C\dot{A}} - \delta_B^D Q_{A\dot{A}C\dot{B}}\right] \delta_{\dot{C}}^{\dot{D}} + \left[\delta_{\dot{A}}^{\dot{D}} Q_{B\dot{B}A\dot{C}} - \delta_{\dot{B}}^{\dot{D}} Q_{A\dot{A}B\dot{C}}\right] \delta_C^D \quad (10)$$

for some $Q_{ab} \equiv Q_{A\dot{A}B\dot{B}} \in V^* \otimes V^*$. Therefore, a torsion-free connection ∇ on an mn-dimensional manifold M with holonomy in $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SL}(m,\mathbb{C}) \cdot \mathrm{SL}(n,\mathbb{C})$ has at an arbitrary point $x \in M$ the curvature tensor of the form (10) for some $Q_{ab}(x) \in \Omega_x M \otimes \Omega_x M$. It is not hard to show that the second Bianchi identities for ∇,

$$\nabla_e R_{abc}{}^d + \nabla_b R_{eac}{}^d + \nabla_a R_{bec}{}^d = 0,$$

imply

$$\begin{aligned} 0 \;=\; & m\left(\nabla_{A\dot{A}} Q_{B\dot{B}C\dot{C}} - \nabla_{B\dot{B}} Q_{A\dot{A}C\dot{C}}\right) + n\left(\nabla_{C\dot{C}} Q_{A\dot{A}B\dot{B}} - \nabla_{A\dot{A}} Q_{C\dot{C}B\dot{B}}\right) \\ & + \left(\nabla_{B\dot{B}} Q_{A\dot{C}C\dot{A}} - \nabla_{A\dot{C}} Q_{B\dot{B}C\dot{A}}\right) + \left(\nabla_{B\dot{A}} Q_{C\dot{C}A\dot{B}} - \nabla_{C\dot{C}} Q_{B\dot{A}A\dot{B}}\right). \end{aligned} \quad (11)$$

[2] One may view indices of the type a, A or $\dot{A}$ as refering to some fixed basis in a relevant vector space or, alternatively, as abstract labels providing us with a transparent notation for such basic tensor operations as (anti)symmetrization, contraction, etc. (cf. [6, 32]).

EXAMPLE 2. Keeping notations of the preceeding paragraph, we consider a subgroup $G_o \subset G$ which is $T_{\mathbb{C}} \cdot SL(m,\mathbb{C}) \cdot SO(n,\mathbb{C})$ represented in $V = V_m \otimes V_n$ with $m, n \geqslant 3$. The G_o-module $K(\mathfrak{g}_o)$ is a subset of $K(\mathfrak{g})$ consisting of all elements $R_{abc}{}^d$ satisfying

$$R_{A\dot{A}B\dot{B}C\dot{C}}{}^{D\dot{D}}\, g_{\dot{E}\dot{D}} + R_{A\dot{A}B\dot{B}C\dot{E}}{}^{D\dot{D}}\, g_{\dot{C}\dot{D}} = 0,$$

where $g_{\dot{E}\dot{D}} \in \odot^2 V_n^*$ is the $SO(n,\mathbb{C})$-invariant quadratic form. Substituting (10) into the above equation, one obtains after elementary algebraic manipulations that

$$Q_{A\dot{A}B\dot{B}} = P_{AB}\, g_{\dot{A}\dot{B}}$$

for some symmetric tensor $P_{AB} \in \odot^2 V_m^*$. [Another way to obtain this result is to note that the Borel-Weil data $(\mathbb{X}, \mathbb{L})$ associated to (G_o, V) is $(\mathbb{CP}_{m-1} \times Q_{n-1}, \pi_1^*(\mathcal{O}(1)) \otimes \pi_2^*(j^*\mathcal{O}(1)))$ implying $\mathfrak{g}_o^{(1)} = H^0(\mathbb{X}, \mathbb{L} \otimes \odot^2 \mathbb{N}^*) = 0$ and $K(\mathfrak{g}_o) = H^1(\mathbb{X}, \mathbb{L} \otimes \odot^3 \mathbb{N}^*) = \odot^2 V_m^* \otimes C \subset V^* \otimes V^*$, where C is the 1-dimensional subspace of $\odot^2 V_n$ spanned by $g_{\dot{A}\dot{B}}$.] Then the second Bianchi identitites (11) imply $\nabla_a Q_{bc} = 0$ which in turn imply $\nabla_m R_{abc}{}^d = 0$. These arguments imply essentially the following

Lemma 3.4 *Let G be the irreducible representation of a subgroup of $SL(m,\mathbb{C}) \cdot SO(n,\mathbb{C})$ in the mn-dimensional vector space $V_m \otimes V_n$. If $m, n \geqslant 3$, then $K^1(\mathfrak{g}) = 0$.*

EXAMPLE 3. Keeping notations of Example 1, consider a subgroup $G_s \subset G$ which is $T_{\mathbb{C}} \cdot SL(m,\mathbb{C}) \cdot Sp(n,\mathbb{C})$ represented in $V = V_m \otimes V_{2n}$ with $m \geqslant 3$, $n \geqslant 2$, and note that $K(\mathfrak{g}_s)$ is a subset of $K(\mathfrak{g})$ consisting of all elements $R_{abc}{}^d$ satisfying

$$R_{A\dot{A}B\dot{B}C\dot{C}}{}^{D\dot{D}}\, \varepsilon_{\dot{E}\dot{D}} - R_{A\dot{A}B\dot{B}C\dot{E}}{}^{D\dot{D}}\, \varepsilon_{\dot{C}\dot{D}} = 0,$$

where $\varepsilon_{\dot{E}\dot{D}} \in \Lambda^2 V_{2n}^*$ is the $Sp(n,\mathbb{C})$-invariant symplectic form. Substituting (10) into this equation, one easily finds

$$Q_{A\dot{A}B\dot{B}} = S_{AB}\, \varepsilon_{\dot{A}\dot{B}}$$

for some antisymmetric tensor $S_{AB} \in \Lambda^2 V_m$. Then the second Bianchi identitites (11) imply $\nabla_a Q_{bc} = 0$ which in turn imply $\nabla_m R_{abc}{}^d = 0$. We may summarize these arguments as follows.

Lemma 3.5 *Let G be the irreducible representation of a subgroup of $GL(m,\mathbb{C}) \cdot Sp(n,\mathbb{C})$ in the $2mn$-dimensional vector space $V_m \otimes V_{2n}$. If $m \geqslant 3$, $n \geqslant 2$, then $K^1(\mathfrak{g}) = 0$.*

An immediate corollary of Lemmas 3.3-3.5 is the following

Proposition 3.6 *Let $G_i \subseteq GL(V_{n_i})$, $i = 1, 2$, be an irreducible complex semisimple matrix Lie group such that the associated Borel-Weil data (X_i, L_i) satisfies $\dim X_i \geqslant 2$. Then $G = T_{\mathbb{C}} \cdot G_1 \cdot G_2 \subset GL(V_{n_1} \otimes V_{n_2})$ can have $K^1(\mathfrak{g}) \neq 0$ only if $G_1 = SL(n_1, \mathbb{C})$ and $G_2 = SL(n_2, \mathbb{C})$.*

3. The case $\mathbb{X} = X \times \mathbb{CP}_1$ with $\dim X \geqslant 2$. Any ample line bundle on $\mathbb{X}$ is of the form $\mathbb{L} = \pi_1^*(L) \otimes \pi_2^*(\mathcal{O}(k))$ for some ample line bundle $L \to X$ and $k \geqslant 1$. We denote in this subsection $V_n := \mathrm{H}^0(X, L)$, $V_2 := \mathrm{H}^0(\mathbb{CP}_1, \mathcal{O}(1))$, $N := J^1 L$, and $\mathfrak{g}$ stands for the Lie algebra $\mathrm{H}^0(X, L \otimes N^*) + sl(2, \mathbb{C})$ represented in $V = V_n \otimes V_2$.

If $k \geqslant 2$, then the associated matrix Lie group $G = \exp(\mathfrak{g})$ is an irreducible matrix subgroup of either $\mathrm{GL}(n, \mathbb{C})\mathrm{SO}(p, \mathbb{C})$ represented in $\mathbb{C}^{np}$ for some $n, p \geqslant 3$ or $\mathrm{GL}(n, \mathbb{C})\mathrm{Sp}(q, \mathbb{C})$ represented in $\mathbb{C}^{2nq}$ for some $n \geqslant 3, q \geqslant 2$. Then, by Lemmas 3.4 and 3.5, $K^1(\mathfrak{g}) = 0$.

So we may assume that $k = 1$.

Proposition 3.7 *Let (X, L) be a pair consisting of a compact complex homogeneous-rational manifold X and an ample line bundle $L \to X$. If $\dim X \geqslant 2$, then*

$$\mathfrak{g}^{(1)} = \mathrm{H}^0(X, N^*) \otimes V_2^*$$

and there is an exact sequence of $\mathfrak{g}$-modules

$$0 \longrightarrow \mathrm{H}^{2,2}(\mathfrak{g}) \longrightarrow \begin{array}{c} \mathrm{H}^0(X, \odot^3 TX \otimes L^{*2}) \otimes \Lambda^2 V_2^* \\ + \\ \mathrm{H}^1(X, TX \otimes L^{*2}) \otimes \odot^2 V_2^* \end{array} \longrightarrow \mathrm{H}^1(X, TX \otimes L^*) \otimes V^* \otimes V_2^*$$

Proof. Since $\dim X \geqslant 2$, the Kodaira vanishing theorem implies $\mathrm{H}^1(X, L^*) = 0$. Then the long exact sequence of

$$0 \longrightarrow L^* \longrightarrow N^* \longrightarrow TX \otimes L^* \longrightarrow 0$$

implies $\mathrm{H}^1(X, N^*) = \mathrm{H}^1(X, TX \otimes L^*)$, while the long exact sequence of (5) with $(X_1, L_1) = (X, L)$ and $(X_2, L_2) = (\mathbb{CP}_1, \mathcal{O}(1))$ implies

$$\mathrm{H}^0(\mathbb{X}, \mathbb{L} \otimes \odot^2 \mathbb{N}^*) = \mathrm{H}^0(X, N^*) \otimes V_2^*,$$

$$\mathrm{H}^1(\mathbb{X}, \mathbb{L} \otimes \odot^2 \mathbb{N}^*) = \mathrm{H}^1(X, N^*) \otimes V_2^* = \mathrm{H}^1(X, TX \otimes L^*) \otimes V_2^*.$$

Analogously, the long exact sequence of (6) implies

$$0 \longrightarrow \mathrm{H}^0(\mathbb{X}, \mathbb{L} \otimes \odot^3 \mathbb{N}^*) \longrightarrow \begin{array}{c} \mathrm{H}^0(X, \odot^2 N^*) \otimes \Lambda^2 V_2^* \\ + \\ 0 \end{array} \longrightarrow \begin{array}{c} \mathrm{H}^0(X, L \otimes \odot^3 N^*) \otimes \Lambda^2 V_2^* \\ + \\ 0 \end{array} \longrightarrow$$

$$\longrightarrow \mathrm{H}^1(\mathbb{X}, \mathbb{L} \otimes \odot^3 \mathbb{N}^*) \longrightarrow \begin{array}{c} \mathrm{H}^1(X, \odot^2 N^*) \otimes \Lambda^2 V_2^* \\ + \\ 0 \end{array} \longrightarrow \begin{array}{c} \mathrm{H}^1(X, L \otimes \odot^3 N^*) \otimes \Lambda^2 V_2^* \\ + \\ \mathrm{H}^2(X, L^* \otimes N^*) \otimes \odot^2 V_2^* \end{array} \longrightarrow \ldots$$

Comparing this with the long exact sequence of

$$0 \longrightarrow \odot^2 N^* \longrightarrow L \otimes \odot^3 N^* \longrightarrow \odot^3 TX \otimes L^{*2} \longrightarrow 0 \tag{12}$$

one obtains

$$\mathrm{H}^1(\mathbb{X}, \mathbb{L} \otimes \odot^3 \mathbb{N}^*) = \mathrm{H}^0(X, \odot^3 TX \otimes L^{*2}) \otimes \Lambda^2 V_2^* \; + \; \mathrm{H}^1(X, TX \otimes L^{*2}) \otimes \odot^2 V_2^*.$$

Then Theorem 2.3 implies the desired result. □

Lemma 3.8 *Let $G \subseteq \mathrm{GL}(V)$ be an irreducible complex representation of a complex reductive Lie group such that the associated Borel-Weil data is of the form $(\mathbb{X} = X \times \mathbb{CP}_1, \mathbb{L})$ with $\dim X \geqslant 2$ and $\mathrm{H}^1(X, TX \otimes L^{*2}) = 0$. Then G can be the holonomy of a non-metric torsion-free affine connection only if G is $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SL}(n, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$ represented in $V = V_n \otimes V_2$.*

Proof. If $(X, L) = (\mathbb{CP}_n, \mathcal{O}(1))$, then G has both modules $K(\mathfrak{g})$ and $K^1(\mathfrak{g})$ non-zero only if it is $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SL}(n, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$ represented in $V_n \otimes V_2$.

Assume now that $(X, L) \neq (\mathbb{CP}_n, \mathcal{O}(1))$. Then

$$\begin{aligned} K(\mathfrak{g}) &\subseteq \mathfrak{g} \otimes \Lambda^2 V^* = \mathfrak{g} \otimes \odot^2 V_n^* \otimes \Lambda^2 V_2^* + \mathfrak{g} \otimes \Lambda^2 V_n^* \otimes \odot^2 V_2^* \\ &\subseteq \mathrm{End}(V_n) \otimes \odot^2 V_n^* \otimes \Lambda^2 V_2^* + \mathrm{End}(V_n) \otimes \Lambda^2 V_n^* \otimes \odot^2 V_2^* + \odot^2 V_n^* \otimes \odot^2 V_2^* \\ &\quad + \Lambda^2 V_n^* \otimes \odot^4 V_2^* \otimes \Lambda^2 V_2 + \Lambda^2 V_n^* \otimes \odot^2 V_2^* + \Lambda^2 V_n^* \otimes \Lambda^2 V_2^*. \end{aligned}$$

On the other hand, by Proposition 3.8,

$$K(\mathfrak{g}) \subseteq \mathrm{H}^0(X, \odot^3 TX \otimes L^{*2}) \otimes \Lambda^2 V_2^* \subseteq \mathrm{End}(V_n) \otimes V_n^* \otimes V_n^* \otimes \Lambda^2 V_2^*.$$

Therefore,

$$K(\mathfrak{g}) \subseteq \mathrm{End}(V_n) \otimes \odot^2 V_n^* \otimes \Lambda^2 V_2^* + \Lambda^2 V_n^* \otimes \Lambda^2 V_2^*.$$

In notations of Example 1, a generic element of $\mathrm{End}(V_n) \otimes \odot^2 V_n^* \otimes \Lambda^2 V_2^* + \Lambda^2 V_n^* \otimes \Lambda^2 V_2^*$ satisfying the first Bianchi identities is, as one may easily check, of the form

$$R_{abc}{}^d = [W_{ABC}{}^D + \epsilon_{BC}\delta_A^D + \epsilon_{AC}\delta_B^D]\varepsilon_{\dot{A}\dot{B}}\delta_{\dot{C}}^{\dot{D}} + [\varepsilon_{\dot{A}\dot{C}}\delta_{\dot{B}}^{\dot{D}} + \varepsilon_{\dot{B}\dot{C}}\delta_{\dot{B}}^{\dot{A}}]\epsilon_{AB}\delta_C^D$$

for some $W_{ABC}{}^D \in V_n \otimes \odot^3 V_n^*$ and $\epsilon_{AB} \in \Lambda^2 V_n^*$. Here $\varepsilon_{\dot{A}\dot{B}} \in \Lambda^2 V_2^*$ denotes the non-degenerate $\mathrm{SL}(2, \mathbb{C})$-invariant symplectic form.

Therefore, if there exist a connection ∇ with holonomy G, its curvature tensor must be of the above form for some tensor fields $W_{ABC}{}^D$ and ϵ_{AB}. However, from the second Bianchi identities it easily follows that

$$\nabla_c(\epsilon_{AB}\varepsilon_{\dot{A}\dot{B}}) = 0.$$

By the Ambrose-Singer theorem, ϵ_{AB} is non-zero. Since G is irreducible, ϵ_{AB} is non-degenerate. Therefore, such a ∇ must preserve the non-degenerate symmetric form $g_{ab} = \epsilon_{AB}\varepsilon_{\dot{A}\dot{B}}$. □

Lemma 3.9 *Let $G \subseteq \mathrm{GL}(V)$ be an irreducible complex representation of a complex reductive Lie group such that the associated Borel-Weil data is of the form $(\mathbb{X} = X \times \mathbb{CP}_1, \mathbb{L})$ with $\dim X \geqslant 2$ and $\mathrm{H}^1(X, TX \otimes L^{*2}) \neq 0$. Then $K^1(\mathfrak{g}) \neq 0$ only if G is either $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SL}(n, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$ or $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SO}(n, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$, both represented in $V_n \otimes V_2$.*

Proof. It follows from Table 3 that $\mathrm{H}^1(X, TX \otimes L^{*2}) \neq 0$ only if $(X, L) = (Q_n, j^*\mathcal{O}(1))$ where Q_n is the n-dimensional quadric and $j : Q_n \hookrightarrow \mathbb{CP}_{n+1}$ is its standard embedding. This together with Fact 2.1 imply that G must be of the form $\mathrm{T}_{\mathbb{C}} \cdot H \cdot \mathrm{SL}(2, \mathbb{C}) \subseteq gl(V_n \otimes V_2)$ where H is one of the following representations

Group H:	$\mathrm{SO}(n, \mathbb{C})$	G_2	$\mathrm{Spin}(7, \mathbb{C})$
Representation space V_n:	$\mathbb{C}^n$	$\mathbb{C}^7$	$\mathbb{C}^8$

Since the Borel-Weil data associated to $G = \mathrm{T}_{\mathbb{C}} \cdot \mathrm{SO}(n, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$ represented in $V_n \otimes V_2$ is $(Q_{n-1} \times \mathbb{CP}_1, \pi_1^*(j^*\mathcal{O}(1)) \otimes \pi_2^*(\mathcal{O}(1))$, one has $\mathfrak{g}^{(1)} = \mathrm{H}^0(\mathbb{X}, \mathbb{L} \otimes \odot^2\mathbb{N}^*) = 0$ and

$$K(\mathfrak{g}) = \mathrm{H}^1(\mathbb{X}, \mathbb{L} \otimes \odot^3\mathbb{N}^*) = \Lambda^2 V_n^* \otimes \Lambda^2 V_2^* + C \otimes \odot^2 V_2^*$$

where C is the 1-dimensional subspace of $\odot^2 V_n$ spanned by the $\mathrm{SO}(n, \mathbb{C})$-invariant metric g_{AB}. Then a generic element of $K(\mathfrak{g})$ must be of the form (cf. [18])

$$\begin{aligned} R_{abc}{}^d \;=\; & [\varepsilon_{\dot{A}\dot{B}} g^{DE}(g_{AB}S_{CE} + g_{AC}S_{BE} + g_{BC}S_{AE} \\ & - g_{AE}S_{BC} - g_{BE}S_{AC}) + \Phi_{\dot{A}\dot{B}}(g_{BC}\delta_A^D - g_{AC}\delta_B^D]\delta_{\dot{C}}^{\dot{D}} \\ & + [g_{AB}\varepsilon_{\dot{A}\dot{B}}\varepsilon^{\dot{D}\dot{E}}\Phi_{\dot{C}\dot{E}} - S_{AB}(\varepsilon_{\dot{B}\dot{C}}\delta_{\dot{A}}^{\dot{D}} + \varepsilon_{\dot{A}\dot{C}}\delta_{\dot{A}}^{\dot{B}})]\delta_C^D \end{aligned} \tag{13}$$

for some $S_{AB} \in \Lambda^2 V_n^*$ and $\Phi_{\dot{A}\dot{B}} \in \odot^2 V_2^*$.

Let $g \subset gl(V)$ be the Lie algebra of the representation of $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{G}_2 \cdot \mathrm{SL}(2, \mathbb{C})$ (resp. $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{Spin}(7, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$) in $V = \mathbb{C}^7 \otimes \mathbb{C}^2$ (resp. in $V = \mathbb{C}^8 \otimes \mathbb{C}^2$). It is a proper matrix subalgebra of the Lie algebra $\mathfrak{g}$ of the representation of $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SO}(7, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$ (resp. $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SO}(8, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$) in $V = \mathbb{C}^7 \otimes \mathbb{C}^2$ (resp. in $V = \mathbb{C}^8 \otimes \mathbb{C}^2$). Then

$$K(g) \subseteq K(\mathfrak{g}) = \Lambda^2 V_n^* \otimes \Lambda^2 V_2^* + C \otimes \odot^2 V_2^*$$

and $K^1(g) \subseteq K^1(\mathfrak{g})$. We claim

$$K(g) \subseteq \Lambda^2 V_n^* \otimes \Lambda^2 V_2^*. \tag{14}$$

If not, then a typical element $R_{abc}{}^d \in K(g)$ contains a non zero term $\Phi_{\dot{A}\dot{B}}(g_{BC}\delta_A^D - g_{AC}\delta_B^D)\delta_{\dot{C}}^{\dot{D}}$ which easily implies that the image of the map

$$\Lambda^2 V \longrightarrow g$$

defined by $R_{abc}{}^d \in g \otimes \Lambda^2 V^*$ contains $\Lambda^2 V_n^* \simeq so(n, \mathbb{C})$. This contradicts to the fact that g is a proper subalgebra of $\mathfrak{g}$.

Finally, it is straightforward to check that the inclusion (14) implies that $K^1(g) = 0$. □

Proposition 3.10 *Let $G \subsetneq \mathrm{GL}(V)$ be an irreducible complex representation of a complex reductive Lie group such that the associated Borel-Weil data is of the form $(\mathbb{X} = X \times \mathbb{CP}_1, \mathbb{L})$ with $\dim X \geqslant 2$. Then G can be the holonomy of a non-metric torsion-free affine connection only if it is either $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SL}(n, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$ or $\mathrm{SO}(n, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$, both represented in $V_n \otimes V_2$.*

Proof. By Lemmas 3.8 and 3.9, one has only to rule out the case $\mathbb{C}^* \cdot \mathrm{SO}(n, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$. But this follows from $R_{abc}{}^c = 0$ which itself follows from (13). □

4. The case $\mathbb{X} = \mathbb{CP}_1 \times \mathbb{CP}_1$. This is the case of $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SL}(2, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$ represented in $\odot^m V_2 \otimes \odot^n V_2$. In the context of the holonomy classification, this class of representations has been studied in [19] and [28] where the following result has been established by two different methods.

Proposition 3.11 *Let $G \subsetneq \mathrm{GL}(V)$ be an irreducible complex representation of a complex reductive Lie group such that the associated Borel-Weil data $(\mathbb{X}, \mathbb{L})$ has $\mathbb{X} = \mathbb{CP}_1 \times \mathbb{CP}_1$. Then $K^1(\mathfrak{g}) \neq 0$ only if G is either the representation of $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SO}(4, \mathbb{C})$ in $\mathbb{C}^4$ or the representation of $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SO}(3, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$ in $\mathbb{C}^6$.*

In fact, for the above representation, $K(\mathbb{C}+so(3, \mathbb{C})+sl(2, \mathbb{C})) = K(so(3, \mathbb{C})+sl(2, \mathbb{C}))$ which means that $\mathbb{C}^* \cdot \mathrm{SO}(3, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$ can not occur as the holonomy of a torsion-free affine connection.

5. Proof of Theorem A. Let $G \subsetneq \mathrm{GL}(V)$ be an irreducible complex representation of a complex reductive Lie group which can be represented as a tensor product of two or more non-Abelian complex representations. Then, by Propositions 3.6, 3.10 and 3.11, G may occur as the holonomy of a non-metric torsion-free affine connection only if it is either $\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SL}(m, \mathbb{C}) \cdot \mathrm{SL}(n, \mathbb{C})$ represented in $\mathbb{C}^m \otimes \mathbb{C}^n$ for $m, n \geqslant 2$, or $\mathrm{SO}(l, \mathbb{C}) \cdot \mathrm{SL}(2, \mathbb{C})$ represented in $\mathbb{C}^l \otimes \mathbb{C}^2$ for $l \geqslant 3$.

Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation of a real reductive Lie group G in a real vector space V and let $\rho : \mathfrak{g} \to gl(V)$ be the associated real irreducible representation of the Lie algebra $\mathfrak{g}$ of G. The latter defines naturally a complex representation $\rho_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to gl(V_{\mathbb{C}})$, where $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ and $V_{\mathbb{C}} = V \otimes \mathbb{C}$. Then two situations may arise [22]:

(i) the complex representation $\rho_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to gl(V_{\mathbb{C}})$ is irreducible; in this case we denote $\rho_{\mathbb{C}}$ by $\tilde{\rho}$;

(ii) there is a complex vector space $W_{\mathbb{C}}$ and an irreducible complex representation $\rho' : \mathfrak{g} \to gl(W_{\mathbb{C}})$ such that V is the underlying real vector space of $W_{\mathbb{C}}$ and ρ is the composition $\rho : \mathfrak{g} \xrightarrow{\rho'} gl(W_{\mathbb{C}}) \longrightarrow gl(V)$, where the second arrow is the natural inclusion of the algebra of all complex automorphisms of V into the algebra of all real automorphisms of V. Then the $\mathfrak{g}_{\mathbb{C}}$-module $V_{\mathbb{C}}$ splits as a direct sum of two irreducible $\mathfrak{g}_{\mathbb{C}}$-submodules $W_{\mathbb{C}} + W_{\mathbb{C}}$ and we denote by $\tilde{\rho} : \mathfrak{g}_{\mathbb{C}} \to gl(W_{\mathbb{C}})$ the restriction of $\rho_{\mathbb{C}}$ to one of these.

In both cases, the $\mathfrak{g}$-modules $K(\rho(\mathfrak{g}))$ and $K^1(\rho(\mathfrak{g}))$ are subsets of $K(\tilde{\rho}(\mathfrak{g}_{\mathbb{C}}))$ and $K^1(\tilde{\rho}(\mathfrak{g}_{\mathbb{C}}))$ respectively. In particular, if $K(\rho(\mathfrak{g}))$ and $K^1(\rho(\mathfrak{g}))$ are non zero, then $K(\tilde{\rho}(\mathfrak{g}_{\mathbb{C}}))$ and $K^1(\tilde{\rho}(\mathfrak{g}_{\mathbb{C}}))$ are non zero as well.

Assume now that the semisimple part of $\mathfrak{g}$ has at least two non-Abelian ideals. Then the Borel-Weil data associated to the irreducible matrix subalgebra $\tilde{\rho}(\mathfrak{g}_{\mathbb{C}})$ must be of the form $(\mathbb{X}, \mathbb{L}) = (X_1 \times X_2, \pi_1^*(L_1) \otimes \pi_2^*(L_2))$ for some compact complex homogeneous-rational manifolds X_1 and X_2 and ample line bundles $L_1 \to X_1$ and $L_2 \to X_2$.

We claim that if $\rho(\mathfrak{g})$ is the holonomy of a torsion-free affine connection ∇ which is not locally symmetric and does not preserve any (pseudo-)Riemannian metric, then $\tilde{\rho}(\mathfrak{g}_{\mathbb{C}})$ is either $t_{\mathbb{C}} + sl(m, \mathbb{C}) + sl(n, \mathbb{C})$ represented in $\mathbb{C}^m \otimes \mathbb{C}^n$, or $so(l, \mathbb{C}) + sl(2, \mathbb{C})$ represented in $\mathbb{C}^m \otimes \mathbb{C}^n$.

Indeed, if

$$\dim X_1 \geqslant 2, \quad \dim X_2 \geqslant 2,$$

or

$$\dim X_1 = \dim X_2 = 1$$

or

$$\dim X_1 \geqslant 2, \ \dim X_2 = 1, \quad \mathrm{H}^1(X_1, TX_1 \otimes L_1^{*2}) \neq 0,$$

then the claim follows from Propositions 3.6 and 3.11 and Lemma 3.9.

Let us next show that the only remaining case

$$\dim X_1 \geqslant 2, \quad \dim X_2 = 1, \qquad \mathrm{H}^1(X_1, TX_1 \otimes L_1^{*2}) = 0$$

implies that $\rho(\mathfrak{g}_{\mathbb{C}})$ is the representation of $t_{\mathbb{C}} + sl(n, \mathbb{C}) + sl(2, \mathbb{C})$ in $\mathbb{C}^n \otimes \mathbb{C}^2$. If $(X_1, L_1) \neq (\mathbb{CP}_n, \mathcal{O}(1))$, then, using the same arguments as in the proof of Lemma 3.8, one may show that ∇ must preserve a non-degenerate complex symmetric form $g_{ab} = \epsilon_{AB}\varepsilon_{\dot{A}\dot{B}}$ and hence its real part and imaginary parts. At least one of these must be non-zero and, by irreducibility of $\rho(\mathfrak{g})$, non-degenerate. Since ∇ is non-metric, this is impossible. Hence the only other option is $(X_1, L_1) = (\mathbb{CP}_n, \mathcal{O}(1))$ and $(X_2, L_2) = (\mathbb{CP}_1, \mathcal{O}(k))$ for some $k \in \mathbb{N}$. By Lemmas 3.4 and 3.5, $k = 1$ implying that $\rho(\mathfrak{g}_{\mathbb{C}})$ is the representation of $t_{\mathbb{C}} + sl(n, \mathbb{C}) + sl(2, \mathbb{C})$ in $\mathbb{C}^n \otimes \mathbb{C}^2$.

Therefore, $\rho(G) \subseteq \mathrm{GL}(V)$ must be of the form $\mathrm{T}_{\mathbb{C}} \cdot G_1 \cdot G_2$, where $\mathrm{T}_{\mathbb{C}}$ is a connected real Lie subgroup of $\mathbb{C}^*$ and $G_i \subseteq \mathrm{GL}(V_i)$, $i = 1, 2$, is one of the following real matrix groups

Group G_i:	$\mathrm{SL}(n, \mathbb{C})$	$\mathrm{SL}(n, \mathbb{R})$	$\mathrm{SU}(n)$	$\mathrm{SL}(m, \mathrm{H})$
Representation space V_i:	$\mathbb{R}^{2n}$	$\mathbb{R}^n$	$\mathbb{R}^{2n}$	$\mathbb{R}^{4m}$
Group G_i:	$\mathrm{SO}(l, \mathbb{C})$	$\mathrm{SO}(p, q)$	$\mathrm{SO}(n, \mathrm{H})$	
Representation space V_i:	$\mathbb{R}^{2l}$	$\mathbb{R}^{p+q}$	$\mathbb{R}^{4n}$	

with $n \geqslant 2$, $m \geqslant 1$, $l \geqslant 3$, $p + q \geqslant 3$.

Since we know $K(\tilde{\rho}(\mathfrak{g}_{\mathbb{C}}))$ explicitly, it is straightforward to check that the only combinations $\rho(G) = \mathrm{T}_{\mathbb{C}} \cdot G_1 \cdot G_2$ which (i) have $K^1(\rho(\mathfrak{g})) \neq 0$, (ii) have no proper subgroup $G' \subset G$ with $K(\rho(\mathfrak{g}')) = K(\rho(\mathfrak{g}))$ and (iii) do not preserve any non-degenerate symmetric bilinear form are the ones given in the table of Theorem A. □

4 Classification of (X, L) with $H^1(X, TX \otimes L^*) \neq 0$

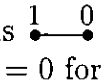
1. Review of the representation theory [6, 21]. Let $\mathfrak{g}$ be a semisimple complex Lie algebra and G the associated simply connected Lie group. Fix a maximally Abelian self-normalizing subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (any two such subalgebras, called Cartan subalgebras, are conjugate under the adjoint action of G). If $\rho : \mathfrak{g} \to gl(V)$ is a representation of $\mathfrak{g}$ in a complex vector space V, then with any $\omega \in \mathfrak{h}^* \equiv \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ one may associate the *weight space* of V by $V_\omega = \{v \in V : \rho(h)v = \omega(h)v \text{ for all } h \in \mathfrak{h}\}$. An element $\omega \in \mathfrak{h}^*$ is called a weight of V if $V_\omega \neq 0$.

In the particular case when $V = \mathfrak{g}$ and $\rho : \mathfrak{g} \to gl(\mathfrak{g})$ is the adjoint representation of $\mathfrak{g}$ on itself, the non-zero weights of $\mathfrak{g}$ are called the *roots* of $\mathfrak{g}$. Thus

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where Φ is the set of all roots of $\mathfrak{g}$ and all sums are direct. A subset $\Delta = \{\alpha_1, \ldots, \alpha_r\} \subset \Phi$ with the property that every $\omega \in \Phi$ may be expressed as a linear combination $\omega = \sum_{i=1}^r a_i \alpha_i$

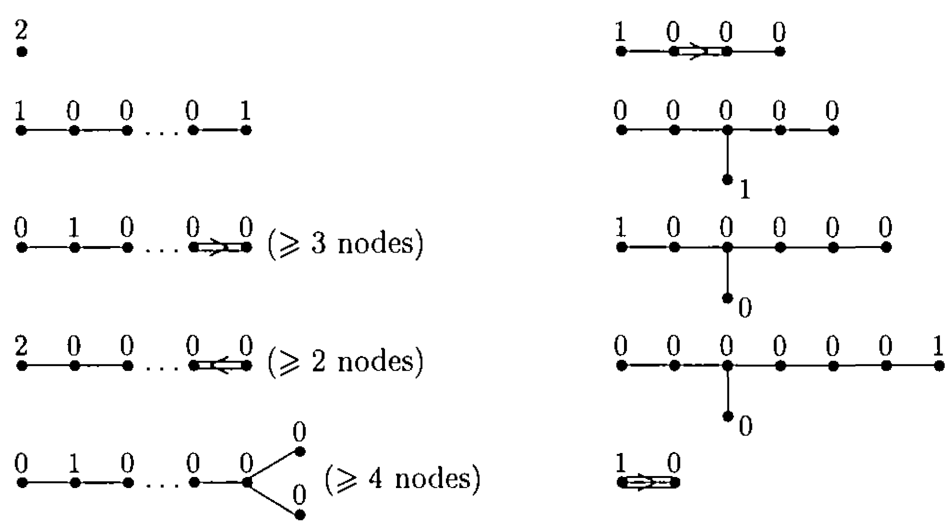
with all a_i being non-negative or all non-positive integers is called a *system of simple roots* of $\mathfrak{g}$. Such Δ exists and any two such Δ's are conjugate under the adjoint action of G. Then $\Phi = \Phi^+ \cup \Phi^-$, where $\Phi^+ = \{\omega \in \Phi : \omega = \sum_{i=1}^r a_i\alpha_i \text{ with } a_i \geqslant 0\}$ is the set of *positive roots* and $\Phi^- = \{\omega \in \Phi : \omega = \sum_{i=1}^r a_i\alpha_i \text{ with } a_i \leqslant 0\}$ is the set of *negative roots* (both with respect to to Δ).

For any root $\alpha \in \Phi^+$ there is a unique element H_α in $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ such that $\alpha(H_\alpha) = 2$. If $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ is the set of simple roots, then the associated set $\{H_{\alpha_1}, \ldots, H_{\alpha_r}\}$ form a basis of $\mathfrak{g}$. Its dual basis $\{\omega_{\alpha_1}, \ldots, \omega_{\alpha_r}\}$ of $\mathfrak{h}^*$ is called the set of *fundamental weights.* One may use it to define the following three importants subsets of $\mathfrak{h}^*$: the set of *(integral) weights* $\Lambda = \{\lambda \in \mathfrak{h}^* : \omega = \sum_{i=1}^r \lambda_i\omega_i \text{ with } \lambda_i \in \mathbb{Z}\}$; the set of *dominant weights* $\Lambda^+ = \{\lambda \in \Lambda : \lambda = \sum_{i=1}^r \lambda_i\omega_i \text{ with } \lambda_i \geqslant 0\}$; and the set of *strongly dominant weights* $\Lambda^{++} = \{\lambda \in \Lambda^+ : \lambda = \sum_{i=1}^r \lambda_i\omega_i \text{ with } \lambda_i > 0\}$. Note that $\lambda_i = \lambda(H_{\alpha_i})$. The minimal integral element $\omega_1 + \omega_2 + \ldots + \omega_r$ in Λ^{++} is denoted by η. Any integral weight λ of $\mathfrak{g}$ can be graphically represented by inscribing the integer λ_i over i-th node of the Dynkin diagram for $\mathfrak{g}$. For example, the fundamental weight ω_1 of $sl(3, \mathbb{C})$ is  .

Let $\lambda \in \Lambda$ be an integral weight. It is called *singular* if $\lambda(H_\alpha) = 0$ for some $\alpha \in \Phi^+$, and *regular* otherwise. The *index* of λ is defined to be the number of positive roots α for which $\lambda(H_\alpha) < 0$ holds; it is denoted by $\mathrm{ind}(\lambda)$.

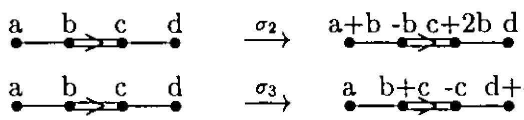
If $\rho : \mathfrak{g} \to gl(V)$ is an irreducible representation of $\mathfrak{g}$, then there exists a unique weight $\omega(V) \in \Lambda^+$ of V, called the *highest weight* of V (relative to the fixed $\mathfrak{h}$ in $\mathfrak{g}$ and Δ in $\mathfrak{h}$) such that $\dim V_\omega = 1$ and $\rho(\mathfrak{g}_\alpha)V_\omega = 0$ for all $\alpha \in \Delta$. This establishes a one-to-one correspondence, $V \Leftrightarrow \omega(V)$, between finite-dimensional irreducible $\mathfrak{g}$-modules and dominant weights; and allows us to use the graphical description of $\omega(V)$ to represent $\rho : \mathfrak{g} \to gl(V)$. For example, the standard representation of $sl(3, C)$ in $\mathbb{C}^3$ gets denoted by  .

If $\mathfrak{g}$ is simple, then the adjoint representation $\rho : \mathfrak{g} \to gl(\mathfrak{g})$ is irreducible. The associated highest weight of $V = \mathfrak{g}$ is a root $\mu \in \Lambda^+$ which is called the *maximal root* of $\mathfrak{g}$. The following is the list of all maximal roots [6, 21]:



(15)

For any simple root $\alpha_i \in \Delta$, denote by σ_i the reflection in the hyperplane perpendicular to α_i. The *Weyl group* W of $\mathfrak{g}$ is the group generated by all the simple reflections σ_i. The action of the simple reflection σ_i on a weight $\lambda \in \Lambda$ can be described by the following rule [6]: to compute $\sigma_i(\lambda)$, let $c = \lambda(H_{\alpha_i})$ be the coefficient of the node associated to α_i; add c to the adjacent coefficients, with multiplicity if there is a multiple edge directed towards the adjacent node, and then replace c by $-c$. For example



For any $w \in W$, there exists a minimal integer $l(w)$ such that w can be expressed as a composition of $l(w)$ simple reflections. This integer is called the *length* of w.

2. Homogeneous manifolds and vector bundles. A maximal solvable subalgebra of a semisimple Lie algebra $\mathfrak{g}$ is called a *Borel subalgebra.* A a subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ is called *parabolic* if it contains a Borel subalgebra. Every Borel subslagebra is G-conjugate to the standard one

$$\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}$$

where $\mathfrak{n} := \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. There is a standard form $\mathfrak{p}$ for any parabolic subalgebra as well. Let $\Delta_\mathfrak{p}$ be a subset of Δ and let $\Phi^+_\mathfrak{p} = \mathrm{span}\,\{\Delta_\mathfrak{p}\} \cap \Phi^+$. Then

$$\mathfrak{p} = \mathfrak{h} + \mathfrak{n} + \sum_{\alpha \in \Phi^+_\mathfrak{p}} \mathfrak{g}_{-\alpha}$$

is the standard parabolic subsalgebra of $\mathfrak{g}$. A useful notation for a standard parabolic $\mathfrak{p} \subseteq \mathfrak{g}$ (and for the associated subgroup $P \subseteq G$) is to cross all nodes in the Dynkin diagram for $\mathfrak{g}$ which correspond to simple roots of $\mathfrak{g}$ in $\Delta \setminus \Delta_\mathfrak{p}$.

It is well known that any compact complex homogeneous-rational manifold X is isomorphic to the quotient space G/P, where G is a simply connected Lie group and $P \subseteq G$ is a parabolic subgroup. It is then very useful to denote X by the same Dynkin diagram as $\mathfrak{p}$, the Lie algebra of P. For example, the odd dimensional quadric Q_{2n-1} gets denoted by $\times\!\!-\!\!\bullet\!\!-\!\!\bullet \ldots \bullet\!\!\Rightarrow\!\!\bullet$.

The number of crossed nodes in the Dynkin diagram for X is called the *rank* of X and is denoted by rank$X$. This number is independent of the representation of X as a quotient G/P.

A vector bundle $E \to X = G/P$ is called *G-homogeneous* if there is a holomorphic representation $\rho : P \to \mathrm{GL}(V)$ such that $E = G \times_\rho V$, i.e. E is the quotient $G \times V/P$, where every $p \in P$ acts on $G \times V$ as follows

$$\begin{array}{ccc} G \times V & \longrightarrow & G \times V \\ (g, v) & \longrightarrow & (g \cdot p, \rho(p^{-1})v). \end{array}$$

If $\rho : P \to \mathrm{GL}(V)$ is irreducible, then E is said to be *irreducible* as well.

The finite-dimensional irreducible representations of P are in one-to-one correspondence with integral weights $\lambda \in \Lambda$ whose Dynkin diagram has non-negative coefficients over the uncrossed nodes for $\mathfrak{p}$. A useful notation for an irreducible homogeneous vector bundle $E \to X$ is to combine the Dynkin diagram for the associated integral weight λ with the Dynkin diagram for $\mathfrak{p}$ into one picture. For example, if $X = \times\!\!-\!\!\bullet$ is the projective plane $\mathbb{CP}_2$, then $\mathcal{O}(-1) = \overset{-1}{\times}\!\!-\!\!\overset{0}{\bullet}$ and $TX = \overset{1}{\times}\!\!-\!\!\overset{1}{\bullet}$.

The cohomology ring $\mathrm{H}^*(X, E)$ of an irreducible homogeneous vector bundle $E \to X$ with integral weight $\lambda \in \Lambda$ can be computed, according to Bott [10], as follows:

(i) if $\lambda + \eta$ is singular, then $\mathrm{H}^*(X, E) = 0$;

(ii) if $\lambda + \eta$ is regular and if $\mathrm{ind}(\lambda + \eta) = p$, then there is a unique element σ_λ (of length p) in the Weyl group of Φ such that $\sigma_\lambda(\lambda + \eta) \in \Lambda^{++}$. Then $\mathrm{H}^*(X, E) = \mathrm{H}^p(X, E)$ and $\mathrm{H}^p(X, E)$ is an irreducible $\mathfrak{g}$-module whose highest weight is $\sigma_\lambda(\lambda + \eta) - \eta$.

For future reference we introduce the following notation: if $\lambda \in \Lambda$ and $\lambda + \eta$ is regular, then $J^k(\lambda)$ denotes the irreducible G-module with highest weight $\sigma_\lambda(\lambda + \eta) - \eta$ if $k = \mathrm{ind}(\lambda + \eta)$ and 0 otherwise; if $\lambda + \eta$ is singular, then $J^k(\lambda) = 0$ for all k.

3. Proof of Theorem B. The statement (i) follows from Proposition 3.1. Let us prove the statement (ii).

If X is reducible, say $X = X_1 \times X_2$ and $L = \pi_1^*(L_1) \otimes \pi_2^*(L_2)$, then

$$\mathrm{H}^1(X, TX \otimes L^*) = \mathrm{H}^0(X_1, TX_1 \otimes L_1^*) \otimes \mathrm{H}^1(X_2, L_2^*) + \mathrm{H}^0(X_2, TX_2 \otimes L_2^*) \otimes \mathrm{H}^1(X_1, L_1^*).$$

This together with statement (i) implies that in the class of reducible X only two bottom lines in Table 3 contribute to the list of all (X, L) with $\mathrm{H}^1(X, TX \otimes L^*) \neq 0$.

Assume from now on that X is irreducible. Though the tangent bundle TX is homogeneous, it is not irreducible in general; even worse, since the parabolic P is not reductive, TX is *not* in general a direct sum of irreducible homogeneous vector bundles. This makes a naive idea of computing $\mathrm{H}^*(X, TX \otimes L^*)$ by the straightforward application of the Bott theorem impractical.

Consider the Atiyah exact sequence

$$0 \longrightarrow Q \longrightarrow \mathfrak{g} \otimes \mathcal{O}_X \longrightarrow TX \longrightarrow 0, \tag{16}$$

where $Q = G \times_{Ad} \mathfrak{p}$. Since the central term of this extension is a trivial vector bundle and $\mathrm{H}^i(X, L^*) = 0$ for $0 \leqslant i \leqslant \dim X - 1$, we have, in the case $\dim X \geqslant 3$,

$$\mathrm{H}^1(X, TX \otimes L^*) = \mathrm{H}^2(X, Q \otimes L^*).$$

An exact sequence of $\mathfrak{p}$-modules

$$0 \longrightarrow \tilde{\mathfrak{n}} \longrightarrow \mathfrak{p} \longrightarrow \mathfrak{p}/\tilde{\mathfrak{n}} \longrightarrow 0,$$

where $\tilde{\mathfrak{n}} = \mathfrak{n} \setminus \sum_{\alpha \in \Phi_{\mathfrak{p}}^+} \mathfrak{g}_\alpha$, gives rise to an exact sequence

$$0 \longrightarrow \Omega^1 X \longrightarrow Q \longrightarrow S \longrightarrow 0$$

of homogeneous vector bundles, where $S = G \times_{Ad} \mathfrak{p}/\tilde{\mathfrak{n}}$ and we used the isomorphism $G \times_{Ad} \tilde{\mathfrak{n}} \simeq \Omega^1 X$. According to Nakano [31], for any compact complex manifold X and any positive line bundle L on X the groups $\mathrm{H}^i(X, \Omega^1 X \otimes L^*) = 0$ vanish for all $i \leqslant \dim X - 2$. Then, in the case $\dim X \geqslant 5$, the long exact sequence of the latter extension implies

$$\mathrm{H}^2(X, Q \otimes L^*) = \mathrm{H}^2(X, S \otimes L^*).$$

which in turn implies

$$\mathrm{H}^1(X, TX \otimes L^*) = \mathrm{H}^2(X, S \otimes L^*)$$

The advantage of working with S instead of TX is that S can always be decomposed into a *direct* sum of irreducible homogeneous subbundles.

The Lie algebra $\mathfrak{s} = \mathfrak{p}/\tilde{\mathfrak{n}}$ is reductive and the adjoint representation of $\mathfrak{p}$ on $\mathfrak{s}$ is semisimple. Under the adjoint representation $\mathfrak{s} \to gl(\mathfrak{s})$ the Lie algebra $\mathfrak{s}$ decomposes into a direct sum of its ideals

$$\mathfrak{s} = \xi_1 + \dots \xi_k + \mathfrak{s}_1 + \dots + \mathfrak{s}_m$$

where ξ_j, $j = 1, \dots, \mathrm{rank}X$, lie in the center of $\mathfrak{s}$ and the non-Abelian ideals $\mathfrak{s}_i$, $i = 1, \dots, m$, are simple. Then, by Bott theorem,

$$\mathrm{H}^2(X, S \otimes L^*) = \bigoplus_{j=1}^{k} J^2(-\lambda) + \bigoplus_{i=1}^{m} J^2(\mu_i - \lambda),$$

where λ is the weight of L and $\mu_1, \dots, \mu_m$ are the maximal roots of the simple ideals $\mathfrak{s}_1, \dots, \mathfrak{s}_m$ [10, 35]. Since, for $\dim X \geqslant 2$, $J^2(-\lambda) = \mathrm{H}^2(X, L^*) = 0$, we obtain the following

Lemma 4.1 *If* $\dim X \geqslant 5$, *then* $\mathrm{H}^1(X, TX \otimes L^*) = \bigoplus_{i=1}^m J^2(\mu_i - \lambda)$.

There are seven irreducible compact complex homogeneous-rational manifolds X with $\dim X \leqslant 4$: projective spaces $\mathbb{CP}_k$ for $k = 1, 2, 3, 4$, quadrics Q_3, Q_4 and the complete flag manifold $F(1, 2; \mathbb{C}^3)$. It is elementary to check that Theorem B is true for this family.

We assume from now on that X is an irreducible complex homogeneous-rational manifold with $\dim X \geqslant 5$.

Let us introduce the following notation: if Γ is a connected subgraph of the Dynkin diagram for X, then the number of simple roots $\{\alpha_j \mid j \in I\}$ in this graph is denoted by $|\Gamma|$; if ω is an integral weight such that $\omega(H_{\alpha_j}) \leqslant 0$ for all $\alpha_j \in \Gamma$, then we write $\omega|_\Gamma \leqslant 0$.

Lemma 4.2 $\mathrm{H}^1(X, TX \otimes L^*) = 0$ *for any ample line bundle* L *on* X *if at least one of the following conditions is satisfied:*

(i) $\mathrm{rank}X \geqslant 3$;

(ii) $\mathrm{rank}X = 2$ *and the crossed nodes are adjacent;*

(iii) $\mathrm{rank}X = 2$, *the crossed nodes are not adjacent and, for each maximal root* μ_i *of the simple ideal* $\mathfrak{s}_i$, $i = 1, \dots, m$, *at least one crossed node is contained in a connected subgraph* Γ *of the Dynkin diagram for* X *such that* $|\Gamma| \geqslant 2$ *and* $\mu_i|_\Gamma \leqslant 0$.

(iv) $\mathrm{rank}X = 1$ *and, for each maximal root* μ_i *of the simple ideal* $\mathfrak{s}_i$, $i = 1, \dots, m$, *the crossed node is contained in a connected subgraph* Γ *of the Dynkin diagram for* X *such that* $|\Gamma| \geqslant 3$ *and* $\mu_i|_\Gamma \leqslant 0$.

Proof. (i) Let λ be the weight of L. Since L is ample, the coefficient of λ over each crossed node is a negative integer (its coefficient over each uncrossed node is, of course, zero). Then $(-\lambda + \mu_i + \eta)(H_{\alpha_j}) \leqslant -\lambda(H_{\alpha_j}) + 1 \leqslant 0$ for all crossed nodes α_j and all $i \in \{1, \dots, m\}$. If the number of crossed nodes is greater than or equal to 3, then either $-\lambda + \mu_i + \eta$ is singular or $\mathrm{ind}(-\lambda+\mu_i+\eta) \geqslant 3$. Whence $\bigoplus_{i=1}^{m} J^2(\mu_i-\lambda)=0$ and the statement follows from Lemma 4.1.

(ii) If α_j and α_{j+1} are adjacent crossed nodes, then $\alpha_j+\alpha_{j+1}$ is a positive root and one has $(-\lambda+\mu_i+\eta)(H_{\alpha_j}) \leqslant 0$, $(-\lambda+\mu_i+\eta)(H_{\alpha_{j+1}}) \leqslant 0$ and hence $(-\lambda+\mu_i+\eta)(H_{\alpha_j+\alpha_{j+1}}) \leqslant 0$. Thus either $-\lambda+\mu_i+\eta$ is singular or $\mathrm{ind}(-\lambda+\mu_i+\eta) \geqslant 3$ for all i and the statement follows from Lemma 4.1.

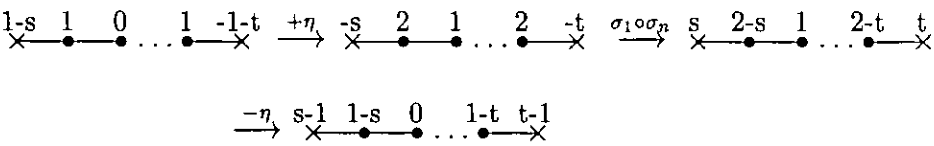
(iii) & (iv) If Γ' is a connected subgraph of the Dynkin diagram for X, then the sum of all simple roots in Γ' is a positive root [11]. Under the conditions stated in (iii) and (iv), one easily finds at least three positive roots α_j such that $(-\lambda+\mu_i+\eta)(H_{\alpha_j}) \leqslant 0$ for all $i \in \{1,\dots,m\}$. Then again either $-\lambda+\mu_i+\eta$ is singular or $\mathrm{ind}(-\lambda+\mu_i+\eta) \geqslant 3$ implying $J^2(-\lambda+\mu_i)=0$. Thus the statement follows from Lemma 4.1. □

Therefore, we can restrict our attention to the cases $\mathrm{rank}X = 1, 2$.

The case $\mathrm{rank}X = 2$. It clear from items (ii) and (iii) of Lemma 4.2 that $\mathrm{H}^1(X, TX \otimes L^*) \neq 0$ only for those X which have the semisimple part $\mathfrak{s}'$ of the parabolic algebra $\mathfrak{p}$ simple, i.e. the number m of simple ideals of $\mathfrak{s}'$ is 1. Therefore, the crossed nodes must be located at the ends of the Dynkin diagram for X. Inspecting the list of maximal roots (15) leaves one with the following three candidates to the list of all (X, L) with $\mathrm{H}^1(X, TX \otimes L^*) \neq 0$:

(1) $(X, L) =$ 

for some s,t$\geqslant$ 1. Computing $\sigma_1 \circ \sigma_n(-\lambda+\mu_1+\eta)-\eta$ as shown in the following diagram

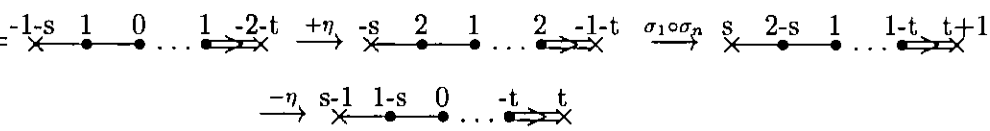
$-\lambda+\mu_1=$ 

one concludes

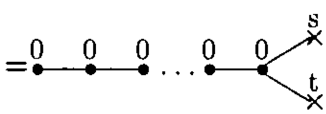
$$\mathrm{H}^1(X, TX \otimes L^*) = J^2(-\lambda+\mu_1) = \begin{cases} \mathbb{C} & \text{s=t=1} \\ 0 & \text{otherwise.} \end{cases}$$

(2) $(X, L) =$ 

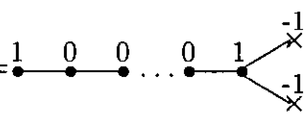
for some s,t$\geqslant$ 1. Then the graph

$-\lambda+\mu_1=$ 

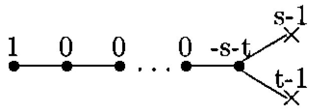
implies $\mathrm{H}^1(X, TX \otimes L^*) = J^2(-\lambda+\mu_1) = 0$ for all s,t$\geqslant$ 1.

(3) $(X, L) =$ 

for some s,t$\geqslant$ 1. In this case

$\mu_1=$ 

The only element of the Weyl group W of length 2 which can, in principle, map $-\lambda+\mu_1+\eta$ to a strictly dominant weight in Λ^{++} is $\sigma_{n-1} \circ \sigma_n$. However, a computation as above shows that

$$\sigma_{n-1} \circ \sigma_n(-\lambda + \mu_1 + \eta) - \eta =$$ 

which implies $\mathrm{H}^1(X, TX \otimes L^*) = J^2(-\lambda + \mu_1) = 0$ for all s,t$\geqslant$ 1.

The case rank$X$ = 1. The number m of simple ideals of the semisimple part of the parabolic algebra $\mathfrak{p}$ can, in principle, be equal to 1, 2 or 3. The case $m = 3$, however, is ruled out by Lemma 4.2(iv). If $m = 2$, then, by Lemma 4.2(iv), at least one of the ideals must be isomorphic to $sl(2, \mathbb{C})$. Therefore, the crossed node must be either an end node (for $m = 1$) or the node adjacent to an end node (for $m = 2$) of the Dynkin diagram for X. Inspecting the list of maximal roots (15) excludes all but the following candidates to (X, L) with $\mathrm{H}^1(X, TX \otimes L^*) \neq 0$:

(1) $(X, L) = (\mathbb{CP}_n, \mathcal{O}(\mathrm{s}))$ =  ($n \geqslant 5$ nodes) for some s $\geqslant$ 1. The odd dimensional projective space has another representation as

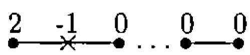
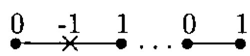
$$(\mathbb{CP}_{2n-1}, \mathcal{O}(s)) =$$  ($n \geqslant 3$ nodes).

The long exact sequence of

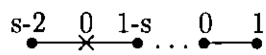
$$0 \longrightarrow \mathcal{O}(-\mathrm{s}) \longrightarrow \mathbb{C}^{n+1} \otimes \mathcal{O}(1 - \mathrm{s}) \longrightarrow TX \otimes L^* \longrightarrow 0$$

implies $\mathrm{H}^1(X, TX \otimes L^*) = 0$ for all s $\geqslant$ 1.

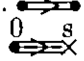
(2) $(X, L) =$  for some s $\geqslant$ 1. There are two maximal roots

$$\mu_1 =$$  , $$\mu_2 =$$  .

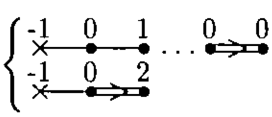
That $J^2(-\lambda + \mu_1) = 0$ for all s $\geqslant$ 1 follows from the proof of Lemma 4.2(iv), while

$$\sigma_1 \circ \sigma_2(-\lambda + \mu_2 + \eta) - \eta =$$ 

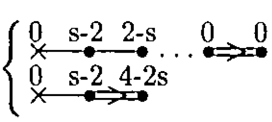
implies $J^2(-\lambda + \mu_2) = 0$ for all s $\geqslant$ 1 as well. Thus $\mathrm{H}^1(X, TX \otimes L^*) = 0$ for all s $\geqslant$ 1.

(3) $(X, L) =$  ($n \geqslant 3$ nodes) for some s $\geqslant$ 1. Note that for $n = 3$ this pair is biholomorphic to  [35] .

The maximal root is

$$\mu_1 =$$ 

Then an easy computation shows

$$\sigma_2 \circ \sigma_1(-\lambda + \mu_1 + \eta) - \eta = =$$ 

which implies

$$\mathrm{H}^1(X, TX \otimes L^*) = J^2(-\lambda + \mu_1) = \begin{cases} \mathbb{C} & \mathrm{s} = 2 \\ 0 & \text{othwerwise.} \end{cases}$$

(4) $(X, L) = \overset{0}{\bullet}\!-\!\overset{s}{\times}\!\Rightarrow\!\overset{0}{\bullet}$ for some s $\geqslant 1$. The maximal roots are

$$\mu_1 = \overset{2}{\bullet}\!-\!\overset{-1}{\times}\!\Rightarrow\!\overset{0}{\bullet} \quad , \qquad \mu_2 = \overset{0}{\bullet}\!-\!\overset{-1}{\times}\!\Rightarrow\!\overset{2}{\bullet}$$

Then

$$\begin{aligned} \sigma_3 \circ \sigma_2(-\lambda + \mu_1 + \eta) - \eta &= \overset{2-s}{\bullet}\!-\!\overset{-s}{\times}\!\Rightarrow\!\overset{2s-2}{\bullet} \\ \sigma_1 \circ \sigma_2(-\lambda + \mu_2 + \eta) - \eta &= \overset{s-2}{\bullet}\!-\!\overset{0}{\times}\!\Rightarrow\!\overset{2-2s}{\bullet} \end{aligned}$$

implying $\mathrm{H}^1(X, TX \otimes L^*) = J^2(-\lambda + \mu_1) + J^2(-\lambda + \mu_2) = 0$ for all s $\geqslant 1$.

(5) $(X, L) = \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!\Rightarrow\!\overset{s}{\times}$ (n nodes) for some s $\geqslant 1$. This pair is biholomorphic to the following one [35]

$$(X, L) = \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!\begin{cases}\!-\overset{0}{\bullet} \\ \!-\overset{s}{\times}\end{cases} \quad (n+1 \text{ nodes}).$$

Then, by Lemma 4.2(iv), $\mathrm{H}^1(X, TX \otimes L^*) = 0$ for all s $\geqslant 1$.

(6) $(X, L) = \overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{s}{\times}\!\Rightarrow\!\overset{0}{\bullet}$ for some s $\geqslant 1$. The maximal roots are

$$\mu_1 = \overset{1}{\bullet}\!-\!\overset{0}{\bullet} \cdots \overset{1}{\bullet}\!-\!\overset{-1}{\times}\!\Rightarrow\!\overset{0}{\bullet} \quad , \qquad \mu_2 = \overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{-1}{\times}\!\Rightarrow\!\overset{2}{\bullet} .$$

The proof of Lemma 4.2(iv) implies $J^2(-\lambda + \mu_2 + \eta) = 0$ for all s $\geqslant 1$, while

$$\sigma_n \circ \sigma_{n-1}(-\lambda + \mu_1 + \eta) - \eta = \overset{1}{\bullet}\!-\!\overset{0}{\bullet} \cdots \overset{1}{\bullet}\!-\!\overset{-s}{\times}\!\Rightarrow\!\overset{2s-2}{\bullet}$$

implies $J^2(-\lambda + \mu_1 + \eta) = 0$ for all s $\geqslant 1$. Whence, by Lemma 4.1, $\mathrm{H}^1(X, TX \otimes L^*) = 0$ for all s $\geqslant 1$.

(7) $(X, L) = \overset{0}{\bullet}\!-\!\overset{s}{\times}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet}$ ($n \geqslant 3$ nodes) for some s $\geqslant 1$. The maximal roots are

$$\mu_1 = \overset{2}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \quad , \qquad \mu_2 = \overset{0}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} .$$

The vanishing of $J^2(-\lambda + \mu_1)$ for all $n \geqslant 4$, s $\geqslant 1$ follows from the proof of Lemma 4.2(iv). That this module vanishes for $n = 3$, s $\geqslant 1$ follows from a simple calculation:

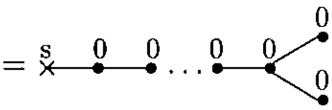
$$\sigma_3 \circ \sigma_2(-\lambda + \mu_1 + \eta) - \eta = \overset{2-s}{\bullet}\!-\!\overset{1-s}{\times}\!\Leftarrow\!\overset{s-2}{\bullet} \quad .$$

Analogously, one finds

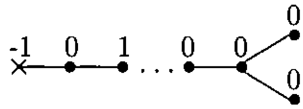
$$\sigma_1 \circ \sigma_2(-\lambda + \mu_2 + \eta) - \eta = \overset{s-1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1-s}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet}$$

implying $J^2(-\lambda+\mu_2)=0$ for all s $\geqslant 2$ and $J^2(-\lambda+\mu_2)=\mathbb{C}$ for s $=1$. Therefore, by Lemma 4.1,

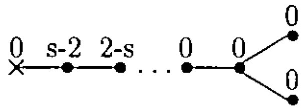
$$\mathrm{H}^1(X,TX\otimes L^*)=\begin{cases}\mathbb{C} & s=1\\ 0 & \text{otherwise.}\end{cases}$$

(8) $(X,L)=$ 

($n\geqslant 5$ nodes) for some s $\geqslant 1$. The maximal root is

$\mu_1=$ 

and an easy calculation shows that

$\sigma_2\circ\sigma_1(-\lambda+\mu_1+\eta)-\eta\ =$ 

.

Therefore,

$$\mathrm{H}^1(X,TX\otimes L^*)=J^2(-\lambda+\mu_1)=\begin{cases}\mathbb{C} & s=2\\ 0 & \text{otherwise.}\end{cases}$$

(9) $(X,L)=$ 

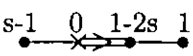
for some s $\geqslant 1$. The maximal roots are

$\mu_1=$ 

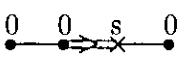
, $\mu_2=$ 

.

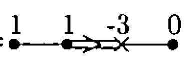
From the proof of Lemma 4.2(iv) it follows that $J^2(-\lambda+\mu_1)=0$ for all s $\geqslant 1$. The only element of the Weyl group of length 2 which can, in principle, make $-\lambda+\mu_2+\eta$ strictly dominant is $\sigma_2\circ\sigma_1$. Since

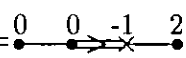
$\sigma_2\circ\sigma_1(-\lambda+\mu_2+\eta)-\eta\ =$ 

the module $J^2(-\lambda+\mu_2)$ vanishes for all s $\geqslant 1$. Therefore, $\mathrm{H}^1(X,TX\otimes L^*)=0$ for all s $\geqslant 1$.

(10) $(X,L)=$ 

for some s $\geqslant 1$. The maximal roots are

$\mu_1=$ 

, $\mu_2=$ 

.

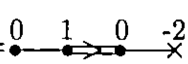
From the proof of Lemma 4.2(iv) it follows that $J^2(-\lambda+\mu_2)=0$ for all s $\geqslant 1$. Since

$\sigma_4\circ\sigma_3(-\lambda+\mu_1+\eta)-\eta\ =$ 

the module $J^2(-\lambda+\mu_2)$ vanishes for all s $\geqslant 1$. Therefore, $\mathrm{H}^1(X,TX\otimes L^*)=0$ for all s $\geqslant 1$.

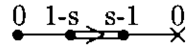
(11) $(X,L)=$ 

for some s $\geqslant 1$. The maximal root is

$\mu_1=$ 

.

Since

$$\sigma_3 \circ \sigma_4(-\lambda + \mu_1 + \eta) - \eta \;=$$ 

we obtain

$$\mathrm{H}^1(X, TX \otimes L^*) = J^2(-\lambda + \mu_1) = \begin{cases} \mathbb{C} & \text{s=1} \\ 0 & \text{otherwise.} \end{cases}$$

(12) $(X, L) =$ 

for some s $\geqslant 1$. The maximal root is $\mu_1 =$ 

. Hence

$$\sigma_2 \circ \sigma_1(-\lambda + \mu_1 + \eta) - \eta \;=$$ 

implying $\mathrm{H}^1(X, TX \otimes L^*) = J^2(-\lambda + \mu_1) = 0$ for all s $\geqslant 1$.

Theorem B is proved. □

Acknowledgement. It is a pleasure to thank Prof. S. Donaldson and Dr L. Schwachhöfer for valuable comments and discussions.

References

[1] Ahiezer, D.N.: Homogeneous complex manifolds. In: Several Complex Variables IV. Springer, 1990

[2] Akivis, M.A.: Webs and almost-Grassmanian structures. Dokl. Akad. Nauk USSR, **252**, 267-270 (1980)

[3] Alekseevski, D.V.: Riemannian spaces with unusual holonomy groups. Funct. Anal. Appl. **2**, 97-105 (1968)

[4] Ambrose, W., Singer, I.M.: A theorem on holonomy. Trans. Amer. Math. Soc. **79**, 423-443 (1953)

[5] Bailey, T.N., Eastwood, M.G.: Complex paraconformal manifolds – their differential geometry and twistor theory. Forum Math. **3**, 61-103 (1991)

[6] Baston, R.J., Eastwood, M.G.: The Penrose transform, its interaction with representation theory. Oxford University Press, 1989.

[7] Berger, M.: Sur les groupes d'holonomie des variétés á connexion affine et des variétés Riemanniennes. Bull. Soc. Math. France **83**, 279-330 (1955)

[8] Berger, M.: Les espaces symétriques noncompacts. Ann. Sci. École Norm. Sup. **74**, 85-177 (1957)

[9] Besse, A.: Einstein manifolds. Springer, 1987

[10] Bott, R.: Homogeneous vector bundles. Ann. Math. **66**, 203-248 (1957)

[11] Bourbaki, N.: Groupes et algébres de Lie. Hermann 1968

[12] Bryant, R.: Metrics with exceptional holonomy. Ann. of Math. **126**, 525-576 (1987)

[13] Bryant, R.: Two exotic holonomies in dimension four, path geometries, and twistor theory. Proc. Symposia in Pure Mathematics **83**, 33-88 (1991)

[14] Bryant, R.: Classical, exceptional, and exotic holonomies: a status report. Preprint, 1995

[15] Cartan, É.: Les groupes de transformations continus, infinis, simples. Ann. Éc. Norm. **26**, 93-161 (1909)

[16] Cartan, É.: Les groups d'holonomie des espaces généralisés. Acta Math. **48**, 1-40 (1926)

[17] Cartan, É.: Sur une classe remarquable d'espaces de Riemann. Bull. Soc. Math. France **54**, 214-264 (1926); **55**, 114-134 (1927)

[18] Chi, Q.-S., Merkulov, S.A., Schwachhöfer, L.J.: On the existence of an infinite series of exotic holonomies. Inv. Math., to appear

[19] Chi, Q.-S., Schwachhöfer, L.J.: Exotic holonomy on moduli spaces of rational curves. Preprint MPI/94-138.

[20] Hano, J., Ozeki, H.: On the holonomy groups of linear connections. Nagoya Math. J. **10**, 97-100 (1956)

[21] Humphreys, J.E.: Introduction to Lie Algebras and Representation Theory. Springer-Verlag, 1987

[22] Iwahori, N.: On real irreducible representations of Lie algebras. Nagoya Math. J. **14**, 59-83 (1959)

[23] Kobayashi, S., Nagano, K. On filtered Lie algebras and geometric structures II. J. Math. Mech. **14**, 513-521 (1965)

[24] Kobayashi, S., Ochiai, T.: On complex manifolds with positive tangent bundles, J. Math. Soc. Japan **22**, 499-525 (1970)

[25] Manin, Yu.I.: New dimensions in geometry. Springer Lect. Notes in Math. 1111, 59-99 (1985)

[26] Manin, Yu.I.: Gauge Field theory and Complex Geometry, 2nd Edition. Springer, in press

[27] Merkulov, S.A.: Paraconformal supermanifolds and non-standard supergravity theories, Class. Quantum Grav. **8**, 557-569 (1991)

[28] Merkulov, S.A.: Existence and geometry of Legendre moduli spaces. Math.Z., to appear

[29] Merkulov, S.A.: Addendum in [26]

[30] under preparation

[31] Nakano, S.: On complex analytic vector bundles, J.Math.Soc.Japan **7**, 1-12 (1955)

[32] Penrose, R.: Non-linear gravitons and curved twistor theory. Gen. Rel. Grav. **7**, 31-52 (1976)

[33] Salamon, S.M.: Riemannian Geometry and Holonomy Groups. Longman, 1989.

[34] Singer, I.M., Sternberg, S.: The infinite groups of Lie and Cartan I. J. Analyse Math. **15**, 1-114 (1965)

[35] Steinsiek, M.: Transformations groups on homogeneous-rational manifolds. Math. Ann. **260**, 423-435 (1982)