HOMOTOPICAL ALGEBRA

AND

ALGEBRAIC HOMOTOPY

by

Hans Joachim Baues

Sonderforschungsbereich 40
Theoretische Mathematik
Beringstraße 4
D-5300 Bonn 1

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3

Preprint MPI/SFB 85-5

(This is the introduction of a
forthcoming book, about 700 typed pages)

To Barbara,
and our children,
Charis and Sarah.

Preface

Homotopical algebra is playing an increasingly important role in topology and algebra. A first fundamental application of homotopical algebra was given by QUILLEN in the development of rational homotopy theory. There exists no textbook on homotopical algebra at any level except the extremely condensed and specific lecture notes of QUILLEN (1967).

The first part of this book is designed to be a mixture between a report and a far reaching elaboration of homotopical algebra.

The second part contains examples and applications towards algebraic homotopy. We describe new results concerning 'towers of categories' which approximate (A) the homotopy category of spaces. These results also imply various basic theorems in the literature; in particular our results on CW-complexes continue the combinatorial homotopy of J.H.C. WHITEHEAD. Also most of the material of the second part is not contained in any textbook on algebraic topology and homotopy theory.

The reader might be a beginning student or a new comer since in the first part homotopical algebra is developed in the presence of a few axioms. Here the author has aimed to provide the reader with sufficient details of all proofs. Many examples and applications in topology and algebra are discussed which illustrate the abstract theory. This way the reader can learn a great deal of ordinary homotopy theory. The axiomatic approach as well offers a new way of organizing a course in homotopy theory which avoids numerous redundant proofs.

The bulk of the book develops methods towards the solution of the homotopy classification problem. The author was tempted to deduce the theory as far as possible from the axioms. Thereafter concrete data like polyhedra, differential algebras or other kinds of objects are pluged in. This yields numerous applications of the abstract theory and it saves a lot of work in the various fields of application.

As prerequisites the reader should know elementary parts of topology and the language of categories. The book can be used also by readers who have only little knowledge of topology and homotopy theory, for example when they want to apply the methods of homotopical algebra in an algebraic context. Yet some knowledge of homotopy theory is helpful since ordinary homotopy theory of topological spaces serves as a leading thread.

There are chapters I,II, ... and the chapters are subdivided into several sections like § 0, § 1, § 1a, § 1b, § 2 ... . Definitions, propositions, remarks etc. are consecutively numbered in each section, each number preceded by the section number, for example (1.5) or (1a.5). A reference like (II. 5.6) points to (5.6) in chapter II, while (5.6) points to (5.6) in the chapter at hand. References to the bibliography are given by the author's name, e.g. J.H.C. WHITEHEAD (1950).

Some of the ideas of this book were presented to the conference "Rational homotopy theory and homotopy theory" in Bonn (1981) and to the conference "Homotopie algebrique et algèbre locale" in Luminy (1982). Also at the invitation of the university of Lille I lectured on this book in 1982. Moreover I presented some results of this book to the "Arbeitstagung Topology" (1984) of the 'Sonderforschungsbereich 170, Geometrie und Analysis' in Göttingen.

I would like to acknowledge the support of the 'Sonderforschungsbereich 40, Theoretische Mathematik' and of the 'Max-Planck-Institut für Mathematik' in Bonn. Moreover I am very grateful to A. GROTHENDIECK for his interest and for a series of letters concerning chapter I and II.

Bonn, January 1985

H.J. Baues

Contents of Part 1

Chapter I: Axioms for homotopy theory

Chapter II: Homotopy theory in a cofibration category

Chapter III: Extensions of categories and the cohomology of a small category

Chapter IV: Maps between mapping cones

Chapter V: Complexes in a cofibration - category

Chapter VI: Maps between complexes and towers of categories

Contents of Part 2:

Chapter VII: The tower of categories for POSTNIKOV decompositions

Chapter VIII: The tower of categories for CW-decompositions under D

Problems of algebraic homotopy

In his lecture at the international congress of mathematicians (1950) J.H.C. WHITEHEAD outlined the idea of algebraic homotopy as follows:

> "In homotopy theory, spaces are classified in terms of homotopy classes of maps, rather than individual maps of one space in another. Thus, using the word category in the sense of S. EILENBERG and Saunders MAC LANE, a homotopy category of spaces is one in which the objects are topological spaces and the "mappings" are not individual maps but homotopy classes of ordinary maps. The equivalences are the classes with two-sided inverses, and two spaces are of the same homotopy type if and only if they are related by such an equivalence. The ultimate object of algebraic homotopy is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that "analytic" is equivalent to "pure" projective geometry."

This object of algebraic homotopy in particular includes the following basic homotopy classification problems:

(1) Classify homotopy types of polyhedra $X, Y \dots$ by algebraic data!
Compute the set of homotopy classes of maps, $[X,Y]$, in terms of the classifying data for X and Y ! Moreover, compute the group of homotopy equivalences, $\mathrm{Aut}(X)$!

There is no restriction on the algebraic theory, which might solve these problems, except the restriction of "effective calculability". Indeed, algebraic homotopy is asking for a theory which, a priori, is not known and which is not uniquely determined by the problem. Moreover, it is not clear whether at all there is a suitable purely algebraic theory for the problem better than the simplicial approach of KAN. For example, in spite of enormous efforts in the last four decades, it is still not possible to compute the <u>homotopy groups of spheres</u>

(2) $$\pi_m S^n = [S^m, S^n]$$

which turned out to have a very rich structure. This example shows that the difficulties for a solution of the homotopy classification problems increase rapidly when, for the spaces involved, the

(3) <u>range</u> = (dimension) - (degree of connectedness)

is growing. On the other hand by a classical result of SERRE the <u>rational homotopy groups of spheres</u>

(4) $$\pi_m(S^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} \;, \; m = n > o \\ \mathbb{Q} \;, \; m = 2n-1 \;, \; n \text{ even} \\ 0 \quad \text{otherwise} \end{cases}$$

are indeed simple objects. These remarks indicate two suitable restrictions for the homotopy classification problem: Consider the problem in a small range (3) or consider the problem for rational spaces.

WHITEHEAD (1949) examined examples of the homotopy classification of polyhedra in <u>a small range</u>. In particular, he classified 3-dimensional homotopy types and simply connected 4-dimensional homotopy types, (we give new proofs of these results in this book). The following related problems ever since remained unsolved though they are just the first steps beyond WHITEHEAD's results:

(5) Compute all homotopy classes of maps between simply connected 4-dimensional polyhedra in terms of WHITEHEAD's

classifying data! Compute the group of homotopy equivalences of such polyhedra! Classify the homotopy types of all simply connected 5-dimensional polyhedra! Classify the homotopy types of all 4-dimensional polyhedra with nontrivial fundamental group π .

These are low dimensional cases of the general homotopy classification problems (1). We illustrate various technical results of homotopical algebra by applying them to WHITEHEAD's problems (5) above; this actually pin points the basic steps for the solution of these problems.

On the other hand QUILLEN (1969) studied the rationalization of the homotopy category of simply connected polyhedra obtained by inverting all maps f which induce isomorphisms $\pi_n(f) \otimes \mathbb{Q}$, $n \geq 2$, on rational homotopy groups. He showed that this rationalization is a category equivalent to the purely algebraic homotopy category of differential LIE algebras over $\mathbb{Q}$. In addition SULLIVAN (1977) obtained the 'dual' result using the DE RHAM algebra. These results form the so far best general algebraic approximations of the homotopy theory of simply connected polyhedra. There is also a slight generalization for nilpotent spaces and for tame spaces respectively.

A disatvantage of rational homotopy theory is the fact that the theory does not apply to polyhedra with arbitrary fundamental groups. We therefore introduce the twisted rationalization of the homotopy category of all polyhedra by inverting all maps f which induce isomorphisms $\pi_1(f)$ and $\pi_n(f) \otimes \mathbb{Q}$, $n \geq 2$. This leads to the following fundamental problem of algebraic homotopy:

(6) Construct a purely algebraic category equivalent to the twisted rationalization of the homotopy category of all polyhedra!

The solution of this problem should imply QUILLEN'S result for the subcategory of simply connected polyhedra and should be compatible with the result of WHITEHEAD on the classification of three dimensional polyhedra.

More generally algebraic homotopy sets us the task:

(7) Construct algebraic model categories M of subcategories T of the homotopy category and measure the difference between M and T !

The theory of towers of categories and of twisted chain complexes in this book is a tool for the problems in(6) and (7). .

Model categories often are obtained by functors which carry polyhedra to algebraic objects like

(8) chain complexes,

(9) differential algebras,

given by the integral chain complex of a loop space,or

(10) differential LIE algebras and DE RHAM algebras respectively.

Moreover KAN introduced

(11) simplicial sets and simplicial free groups

which actually form categories equivalent to the homotopy category of polyhedra.

The categories, defined by the objects in (8) ... (11) respectively, are as well homotopy categories in which the "mappings" are not individual maps but homotopy classes of maps. There are actually many more algebraic homotopy categories some of them not related to spaces at all. In each of them one has homotopy classification problems as in (1). It turned out that there is a striking similarity of properties of such homotopy categories. This fact and the large number of homotopy categories and of model categories (7) make it necessary to develop a theory based on axioms which are in force in most of the homotopy categories.

We show that even part of the homotopy classification problem is of an abstract nature which does not depend on the underlying homotopy category. This part is longing for the elaboration of homotopical algebra. The abstract theory of part 1 originates from the authors interest in the concrete problems (5), (6) and (7) above. We describe numerous examples and applications of the abstract theory refering to these problems. Further applications will appear elsewhere.

There are indeed many more examples and applications of the abstract theory of this book in topology and algebra. The reader will understand that we cannot discuss such applications in all the many different cofibration categories and fibration categories described in chapter I. We only picked out some applications relevant for the problems of algebraic homotopy.

Introduction of part 1:

In this part homotopy theory is developed abstractly in the presence of only four axioms which define a cofibration category. Many applications of the abstract theory and numerous examples in topology and algebra are described. The axioms have been chosen according to the following two criteria:

(12) The axioms should be sufficiently strong to permit the basic constructions of homotopy theory.

(13) The axioms should be as weak (and as simple) as possible, so that the constructions of homotopy theory are available in as many contexts as possible.

There is indeed a wide variety of contexts where the techniques of homotopy theory are useful. Therefore the unification due to the abstract development of the theory posesses major advantages: One proof replaces many, in addition an interplay takes place among the various applications. This is very fruitful for many topological and algebraic contexts. We derive from the axioms a sophisticated theory which in topology can be compared with the combinatorial homotopy theory in the sense of J.H.C. WHITEHEAD. This leads further than the results on abstract homotopy theory previously obtained in the literature. Our development of the abstract theory is mainly directed to the homotopy classification problems (1).

The idea of axiomatizing homotopy is used implicitly by ECKMANN-HILTON in studying the phenomena of duality in homotopy theory. HILTON (1965), (p. 168), actually draws up a programme by mentioning:

"Finally we remark that one would try to define the notions of cone, suspension, loop space, etc. for the category C and thus place the duality on a strict logical basis. It would seem therefore that we should consider an abstract system formalizing the category of spaces, its homotopy category and the homotopy functors connecting them."

To carry out this programme is part of homotopical algebra. A specific approach is due to QUILLEN (1967) who introduced the notion of a closed model category which is defined by axioms on cofibrations , fibrations and weak equivalences. This notion is self dual, that is, the dual category is once more a closed model category where the roles of fibrations and cofibrations are interchanged. The homotopy theory of topological spaces, however, does not satisfy such a strict duality. For this reason and for the sake of more generality and more simplicity we introduce the notion of a cofibration category which extracts from a closed model category the essential features of cofibrations and weak equivalences. The axioms of a fibration category are obtained by formally dualizing the axioms of a cofibration category.

We now discuss briefly the contents of the chapters in part 1. For a further discussion see chapter introductions.

Chapter I: The foundational part of homotopical algebra tries to find the abstract notion of a homotopy theory in sufficient generality to cover the different homotopy theories encountered. For this reason we compare the various systems of axioms on a homotopy theory which appeared in the literature. It turns out that the axioms of a cofibration category form a good compromise with respect to the criteria (12) and (13) above. Moreover we describe a long list of examples of cofibration categories.

In topology cofibrations (and dually fibrations) are introduced by use of the cylinder $I \times X$ where I is the unit interval.

We define an I-category by certain obvious axioms on an abstract cylinder functor in the sense of KAN and we show that an I-category is a cofibration category. In particular, the category of topological spaces is easily seen to be an I-category. Therefore cofibrations in topology satisfy our axioms of a cofibration category and by strict duality fibrations in topology satisfy the axioms of a fibration category. These results show that the abstract theory actually implies a lot of ordinary homotopy theory, in particular many results discussed by DIECK-KAMPS-PUPPE (1970) or JAMES (1985). For example a theorem of DOLD is available in any cofibration category and thus in any I-category and thus in topology.

Cofibrant objects and fibrant objects are defined by the structure of a cofibration category. The role of such objects is illustrated by examples: The singular set of a space and a resolution of an algebra correspond to cofibrant objects. Localizations, completions, and QUILLEN's (+)-constructions respectively are fibrant objects.

Chapter II. The technical part of homotopical algebra deduces from the axioms results relevant in homotopy theory. This is rewarding since it avoids redundant proofs in the many fields where the axioms are valid. In particular, it avoids all 'dual' proofs in a fibration category. We emphasize that the abstract theory of part 1 is worked out for applications on concrete problems like the problems of WHITEHEAD in (5). We intend, however, to deduce the theory as far as possible from the axioms. Thereafter we apply the results in a concrete situation. Our intention leads to a far reaching and sophisticated homotopy theory which is available in any cofibration category:

We introduce homotopy groups, the action of the fundamental group, homotopy groups of pairs, and the exact sequence for such pairs. We study (general) mapping cones and the (general) cofiber sequence with respect to based objects in a cofibration category. This indeed is a useful generalization of the usual treatment of the cofiber sequence in the sense of D. PUPPE. For example the EILENBERG-MAC LANE fibrations, which classify cohomology with local coefficients, are based objects in a fibration category and the fibrations in a POSTNIKOV - decomposition of a nonsimple space are dual to a general mapping cone. Therefore the general fiber sequence (dual to the general cofiber sequence) can be applied here while the usual fiber sequence does not work.

Though we do not construct function spaces we consider groups which correspond to the homotopy groups of function spaces. The exact sequence for such groups shows that results of BARCUS-BARRATT and JAMES-THOMAS are strictly dual and that these results hold in any cofibration category. Here the difference construction, the partial suspension and the functional suspension in a cofibration category are of importance. These technical constructions are crucial in the proofs of the following chapters.

We also discuss products in a cofibration category and we use them to define WHITEHEAD-products and the HOPF-construction. It turns out that the cofibration category of spaces and the fibration category of spaces have very different properties with respect to products and sums respectively. This phenomenon contradicts a global assumption of duality in topology.

Finally we describe some examples on the classification of maps in topology which we deduce from the abstract theory. For example we describe the set of homotopy classes of maps $[M,U]$ where M is a simply connected 4-dimensional manifold. Dually we obtain the classical result of PONTRJAGIN on the set $[X,S^2]$ where X is a 3-dimensional polyhedron. Also the result of DOLD-WHITNEY on sphere bundles over a 4-dimensional polyhedron is an illustrating example.

Chapter III: Our study of the homotopy classification problem leads to certain new concepts having to do with general categories and functors. In particular we introduce linear extensions of categories which generalize the classically considered extensions of groups. Numerous examples of such linear extensions are described in this and in the following chapters. It is an old result that extensions of a group are classified by a second cohomology group. In a similar way we classify the linear extensions of a small category. Here we use new cohomology groups of a small category which generalize those of HOCHSCHILD-MITCHELL which in turn generalize the singular cohomology of the classifying space of the category. We introduce a cup product for our cohomology groups. Moreover, we show that the HOPF-construction yields a canonical element in the first cohomology of the category of free abelian groups; we call this element the HOPF-class. It seems that such cohomology classes are crucial ingredients of algebraic homotopy.

Chapter IV: We consider maps between mapping cones in a cofibration category. In particular principal maps and twisted maps between mapping cones yield subcategories PRIN and TWIST of the homotopy category. These categories are shown to be linear extensions of model categories $Prin/\simeq$ and $Twist/\simeq$ respectively. In many applications in topology it is possible to compute the model categories Prin and Twist and also the homotopy relation $\simeq$ on these categories. It is much harder to compute the categories PRIN and TWIST since there is the extension problem. We also have categories of tracks $\overline{PRIN}$ and $\overline{TWIST}$ from which we derive PRIN and TWIST respectively as quotient categories. As an example we consider the extension problem for the category of tracks of mappings between one point unions of 2-dimensional spheres. It turns out that the cup product of the HOPF-class and of the cohomology class, deduced from the lower central series of free groups, describes this extension problem.

Various examples and applications of the abstract results in topology are described in the second part of this chapter. In particular, twisted maps are applied to WHITEHEAD'S problem (5). This indeed is a good test for the feasebility of the abstract theory in concrete cases. As an example we compute the group of homotopy equivalences of a simply connected 4-dimensional manifold and of the connected sum $(S^1 \times S^3) \# (S^2 \times S^2)$ which is not simply connected.

Moreover two crucial results are proved, the general suspension theorem and the general loop theorem. These yield criteria for twisted maps in topology and thus they allow a new approach to the following problems:

(14) Describe conditions on the maps f and g which imply that the mapping cones C_f and C_g are homotopy equivalent! Compute $[C_f, C_g]$ and $\mathrm{Aut}(C_f)$ in terms of the homotopy classes of f and g !

(15) Describe conditions on the maps f and g which imply that the homotopy theoretic fibers P_f and P_g are homotopy equivalent! Compute $[P_f, P_g]$ and $\mathrm{Aut}(P_f)$ in terms of the homotopy classes of f and g .

Our results on these problems are valid in a much better range than the known results in the literature. We also study (14) and (15) in the category of spaces under D and in the category of spaces over D respectively.

Chapter V: The category of filtered objects in a cofibration category is again a cofibration category. For a filtered object X the homotopy groups of the function space U^X are embedded in a short exact $(\lim^1)$-sequence in the sense of MILNOR. The main part of this chapter is concerned with complexes which are filtered objects obtained by a succession of attaching cones. In topology CW-compleces (and dually POSTNIKOV-decompositions) are examples of such complexes. A spectral sequence converging to the homotopy groups of the function space U^X is constructed. The E_2-term is given by twisted cohomology groups of the complex X . This cohomology is defined in terms of a functor which carries complexes to twisted chain complexes. In topology the twisted chain complex of a CW-complex is given by the cellular chain complex of the universal covering. Therefore the twisted cohomology in topology yields the cohomology with local coefficients. On the other hand in the category of differential algebras the twisted cohomology corresponds to the HOCHSCHILD cohomology.

Complexes, twisted chain complexes, and twisted cohomology groups are defined in any cofibration category. These are features of homotopical algebra related to classical homological algebra. At this point we compare our concept with the one of QUILLEN (1967) who states:

> "Homotopical algebra or non-linear homological algebra is the generalization of homological algebra to arbitrary categories which results by considering a simplicial object as being a generalization of a chain complex."

We alter this concept by considering a complex instead of a simplicial object as being the generalization of a chain complex.

<u>Chapter VI</u>: Nonlinearity of homotopy theory is caused by the trouble that homotopy groups do not satisfy the <u>excision axiom</u>. The theorem of BLAKERS-MASSEY, however, shows that excision is satisfied in a small range, see (3). This actually implies a certain amount of linearity which in topology leads to stable homotopy theory. Using linearity conditions, corresponding to the BLAKERS-MASSEY theorem, we define a (very) good class of complexes in a cofibration category. In topology CW-complexes and dually POSTNIKOV-decompositions form examples of such classes of complexes. Moreover the r-step structure of CW-complexes (and dually of POSTNIKOV-decompositions) yields such a class for each r, $r \geq 1$.

Our main result is the tower of categories, denoted by TWIST_*, which approximates the homotopy category of maps between complexes in a (very) good class.

A tower of categories essentially is a succession of linear extensions of categories. As an illustration we describe a <u>tower of groups</u> $(n \geq 2)$

(16)
$$\begin{array}{ccccccccccc} & & A_n & & & & & & A_2 & & \\ & & \downarrow i & & & & & & \downarrow & & \\ G & \longrightarrow \dots \longrightarrow & G_{n+1} & \xrightarrow{\lambda} & G_n & \longrightarrow \dots \longrightarrow & G_3 & \longrightarrow & G_2 = K & & \\ & & & & \downarrow o & & & & \downarrow & & \\ & & & & H_n & & & & H_2 & & \end{array}$$

which is characterized by the following properties:

a) $G = \varprojlim G_n$ is the inverse limit of groups

b) The sequence

$0 \longrightarrow A_n \longrightarrow G_{n+1} \longrightarrow G_n \longrightarrow H_n$ is exact for all n.

(c) A_n and H_n are G_n-bimodules with the action denoted by

$$(\xi, a, \eta) \longmapsto \xi_* \eta^*(a) \quad , \quad (\xi, \eta \in G_n)$$

(d) i is equivariant, that is

$$i(\xi_*(\xi^{-1})^*(a)) = x(ia)x^{-1} \quad , \quad x \in \lambda^{-1}(\xi) \quad .$$

(e) O is a derivation, that is

$$O(\xi \cdot \eta) = \xi_* O(\eta) + \eta^* O(\xi) \quad .$$

Towers of categories are the canonical extension of such towers of groups in the language of categories.

In the tower $TWIST_*$, constructed in this chapter, the top-category G corresponds to the homotopy category of finite complexes and the bottom category K corresponds to the homotopy category of twisted chain complexes. Therefore each group $G(X) = Aut(X)$ of homotopy equivalences of a complex X has the structure of a tower of groups as in (16) where $K(X)$ is the group of homotopy equivalences of the twisted chain complex. The tower $TWIST_*$ describes precisely the connection between the category of complexes and the category of twisted chain complexes.

There are many important applications of the tower $TWIST_*$. For example, the tower implies a WHITEHEAD-theorem available in any cofibration category. In addition, as we will see in part 2, the tower has essential features of WHITEHEAD's combinatorial homotopy theory of polyhedra.

Introduction of part 2:

This part contains applications of the abstract theory in part 1, in particular the tower of categories $TWIST_*$ is studied in concrete examples. We consider two topological examples derived from

(17) the POSTNIKOV decompositions of fibrations over D , and from

(18) the CW-decompositions of cofibrations under D .

Here D is a fixed space. The tower of categories of POSTNIKOV decomositions (17) is a (dual) example in a fibration category. It will be very helpful for the reader to study first these applications which illustrate and motivate the abstract (and technical) theory on complexes and towers of categories in chapter V and chapter VI. These applications yield many new results on the homotopy classification problems (1) for polyhedra.

Chapter VII: The tower of categories for POSTNIKOV decompositions approximates the homotopy category of topological fibrations and of fiber preserving maps over a fixed base space D , (the result is also of interest when $D = *$ is a point.)

There are some important consequences. For example the group of fiber preserving homotopy equivalences of a fibration has the structure of a tower of groups as in (16). One can derive easely results of DROR-ZABRODSKY on the nilpotency of certain subgroups from this fact.

Moreover, the tower yields an obstruction theory for the realizability of homomorphisms between the homotopy groups of fibers by a fiber preserving map. This improves the method of ADAMS (1956). The obstructions are useful since by the WHITEHEAD theorem they are connected with the classification of homotopy types.

The spectral sequence for homotopy groups of function spaces in a cofibration category, applied to CW-complexes (or dually applied to POSTNIKOV-decompositions), yields the spectral sequences of FEDERER and ATIYAH-HIRZEBRUCH. We also prove a result of G.W. WHITEHEAD on the nilpotency of the groups $[\Sigma X, Y]$ and we prove results of SULLIVAN and SCHEERER on the nilpotency of function spaces. We actually show that these are special cases of results which hold in any cofibration category.

Chapter VIII: This chapter continues the work of J.H.C. WHITEHEAD on the combinatorial homotopy theory of CW-complexes. In fact the essence of combinatorial homotopy can be described by the tower of categories for CW-decompositions. Most of the results of WHITEHEAD's paper "Combinatorial homotopy II" are immediate and very special consequences. We also deduce easely the final theorem in WHITEHEAD's paper "Simple homotopy types" which constructs small CW-decompositions by small models of the chain complex of the universal covering. We give new and conceptually easy proofs of these results. Actually we prove generalizations to the relative case under a space D .

This as well yields finiteness obstructions for relative CW-complexes under D which for D = * coincide with those of WALL (1966).

The ECKMANN-HILTON homology decomposition of a cofibration and also a result of MILNOR on the minimal number of cells in a CW-decomposition are easy examples of small CW-decompositions. To this end we point out that actually most of the small models in homotopy theory (for example minimal models) can be derived from towers of categories along the same lines as our construction of small CW-decompositions. We describe this procedure for differential algebras elsewhere. Also an old result of KAN (1959) on a relation between CW-complexes and free simplicial groups can be obtained by this method.

The tower of categories for CW-decompositions approximates the homotopy category of spaces under D. The bottom category of this tower is the homotopy category of chain complexes of the universal coverings. In low dimensions the tower is very efficient for the homotopy classification problem (1). For example one deduces that homotopy types of 3-dimensional CW-complexes X under D are 1-1 corresponded to purely algebraic homotopy types of 3-dimensional crossed chain complexes under the group $\pi_1(D)$ (provided $\pi_2(D) = 0$). For D = * this is a result of WHITEHEAD. We study in detail the low dimensional part of the tower and we describe the connections with the algebraic homotopy category of crossed chain complexes which are essentially WHITEHEAD's homotopy systems and which are special crossed complexes as studied by BROWN-HIGGINS.

The tower for CW-decompositions shows that the group of homotopy equivalences (in the category of spaces under D) has the structure of a tower of groups as in (16). Moreover, one obtains an obstruction theory for the realizability of chain maps and of homomorphisms between homology groups. This, in particular, is relevant for the classification of 4-dimensional homotopy types; compare WHITEHEAD's problems described in (5) above.

The twisted localization of the homotopy category of spaces with basepoint as well is approximated by a tower of categories which has all the structure of the tower for CW-decompositions discussed above. For example this yields finiteness obstructions for the twisted localizations. Moreover this is relevant for the computation of the twisted rationalization in problem (6).

BIBLIOGRAPHY

ADAMS, J.F.: Four applications of self-obstruction invariants. J. London Math. Soc. 31 (1956) 148 - 159.

ADAMS, J.F. and **HILTON, P.J.:** On the chain algebra of a loop space. Comment. Helv. 30 (1956) 305 - 330.

ANDERSON, D.W.: Fibrations and geometric realizations. Bull. AMS 54 (1978) 765 - 788.

BARCUS, W.D. and **BARRATT, M.G.:** On the homotopy classifications of the extensions of a fixed map. Trans. AMS 88 (1958) 57 - 74.

BARRATT, M.G.: Homology ringoids and homotopy groups Q.J. Math. Oxford (2)5 (1954) 271 - 290.

BAUES, H.J.: Relationen für premäre Homotopieoperationen und eine verallgemeinerte EHP-Sequenz. Ann. scien. Ec. Norm. Sup. 8 (1975) 509 - 533.

Obstruction theory, Springer Lecture Notes in Math. 628 (1977), 387 pages.

Commutator calculus and groups of homotopy classes. London Math. Soc. lecture Note Series 50 (1981), 160 pages. Cambridge University Press.

On the homotopy classification problem, preprint (1983), Max-Planck-Institut für Mathematik in Bonn, MPI/SFB 83 - 25.

The chains on the loops and 4-dimensional homotopy types. Astêrisque 113 - 114 (Homotopie algêbrique and algêbre locale) (1984) 44 - 59.

Geometry of loop spaces and the cobar construction. Memoirs of the AMS 230 (1980) 170 pages.

BAUES, H.J. and **LEMAIRE, J.M.:** Minimal models in Homotopy theory. Math. Ann. 225 (1977) 219 - 42.

BAUES, H.J. and **WIRSCHING, G.:** The cohomology of small categories, preprint (1984) Max-Planck-Institut für Mathematik Bonn, MPI/SFB 84 - 44.

BLAKERS, A.L. and **MASSEY, W.S.:** The homology groups of a triad II. Ann. of Math. 55 (1952) 192 - 201.

BOUSFIELD, A.K.: The localization of spaces with respect to homology. Topology 14 (1975) 133 - 150.

BOUSFIELD, A.K. and **GUGENHEIM, V.K.A.M.:** On PL De Rham theory and rational homotopy type. Memoirs of the AMS 179 (1976).

BOUSFIELD, A.K. and **KAN, D.M.:** Homotopy Limits, completions and localizations Lecture Notes in Math. 304 Springer, Berlin-Heidelberg-New York (1972).

BROWN, K.S.: Abstract homotopy theory and generalized sheaf cohomology. Transactions of the AMS 186 (1973) 419 - 458.

BROWN, R.: Non-abelian cohomology and the homotopy classification of maps Astèrisque, 113 - 114 (1984) 167 - 172.

BROWN, R. and HEATH P.R.: Coglueing homotopy equivalences Math. Z. 113 (1970) 313 - 325.

BROWN, R. and HIGGINGS, P.J.: Crossed complexes and chain complexes with operators. Preprint, University College of North Wales Bangor, Gwynedd. U.K. (1984).

Crossed complexes and non-abelian extensions. Proc. Int. Conf. on Category Theory Gummerbach 1981 Springer Lecture Notes.

BROWN, R. and HUEBSCHMANN, J.: Identities among relations. "Brown and Thickstun (1982), Low dimensional topology" Proceedings 1 Cambridge University Press.

CAPPELL, S.E. and SHANESON, J.L.: On 4-dimensional surgery and applications. Comm. Math. Helv. 46 (1971) 500 - 528.

CARTAN, H. and EILENBERG, S.: Homological algebra. Princeton University Press 1956.

DIDIERJEAN, G. Groupes d'homotopie du monoide des èquivalences d'homotopie fibrèes. C.R. Acad. Sc. Paris t. 292 (1981).

DIECK, tom T., KAMPS, K.H. and PUPPE, D.: Homotopietheorie, Lecture Notes in Math. 157 Springer Verlag (1970).

DOLD, A.: Partitions of unity in the theory of fibrations. Ann. of Math. 78 (1963) 223 - 255.

Halbexakte Homotopiefunktoren. Lecture Notes in Math. 12 Springer Verlag (1966).

DOLD, A. and LASHOF, R.: Principal quasifiberings and fiber homotopy equivalence of bundles Ill. J. Math. 3 (1959) 285 - 305.

DOLD, A. and WHITNEY, H.: Classification of oriented sphere bundles over a 4-complex. Ann. of Math. 69 (1959) 667 - 677.

DROR, E. and ZABRODSKY, A.: Unipotency and nilpotency in homotopy equivalences. Topology 18 (1979) 187 - 197.

DWYER, W. and FRIEDLANDER, E.: Étale K-theory and arithmetic. Bull. AMS 6 (1982) 453 - 455.

DWYER, W.G. and KAN, D.M.: Function complexes for diagrams of simplicial sets. Proc. Konink. Neder. Akad. 86 (1983) 139 - 150.

Function complexes in homotopical algebra. Topology 19 (1980), 427 - 440.

Calculating simplicial locatizations. J. pure appl. Algebra. 18 (1980) 17 - 35.

DWYER, W.G. and KAN, D.M.: Simplicial localizations of categories J. pure appl. Algebra 17 (1980) 267 - 284.

Homotopy theory and semiplicial groupoids Proc. Konink. Neder. Akad. 87 (1984), 379 - 389.

DYER, N.M.: Homotopy classification of (π,m)-complexes. J. pure appl. Algebra 7 (1976) 249 - 282.

EDWARDS, D.A. and HASTINGS, H.M.: Čech and Steenrod homotopy theories. Lect. notes in Math. 542 (1976).

EGGAR, M.H.: Ex-Homotopy theory, Compositio Math. 27 (2) (1973), 185 - 195.

ELLIS, G.J.: Crossed modules and their higher dimensional analogues University of Wales Ph. D. thesis.

FEDERER, H.: A study of function spaces by spectral sequences. Trans. AMS (1956) 340 - 361.

FRITSCH, R. and LATCH, D.M.: Homotopy inverses for Nerve Math. Z. 177 (2) (1981), 147 - 180.

GABRIEL, P. and ZISMAN, M.: Calculus of fractions and homotopy theory, Ergebnisse der Mathematik und ihrer Grenzgebiete 35 Springer-Verlag 1967.

GRAY, B.: Homotopy theory. Acadamic press (1975).

GROTHENDIECK, A.: Pursuing stacks (preprint).

GUGENHEIM, V.K.A.M. and MUNKHOLM, H.J.: On the extended functoriality of Tor and Cotor. J. pure and appl. Algebra 4 (1974) 9 - 29.

HALPERIN, S.: Lectures on minimal models. Lille Publ. (1978) (1981) IRMA. Together with WATKISS, C.: Relative homotopical algebra. Lille Publ. IRMA.

HARTL. M.: The secondary cohomology of the category of finitely generated abelion groups Preprint Bonn (1985), Diplomarbeit Mathematisches Institut der Universität Bonn.

HASTINGS, M.H.: Fibrations of compactly generated spaces. Mich. J. Math. 21 (1974) 243 - 251.

HELLER, A.: Stable homotopy categories. Bull. AMS 74 (1968) 28 - 63.

HELLING, B.: Homotopieklassifikation in der Kategorie der differentiellen Algebren. Diplomarbeit, Math. Inst. der Universität Bonn (1984).

HILTON, P.: Homotopy theory and duality Nelson (1965) Gordon Breach.

General cohomology theory and K-theory. London Math. Soc. Lecture Note 1 (1971).

An introduction to homotopy theory. Cambridge University press (1953).

HILTON, P.J. and MILSLIN, G. and ROITBERG, J.: Localizations of nilpotent groups and spaces. North Holland Math. Studies 15 Amsterdam (1975).

HILTON, P.J. and MISLIN, G. and ROITBERG, J. and STEINER, R.: On free maps and free homotopies into nilpotent spaces. Lecture Notes in Math. 673. Algebraic Topology, Proceedings, Vancover 1977 202 - 218, Springer Verlag 1978.

HILTON, P.J. and STAMMBACH, U.: A Course in Homological Algebra. Springer GTM 4, New York 1971.

HU, S.T.: Homotopy theory, Academic press, New York and London 1959.

HUBER, M. and MEYER, W.: Cohomology theories and infinite CW-complexes. Comment. Math. Helv. 53 (1978) 239 - 257.

HUSEMOLLER, D., MOORE, J.C. and STASHEFF, J.: Differential homological algebra and homogeneous spaces. J. pure and appl. Algebra 5 (1974) 113 - 185.

ILLUSIE, L.: Complexe cotangent et deformations II. Lect. Notes in Math. 283 Springer Verlag.

JAMES, I.M.: General topology and homotopy theory Springer Verlag Berlin (1984).

Ex Homotopy theory I. Illinois J. of Math. 15 (1971) 329 - 345.

JAMES, I.M. and THOMAS, E.: On the enumeration of cross sections. Topology 5 (1966) 95 - 114.

JOHNSTONE, P.T.: Topos Theory. Acadmic Press London 1977.

KAMPS, K.H.: Kan-Bedingungen und abstrakte Homotopietheorie. Math. Z. 124 (1972), 215 - 236.

Fundamentalgruppoid und Homotopien. Arch. Math. 24, 456 - 460.

KAHN, P.J.: Self-equivalences of (n-1)-connected 2n-manifolds. Bull. Amer. Math. Soc. 72 (1966) 562 - 566.

KAN, D.M.: Abstract homotopy I, II. Proc. Nat. Acad. Sci. USA 41 (1955) 1092 - 1096, 42 (1956) 255 - 258.

On homotopy theory and C.s.s. groups. Ann. of Math. 68 (1958) 38 - 53.

A relation between CW-complexes and free C.s.s. groups. Amer. J. Math. 81 (1959) 512 - 528.

KELLY, G.M. and STREET, R.: Review of the elements of 2-categories. Lecture Notes in Math. 420 (1974) 75 - 103 (Springer-Verlag).

LEGRAND, A.: Homotopie des espaces de sections. Lecture Notes in Math. 941 Springer Verlag 1982.

LODAY, J.L.: K-théorie algébrique et représentations de groupes. Ann. Sc. Ec. Norm. Sup. 9 (1976) 309 - 377.

MAC LANE, S.: Homology, Grundlehren 114, Springer 1967.

MAGNUS, W., KARRASS, A. and SOLITAR, D.: Combinatorial group theory Pure and Appl. Math. vol 13 Interscience, New York 1966.

MARCUM, J.H.: Parameter constructions in homotopy theory, An. Acad. brasil. Cienc. (1976) 48 387 - 402.

MASSEY, W.S.: Exact couples in algebraic topology I - V Ann. of Math. 56 (1952) 363 - 396, 57 (1953) 248 - 286.

Mc CLENDON, J.F.: Higher order twisted cohomology operations, Inventiones math. 7 (1969) 183 - 214.

Reducing towers of principal fibrations. Nagoya Math. J. 54 (1974) 149 - 164.

MILGRAM, J.R.: The bar construction and abelian H-spaces Ill. J. Math. 11 (1967) 242 - 250.

MILNOR, J.: On spaces having the homotopy type of a CW-complex. Trans. Amer. Soc. 90 (1959) 272 - 280.

On axiomatic homology theory. Pacific J. Math. 12 (1962) 337 - 341.

Morse theory, Princeton Univ. Press (1963).

MIMURA, M., NISHIDA, G. and TODA: Localization of CW-complexes and its applications, J. Math. Soc. Japan 23 (1971) 593 - 624.

MITCHELL, B.: Rings with several objects, Advances in Math. 8 (1972) 1 - 161.

MOORE, J.C.: Seminaire H. Carlan 1954 - 55 Exp. 3

Differential homological algebra. Proc. Int. Cong. Math. I. 1 - 5 (1970).

MUNKHOLM, H.J.: DGA algebras as in Quillen model category, relations to shm maps, J. pure appl. Alg. 13 (1978) 221 - 232.

NOMURA, Y.: A non-stable secondary operation and homotopy classification of maps. Osaka J. Math. 6 (1969) 117 - 134.

Homotopy equivalences in a principal fiber space. Math. Z. ? (1966) 380 - 388.

OKA, S. and SAWASHITA, N. and SUGAWARA, M.: On the group of self equivalences of a mapping cone. Hiroshima Math. J. 4 (1974) 9 - 28.

PONTRJAGIN, L.: A classification of mappings of the 3-dimensional complex into the 2-dimensional sphere. Rec. Math. (Math. Sbornik) N.S. 9 (51) (1941) 331 - 363.

PUPPE, D.: Homotopiemengen und ihre induzierten Abbildungen I. Math. Z. 69 (1958) 299 - 344.

QUILLEN, D.G.: Homotopical algebra, Lecture Notes in Math. 43 Springer Verlag 1967.

QUILLEN, D.G.: Higher algebraic K-theory I. Springer Lecture Notes 341 (1973) 85 - 147.

An application of simplicial profinite groups. Comment. Math. Helvet. 44 (1969) 45 - 60.

Rational homotopy theory. Ann. of Math. 90 (1969) 205 - 295.

RATCLIFFE, J.G.: Free and projective crossed modules J. London Math. Soc. (2) 22 (1980) 66 - 74.

ROOS, J.E.: Sur les foncteurs dèrivès de $\varprojlim$. Applications. Compte rendue Acad. Sc. Paris 252 (1961) 3702 - 3704.

RUTTER, J.W.: A homotopy classification of maps into an induced fibre space. Topology 6 (1967) 379 - 403.

Self equivalences and principal morphisms. Proc. London Math. Soc. 20 (1970) 644 - 658.

Groups of self homotopy equivalences of induced spaces. Comment. Math. Helv. 45 (1970) 236 - 255.

SAWASHITA, N.: On the group of self equivalences of the product of spheres, Hiroshima Math. J. 5 (1970) 69 - 86.

SCHEERER, H.: Lokalisierungen von Abbildungsräumen. Comp. Math. 40 (1980) 269 - 281.

SHIH, W.: On the group E(X) of Homotopy Equivalence maps. Bull. Amer. Math. Soc. 70 (1964).

SHITANDA, Y.: Sur la thèorie d'homotopie abstract Memoirs of the faculty of science, Kyushu university Ser. A 38 (1984), 183 - 198.

SPANIER, E.: Algebraic topology. Mc Graw Hill (1966).

STEENROD, N.E.: Products of cocycles and extensions of mappings. Ann. of Math. 48 (1947) 290 - 320.

SRRØM, A.: Note on cofibrations II. Math. Scand . 22 (1968) 130 - 142.

The homotopy category is a homotopy category, Arch. Math. 23 (1972) 435 - 441.

SULLIVAN, D.: Geometric Topology - 1: Localization, Periodicity and Galois symmetry. MIT Press (1970).

Infinitesimal computations in topology. Public. Math. 47 (1977) Inst. Hautes Etudes Sc. Paris.

SWITZER, R.M.: Counting elements in homotopy sets. Math. Z. 178 (1981) 527 - 554.

THOMASON, R.W.: Cat as a closed model category, MIT Cambridge Massachusetts.

TSUKIYAMA, K.: On the group os fibre homotopy-equivalences. Hiroshima Math. J. 12 (1982), 349 - 376.

TSUKIYAMA, K.: Self-homotopy-equivalences of a space with two nonvanishing homotopy groups. Proceedings of AMS 79 (1980), 134 - 138.

VARADARAJAN, K.: Numerical invariants in homotopical algebra I,II, Can. J. Math. 27 (1975) 901 - 934, 935 - 960.

VOGT, R.M.: Homotopy limits and colimits. Math. Z. 134 (1973) 11 - 52.

WALDHAUSEN, F.: Algebraic K-theory of spaces. Preprint (1984) Bielefeld.

WALL, C.T.C.: Finiteness conditions for CW-complexes II. Proc. Roy. Soc. 295 (1966) 129 - 139.

WATTS, Ch. E.: A Homology Theory for Small Categories. Proc. of the Conf. on Categorical Algebra. La Jolla. CA 1965.

WHITEHEAD, G.W.: On mappings into group like spaces. Comment. Math. Helv. 21 (1954) 320 - 338.

WHITEHEAD, J.H.C.: On simply connected 4-dimensional polyhedra. Comm. Math. Helv. 22 (1949) 48 - 92.

A certain exact sequence. Ann. Math. 52 (1950) 51 - 110.

Combinatorial homotopy II. Bull. AMS 55 (1949)' 213 - 245.

Algebraic homotopy theory. Proc. Int. Congress of Mathematicians, Harvard, (1950)'. Vol.II 354 - 357.

Simple homotopy types. Amer. J. Math. 72 (1950)" 1 - 57.

WIRSCHING, G.: Kohomologie kleiner Kategorien. Diplomarbeit, Math. Institut der Universität Bonn (1984).