Anisotropic Edge Pseudo-Differential Operators with Discrete Asymptotics

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0 Introduction

This paper studies anisotropic pseudo–differential operators as they are necessary for solving parabolic differential equations. Such a concept for scalar symbols was studied by PIRIOU [PIR1], [PIR2] for operators on $\mathbb{R} \times W$ with the time variable $t \in \mathbb{R}$ and a space region W that may be a compact C^∞–manifold (with or without boundary). If W is allowed to have singularities, e.g., conical ones or edges, then it becomes necessary to perform a more specific pseudo–differential analysis close to the singularities. Our calculus here will refer to the case of edge singularities.

The local model of W near an edge of dimension q is $X^\Delta \times \mathbb{R}^q$ with the model cone $X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$, where the base X is a closed compact C^∞–manifold of dimension n. Note that the case $n = 0$ corresponds to standard boundary or transmission problems. Together with the t–axis we obtain the wedge $X^\Delta \times \mathbb{R} \times \mathbb{R}^q$, where $\mathbb{R} \times \mathbb{R}^q \ni (t, y)$ is now to be regarded as an anisotropic edge.

The analysis of pseudo–differential operators itself will be formulated on $X^\wedge \times \mathbb{R} \times \mathbb{R}^q$ with the open stretched model cone $X^\wedge := \mathbb{R}_+ \times X \ni (r, x)$. As it is known from the theory of elliptic operators on manifolds with edges it is not always adequate to localize the objects on X but to preserve the global descriptions. Moreover, similarly to boundary value problems, there is an analogue of the SHAPIRO–LOPATINSKIJ condition for elliptic edge problems, that suggests to interpret the operators as pseudo–differential ones on the edge with operator–valued symbols.

The values of the symbols belong to the algebra of pseudo–differential operators on the stretched cone $X^\wedge$ as it may be found in SCHULZE [SCH1], [SCH2]. The operator–valued analogue of the scalar pseudo–differential calculus will be understood here in the FOURIER–edge–approach that was elaborated in SCHULZE [SCH3] for the elliptic theory.

The parabolicity on $X^\wedge \times \mathbb{R}^{1+q} \ni (r, x, t, y)$ is to some extent as anisotropic ellipticity. For inverting parabolic operators within a corresponding pseudo–differential algebra it is a necessary step to introduce and to study the algebra itself. This is just the content of the present paper. The algebra will include the trace and potential operators as edge conditions that play an analogous role as boundary conditions in the ellipticity.

The anisotropic ellipticity as another necessary part for etablishing the parabolicity will be presented in BUCHHOLZ, SCHULZE [BUC1]. In a forthcoming paper, cf. [BUC2], we will pass to a subalgebra of VOLTERRA operators, analogously to the concept of PIRIOU [PIR1], who has treated the case of scalar symbols.

The singularities in our context will make it necessary from the very beginning to allow symbols and distribution spaces with cone and edge asymptotics. The regularity of solutions should (and finally will) contain the asymptotics, similarly to those of the edge theory, cf. [SCH1], [SCH2]. The most classical asymptotics of distributions on $X^\wedge$, here called discrete (in contrast to the continuous ones, cf. [SCH1], [SCH2]) are of the form

$$u(r,x) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{m_j} c_{jk}(x) r^{-p_j} \ln^k r \quad \text{as} \quad r \to 0,$$

cf. [KON1]. Here $p_j \in \mathbb{C}$, $j \in \mathbb{N}$, is a sequence with $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$, and $\operatorname{Re} p_j > \frac{n+1}{2} - \gamma$ for all j with some weight $\gamma \in \mathbb{R}$. Moreover, the coefficients $c_{jk}(x)$ belong to finite–dimensional subspaces $L_j \subset C^\infty(X)$ for $0 \le k \le m_j$, $j \in \mathbb{N}$.

It will be also convenient to talk about finite asymptotic expansions

$$\sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk}(x) r^{-p_j} \ln^k r \tag{1}$$

with $\frac{n+1}{2} - \gamma + \theta < \operatorname{Re} p_j < \frac{n+1}{2} - \gamma$ for $j = 1, \dots, N$, where $\theta < 0$ is given. We set $\Gamma_\beta := \{w \in \mathbb{C} : \operatorname{Re} w = \beta\}$ and then the half open interval $\Theta = (\theta, 0]$ will play the role of a *weight strip* on the left of the *weight line* $\Gamma_{\frac{n+1}{2}-\gamma}$ in the complex plane of the MELLIN covariable $w \in \mathbb{C}$. For formulating our spaces with asymptotics we will call the sequence of data $P = \{(p_j, m_j, L_j)\}_{j=1,\dots,N}$, $N = N(P)$, an *asymptotic type* associated with the weight data $\mathbf{g} = (\gamma, \Theta)$. The set of all such P (for which $\pi_{\mathbb{C}} P = \{p_j\}_{j=1,\dots,N}$ is contained in the indicated weight strip) will be denoted by $\operatorname{As}(\mathbf{g})$. Moreover, for simplicity we will always assume that all the involved asymptotic types satisfy the shadow condition, i.e.

$$(p, m, L) \in P \Rightarrow (p - j, m(j), L(j)) \in P \tag{2}$$

holds for all $j \in \mathbb{N}$ with $\operatorname{Re} p - j > \frac{n+1}{2} - \gamma + \theta$ for certain $m(j) \ge m$ and $L(j) \supseteq L$. Now we define $\mathcal{E}_P$ as the linear span of all functions having the form (1) with the above conditions. Of course, $\mathcal{E}_P$ is finite–dimensional.

The pseudo–differential operators shall be formulated in r–direction close to $r = 0$ in terms of the MELLIN transformation $\mathcal{M} = \mathcal{M}_{r \to w}$, $w \in \mathbb{C}$,

$$\mathcal{M}u(w) = \int_0^\infty r^{w-1} u(r)\, dr.$$

The MELLIN transformation is first defined for $u \in C_0^\infty(\mathbb{R}_+)$ and then extended in a standard way to the various distribution spaces. In particular, $\mathcal{M}$ extends to an isomorphism $\mathcal{M} : L^2(\mathbb{R}_+) \to L^2(\Gamma_{\frac{1}{2}})$ between the corresponding spaces of square integrable functions, where L^2 on the weight line $\{\operatorname{Re} w = \frac{1}{2}\}$ refers to the image of the LEBESGUE measure on $\mathbb{R}$ under the map $\mathbb{R} \to \Gamma_{\frac{1}{2}}$, $\rho \mapsto \operatorname{Im} w$. Then the inverse MELLIN transform is given by

$$(\mathcal{M}^{-1} g)(r) = \frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{2}}} r^{-w} g(w)\, dw.$$

We will also employ the weighted MELLIN transform $(\mathcal{M}_\gamma u)(w) = (\mathcal{M}(r^{-\gamma} u))(w + \gamma)$, $\gamma \in \mathbb{R}$, which induces an isomorphism $\mathcal{M}_\gamma : r^\gamma L^2(\mathbb{R}_+) \to L^2(\Gamma_{\frac{1}{2}-\gamma})$.

Let $\mathcal{F}$ denote the FOURIER transformation in $\mathbb{R}^n$, $\mathcal{F} = \mathcal{F}_{x\to\xi}$, defined by $(\mathcal{F}v)(\xi) = \int e^{-ix\xi}v(x)dx$. Then $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)$ for $s,\gamma \in \mathbb{R}$ will denote the closure of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ with respect to the norm

$$\|u\|_{s,\gamma} = \left\{\int\int (1 + \left|\frac{n+1}{2} - \gamma + i\rho\right|^2 + |\xi|^2)^s \left|\left(\mathcal{M}_{\frac{n+1}{2}-\gamma}\mathcal{F}u\right)(\gamma - \frac{n}{2} + i\rho, \xi)\right|^2 đ\rho đ\xi\right\}^{\frac{1}{2}},$$

where we set $đ\rho = \frac{d\rho}{2\pi}$ and $đ\xi = \frac{d\xi}{(2\pi)^n}$. Further $\mathcal{H}^{s,\gamma}(X^\wedge)$ is defined as the subspace of all $u \in \mathcal{D}'(X^\wedge)$ for which $\chi_*(\phi u) \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)$ for every chart $\chi : \tilde{V} \to \mathbb{R}^n$ on X and every $\phi \in C_0^\infty(\tilde{V})$.

In this paper a cut–off function will be any $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ with $\omega \equiv 1$ in a neighbourhood of $r = 0$. Let $\tilde{V} \subset X$ be an arbitrary coordinate neighbourhood and $\chi'_{\tilde{V}} : \tilde{V} \to V$ a diffeomorphism to an open set $V \subset S^n$. Define $\chi_{\tilde{V}} : \mathbb{R}_+ \times \tilde{V} \to \mathbb{R}^{1+n}$ by $\chi_{\tilde{V}}(r,x) = \tilde{x}$ with $|\tilde{x}| = r$ and $\chi'_{\tilde{V}}(x) = \tilde{x}/|\tilde{x}|$. Then $\mathcal{K}^{s,\gamma}(X^\wedge)$ for $s,\gamma \in \mathbb{R}$ will denote the space of all $u \in \mathcal{D}'(X^\wedge)$ such that $\omega(r)u \in \mathcal{H}^{s,\gamma}(X^\wedge)$ and $(\chi_{\tilde{V}})_*((1-\omega(r))\phi u) \in H^s(\mathbb{R}^{1+n})$ for every $\tilde{V}$ and arbitrary $\phi \in C_0^\infty(\tilde{V})$.

$\mathcal{H}^{s,\gamma}(X^\wedge)$ and $\mathcal{K}^{s,\gamma}(X^\wedge)$ are BANACH spaces in a natural way (even HILBERTizable), where $\mathcal{H}^{0,0}(X^\wedge) = \mathcal{K}^{0,0}(X^\wedge)$ coincide with $r^{-\frac{n}{2}}L^2(X^\wedge)$, where $L^2(X^\wedge)$ refers to $drdx$ with dx being associated with a RIEMANNian metric on X.

If E_1 and E_2 are BANACH spaces that are contained in a HAUSDORFF vector space we can define the (non–direct) sum $E = E_1 + E_2$, consisting of all $e = e_1 + e_2$ with $e_1 \in E_1$ and $e_2 \in E_2$. If $\Delta = \{(e,-e) : e \in E_1 \cap E_2\}$, then $E \cong (E_1 \oplus E_2)/\Delta$ allows to introduce a BANACH structure in E. Further if a BANACH space B is a module over an algebra A, then $[a]B$ for $a \in A$ will denote the closure of $\{ab : b \in B\}$ in B. Analogous notations will be used for FRÉCHET spaces E_1, E_2 and B, respectively. In particular, the spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ and $\mathcal{K}^{s,\gamma}(X^\wedge)$ are modules over $C_0^\infty(\overline{\mathbb{R}}_+ \times X)$ as well as over the algebra of all $\phi(r,x) \in C^\infty(X^\wedge)$ vanishing near $r = 0$ and for which $1 - \phi \in C_0^\infty(\overline{\mathbb{R}}_+ \times X)$.

Since $\mathcal{H}^{s,\gamma}(X^\wedge) \subset H^s_{loc}(X^\wedge)$ and $\mathcal{K}^{s,\gamma}(X^\wedge) \subset H^s_{loc}(X^\wedge)$ we can write

$$\mathcal{K}^{s,\gamma}(X^\wedge) = [\omega]\mathcal{H}^{s,\gamma}(X^\wedge) + [1-\omega]H^s(X^\wedge) \tag{3}$$

if $H^s(X^\wedge)$ is defined as the subspace of all $u \in H^s_{loc}(X^\wedge)$ with $(\chi_{\tilde{V}})_*((1-\omega(r))\phi u) \in H^s(\mathbb{R}^{1+n})$ for every $\tilde{V}, \phi$, and (3) is independent of the concrete cut–off function ω.

From the definitions it follows easily $\mathcal{H}^{s,\gamma}(X^\wedge) = r^\gamma\mathcal{H}^{s,0}(X^\wedge)$ and $\mathcal{K}^{s,\gamma}(X^\wedge) = k^\gamma(r)\mathcal{K}^{s,0}(X^\wedge)$ with any $k^\gamma(r) \in C^\infty(\mathbb{R}_+)$ that is strictly positive and equals r^γ for $0 < r < c_0$ and 1 for $c_1 < r < \infty$ with constants $c_0 < c_1$. Thus it often suffices to look at the case $\gamma = 0$, where we also write $\mathcal{H}^s(X^\wedge) = \mathcal{H}^{s,0}(X^\wedge)$ and $\mathcal{K}^s(X^\wedge) = \mathcal{K}^{s,0}(X^\wedge)$.

We have $\mathcal{K}^{s',\gamma'}(X^\wedge) \subseteq \mathcal{K}^{s,\gamma}(X^\wedge)$ for every $s' \geq s$ and $\gamma' \geq \gamma$. Let us set

$$\mathcal{K}^{s,\gamma}_\Theta(X^\wedge) = \bigcap_{\epsilon>0} \mathcal{K}^{s,\gamma-\theta-\epsilon}(X^\wedge)$$

for $\Theta = (\theta, 0]$, endowed with the corresponding FRÉCHET topology of the projective limit. Further for every $P \in \mathrm{As}(\gamma, \Theta)$ we form

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) = \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) + [\omega]\mathcal{E}_P$$

with the FRÉCHET topology of the direct sum (recall that $[\omega]\mathcal{E}_P$ is a finite–dimensional subspace of $\mathcal{K}^{\infty,\gamma}(X^\wedge)$). Alternatively we could set

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) = \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) + \mathcal{K}_P^{\infty,\gamma}(X^\wedge)$$

with the topology of the non–direct sum. Finally we define

$$\mathcal{H}_P^{s,\gamma}(X^\wedge) = [\omega]\mathcal{K}_P^{s,\gamma}(X^\wedge) + [1-\omega]\mathcal{H}^{s,\gamma}(X^\wedge)$$

and

$$\mathcal{S}_P^{\gamma}(X^\wedge) = [\omega]\mathcal{K}_P^{\infty,\gamma}(X^\wedge) + [1-\omega]\mathcal{S}(X^\wedge),$$

also with the FRÉCHET topology of the non–direct sum, where $\mathcal{S}(X^\wedge) := \mathcal{S}(\mathbb{R}, C^\infty(X))|_{\mathbb{R}_+}$. This is independent of the concrete choice of the cut–off function ω.

Let us set for a moment $\hat{H}^s(\mathbb{R}\times X) := \mathcal{F}_{\lambda\to\rho}H^s(\mathbb{R}\times X)$ with the standard SOBOLEV space $H^s(\mathbb{R}\times X)$ and the one–dimensional FOURIER transformation $\mathcal{F}_{\lambda\to\rho}$. Further let $\hat{H}^s(\Gamma_{\frac{n+1}{2}-\gamma}\times X)$ be the preimage of $\hat{H}^s(\mathbb{R}\times X)$ under the map $\Gamma_{\frac{n+1}{2}-\gamma} \to \mathbb{R}$, $w \mapsto \mathrm{Im}\, w = \rho$. Then it is well–known (cf., e.g., SCHULZE [SCH2]) that $\mathcal{M}_{\gamma-\frac{n}{2}}\mathcal{H}^{s,\gamma}(X^\wedge) = \hat{H}^s(\Gamma_{\frac{n+1}{2}-\gamma}\times X)$ holds.

If $\Omega \subseteq \mathbb{C}$ is an open set and E a FRÉCHET space we will denote by $\mathcal{A}(\Omega, E)$ the space of all holomorphic E–valued functions in Ω. Let $\omega(r)$ be a cut–off function and $\delta > 0$. Then $r^\delta\omega\mathcal{H}^{s,\gamma}(X^\wedge)$ is a subspace of $\mathcal{H}^{s,\gamma}(X^\wedge)$ and $\mathcal{M}_{\gamma-\frac{n}{2}}(r^\delta\omega u)$ has for every $u \in \mathcal{H}^{s,\gamma}(X^\wedge)$ an extension to an element in $\mathcal{A}(\{\mathrm{Re}\, w > \frac{n+1}{2} - \gamma - \delta\}, H^s(X))$ for which

$$\mathcal{M}_{\gamma-\frac{n}{2}}(r^\delta\omega u)|_{\Gamma_\beta} \in \hat{H}^s(\Gamma_\beta \times X)$$

for all $\beta > \frac{n+1}{2} - \gamma - \delta$, uniformly in $\frac{n+1}{2} - \gamma - \delta < \beta < c$ for every $c > \frac{n+1}{2} - \gamma - \delta$. Now let P be an element in $\mathrm{As}(\gamma, \Theta)$ for $\Theta = (\theta, 0]$. A function $\chi(w) \in C^\infty(\mathbb{C})$ is called a $\pi_{\mathbb{C}}P$–excision function if $\chi(w) = 0$ for $\mathrm{dist}(w, \pi_{\mathbb{C}}P) < \varepsilon_0$, $\chi(w) = 1$ for $\mathrm{dist}(w, \pi_{\mathbb{C}}P) > \varepsilon_1$ with constants $0 < \varepsilon_0 < \varepsilon_1 < \infty$.

Let us denote by $\mathcal{A}_P^{s,\gamma}(X)$ the space of all $f(w) \in \mathcal{A}(\{\mathrm{Re}\, w > \frac{n+1}{2} - \gamma + \theta\} \setminus \pi_{\mathbb{C}}P; H^s(X))$ such that

(i) $f(w)$ is meromorphic with poles at p_j of multiplicities $m_j + 1$ and LAURENT coefficients at $(w - p_j)^{-(k+1)}$ in L_j for $0 \le k \le m_j$ and all j,

(ii) for every $\pi_{\mathbb{C}}P$–excision function $\chi(w)$ we have $\chi(w)f(w)|_{\Gamma_\beta} \in \hat{H}^s(\Gamma_\beta \times X)$ for every $\beta > \frac{n+1}{2} - \gamma + \theta$ and uniformly in $c_0 < \beta < c_1$ for every $c_1 > c_0 > \frac{n+1}{2} - \gamma + \theta$.

A basic observation that will currently be used below is

$$\mathcal{M}_{\gamma-\frac{n}{2}}(\omega\mathcal{K}_P^{s,\gamma}(X^\wedge)) \subset \mathcal{A}_P^{s,\gamma}(X) \text{ and } \omega\mathcal{M}_{\gamma-\frac{n}{2}}^{-1}(\mathcal{A}_P^{s,\gamma}(X)) \subset \mathcal{K}_P^{s,\gamma}(X^\wedge).$$

In other words the asymptotics can be controled in the MELLIN image in terms of the poles and of the growth for $|\mathrm{Im}\, w| \to \infty$.

Let $S^\mu(\Gamma_\beta)$ for $\mu \in \mathbb{R}$ be the HÖRMANDER symbol class on the line $\Gamma_\beta \cong \mathbb{R} \ni \rho$ of order μ for which the usual estimates hold, namenly $S^\mu(\Gamma_\beta)$ is the space of all $a(w) \in C^\infty(\Gamma_\beta)$ with $|D_\rho^j a(w)| \le c_j(1+|\rho|)^{\mu-j}$ for all $\rho \in \mathbb{R}$ with constants $c_j > 0$, for all $j \in \mathbb{N}$. Then we can form MELLIN pseudo-differential operators $\mathrm{op}_M^\gamma(a) \,:\, C_0^\infty(\mathbb{R}_+) \to C^\infty(\mathbb{R}_+)$, $a(w) \in S^\mu(\Gamma_{\frac{1}{2}-\gamma})$, by

$$\mathrm{op}_M^\gamma(a)u(r) = r^\gamma \mathcal{M}_{w\to r}^{-1}\{(T^{-\gamma}a)(w)\mathcal{M}_{r'\to w}\{r'^{-\gamma}u(r')\}(w)\}, \tag{4}$$

where $(T^\delta a)(w) := a(w+\delta)$ for any $\delta \in \mathbb{R}$. Pseudo-differential operators of that type were considered, e.g., in SCHULZE [SCH1], [SCH2]. We will write, in particular, $\mathrm{op}_M^0(a) = \mathrm{op}_M(a)$. Analogously we can form MELLIN pseudo-differential operators with symbols of the classes $a(r,r',w) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, S^\mu(\Gamma_{\frac{1}{2}-\gamma}))$, $C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, S^\mu(\Gamma_{\frac{1}{2}-\gamma}))$, $C^\infty(\overline{\mathbb{R}}_+, S^\mu(\Gamma_{\frac{1}{2}-\gamma}))$ and so on.

We will be in fact interested in operator-valued MELLIN symbols with values in $\Psi^\mu(X)$, which is the class of all pseudo-differential operators of order $\mu \in \mathbb{R}$ on X. By $\Psi_{cl}^\mu(X)$ we shall denote the subspace of classical pseudo-differential operators. There is a well-known extension of the concept of pseudo-differential operators to the case of parameter dependence. If $\mathbb{R}^k \ni \lambda$ is the parameter space we obtain $\Psi^\mu(X;\mathbb{R}^k)$ and $\Psi_{cl}^\mu(X;\mathbb{R}^k)$ as the space of corresponding operator families $a(\lambda)$ that are mod $\Psi^{-\infty}(X;\mathbb{R}^k)$ locally defined by symbols $p(x,\xi,\lambda)$ in the S^μ or S_{cl}^μ classes with respect to (ξ,λ), treated as an $(n+k)$-dimensional covariable. Here $\Psi^{-\infty}(X;\mathbb{R}^k) = \mathcal{S}(\mathbb{R}^k, \Psi^{-\infty}(X))$ is the SCHWARTZ space on $\mathbb{R}^k$ with values in $\Psi^{-\infty}(X) \cong C^\infty(X \times X)$.

We will denote by $\Psi^\mu(X;\Gamma_\beta)$ the space of all parameter-dependent operators $a(w)$, for which $a(\beta+i\rho)$ belongs to $\Psi^\mu(X;\mathbb{R}_\rho)$. Analogously we have $\Psi_{cl}^\mu(X;\Gamma_\beta)$. All our spaces are endowed with canonical FRÉCHET topologies. If we form $\mathrm{op}_M^\gamma(a)$ with $a \in \Psi^\mu(X;\Gamma_\beta)$ then the pseudo-differential action on X is automatically carried out through the values of the symbol $a(w)$. The operator $\mathrm{op}_M^\gamma(a) \,:\, C_0^\infty(X^\wedge) \to C^\infty(X^\wedge)$ then extends by continuity to a continuous operator

$$\mathrm{op}_M^\gamma(a) \,:\, \mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge)$$

for every $s \in \mathbb{R}$. These properties as well as the following definition may be found in SCHULZE [SCH1], [SCH2].

Let $R = \{(r_j, n_j, L_j)\}_{j\in\mathbb{Z}}$ be an arbitrary sequence with $r_j \in \mathbb{C}$, $|\mathrm{Re}\, r_j| \to \infty$ as $j \to \infty$, $n_j \in \mathbb{N}$ and L_j being a finite-dimensional subspace of finite-dimensional operators in $\Psi^{-\infty}(X)$. We set $\pi_{\mathbb{C}} R = \bigcup_{j\in\mathbb{Z}} \{r_j\}$.

For every triple (p,m,l) with $p \in \mathbb{C}$, $m \in \mathbb{N}$, $l \in \Psi^{-\infty}(X)$ being a finite-dimensional operator we can form the meromorphic operator function

$$f(p,m,l)(w) = l\mathcal{M}_{r\to w}(\omega(r)r^{-p-\delta}\log^m r)(w+\delta).$$

Here $\delta \in \mathbb{R}$ is arbitrary with Re $(p+\delta) < \frac{1}{2}$, such that the MELLIN transform in the usual form makes sense, and $\omega(r)$ is a fixed cut-off function. Note that $f(p,m,l)(w)$ has its pole at p of multiplicity $m+1$.

Then $N_R^\mu(X)$ for $\mu \in \mathbb{R}$ is defined as the space of all $h(w) \in \mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}} R; \Psi^\mu(X))$ with the following properties

(i) $h(w)$ is meromorphic with poles at r_j of multiplicities $n_j + 1$ and LAURENT coefficients at $(w-r_j)^{-(k+1)}$ in L_j for $0 \le k \le n_j$ and all j,

(ii) for all reals $c_0 < c_1$ and arbitrary $(r_{j_m}, n_{j_m}, L_{j_m}) \in R$, $m = 1,\dots,N$, running over all triples in R with $c_0 <$ Re $r_j < c_1$, there are elements $l_{j_m,k} \in L_{j_m}$, $0 \le k \le n_{j_m}$, such that

$$h_{(c_0,c_1)}(w) := h(w) - \sum_{m=1}^{N}\sum_{k=0}^{n_{j_m}-1} f(r_{j_m}, k, l_{j_m,k})(w)$$

belongs to $\mathcal{A}(\{c_0 < \text{Re } w < c_1\}, \Psi^\mu(X))$ with $h_{(c_0,c_1)}|_{\Gamma_\beta} \in \Psi^\mu(X;\Gamma_\beta)$ for every $\beta \in (c_0,c_1)$ and uniformly in $\tilde{c}_0 < \beta < \tilde{c}_1$ for every $c_0 < \tilde{c}_0 < \tilde{c}_1 < c_1$.

If $h \in N_R^\mu(X)$ for some MELLIN asymptotic type R holds we will also write sg$(h) \subseteq \pi_{\mathbb{C}} R$, i.e., sg(h) indicates the system of poles of the meromorphic operator function . Replacing Ψ^μ by Ψ^μ_{cl} we obtain by definition $M_R^\mu(X)$. The spaces $N_R^\mu(X)$ and $M_R^\mu(X)$ are FRÉCHET spaces in a natural way. In particular, $M_R^{-\infty}(X) = N_R^{-\infty}(X)$ is a nuclear FRÉCHET space. For $\pi_{\mathbb{C}} R = \emptyset$ we denote the corresponding spaces by $N_O^\mu(X)$ and $M_O^\mu(X)$, respectively. Below we also use the decompositions $N_R^\mu(X) = N_O^\mu(X) + N_R^{-\infty}(X)$ and $M_R^\mu(X) = M_O^\mu(X) + M_R^{-\infty}(X)$ with the non-direct sum of FRÉCHET spaces. It will also be interesting to consider the spaces $C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, N_R^\mu(X))$, $C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, M_R^\mu(X)) \ni h(r,r',w)$.

It is clear that we have $C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, N_R^\mu(X))|_{\Gamma_\beta} \subset C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, \Psi^\mu(X;\Gamma_\beta))$ for every $\beta \in \mathbb{R}$ for which $\pi_{\mathbb{C}} R \cap \Gamma_\beta = \emptyset$. For two cut-off functions $\omega(r), \tilde{\omega}(r)$ and $h(r,r',w) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, \Psi^\mu(X;\Gamma_\beta))$ the operator

$$\omega \mathrm{op}_M^\gamma(h)\tilde{\omega} \; : \; \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge)$$

is continuous for every $s \in \mathbb{R}$. Moreover, if $h(r,r',w) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, N_R^\mu(X))$ and $\pi_{\mathbb{C}} R \cap \Gamma_\beta = \emptyset$, then for every $P \in \mathrm{As}(\gamma + \frac{n}{2}, \Theta)$ there is a $Q \in \mathrm{As}(\gamma + \frac{n}{2}, \Theta)$, dependent on P and R, such that

$$\omega \mathrm{op}_M^\gamma(h)\tilde{\omega} \; : \; \mathcal{K}_P^{s,\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{K}_Q^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge)$$

is continuous for every $s \in \mathbb{R}$.

1 The ideal of smoothing edge symbols

1.1 General anisotropic operator–valued symbols

In this section we define anisotropic symbols of pseudo–differential operators with values in the space of linear continuous operators between BANACH or FRÉCHET spaces E and $\tilde{E}$. Let us first assume that E and $\tilde{E}$ are BANACH spaces. Later we will extend the theory to the case of FRÉCHET spaces.

With E and $\tilde{E}$ we associate strongly continuous groups of isomorphisms $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}}$, respectively. Note that for every such group there are constants $M, c > 0$ such that

$$\|\kappa_\lambda\|_{\mathcal{L}(E)} \leq \begin{cases} c\lambda^{-M} & \text{for} \quad \lambda \leq 1, \\ c\lambda^{M} & \text{for} \quad \lambda \geq 1. \end{cases} \tag{1}$$

We describe anisotropy with any fixed $1 \leq l \in \mathbb{N}$. The isotropic case is contained in the anisotropic theory with $l = 1$.

For $(\tau, \eta) \in \mathbb{R}_\tau \times \mathbb{R}^q_\eta = \mathbb{R}^{1+q}$ we define the anisotropic norm function

$$|\tau, \eta|_l = (|\tau|^2 + |\eta|^{2l})^{\frac{1}{2l}}. \tag{2}$$

Furthermore, we fix an anisotropic smoothed norm function

$$[\tau, \eta]_l := \omega(\tau, \eta) + (1 - \omega(\tau, \eta))|\tau, \eta|_l, \tag{3}$$

where we suppose $\omega(\tau, \eta) \in C_0^\infty(\mathbb{R}^{1+q}, [0, 1])$, such that there are real numbers $1 < c < c'$ with

$$\omega(\tau, \eta) = \begin{cases} 1 & \text{for} \quad 0 \leq |\tau, \eta|_l \leq c, \\ 0 & \text{for} \quad |\tau, \eta|_l \geq c'. \end{cases} \tag{4}$$

For abbreviation we set $\kappa(\tau, \eta) := \kappa_{[\tau,\eta]_l}$ and $\tilde{\kappa}(\tau, \eta) := \tilde{\kappa}_{[\tau,\eta]_l}$.

Definition 1 *Let $\nu \in \mathbb{R}$ and $U = U_0 \times U'$ with open sets $U_0 \subseteq \mathbb{R}^{p_0}, U' \subseteq \mathbb{R}^p$; then the space of anisotropic operator–valued symbols*

$$S^{\nu,l}(U \times \mathbb{R}^{1+q}; E, \tilde{E}) \tag{5}$$

is defined as the set of all $a(t, y, \tau, \eta) \in C^\infty(U \times \mathbb{R}^{1+q}, \mathcal{L}(E, \tilde{E}))$ such that

$$\|\tilde{\kappa}^{-1}(\tau, \eta)\{D^\alpha_{t,y} D^\beta_{\tau,\eta} a(t, y, \tau, \eta)\}\kappa(\tau, \eta)\|_{\mathcal{L}(E,\tilde{E})} \leq c[\tau, \eta]_l^{\nu-|\beta|_l} \tag{6}$$

holds for all $\alpha = (\alpha_0, \alpha') \in \mathbb{N}^{p_0} \times \mathbb{N}^p$, $\beta = (\beta_0, \beta') \in \mathbb{N} \times \mathbb{N}^q$ and all $(t, y) \in K$ with $K \subset\subset U$ and all $(\tau, \eta) \in \mathbb{R}^{1+q}$ with constants $c = c(\alpha, \beta, K) \geq 0$. Here we denote by $|\beta|_l$ for multi–indices $\beta \in \mathbb{N}^{1+q}$ the number $|\beta|_l = l\beta_0 + |\beta'|$.

The best constants $c = c(\alpha, \beta, K)$ in (6) form a system of semi–norms, which gives a FRÉCHET topology on (5). With this definition the space (5) depends on the concrete choice of the $\kappa_\lambda, \tilde{\kappa}_\lambda$. They are always fixed in our applications; for abbreviation we omit them in the notation. We only discuss the cases $U = U_0 \times U'$ with open $U_0 \subseteq \mathbb{R}$ and $U' \subseteq \mathbb{R}^q$ or $U = (U_0 \times U') \times (U_0 \times U')$, where in the latter case we also denote the symbols as amplitude functions with variables t, y, t', y', τ, η.

Of course, we also allow the special case $E = \mathbb{C}$ or $\tilde{E} = \mathbb{C}$. Then we set $\kappa_\lambda = \mathrm{id}$ and $\tilde{\kappa}_\lambda = \mathrm{id}$ for all $\lambda \in \mathbb{R}_+$. With $E = \tilde{E} = \mathbb{C}$ we get the scalar anisotropic symbols and write shortly $S^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathbb{C}, \mathbb{C}) = S^{\nu,l}(U \times \mathbb{R}^{1+q})$ (see also [PIR1]).

We denote by $S^{\nu,l}(\mathbf{R}^{1+q}; E, \tilde{E})$ the closed subspace of anisotropic operator–valued symbols with constant coefficients, that means they do not depend on (t, y, t', y').

Example 2 As mentioned in the introduction the spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ are BANACH spaces for all $s, \gamma \in \mathbf{R}$. Setting $(\kappa_\lambda u)(r, x) = (\tilde{\kappa}_\lambda u)(r, x) = \lambda^{\frac{n+1}{2}} u(\lambda r, x)$ for all $u(r, x) \in \mathcal{K}^{s,\gamma}(X^\wedge)$ we may define the symbol spaces $S^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\nu,\gamma-\mu}(X^\wedge))$.

For example, the operator valued function $a(\tau, \eta) = (i\tau + |\eta|^2) \otimes e$, where e denotes the embedding $\mathcal{K}^{s,\gamma}(X^\wedge) \hookrightarrow \mathcal{K}^{s',\gamma'}(X^\wedge)$ with $s \geq s'$ and $\gamma \geq \gamma'$, belongs to $S^{2,2}(\mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s',\gamma'}(X^\wedge))$.

Another essential example of an operator–valued symbol is given by the following

Lemma 3 *The operator M_ϕ of multiplication by $\phi(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ belongs to $S^{0,l}(\mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s,\gamma}(X^\wedge))$ for arbitrary $s, \gamma \in \mathbb{R}$.*

Moreover, $M : \phi \mapsto M_\phi$ induces for all $s \in \mathbb{R}$ a continuous map

$$M : C_0^\infty(\overline{\mathbb{R}}_+) \to S^{0,l}(\mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s,\gamma}(X^\wedge)). \tag{7}$$

Proof: Since M_ϕ does not depend on the covariables $(\tau, \eta) \in \mathbf{R}^{1+q}$ we only have to find a constant $c_\phi > 0$ tending to zero for $\phi \to 0$ in $C_0^\infty(\overline{\mathbb{R}}_+)$ such that the inequality

$$\|\kappa^{-1}(\tau, \eta) M_\phi \kappa(\tau, \eta)\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge))} \leq c_\phi \tag{8}$$

holds uniformly in $(\tau, \eta) \in \mathbb{R}^{1+q}$.

Because of $\kappa^{-1}(\tau, \eta) M_\phi \kappa(\tau, \eta) = M_{\phi(r[\tau,\eta]_l^{-1})}$ we can prove (8) in the same way as SCHULZE [SCH1] 3.2.1 Proposition 5.

Note that the essential observation for choosing c_ϕ independent of (τ, η) is $|\mathcal{M}_0\{\phi(r[\tau, \eta]_l^{-1})\}(i\rho)| = |[\tau, \eta]_l^{i\rho}| \; |\mathcal{M}_0\phi(i\rho)| = |\mathcal{M}_0\phi(i\rho)|$.

The second part of the lemma follows from $c_\phi \to 0$ for $\phi \to 0$ in (8). □

Now we formulate some assertions that are valid analogously in the isotropic theory and can be proved by the same techniques as in [SCH1]. Therefore we will omit the proofs here for abbreviation.

The definition of the symbol classes and the nuclearity of $C^\infty(U)$ give the following

Lemma 4 *We have for an open set* $U = U_0 \times U' \subseteq \mathbb{R}^{1+q}$

$$S^{\nu,l}(U^2 \times \mathbb{R}^{1+q}; E, \widetilde{E}) = C^\infty(U^2, S^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E})) = C^\infty(U^2)\hat{\otimes}_\pi S^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E})),$$

such that every $a(t,y,t',y',\tau,\eta) \in S^{\nu,l}(U^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$ *can be written as a convergent sum*

$$a(t,y,t',y',\tau,\eta) = \sum_{j=0}^{\infty} c_j b_j(t,y) a_j(\tau,\eta) d_j(t',y') \tag{9}$$

with a sequence $(c_j) \in l_1$, $b_j, d_j \to 0$ *in* $C^\infty(U)$ *and* $a_j \to 0$ *in* $S^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E})$ *for* $j \to \infty$.

Lemma 5 *Let* $E, \widetilde{E}$ *and* $\widetilde{\widetilde{E}}$ *be* Banach *spaces and* $\{\kappa_\lambda\}$, $\{\tilde{\kappa}_\lambda\}$ *and* $\{\tilde{\tilde{\kappa}}_\lambda\}$ *the associated group actions. Then we have for arbitrary* $\nu, \tilde{\nu} \in \mathbb{R}$

(i) for all $\nu \leq \tilde{\nu}$ *there are continuous embeddings*

$$S^{\nu,l}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}) \hookrightarrow S^{\tilde{\nu},l}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}), \tag{10}$$

(ii)

$$S^{\nu,l}(U \times \mathbb{R}^{1+q}; \widetilde{\widetilde{E}}, \widetilde{E}) S^{\tilde{\nu},l}(U \times \mathbb{R}^{1+q}; E, \widetilde{\widetilde{E}}) \subseteq S^{\nu+\tilde{\nu},l}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}), \tag{11}$$

with the point-wise composition of the operator-valued symbols,

(iii)

$$\left.\begin{array}{l} S^{\nu,l}(U \times \mathbb{R}^{1+q}) S^{\tilde{\nu},l}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}) \\ S^{\nu,l}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}) S^{\tilde{\nu},l}(U \times \mathbb{R}^{1+q}) \end{array}\right\} \subseteq S^{\nu+\tilde{\nu},l}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}), \tag{12}$$

with the point-wise multiplication of the scalar symbols by the operator-valued ones, and

(iv) for all $\alpha \in \mathbb{N}^{p_0+p}$, $\beta \in \mathbb{N}^{1+q}$ *and arbitrary* $\nu \in \mathbb{R}$ *we have*

$$D^\alpha_{t,y} D^\beta_{\tau,\eta} S^{\nu,l}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}) \subseteq S^{\nu-|\beta|_l,l}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}),\ |\beta|_l = l\beta_0 + |\beta'|. \tag{13}$$

Equation (11) is the composition law for operator-valued symbols.

Remark 6 The space $S^{-\infty}(U \times \mathbb{R}^{1+q}; E, \widetilde{E})$ has a representation as projective limit of the form

$$S^{-\infty}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}) = \varprojlim_{\nu \in \mathbb{R}} S^{\nu,l}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}), \tag{14}$$

with the projective inclusions spectrum of the Fréchet spaces $(S^{\nu,l}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}))_{\nu \in \mathbb{R}}$.

This space is independent of the concrete choice of $\{\kappa_\lambda\}$ and $\{\tilde{\kappa}_\lambda\}$ and does not depend on the anisotropy l.

Furthermore, the inequalities (6) imply

$$S^{-\infty}(\mathbb{R}^{1+q}; E, \widetilde{E}) = \mathcal{S}(\mathbb{R}^{1+q}, \mathcal{L}(E, \widetilde{E})), \tag{15}$$

where we have on the right the SCHWARTZ space of rapidly decreasing functions in $\mathbf{R}^{1+q}$ with values in $\mathcal{L}(E,\tilde{E})$.

Thus Lemma 4 and the stability of the projective tensor product under the projective limit gives

$$S^{-\infty}(U\times\mathbb{R}^{1+q};E,\tilde{E})=C^{\infty}(U,\mathcal{S}(\mathbb{R}^{1+q},\mathcal{L}(E,\tilde{E}))). \tag{16}$$

Proposition 7 *Let $a_j(t,y,\tau,\eta)\in S^{\nu_j,l}(U\times\mathbf{R}^{1+q};E,\tilde{E})$, $j\in\mathbb{N}$, be any sequence of anisotropic operator-valued symbols with $\nu_j\to-\infty$ for $j\to\infty$. Then there exists an $a(t,y,\tau,\eta)\in S^{\nu,l}(U\times\mathbb{R}^{1+q};E,\tilde{E})$ with $\nu=\max\{\nu_j:j\in\mathbb{N}\}$ such that for every $M\in\mathbb{N}$ there is an $N\in\mathbb{N}$ with*

$$a(t,y,\tau,\eta)-\sum_{j=0}^{N}a_j(t,y,\tau,\eta)\in S^{\nu-M,l}(U\times\mathbb{R}^{1+q};E,\tilde{E}), \tag{17}$$

and $a(t,y,\tau,\eta)$ is uniquely determined mod $S^{-\infty}(U\times\mathbb{R}^{1+q};E,\tilde{E})$.

We write $a\sim\sum a_j$ and call a asymptotic sum of $(a_j)_{j=0}^{\infty}$.

Definition 8 *A function $f\in C^{\infty}(\mathbf{R}^{1+q}\setminus\{0\},\mathcal{L}(E,\tilde{E}))$ is called anisotropic ν-homogeneous in the operator-valued sense if it satisfies*

$$f(\lambda^l\tau,\lambda\eta)=\lambda^{\nu}\tilde{\kappa}_{\lambda}f(\tau,\eta)\kappa_{\lambda}^{-1} \tag{18}$$

for all $\lambda\in\mathbf{R}_+$ and every $(\tau,\eta)\in\mathbf{R}^{1+q}\setminus\{0\}$.

We call the function $f\in C^{\infty}(\mathbb{R}^{1+q},\mathcal{L}(E,\tilde{E}))$ anisotropic ν-homogeneous in the operator-valued sense for large $|\tau,\eta|_l$ if equation (18) holds for all $\lambda\geq 1$ and every $(\tau,\eta)\in\mathbb{R}^{1+q}$ with $|\tau,\eta|_l>c$ for some constant $c>0$. The smallest constant c with this property is called homogeneity constant of f.

Lemma 9 *Every $a(\tau,\eta)\in C^{\infty}(\mathbb{R}^{1+q},\mathcal{L}(E,\tilde{E}))$ which is anisotropic ν-homogeneous in the operator-valued sense for large $|\tau,\eta|_l$ belongs to $S^{\nu,l}(\mathbf{R}^{1+q};E,\tilde{E})$.*

Proof: Because of the (t,y)-independence of the function a we only have to check the inequalities

$$\|\tilde{\kappa}^{-1}(\tau,\eta)\{D^{\beta}_{\tau,\eta}a(\tau,\eta)\}\kappa(\tau,\eta)\|\leq C[\tau,\eta]_l^{\nu-|\beta|_l} \tag{19}$$

for every $\beta\in\mathbb{N}^{1+q}$ and $(\tau,\eta)\in\mathbb{R}^{1+q}$. We get (19) for $|\tau,\eta|_l\leq c$ by compactness of this anisotropic ball. Outside we get by the homogeneity

$$\begin{aligned}&\|\tilde{\kappa}^{-1}(\tau,\eta)D^{\beta}_{\tau,\eta}a(\tau,\eta)\kappa(\tau,\eta)\|\\ =\;&\|\tilde{\kappa}^{-1}(\tau,\eta)[\tau,\eta]_l^{\nu-|\beta|_l}\tilde{\kappa}(\tau,\eta)(D^{\beta}_{\tau,\eta}a)\left(\frac{\tau}{[\tau,\eta]_l^l},\frac{\eta}{[\tau,\eta]_l}\right)\kappa^{-1}(\tau,\eta)\kappa(\tau,\eta)\|\\ =\;&[\tau,\eta]_l^{\nu-|\beta|_l}\left\|(D^{\beta}_{\tau,\eta}a)\left(\frac{\tau}{[\tau,\eta]_l^l},\frac{\eta}{[\tau,\eta]_l}\right)\right\|\\ \leq\;&C[\tau,\eta]_l^{\nu-|\beta|_l},\end{aligned}$$

with $C = \sup\{\|D^{\beta}_{\tau,\eta}a(\tau,\eta)\| \ : \ |\tau,\eta|_l = 1\}$.

In the above estimate we used that the derivatives of homogeneous functions are homogeneous of the corresponding diminished order, again. □

Next we define classical symbols. The scalar versions of these spaces are treated in [PIR1].

We denote by $S^{(\nu),l}(U \times (\mathbb{R}^{1+q}\setminus\{0\}); E, \tilde{E})$ the subspace of all operator-valued functions $f_{(\nu)}(t,y,\tau,\eta) \in C^\infty(U \times (\mathbb{R}^{1+q}\setminus\{0\}), \mathcal{L}(E,\tilde{E}))$, that are anisotropic ν-homogeneous in (τ,η) for all $(t,y) \in U$.

From $|\tau,\eta|_l^{-\nu}\tilde{\kappa}^{-1}_{|\tau,\eta|_l} f_{(\nu)}(t,y,\tau,\eta)\kappa_{|\tau,\eta|_l} = f_{(\nu)}(t,y,\frac{\tau}{|\tau,\eta|_l^l},\frac{\eta}{|\tau,\eta|_l})$ we get isomorphisms

$$S^{(\nu),l}(U \times (\mathbb{R}^{1+q}\setminus\{0\}); E, \tilde{E}) \to C^\infty(U \times \{(\tau,\eta) \in \mathbb{R}^{1+q} : |\tau,\eta|_l = 1\}, \mathcal{L}(E,\tilde{E})). \qquad (20)$$

The inverse operator is the anisotropic ν-homogeneous extension of the given element.

Furthermore, we denote by $S^{[\nu],l}(U \times \mathbb{R}^{1+q}; E, \tilde{E})$ the subspace of all operator-valued functions $f(t,y,\tau,\eta) \in C^\infty(U \times \mathbb{R}^{1+q}, \mathcal{L}(E,\tilde{E}))$ that are ν-homogeneous for large $|\tau,\eta|_l$. For an arbitrary excision function $\chi(\tau,\eta)$ we have

$$\chi(\tau,\eta)S^{(\nu),l}(U \times (\mathbb{R}^{1+q}\setminus\{0\}); E, \tilde{E}) \subset S^{[\nu],l}(U \times \mathbb{R}^{1+q}; E, \tilde{E}).$$

From Lemma 9 it follows $S^{[\nu],l}(U \times \mathbb{R}^{1+q}; E, \tilde{E}) \subset S^{\nu,l}(U \times \mathbb{R}^{1+q}; E, \tilde{E})$.

Definition 10 *Let $\nu \in \mathbb{R}$ and $U = U_0 \times U$ with open $U_0 \times U \in \mathbb{R}^{p_0+p}$ be given as in Definition 1. Then the space of classical anisotropic operator-valued symbols of order ν*

$$S^{\nu,l}_{cl}(U \times \mathbb{R}^{1+q}; E, \tilde{E}) \qquad (21)$$

is defined as the set of all $a(t,y,\tau,\eta) \in S^{\nu,l}(U \times \mathbb{R}^{1+q}; E, \tilde{E})$ such that there is a sequence $(a_{(\nu-j)}(t,y,\tau,\eta))_{j=0}^\infty$ with $a_{(\nu-j)}(t,y,\tau,\eta) \in S^{(\nu-j),l}(U \times \mathbb{R}^{1+q}; E, \tilde{E})$ for every j, such that for any excision function $\chi(\tau,\eta)$ we have

$$a(t,y,\tau,\eta) \sim \sum_{j=0}^{\infty} \chi(\tau,\eta)a_{(\nu-j)}(t,y,\tau,\eta) \qquad (22)$$

in the sense of Proposition 7.

Analogously to the isotropic case we call the function $\sigma^\nu_\wedge(a)(t,y,\tau,\eta) := a_{(\nu)}(t,y,\tau,\eta)$ anisotropic ν-homogeneous principal symbol of a.

Remark 11 The anisotropic $(\nu - j)$-homogeneous components $a_{(\nu-j)}(t,y,\tau,\eta)$ of some $a(t,y,\tau,\eta) \in S^{\nu,l}_{cl}(U \times \mathbb{R}^{1+q}; E, \tilde{E})$ are uniquely determined.

Remark 12 It follows that any function $a(t,y,\tau,\eta) \in S^{[\nu],l}(U \times \mathbb{R}^{1+q}; E, \tilde{E})$ belongs to $S^{\nu,l}_{cl}(U \times \mathbb{R}^{1+q}; E, \tilde{E})$. Taking the uniquely determined anisotropic ν-homogeneous extension

$a_{(\nu)}(t,y,\tau,\eta) \in S^{(\nu),l}(U \times \mathbb{R}^{1+q}; E, \widetilde{E})$ of $a(t,y,\tau,\eta)|_{\{[\tau,\eta]_l = K\}}$ with some constant $K > 0$ which is larger than the homogeneity constant of a, we have

$$a(t,y,\tau,\eta) - \chi(\tau,\eta)a_{(\nu)}(t,y,\tau,\eta) \in C^\infty(U, \mathcal{S}(\mathbb{R}^{1+q}, \mathcal{L}(E,\widetilde{E})))$$

for every excision function $\chi(\tau,\eta)$ such that $a(t,y,\tau,\eta) \sim \chi(\tau,\eta)a_{(\nu)}t,y,\tau,\eta)$.

Next we want to define a FRÉCHET topology in the space of classical anisotropic operator-valued symbols. By definition we have a canonical embedding

$$S^{\nu,l}_{cl}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}) \hookrightarrow S^{\nu,l}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}) \tag{23}$$

for every $\nu \in \mathbb{R}$. The relation (20) induces in $S^{(\nu),l}(U \times (\mathbb{R}^{1+q}\setminus\{0\}); E, \widetilde{E})$ a FRÉCHETtopology.

Then Remark 11 gives by $a \mapsto \chi(\tau,\eta)\sum_{j=0}^{k} a_{(\nu-j)}$ a well-defined sequence of linear mappings

$$\alpha_k \;:\; S^{\nu,l}_{cl}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}) \longrightarrow S^{\nu-k-1,l}_{cl}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}),\; k \in \mathbb{N}. \tag{24}$$

Moreover, we get by $a \mapsto a_{(\nu-j)}$ a sequence of linear mappings

$$\beta_j \;:\; S^{\nu,l}_{cl}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}) \longrightarrow S^{(\nu-j),l}_{cl}(U \times \mathbb{R}^{1+q}; E, \widetilde{E}),\; j \in \mathbb{N}. \tag{25}$$

Now we topologize (21) with the weakest local convex topology such that all the mappings (23), (24) and (25) are continuous.

Let us now deal with the case of FRÉCHET spaces E and $\widetilde{E}$. We further assume that there are representations

$$E = \varprojlim_{j\to\infty} E_j \text{ and } \widetilde{E} = \varprojlim_{k\to\infty} \widetilde{E}_k$$

with projective inclusion spectra of BANACH spaces $E_0 \hookleftarrow E_1 \hookleftarrow \cdots$ and $\widetilde{E}_0 \hookleftarrow \widetilde{E}_1 \hookleftarrow \cdots$ such that the associated sets of $\{\kappa^{(j)}_\lambda\}_{j=0}^\infty$ and $\{\tilde{\kappa}^{(k)}_\lambda\}_{k=0}^\infty$ satisfy the compatibility conditions $\kappa^{(j)}_\lambda|_{E_{j+1}} = \kappa^{(j+1)}_\lambda$ and $\tilde{\kappa}^{(k)}_\lambda|_{\widetilde{E}_{k+1}} = \tilde{\kappa}^{(k+1)}_\lambda$ for all $j,k \in \mathbb{N}$. Without loss of generality we may also assume, that the corresponding norms q_j in E_j and $\tilde{q}_k$ in $\widetilde{E}_k$ give ordered semi-norm systems in E and $\widetilde{E}$.

Definition 13 *Under the above assumptions we define for $\nu \in \mathbb{R}$ and open $U = U_0 \times U' \in \mathbb{R}^{p_0+p}$ the space $S^{\nu,l}(U \times \mathbb{R}^{1+q}; E, \widetilde{E})$ of anisotropic symbols of order ν with values in $\mathcal{L}(E,\widetilde{E})$ as the set of all $a(t,y,\tau,\eta) \in C^\infty(U \times \mathbb{R}^{1+q}, \mathcal{L}(E,\widetilde{E}))$ such that for every $k \in \mathbb{N}$ there is some $j = j(k) \in \mathbb{N}$ with $a(t,y,\tau,\eta)$ belonging to $S^{\nu,l}(U \times \mathbb{R}^{1+q}; E_j, \widetilde{E}_k)$.*

Remark 14 With this definition the assertions of Lemma 5 are also fulfilled for FRÉCHET spaces E and $\widetilde{E}$.

In the same manner we define the corresponding subspaces of classical anisotropic symbols $S^{\nu,l}_{cl}(U \times \mathbb{R}^{1+q}; E, \widetilde{E})$ with $S^{\nu,l}_{cl}(U \times \mathbb{R}^{1+q}; E_j, \widetilde{E}_k)$ for every $k \in \mathbb{N}$ with some $j = j(k) \in \mathbb{N}$.

Example 15 For $P \in \mathrm{As}(\gamma, \Theta)$ with $\Theta = (\theta, 0]$ and $\gamma \in \mathbb{R}$ we have a representation as projective limit

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) = \varprojlim_{j\to\infty} \mathcal{K}^{s,\gamma-\theta-\varepsilon_j}(X^\wedge) + [\omega]\mathcal{E}_P$$

with $\varepsilon_0 > 0$ such that $\pi_{\mathbb{C}} P \cap \{z \in \mathbb{C} : \frac{n+1}{2} - \gamma + \theta \leq \mathrm{Re}\, z \leq \frac{n+1}{2} - \gamma + \theta + \varepsilon_0\} = \emptyset$ and $\varepsilon_j = \varepsilon_0 2^{-j}$, $j \in \mathbb{N}$. Analogously we take for $Q \in \mathrm{As}(\gamma - \mu, \Theta)$ a sequence $(\delta_k = \delta_0 2^{-k})_{k=0}^{\infty}$ where $\delta_0 > 0$ is chosen such that $\pi_{\mathbb{C}} Q \cap \{z \in \mathbb{C} : \frac{n+1}{2} - \gamma + \mu + \theta \leq \mathrm{Re}\, z \leq \frac{n+1}{2} - \gamma + \mu + \theta + \delta_0\} = \emptyset$ holds. Then we have

$$\mathcal{K}_Q^{s-\nu,\gamma-\mu}(X^\wedge) = \varprojlim_{k\to\infty} \mathcal{K}^{s-\nu,\gamma-\mu-\theta-\delta_k}(X^\wedge) + [\omega]\mathcal{E}_Q.$$

Further we set $(\kappa_\lambda^{(j)} u)(r, x) = (\tilde{\kappa}_\lambda^{(k)} u)(r, x) = \lambda^{\frac{n+1}{2}} u(\lambda r, x)$ for all $j, k \in \mathbb{N}$ which obviously satisfy the compatibility conditions. Then by definition the symbol space

$$S^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}_P^{s,\gamma}(X^\wedge), \mathcal{K}_Q^{s-\nu,\gamma-\mu}(X^\wedge)) \tag{26}$$

is the set of all $a(t, y, \tau, \eta) \in C^\infty(U \times \mathbb{R}^{1+q}, \mathcal{L}(\mathcal{K}_P^{s,\gamma}(X^\wedge), \mathcal{K}_Q^{s-\nu,\gamma-\mu}(X^\wedge)))$ such that for every $k \in \mathbb{N}$ there is some $j = j(k) \in \mathbb{N}$ such that $a(t, y, \tau, \eta)$ belongs to $S^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma-\theta-\varepsilon_j}(X^\wedge) + [\omega]\mathcal{E}_P, \mathcal{K}^{s-\nu,\gamma-\mu-\theta-\delta_k}(X^\wedge) + [\omega]\mathcal{E}_Q)$.

Lemma 16 *Let E and $\widetilde{E}$ be* FRÉCHET *spaces, that have representations as projective limits of* BANACH *spaces and further we assume, that the associated group actions satisfy the corresponding compatibility conditions.*

Then every $a(\tau, \eta) \in C^\infty(\mathbb{R}^{1+q}, \mathcal{L}(E, \widetilde{E})) := \bigcap_{k=0}^{\infty} \bigcup_{j=0}^{\infty} C^\infty(\mathbb{R}^{1+q}, \mathcal{L}(E_j, \widetilde{E}_k))$ which is anisotropic ν-homogeneous in the operator-valued sense for large $|\tau, \eta|_l$ belongs to $S^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E})$.

Proof: As mentioned above we have to find for every $k \in \mathbb{N}$ a number $j = j(k)$ such that the function $a(\tau, \eta)$ belongs to $S^{\nu,l}(U \times \mathbb{R}^{1+q}; E_j, \widetilde{E}_k)$. Because of Lemma 9 it remains only to check $a(\tau, \eta) \in C^\infty(\mathbb{R}^{1+q}, \mathcal{L}(E_j, \widetilde{E}_k))$. But from

$$a(\tau, \eta) \in \bigcap_{k=0}^{\infty} \bigcup_{j=0}^{\infty} C^\infty(\mathbb{R}^{1+q}, \mathcal{L}(E_j, \widetilde{E}_k))$$

we get for every k some $j = j(k)$ with this property. □

Remark 17 In view of Remark 12 we obtain that $a(\tau, \eta)$ under the conditions of Lemma 16 is a classical anisotropic operator-valued symbol.

Lemma 18 *The operator M_ϕ of multiplication by $\phi(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ belongs to $S^{0,l}(\mathbb{R}^{1+q}; \mathcal{K}_P^{s,\gamma}(X^\wedge), \mathcal{K}_P^{s,\gamma}(X^\wedge))$ for arbitrary $s, \gamma \in \mathbb{R}$ and every asymptotic type $P \in \mathrm{As}(\gamma, \Theta)$ with $\Theta = (\theta, 0]$, $-\infty < \theta < 0$ satisfying the shadow condition, cf. (0,(2)).*

Moreover, the mapping

$$M \; : \; C_0^\infty(\overline{\mathbb{R}}_+) \ni \phi \mapsto M_\phi \in \bigcap_{s,\gamma\in\mathbb{R}} S^{0,l}(\mathbb{R}^{1+q}; \mathcal{K}_P^{s,\gamma}(X^\wedge), \mathcal{K}_P^{s,\gamma}(X^\wedge)) \tag{27}$$

is continuous.

By analogous arguments as in the proof of Lemma 3 we see that the proof does not depend on the anisotropy l. Therefore we will omit it for abbreviation.

1.2 Operators in anisotropic wedge SOBOLEV spaces

As in Section 1.1 we first fix BANACH spaces E and $\widetilde{E}$ and associated strongly continuous groups κ_λ and $\tilde{\kappa}_\lambda$, respectively. Now let $\mathcal{S}(\mathbb{R}^{1+q}, E) = \mathcal{S}(\mathbb{R}^{1+q})\widehat{\otimes}_\pi E$ be the SCHWARTZ space of rapidly decrasing E–valued functions. Then we define the FOURIER transformation

$$\mathcal{F} \; : \; \mathcal{S}(\mathbb{R}^{1+q}, E) \to \mathcal{S}(\mathbb{R}^{1+q}, E)$$

by $\mathcal{F} = F \otimes \mathrm{id}_E$, where F denotes the standard FOURIER transformation. Setting

$$\mathcal{S}'(\mathbb{R}^{1+q}, E) := \mathcal{L}(\mathcal{S}(\mathbb{R}^{1+q}), E)$$

we define for $T \in \mathcal{L}(\mathcal{S}(\mathbb{R}^{1+q}_{t,y}), E)$ the FOURIER transform $\mathcal{F}T = T \circ F \in \mathcal{L}(\mathcal{S}(\mathbb{R}^{1+q}_{\tau,\eta}), E)$. Then for $S \in \mathcal{L}(\mathcal{S}(\mathbb{R}^{1+q}_{\tau,\eta}), E)$ we get the inverse FOURIER transform by $\mathcal{F}^{-1}S = S \circ F^{-1}$. We then obviously obtain $\mathcal{F}^{-1}\mathcal{F}T = T$. Note that we get $\mathcal{F} = F$ for $E = \mathbb{C}$ as well as $\mathcal{F}^{-1} = F^{-1}$, that justifies the use of the same letter for the above defined FOURIER transformation for vector–valued distributions and the scalar FOURIER transformation.

Definition 1 *Let $a(t,y,t',y',\tau,\eta) \in S^{\nu,l}(U^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$ with an open $U = U_0 \times U' \subset \mathbb{R}^{1+q}$ be given; $U^2 = U \times U$. Then we define*

$$\begin{aligned} \mathrm{Op}(a)u(t,y) &= \mathcal{F}^{-1}_{(\tau,\eta)\to(t,y)}\mathcal{F}_{(t',y')\to(\tau,\eta)}\{a(t,y,t',y',\tau,\eta)u(t',y')\} \qquad (1)\\ &= \iint e^{i(t-t')\tau+i(y-y')\eta}a(t,y,t',y',\tau,\eta)u(t',y')dt'\,dy'\,đ\tau\,đ\eta \end{aligned}$$

for $u(t',y') \in C_0^\infty(U,E)$.

Remark 2 Similarly to the scalar theory $\mathrm{Op}(a)$ is a continuous operator

$$\mathrm{Op}(a): \;\; C_0^\infty(U,E) \to C^\infty(U,\widetilde{E})$$

for every $a(t,y,t',y',\tau,\eta) \in S^{\nu,l}(U^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$.

Definition 3 *We denote by $\Psi^{\nu,l}(U;E,\widetilde{E})$ the FRÉCHET space of all operators $\mathrm{Op}(a)$ with $a(t,y,t',y',\tau,\eta) \in S^{\nu,l}(U^2 \times \mathbb{R}^{1+q}; E, \widetilde{E})$.*

Furthermore, we write $\Psi^{\nu,l}_{cl}(U;E,\widetilde{E})$ for the subspace of all $\mathrm{Op}(a)$ with classical symbols.

The elements of $\Psi^{\nu,l}(U;E,\widetilde{E})$ are called anisotropic pseudo–differential operators, those in $\Psi^{\nu,l}_{cl}(U;E,\widetilde{E})$ classical anisotropic pseudo–differential operators of order ν.

Analogously to the isotropic theory we use the notation

$$\Psi^{\infty,l}(U;E,\tilde{E}) = \bigcup_{\nu\in\mathbb{R}} \Psi^{\nu,l}(U;E,\tilde{E}), \quad (2)$$

$$\Psi^{-\infty}(U;E,\tilde{E}) = \bigcap_{\nu\in\mathbb{R}} \Psi^{\nu,l}(U;E,\tilde{E}), \quad (3)$$

where the space (3) is isomorphic to the space of integral operators with C^∞-kernel.

Let us now introduce the anisotropic wedge SOBOLEV spaces of E-valued distributions.

Definition 4 *For every $s \in \mathbb{R}$ we get by*

$$u \mapsto \|u\|_{s,l} := \left(\int_{\mathbb{R}^{1+q}} [\tau,\eta]_l^{2s} \|\kappa^{-1}(\tau,\eta)\mathcal{F}u(\tau,\eta)\|_E^2 d\tau d\eta\right)^{1/2} \quad (4)$$

a norm in $\mathcal{S}(\mathbb{R}^{1+q},E)$. The anisotropic wedge SOBOLEV *space $\mathcal{W}^{s,l}(\mathbb{R}^{1+q},E)$ of order s is defined as the completion of $\mathcal{S}(\mathbb{R}^{1+q},E)$ with respect to the norm (4).*

Note that the anisotropic wedge SOBOLEV spaces are BANACH spaces. As usually we write $H^{s,l}(\mathbb{R}^{1+q},E)$ if we have $\kappa_\lambda = \mathrm{id}_E$ for all $\lambda \in \mathbb{R}_+$. With $E = \mathbb{C}$ we get the scalar version of anisotropic SOBOLEV spaces, which we denote by $H^{s,l}(\mathbb{R}^{1+q})$.

Remark 5 Analogously to the isotropic case (cf. [SCH1], Section 3.1), the operator $T = \mathcal{F}^{-1}_{(\tau,\eta)\to(t,y)}\kappa^{-1}(\tau,\eta)\mathcal{F}_{(t,y)\to(\tau,\eta)}$ extends by continuity to an isometry

$$T \,:\, \mathcal{W}^{s,l}(\mathbb{R}^{1+q},E) \to H^{s,l}(\mathbb{R}^{1+q},E). \quad (5)$$

This gives us the possibility to define $\mathcal{W}^{s,l}(\mathbb{R}^{1+q},\mathcal{V})$ for subspaces $\mathcal{V} \subset E$ which are not necessary preserved under κ_λ by $\mathcal{W}^{s,l}(\mathbb{R}^{1+q},\mathcal{V}) := T^{-1}H^{s,l}(\mathbb{R}^{1+q},\mathcal{V})$ (cf. Example 15 below).

Lemma 6 *For all $s \in \mathbb{R}$ the space $C_0^\infty(\mathbb{R}^{1+q},E)$ is dense in $\mathcal{W}^{s,l}(\mathbb{R}^{1+q},E)$.*

Proof: By definition we only have to prove, that $C_0^\infty(\mathbb{R}^{1+q},E)$ is dense with respect to the norm (4) in $\mathcal{S}(\mathbb{R}^{1+q},E)$. The isotropic case $l = 1$ was treated, for instance, in [SCH1], such that we have to show the assertion for $l > 1$.

Using the inequality $[\tau,\eta]_l \le c[\tau,\eta]_1$ we have $\|u\|_{s,l} \le \|u\|_{s,1}$ for all $s \in \mathbb{R}$ and every $u \in \mathcal{S}(\mathbb{R}^{1+q},E)$. Thus the isotropic case implies the anisotropic case. □

Example 7 In our applications we are dealing with the case of $\mathcal{K}^{s,\gamma}(X^\wedge)$-valued distributions. Like in Section 1.1 we take $\kappa_\lambda u(r) = \lambda^{\frac{n+1}{2}}u(\lambda r)$ as associated group action. Then we have from Definition 4 the BANACH spaces $\mathcal{W}^{s,l}(\mathbb{R}^{1+q},\mathcal{K}^{s,\gamma}(X^\wedge))$. Setting $s = \gamma = 0$ we get the HILBERT space $\mathcal{W}^{0,l}(\mathbb{R}^{1+q},\mathcal{K}^{0,0}(X^\wedge)) = \mathcal{W}^0(\mathbb{R}^{1+q},\mathcal{K}^0(X^\wedge))$, which is independent of the anisotropy l. The corresponding scalar product is given by

$$(u,v)_0 = \int (\mathcal{F}u(\tau,\eta),\mathcal{F}v(\tau,\eta))_{\mathcal{K}^0} d\tau d\eta.$$

Of course, the space $C_0^\infty(\mathbb{R}^{1+q}, C_0^\infty(X^\wedge)) = C_0^\infty(\mathbb{R}^{1+q} \times X^\wedge)$ is dense in $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, \mathcal{K}^{s,\gamma}(X^\wedge))$ for every $s, \gamma \in \mathbb{R}$, such that the form $(\cdot,\cdot)_0 : C_0^\infty(\mathbb{R}^{1+q} \times X^\wedge) \times C_0^\infty(\mathbb{R}^{1+q} \times X^\wedge) \to \mathbb{C}$ extends for all $s, \gamma \in \mathbb{R}$ to a non-degenerate sesquilinear form

$$(\cdot,\cdot) : \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, \mathcal{K}^{s,\gamma}(X^\wedge)) \times \mathcal{W}^{-s,l}(\mathbb{R}^{1+q}, \mathcal{K}^{-s,-\gamma}(X^\wedge)) \to \mathbb{C}.$$

This allows to introduce formal adjoints A^* of operators

$$A : \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, \mathcal{K}^{s,\gamma}(X^\wedge)) \to \mathcal{W}^{s-\nu,l}(\mathbb{R}^{1+q}, \mathcal{K}^{s-\nu,\gamma-\mu}(X^\wedge))$$

that for all $s \in \mathbb{R}$ are continuous operators

$$A^* : \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, \mathcal{K}^{s,-\gamma+\mu}(X^\wedge)) \to \mathcal{W}^{s-\nu,l}(\mathbb{R}^{1+q}, \mathcal{K}^{s-\nu,-\gamma}(X^\wedge)).$$

Lemma 8 *There is a canonical embedding $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) \hookrightarrow \mathcal{S}'(\mathbb{R}^{1+q}, E)$ given by $\langle \phi, u \rangle = \int_{\mathbb{R}^{1+q}} \phi(t,y) u(t,y)\, dt\, dy$ with $\phi(t,y) \in \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$ and $u(t,y) \in \mathcal{S}'(\mathbb{R}^{1+q}, E)$.*

We omit for abbreviation the proof, which is completely analogous to the isotropic case (cf. [HIR1]).

Corollary 9 *An E–valued tempered distribution $u(t,y)$ belongs to the anisotropic wedge* SOBOLEV *space $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$ iff $\mathcal{F}u(\tau,\eta)$ is measurable and $\|u\|_{s,l} < \infty$.*

Definition 10 *Let $U \subseteq \mathbb{R}^{1+q}$ be open and $K \subset\subset U$ compact. Then we define*

$$\begin{aligned}
\mathcal{W}^{s,l}_K(U,E) &:= \{u \in \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) \ : \ \operatorname{supp} u \subseteq K\}, \\
\mathcal{W}^{s,l}_{comp}(U,E) &:= \bigcup_{K \subset\subset U} \mathcal{W}^{s,l}_K(U,E), \\
\mathcal{W}^{s,l}_{loc}(U,E) &:= \{u \in \mathcal{S}'(\mathbb{R}^{1+q}, E) \ : \ \omega u \in \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) \text{ for all } \omega \in C_0^\infty(U)\}.
\end{aligned}$$

Theorem 11 *Let $U \subset \mathbb{R}^{1+q}$ be an open set and $a(t,y,t',y',\tau,\eta) \in S^{\nu,l}(U^2 \times \mathbb{R}^{q+1}; E, \widetilde{E})$ be an anisotropic operator–valued symbol of order $\nu \in \mathbb{R}$ and $\mathrm{Op}(a) \in \Psi^{\nu,l}(U; E, \widetilde{E})$ the associated pseudo–differential operator. Then*

$$\mathrm{Op}(a) : \mathcal{W}^{s,l}_{comp}(U,E) \to \mathcal{W}^{s-\nu,l}_{loc}(U, \widetilde{E})$$

is continuous for all $s \in \mathbb{R}$.

The proof will be given in terms of a tensor product argument using the two following lemmata, which can be obtained analogously to the isotropic case, see for instance [SCH1] or [HIR1].

Lemma 12 *For every $v \in \mathcal{S}(\mathbb{R}^{1+q})$ the operator $M_v : C_0^\infty(\mathbb{R}^{1+q}, E) \ni \phi \mapsto v\phi \in C_0^\infty(\mathbb{R}^{1+q}, E)$ has a unique continuous extension to $M_v : \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) \to \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E)$. Furthermore, the map $M : \mathcal{S}(\mathbb{R}^{1+q}) \ni v \mapsto M_v \in \mathcal{L}(\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E))$ is continuous for all $s \in \mathbb{R}$.*

Lemma 13 *Let $a(\tau,\eta) \in S^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E})$ be an anisotropic operator-valued symbol with constant coefficients of order $\nu \in \mathbb{R}$ and $\mathrm{Op}(a) \in \Psi^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E})$ the associated pseudo-differential operator. Then $\mathrm{Op}(a) : \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) \to \mathcal{W}^{s-\nu,l}(\mathbb{R}^{1+q}, \widetilde{E})$ is continuous for all $s \in \mathbb{R}$ and $\|\mathrm{Op}(a)\| \le \sup_{(\tau,\eta)\in\mathbb{R}^{1+q}} [\tau,\eta]_l^{-\nu} \|\tilde{\kappa}^{-1}(\tau,\eta) a(\tau,\eta) \kappa(\tau,\eta)\|_{\mathcal{L}(E,\widetilde{E})} =: p_{0,0}^{(\nu)}(a)$.*

Proof: (of Theorem 11) For every $K \subset\subset U$ there exists a function $\phi \in C_0^\infty(U)$ such that $\phi(t,y) \equiv 1$ in a neighbourhood of K. We want to prove that for every fixed $K \subset\subset U$ with such a $\phi \in C_0^\infty(U)$ and arbitrary $\psi \in C_0^\infty(U)$ the operator

$$M_\psi \mathrm{Op}(a) M_\phi \ : \ \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) \to \mathcal{W}^{s-\nu,l}(\mathbb{R}^{1+q}, \widetilde{E}) \tag{6}$$

is continuous for all $a(t,y,t',y',\tau,\eta) \in S^{\nu,l}(U^2 \times \mathbb{R}^{q+1}; E, \widetilde{E})$.

Because of $S^{\nu,l}(U^2 \times \mathbb{R}^{1+q}; E, \widetilde{E}) = C^\infty(U) \hat{\otimes}_\pi S^{\nu,l}(\mathbb{R}^{1+q}; E, \widetilde{E}) \hat{\otimes}_\pi C^\infty(U)$ we have the representation $a(t,y,t',y',\tau,\eta) = \sum_{j=0}^\infty \lambda_j b_j(t,y) a_j(\tau,\eta) d_j(t',y')$, where $b_j \to 0$ and $d_j \to 0$ in $C^\infty(U)$, $a_j \to 0$ in $S^{\nu,l}(\mathbb{R}^{q+1}; E, \widetilde{E})$ and $\{\lambda_j\}_{j=1}^\infty \in l_1$.

Thus we obtain

$$M_\psi \mathrm{Op}(a) M_\phi = \sum_{j=1}^\infty \lambda_j M_{\psi b_j} \mathrm{Op}(a_j) M_{\phi d_j},$$

where $\psi b_j \to 0$ and $\phi d_j \to 0$ in $C_0^\infty(U)$. Therefore, we get by Lemma 12 and Lemma 13

$$\begin{aligned}
\|M_\psi \mathrm{Op}(a) M_\phi u\|_{s-\nu,l} &= \|\sum_{j=1}^\infty \lambda_j M_{\psi b_j} \mathrm{Op}(a_j) M_{\phi d_j} u\|_{s-\nu,l} \\
&\le \sum_{j=1}^\infty |\lambda_j| \, \|M_{\psi b_j}\| \, \|\mathrm{Op}(a_j)\| \, \|M_{\phi d_j}\| \, \|u\|_{s,l} \\
&\le \sum_{j=1}^\infty |\lambda_j| c_{\psi b_j} p_{0,0}^{(\nu)}(a_j) c_{\phi d_j} \|u\|_{s,l}.
\end{aligned}$$

The convergence of $\{c_{\psi b_j}\}$, $\{c_{\phi d_j}\}$ and $\{p_{0,0}^{(\nu)}(a_j)\}$ implies, in particular, boundedness, which gives the continuity of (6). □

Like in Section 1.1 we deal with the case of Fréchet spaces E and $\widetilde{E}$, which have representations as projective limits of Banach spaces, where the associated group actions fulfill the corresponding compatibility conditions.

In this case we also set $\Psi^{\nu,l}(U; E, \widetilde{E}) := \mathrm{Op}(S^{\nu,l}(U^2 \times \mathbb{R}^{1+q}; E, \widetilde{E}))$ for arbitrary open $U \subset \mathbb{R}^{1+q}$ and define $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) = \varprojlim_{k\to\infty} \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E_k)$ as well as

$$\mathcal{W}^{s,l}_{comp}(U,E) = \varprojlim_{k\to\infty} \mathcal{W}^{s,l}_{comp}(U,E_k) \text{ and } \mathcal{W}^{s,l}_{loc}(U,E) = \varprojlim_{k\to\infty} \mathcal{W}^{s,l}_{loc}(U,E_k).$$

Corollary 14 *Every $A \in \Psi^{\nu,l}(U; E, \widetilde{E})$ has a unique extension to a linear continuous operator $A : \mathcal{W}^{s,l}_{comp}(U,E) \to \mathcal{W}^{s-\nu,l}_{loc}(U,\widetilde{E})$ for all $s \in \mathbb{R}$.*

Example 15 In our applications we also need the spaces

$$E = \mathcal{K}_P^{s,\gamma}(X^\wedge) = \varinjlim_{k\to\infty} \mathcal{K}^{s,\gamma-\theta-2^{-k}\varepsilon_0}(X^\wedge) + [\omega]\mathcal{E}_P$$

for any asymptotic type $P \in \mathrm{As}(\gamma, \Theta)$. Then we have by definition

$$\begin{aligned} \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, E) &= \varinjlim_{k\to\infty} \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, \mathcal{K}^{s,\gamma-\theta-2^{-k}\varepsilon_0}(X^\wedge) + [\omega]\mathcal{E}_P) \\ &= \varinjlim_{k\to\infty} \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, \mathcal{K}^{s,\gamma-\theta-2^{-k}\varepsilon_0}(X^\wedge)) + \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, [\omega]\mathcal{E}_P), \end{aligned}$$

where $\mathcal{F}\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, [\omega]\mathcal{E}_P)$ is spanned by distributions of the form

$$\omega(r[\tau,\eta]_l)c_{jk}(x)[\tau,\eta]_l^{-p_j} r^{-p_j} \ln^k(r[\tau,\eta]_l)\mathcal{F}v(\tau,\eta)$$

with $v(t,y) \in H^{s,l}(\mathbb{R}^{1+q})$. These are the singular functions (in the FOURIER image) of the anisotropic discrete edge asymptotics. $\mathcal{W}^{s,l}(\mathbb{R}^{1+q}, [\omega]\mathcal{E}_P)$ is to be understood in the sense $T^{-1}H^{s,l}(\mathbb{R}^{1+q}, [\omega]\mathcal{E}_P)$ with the above isometry (5) restricted to $H^{s,l}(\mathbb{R}^{1+q}, [\omega]\mathcal{E}_P)$ with the subspace $[\omega]\mathcal{E}_P$ of $\mathcal{K}^{s,\gamma}(X^\wedge)$.

Remark 16 For our applications it will be convenient to use shorter notations for the anisotropic wedge SOBOLEV spaces with values in $\mathcal{K}^{s,\gamma}(X^\wedge)$ or the subspaces with asymptotics $\mathcal{K}_P^{s,\gamma}(X^\wedge)$. For that reason we will write

$$\begin{aligned} \mathcal{W}^{s,\gamma;l}(X^\wedge \times \mathbb{R}^{1+q}) &:= \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, \mathcal{K}^{s,\gamma}(X^\wedge)) \quad \text{and} \\ \mathcal{W}_P^{s,\gamma;l}(X^\wedge \times \mathbb{R}^{1+q}) &:= \mathcal{W}^{s,l}(\mathbb{R}^{1+q}, \mathcal{K}_P^{s,\gamma}(X^\wedge)). \end{aligned}$$

Moreover, for some open $U = U_0 \times U' \subseteq \mathbb{R}^{1+q}$ we may form the *comp*, *loc* versions of these spaces, which are written as

$$\mathcal{W}^{s,\gamma;l}_{comp(t,y)}(X^\wedge \times U) := \mathcal{W}^{s,l}_{comp}(U, \mathcal{K}^{s,\gamma}(X^\wedge)) \;\text{ and }\; \mathcal{W}^{s,\gamma;l}_{loc(t,y)}(X^\wedge \times U) := \mathcal{W}^{s,l}_{loc}(U, \mathcal{K}^{s,\gamma}(X^\wedge)).$$

Note that we observe $\mathcal{W}^{s,\gamma;l}(X^\wedge \times \mathbb{R}^{1+q}) \subset H^{s,l}_{loc}(X^\wedge \times \mathbb{R}^{1+q})$ for every $s, \gamma \in \mathbb{R}$.

1.3 GREEN symbols and operators

In the following we shall introduce an ideal in the anisotropic edge symbol algebra, namely the GREEN edge symbols. They will play the role of the smoothing elements in the edge symbol algebra. However they will be not negligible, since the associated pseudo-differential operators will not be compact.

The elements of the edge symbol algebra are $\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{N_-}, \mathcal{K}^{s',\gamma'}(X^\wedge) \oplus \mathbb{C}^{N_+})$- or $\mathcal{L}(\mathcal{K}_P^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{N_-}, \mathcal{K}_Q^{s',\gamma'}(X^\wedge) \oplus \mathbb{C}^{N_+})$-valued symbols. Here $s, s', \gamma, \gamma' \in \mathbb{R}$, N_-, N_+ are non-negative integers and P, Q are asymptotic types. Moreover, we fix once and for all the group action $\kappa_\lambda(u(r,x) \oplus z) = \lambda^{\frac{n+1}{2}} u(\lambda r, x) \oplus z$ for all $u(r,x) \oplus z \in \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^M$, $M = N_-, N_+$. From now on for abbreviation we will drop $X^\wedge$ in the notation and write, for instance, $\mathcal{K}^{s,\gamma}$ instead of $\mathcal{K}^{s,\gamma}(X^\wedge)$.

Definition 1 *Let $\mu, \nu \in \mathbb{R}$ and $U = U_0 \times U'$ with open $U_0 \subseteq \mathbb{R}^{p_0}$ and $U' \subseteq \mathbb{R}^{p'}$. For given weight data $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$, $\Theta = (\theta, 0]$, $-\infty < \theta < 0$, we denote by $R_G^{\nu,l}(U \times \mathbb{R}^{1+q}, \mathbf{g})$ the space of all*

$$g(t, y, \tau, \eta) \in \bigcap_{s \in \mathbb{R}} S_{cl}^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}, \mathcal{K}^{\infty,\gamma-\mu}),$$

such that there are asymptotic types $P \in \mathrm{As}(\gamma - \mu, \Theta)$ and $Q \in \mathrm{As}(-\gamma, \Theta)$, with

$$g(t, y, \tau, \eta) \in \bigcap_{s \in \mathbb{R}} S_{cl}^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}, \mathcal{S}_P^{\gamma-\mu}) \tag{1}$$

and

$$g^*(t, y, \tau, \eta) \in \bigcap_{s \in \mathbb{R}} S_{cl}^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}^{s,-\gamma+\mu}, \mathcal{S}_Q^{-\gamma}). \tag{2}$$

Here $\cdot^*$ means the point-wise formal adjoint with respect to the (non-degenerate) sesquilinear form $(\cdot,\cdot) : \mathcal{K}^{s,\gamma} \times \mathcal{K}^{-s,-\gamma} \to \mathbb{C}$. The elements of $R_G^{\nu,l}(U \times \mathbb{R}^{1+q}, \mathbf{g})$ are so-called anisotropic GREEN *edge symbols.*

Sometimes we need the space of modified anisotropic GREEN edge symbols denoted by $R_{(G)}^{\nu,l}(U \times \mathbb{R}^{1+q}, \mathbf{g})$, which contains all

$$g(t, y, \tau, \eta) \in \bigcap_{s \in \mathbb{R}} S_{cl}^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}, \mathcal{K}^{\infty,\gamma-\mu}),$$

such that there are asymptotic types $P \in \mathrm{As}(\gamma - \mu, \Theta)$ and $Q \in \mathrm{As}(-\gamma, \Theta)$, with

$$g(t, y, \tau, \eta) \in \bigcap_{s \in \mathbb{R}} S_{cl}^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}, \mathcal{K}_P^{\infty,\gamma-\mu}) \tag{3}$$

and

$$g^*(t, y, \tau, \eta) \in \bigcap_{s \in \mathbb{R}} S_{cl}^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}^{s,-\gamma+\mu}, \mathcal{K}_Q^{\infty,-\gamma}). \tag{4}$$

Moreover, we indicate by P, Q the subspaces of anisotropic GREEN edge symbols with fixed asymptotic types P and Q, such that we have the spaces $R_G^{\nu,l}(U \times \mathbb{R}^{1+q}, \mathbf{g})_{P,Q}$ and $R_{(G)}^{\nu,l}(U \times \mathbb{R}^{1+q}, \mathbf{g})_{P,Q}$, respectively.

Example 2 Taking $c(r,x) \in \mathcal{S}_P^{\gamma-\mu}$ and $d(r,x) \in \mathcal{S}_Q^{-\gamma}$ we get from $u(r,x) \mapsto Gu(r,x) := c(r,x)(u,d)$ a GREEN operator $G : \mathcal{K}^{\infty,\gamma} \to \mathcal{S}_P^{\gamma-\mu}$ on $X^\wedge$ with the formal adjoint operator $G^* : \mathcal{K}^{\infty,\gamma} \to \mathcal{S}_Q^{-\gamma}$ defined by $v(r,x) \mapsto G^*u(r,x) = d(r,x)(v,c)$.

Now setting $g(\tau,\eta) = \kappa(\tau,\eta) G \kappa^{-1}(\tau,\eta)$ we obtain in view of the anisotropic 0–homogeneity for $|\tau,\eta|_l > c$

$$g(\tau,\eta) \in R_G^{0,l}(\mathbb{R}^{1+q}, \mathbf{g})$$

with $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$. Then for every $\nu \in \mathbb{R}$ we have with $\omega(r[\tau,\eta]_l)[\tau,\eta]_l^\nu g(\tau,\eta)\omega_0(r[\tau,\eta]_l)$ a typical anisotropic GREEN edge symbol for every two cut-off functions ω, ω_0.

Definition 3 *We denote by* $\mathfrak{R}_G^{\nu,l}(U \times \mathbb{R}^{1+q}, \mathbf{g}; N_-, N_+)$ *the space of all*

$$\mathfrak{g}(t,y,\tau,\eta) \in \bigcap_{s\in\mathbb{R}} S_{cl}^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma} \oplus \mathbb{C}^{N_-}, \mathcal{K}^{\infty,\gamma-\mu} \oplus \mathbb{C}^{N_+}),$$

such that there are asymptotic types P and Q, with

$$\mathfrak{g}(t,y,\tau,\eta) \in \bigcap_{s\in\mathbb{R}} S_{cl}^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma} \oplus \mathbb{C}^{N_-}, \mathcal{S}_P^{\gamma-\mu} \oplus \mathbb{C}^{N_+}) \tag{5}$$

and

$$\mathfrak{g}^*(t,y,\tau,\eta) \in \bigcap_{s\in\mathbb{R}} S_{cl}^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}^{s,-\gamma+\mu} \oplus \mathbb{C}^{N_+}, \mathcal{S}_Q^{-\gamma} \oplus \mathbb{C}^{N_-}). \tag{6}$$

Here $\cdot^*$ indicates the point-wise formal adjoint in the sense

$$(\mathfrak{g}u, v)_{\mathcal{K}^0\oplus\mathbb{C}^{N_+}} = (u, \mathfrak{g}^*v)_{\mathcal{K}^0\oplus\mathbb{C}^{N_-}}$$

for all $u \in C_0^\infty(X^\wedge) \oplus \mathbb{C}^{N_-}$, $v \in C_0^\infty(X^\wedge) \oplus \mathbb{C}^{N_+}$.

The elements $\mathfrak{g} \in \mathfrak{R}_G^{\nu,l}(U \times \mathbb{R}^{1+q}, \mathbf{g}; N_-, N_+)$ are block-matrices like $\mathfrak{g} = (g_{jk})_{j,k=1,2}$, where g_{11} belongs to $R_G^{\nu,l}(U \times \mathbb{R}^{1+q}, \mathbf{g})$, and the other elements are families of finite dimensional operators. g_{12} and g_{21} are symbols of the trace and potential operators, respectively, and g_{22} is a $N_+ \times N_-$–matrix of scalar anisotropic classical symbols of order ν.

The formal properties of the entries g_{12}, g_{21} and g_{22} are analogous to those of g_{11}; therefore we will mainly discuss the left upper corner of elements in $\mathfrak{R}_G^{\nu,l}(U \times \mathbb{R}^{1+q}, \mathbf{g}; N_-, N_+)$.

In the same manner we define the modified classes $\mathfrak{R}_{(G)}^{\nu,l}$ with left upper corners in $R_{(G)}^{\nu,l}(U \times \mathbb{R}^{1+q}, \mathbf{g})$.

Remark 4 From the definition and the composition law for operator–valued symbols it follows immediately that the product of two GREEN symbols is a GREEN symbol, again, where the order of the product is the sum of the orders of the factors.

Of course, also the formal adjoint of an GREEN symbol has the GREEN property.

Remark 5 Multiplication by arbitrary r–powers does not destroy the GREEN property of symbols, i.e. for all $a, b \in \mathbb{R}$ we have $r^a R_G^{\nu,l}(U \times \mathbb{R}^{1+q}, \mathbf{g}) r^{-b} = R_G^{\nu-a+b,l}(U \times \mathbb{R}^{1+q}, \mathbf{g})$ as well as

$$\begin{pmatrix} r^a & 0 \\ 0 & [\tau,\eta]_l^{-a} \end{pmatrix} \mathfrak{R}_G^{\nu,l}(U \times \mathbb{R}^{1+q}, \mathbf{g}; N_-, N_+) \begin{pmatrix} r^{-b} & 0 \\ 0 & [\tau,\eta]_l^{b} \end{pmatrix} = \mathfrak{R}_G^{\nu-a+b,l}(U \times \mathbb{R}^{1+q}, \mathbf{g}; N_-, N_+).$$

Lemma 6 *Let $\mathfrak{g}_j(t,y,\tau,\eta) \in \mathfrak{R}_G^{\nu-j,l}(U \times \mathbb{R}^{1+q}, \mathbf{g}; N_-, N_+)_{P,Q}$, $j \in \mathbb{N}$, be any sequence of anisotropic* GREEN *edge symbols. Then there exists a $\mathfrak{g}(t,y,\tau,\eta) \in \mathfrak{R}_G^{\nu,l}(U \times \mathbb{R}^{1+q}, \mathbf{g}; N_-, N_+)_{P,Q}$, which is uniquely determined* mod $\mathfrak{R}_G^{-\infty}(U \times \mathbb{R}^{1+q}, \mathbf{g}; N_-, N_+)_{P,Q}$, *such that*

$$\mathfrak{g}(t,y,\tau,\eta) \sim \sum_{j=0}^{\infty} \mathfrak{g}_j(t,y,\tau,\eta), \tag{7}$$

where $\sim$ means, that for every $M \in \mathbb{N}$ there is an $N \in \mathbb{N}$ with

$$\mathfrak{g}(t,y,\tau,\eta) - \sum_{j=0}^{N} \mathfrak{g}_j(t,y,\tau,\eta) \in R_G^{\nu-M,l}(U \times \mathbb{R}^{1+q}, \mathbf{g}; N_-, N_+)_{P,Q}.$$

An analogous result holds for the $\mathfrak{R}_{(G)}$-classes, where we write $\sim_G$ instead of $\sim$.

Proof: If $\chi(\tau,\eta)$ is some excision function and $(d_j)_{j=0}^{\infty}$ tends to ∞ if $j \to \infty$ sufficiently fast, then the sum $\sum_{j=0}^{\infty} \chi\left(\frac{\tau,\eta}{d_j}\right) \mathfrak{g}_j(t,y,\tau,\eta)$ converges absolutely in $S_{cl}^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma} \oplus \mathbb{C}^{N_-}, \mathcal{S}_P^{\gamma-\mu} \oplus \mathbb{C}^{N_+})$ for every $s \in \mathbb{R}$. Analogously there is some $(d_j^*)_{j=0}^{\infty}$ tending to ∞ for $j \to \infty$ such that the sum $\sum_{j=0}^{\infty} \chi\left(\frac{\tau,\eta}{d_j^*}\right) \mathfrak{g}_j^*(t,y,\tau,\eta)$ converges absolutely in $S_{cl}^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}^{s,-\gamma+\mu} \oplus \mathbb{C}^{N_+}, \mathcal{S}_Q^{-\gamma-\mu} \oplus \mathbb{C}^{N_-})$ for every $s \in \mathbb{R}$. Setting $c_j := \max\{d_j, d_j^*\}$ we have the convergence of $\mathfrak{g} = \sum_{j=0}^{\infty} \chi\left(\frac{\tau,\eta}{c_j}\right) \mathfrak{g}_j(t,y,\tau,\eta)$ as well as $\mathfrak{g}^* = \sum_{j=0}^{\infty} \chi\left(\frac{\tau,\eta}{c_j}\right) \mathfrak{g}_j^*(t,y,\tau,\eta)$, simultaneously, where $\mathfrak{g}^*$ is the formal adjoint of $\mathfrak{g}$ and (refgsum) holds. The assertion for the $\mathfrak{R}_{(G)}$-classes follows in the same way. □

Let us now define the GREEN operators of the anisotropic edge calculus. For that we first introduce the smoothing GREEN operators.

Definition 7 *$Y_G^{-\infty}(U,\mathbf{g})$ for open $U \subset \mathbb{R}^{1+q}$ and weight data $\mathbf{g} = (\gamma, \gamma-\mu, \Theta)$ denotes the subspace of all $G \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{W}_{comp(t,y)}^{s,\gamma;l}(X^\wedge \times U), \mathcal{W}_{loc(t,y)}^{\infty,\gamma-\mu}(X^\wedge \times U))$ such that there are asymptotic types $P \in \mathrm{As}(\gamma-\mu,\Theta)$ and $Q \in \mathrm{As}(-\gamma,\Theta)$ with*

$$\begin{aligned} \phi G \psi &\in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{W}^{s,\gamma;l}(X^\wedge \times \mathbb{R}^{1+q}), \mathcal{W}_P^{\infty,\gamma-\mu}(X^\wedge \times \mathbb{R}^{1+q})) \quad \text{and} \\ \phi G^* \psi &\in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{W}^{s,-\gamma+\mu;l}(X^\wedge \times \mathbb{R}^{1+q}), \mathcal{W}_Q^{\infty,-\gamma}(X^\wedge \times \mathbb{R}^{1+q})) \end{aligned}$$

for all functions $\phi, \psi \in C_0^\infty(U)$.

Here $\cdot^*$ denotes the formal adjoint operator with respect to the sesquilinear form defined in 1.2 Example 7.

Definition 8 *With the notations of Definition 7 and for $N_-, N_+ \in \mathbb{N}$ we denote by $\mathfrak{Y}_G^{-\infty}(U,\mathbf{g};N_-,N_+)$ the subspace of all*

$$\mathfrak{G} \in \bigcap_{s\in\mathbb{R}} \mathcal{L}\begin{pmatrix} \mathcal{W}_{comp(t,y)}^{s,\gamma;l}(X^\wedge \times U) & & \mathcal{W}_{loc(t,y)}^{\infty,\gamma-\mu}(X^\wedge \times U) \\ \oplus & , & \oplus \\ H_{comp}^{s,l}(U,\mathbb{C}^{N_-}) & & H_{loc}^{\infty}(U,\mathbb{C}^{N_+}) \end{pmatrix}$$

such that there are asymptotic types $P \in \mathrm{As}(\gamma-\mu,\Theta)$ and $Q \in \mathrm{As}(-\gamma,\Theta)$ with

$$\Phi\mathfrak{G}\Psi \in \bigcap_{s\in\mathbb{R}} \mathcal{L}\begin{pmatrix} \mathcal{W}^{s,\gamma;l}(X^\wedge \times \mathbb{R}^{1+q}) & & \mathcal{W}_P^{\infty,\gamma-\mu}(X^\wedge \times \mathbb{R}^{1+q}) \\ \oplus & , & \oplus \\ H^{s,l}(\mathbb{R}^{1+q},\mathbb{C}^{N_-}) & & H^{\infty}(\mathbb{R}^{1+q},\mathbb{C}^{N_+}) \end{pmatrix} \quad \text{and}$$

$$\Psi\mathfrak{G}^*\Phi \in \bigcap_{s\in\mathbb{R}} \mathcal{L}\begin{pmatrix} \mathcal{W}^{s,-\gamma+\mu;l}(X^\wedge\times\mathbb{R}^{1+q}) & & \mathcal{W}_Q^{\infty,-\gamma}(X^\wedge\times\mathbb{R}^{1+q}) \\ \oplus & , & \oplus \\ H^{s,l}(\mathbb{R}^{1+q},\mathbb{C}^{N_+}) & & H^\infty(\mathbb{R}^{1+q},\mathbb{C}^{N_-}) \end{pmatrix}$$

for all $\Phi = \mathrm{diag}(\phi_1,\phi_2)$ *and* $\Psi = \mathrm{diag}(\psi_1,\psi_2)$ *with functions* $\phi_1,\phi_2,\psi_1,\psi_2 \in C_0^\infty(U)$.

Here $\cdot^*$ *denotes the formal adjoint operator with respect to*

$$(\mathfrak{G}u,v)_{\mathcal{W}^{0,0}(X^\wedge\times\mathbb{R}^{1+q})\oplus H^{0,l}(\mathbb{R}^{1+q},\mathbb{C}^{N_+})} = (u,\mathfrak{G}^*v)_{\mathcal{W}^{0,0}(X^\wedge\times\mathbb{R}^{1+q})\oplus H^{0,l}(\mathbb{R}^{1+q},\mathbb{C}^{N_-})}.$$

The elements of $\mathfrak{Y}_G^{-\infty}(U,\mathbf{g};N_-,N_+)$ *are called smoothing* GREEN *operators of the anisotropic edge calculus (to be treated below).*

Definition 9 *For* $\nu\in\mathbb{R}$*, open* $U\in\mathbb{R}^{1+q}$ *and given weight data* $\mathbf{g}=(\gamma,\gamma-\mu,\Theta)$ *we define the space*

$$Y_G^{\nu,l}(U,\mathbf{g}) := \mathrm{Op}(R_G^{\nu,l}(U\times\mathbb{R}^{1+q},\mathbf{g})) + Y_G^{-\infty}(U,\mathbf{g}) \tag{8}$$

and analogously $\mathfrak{Y}_G^{\nu,l}(U,\mathbf{g};N_-,N_+)$.

Corollary 10 *Every* $\mathfrak{G}\in\mathfrak{Y}_G^{\nu,l}(U,\mathbf{g};N_-,N_+)$ *extends for arbitrary* $s\in\mathbb{R}$ *to a continuous operator*

$$\Phi\mathfrak{G}\Psi : \begin{matrix} \mathcal{W}^{s,\gamma;l}(X^\wedge\times\mathbb{R}^{1+q}) \\ \oplus \\ H^{s,l}(\mathbb{R}^{1+q},\mathbb{C}^{N_-}) \end{matrix} \longrightarrow \begin{matrix} \mathcal{W}_P^{s-\nu,\gamma-\mu;l}(X^\wedge\times\mathbb{R}^{1+q}) \\ \oplus \\ H^{s-\nu,l}(\mathbb{R}^{1+q},\mathbb{C}^{N_+}) \end{matrix},$$

with Φ *and* Ψ *as in Definition 8.*

1.4 Smoothing MELLIN symbols with discrete asymptotics

A further ingredient of the edge symbols which occur in the isotropic wedge theory is a finite sum of so-called smoothing MELLIN symbols. In this section we deal with the anisotropic equivalent of this symbols.

Definition 1 *Let* $\omega(r)$ *and* $\omega_0(r)$ *be arbitrary cut-off functions and choose an* $h(t,y,w)\in C^\infty(U,M_R^{-\infty}(X))$ *with some asymptotic type* R *for* MELLIN *symbols. Assume further* $\mu,\nu,\gamma\in\mathbb{R}$, $\mu\geq\nu$, $j\in\mathbb{N}$ *and* $\alpha\in\mathbb{N}^{1+q}$*. Then a smoothing* MELLIN *edge symbol is an* (t,y,τ,η)*-dependent operator-valued function of the form*

$$a(t,y,\tau,\eta) = \omega(r[\tau,\eta]_l)r^{-\nu+j}\mathrm{op}_M^{\delta-\frac{n}{2}}(h)(t,y)(r^l\tau,r\eta)^\alpha\omega_0(r[\tau,\eta]_l) \tag{1}$$

with some $\delta\in\mathbb{R}$ *satisfying* $\mathrm{sg}(h)(t,y)\cap\Gamma_{\frac{n+1}{2}-\delta}=\emptyset$ *and* $\delta+j+\mu-\nu\geq\gamma\geq\delta$.

Note that $\mathrm{op}_M^{\delta-\frac{n}{2}}$ in (1) acts on the r-dependence of all functions on the right of this expression. Below we shall see, how the r-powers of the covariables also can be shifted to the left modulo GREEN symbols.

Proposition 2 *Every smoothing* MELLIN *symbol* $a(t,y,\tau,\eta)$ *belongs to* $S^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge))$ *for all* $s \in \mathbb{R}$ *and all allowed* $\gamma \in \mathbb{R}$.

Moreover, for every given asymptotic type $P \in \mathrm{As}(\gamma,\Theta)$ *there exists an asymptotic type* $Q \in \mathrm{As}(\gamma-\mu,\Theta)$ *such that the smoothing* MELLIN *symbol* $a(t,y,\tau,\eta)$ *belongs to* $S^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}_P^{s,\gamma}(X^\wedge), \mathcal{K}_Q^{\infty,\gamma-\mu}(X^\wedge))$ *for all* $s \in \mathbb{R}$.

Proof: From the cone calculus (cf. [SCH1], [SCH3]) it follows immediately that the symbol $a(t,y,\tau,\eta)$ is a C^∞–function in $U \times \mathbb{R}^{1+q}$ with values in $\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge))$ and $\mathcal{L}(\mathcal{K}_P^{s,\gamma}(X^\wedge), \mathcal{K}_Q^{\infty,\gamma-\mu}(X^\wedge))$, respectively, with some asymptotic type Q depending on P and the MELLIN symbol h.

Next we observe for $\lambda > 1$ and $|\tau,\eta|_l > c'$ with the constant $c' \in \mathbb{R}$ from (1.1,(4))

$$\begin{aligned} &\kappa_\lambda^{-1} a(t,y,\lambda^l\tau,\lambda\eta)\kappa_\lambda u(r,x) \\ =\;& \kappa_\lambda^{-1}\omega(r[\lambda^l\tau,\lambda\eta]_l)r^{-\nu+j}\mathrm{op}_M^{\delta-\frac{n}{2}}(h)(t,y)(r^l\lambda^l\tau, r\lambda\eta)^\alpha\omega_0(r[\lambda^l\tau,\lambda\eta]_l)\kappa_\lambda u(r,x) \\ =\;& \kappa_\lambda^{-1}\omega(r\lambda[\tau,\eta]_l)r^{-\nu+j}\mathrm{op}_M^{\delta-\frac{n}{2}}(h)(t,y)((r\lambda)^l\tau, r\lambda\eta)^\alpha\omega_0(r\lambda[\tau,\eta]_l)\lambda^{\frac{n+1}{2}}u(\lambda r,x) \\ =\;& \lambda^{\nu-j}\omega(r[\tau,\eta]_l)r^{-\nu+j}\mathrm{op}_M^{\delta-\frac{n}{2}}(h)(t,y)(r^l\tau, r\eta)^\alpha\omega_0(r[\tau,\eta]_l)u(r,x) \\ =\;& \lambda^{\nu-j}a(t,y,\tau,\eta)u(r,x) \end{aligned}$$

for every $u(r,x) \in \mathcal{K}^{s,\gamma}(X^\wedge)$, which implies that $a(t,y,\tau,\eta)$ is anisotropic $(\nu-j)$–homogeneous in the operator–valued sense for large $|\tau,\eta|_l$.

Thus by 1.1 Lemma 9 and 1.1 Lemma 16 $a(t,y,\tau,\eta)$ belongs to $S^{\nu-j,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge))$ as well as to $S^{\nu-j,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}_P^{s,\gamma}(X^\wedge), \mathcal{K}_Q^{\infty,\gamma-\mu}(X^\wedge))$ for all $s \in \mathbb{R}$. From 1.1 Lemma 5 (i) we then obtain the assertion. □

Example 3 Let $L \subset C^\infty(X \times X)$ be any finite dimensional subspace, representing finite dimensional smoothing operators, and define for $p \in \mathbb{C}$ with $\mathrm{Re}\, p < \frac{1}{2}$ and $k \in \mathbb{N}$ a function $f(r,x,x')$ by $f(r,x,x') = \omega(r)r^{-p}\ln^k r c(x,x')$ with any cut–off function $\omega(r)$ and $c(x,x') \in L$. Then we get by $h(w,x,x') = \mathcal{M}_{r\to w}\{f(r,x,x')\}(w,x,x')$ a MELLIN symbol belonging to $M_R^{-\infty}(X)$. If f also depends C^∞ of (t,y) then

$$h(t,y,w,x,x') = \mathcal{M}_{r\to w}\{f(t,y,r,x,x')\}(t,y,w,x,x')$$

will belong to $C^\infty(U, M_R^{-\infty}(X))$.

Our next aim is it to prove some useful properties of smoothing MELLIN–symbols. The assertions often hold modulo GREEN operators, such that it is nessecary to characterize the formal adjoint of a smoothing MELLIN symbol.

Proposition 4 *For every smoothing* MELLIN *symbol* $h \in C^\infty(U, M_R^{-\infty}(X))$ *there exists an adjoint symbol* $h^* \in C^\infty(U, M_P^{-\infty}(X))$ *with some asymptotic type* P *for* MELLIN *symbols, satisfying the relation*

$$\left(\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\right)^* = \mathrm{op}_M^{-\gamma-\frac{n}{2}}(h^*), \tag{2}$$

where $\cdot^*$ *denotes the point-wise formal adjoint operator with respect to the (non-degenerate) sesquilinear form*

$$(\cdot,\cdot) \;:\; \mathcal{K}^{s,\gamma}(X^\wedge) \times \mathcal{K}^{-s,-\gamma}(X^\wedge) \to \mathbb{C}.$$

Proof: The sesquilinear form $(\cdot,\cdot)$ has for $u, v \in C_0^\infty(X^\wedge)$ the form

$$\begin{aligned}(u,v) &= \int_X \int_0^\infty r^{-\frac{n}{2}} u(r,x) r^{-\frac{n}{2}} \overline{v(r,x)} dr dx \\ &= (r^{-\frac{n}{2}} u(r,x), r^{-\frac{n}{2}} v(r,x))_{L^2(X^\wedge)}.\end{aligned}$$

We may construct h^* point-wise, such that we fix $(t,y) \in U$, and for abbreviation we omit it in the notation.

Now setting $A = \mathrm{op}_M^{\gamma-\frac{n}{2}}(h) = r^{\gamma-\frac{n}{2}} \mathrm{op}_M(T^{-\gamma+\frac{n}{2}} h) r^{-\gamma+\frac{n}{2}}$ we have to find A^* such that

$$(Au, v) = (r^{-\frac{n}{2}} Au, r^{-\frac{n}{2}} v)_{L^2} = (r^{-\frac{n}{2}} u, r^{-\frac{n}{2}} A^* v)_{L^2} = (u, A^* v)$$

holds.

We obtain for $f = T^{-\gamma+\frac{n}{2}} h$ and $u, v \in C_0^\infty(X^\wedge)$

$$\begin{aligned}(Au, v) &= (r^{-\frac{n}{2}} r^{\gamma-\frac{n}{2}} \mathrm{op}_M(f) r^{-\gamma+\frac{n}{2}} u, r^{-\frac{n}{2}} v)_{L^2} \\ &= (\mathrm{op}_M(f) r^{-\gamma+\frac{n}{2}} u, r^{-n} r^{\gamma-\frac{n}{2}} v)_{L^2} \\ &= (r^{-\gamma+\frac{n}{2}} u, \mathrm{op}_M(f^*) r^{-n} r^{\gamma-\frac{n}{2}} v)_{L^2} \\ &= (r^{-\frac{n}{2}} u, r^{-\frac{n}{2}} r^{-\gamma-\frac{n}{2}} r^n \mathrm{op}_M(T^n f^*) r^{-n} r^{\gamma+\frac{n}{2}} v)_{L^2} \\ &= (r^{-\frac{n}{2}} u, r^{-\frac{n}{2}} r^{-\gamma-\frac{n}{2}} \mathrm{op}_M(T^{\gamma+\frac{n}{2}} T^{-\gamma+\frac{3n}{2}} f^*) r^{\gamma+\frac{n}{2}} v)_{L^2} \\ &= (r^{-\frac{n}{2}} u, r^{-\frac{n}{2}} \mathrm{op}_M^{-\gamma-\frac{n}{2}}(T^{-\gamma+\frac{3n}{2}} f^*) v)_{L^2}.\end{aligned}$$

But $f^*(w) = f^{(*)}(1-\overline{w})$, (cf. [SCH1]), where $\cdot^{(*)}$ means the formal adjoint in x-variables, such that we have $h^*(w) = T^n f^{(*)}(1-\overline{w})$.

An evident modification of the arguments then yields the assertion also in the (t,y)-dependent case. □

Proposition 5 *Let* $m(t,y,\tau,\eta) = r^{-\nu+j} \mathrm{op}_M^{\delta-\frac{n}{2}}(h)(t,y)(r^l\tau, r\eta)^\alpha$ *be given with some* MELLIN *symbol* $h(t,y) \in C^\infty(U, M_R^{-\infty})$, *where* R *is some asymptotic type for* MELLIN *symbols. Then under the conditions of Definition 1 and for every* $\phi(r) \in C_0^\infty(\mathbb{R}_+)$ *we obtain that* $\phi(r[\tau,\eta]_l) m(t,y,\tau,\eta) \omega_0(r[\tau,\eta]_l)$ *as well as* $\omega(r[\tau,\eta]_l) m(t,y,\tau,\eta) \phi(r[\tau,\eta]_l)$ *belongs to* $R_G^{\nu,l}(U \times \mathbb{R}^{1+q}, \mathbf{g})$ *for* $\mathbf{g} = (\gamma, \gamma-\mu, \Theta)$.

For the proof we need the following

Lemma 6 *Let $m(t,y,\tau,\eta)$ be given as in Proposition 5 and set for $\beta \geq 0$ $m_\beta(t,y,\tau,\eta) = r^{-\nu+j}\mathrm{op}_M^{\delta-\frac{n}{2}}(T^{-\beta}h)(t,y)(r^l\tau, r\eta)^\alpha$, where we assume $\mathrm{sg}(T^{-\beta}h)(t,y) \cap \Gamma_{\frac{n+1}{2}-\delta} = \emptyset$. Then we have*

$$\phi(r[\tau,\eta]_l)(m(t,y,\tau,\eta)r^\beta - r^\beta m_\beta(t,y,\tau,\eta))\psi(r[\tau,\eta]_l) \in R_G^{\nu+\beta,l}(U \times \mathbb{R}^{1+q}, \mathbf{g}) \tag{3}$$

for every $\phi, \psi \in C_0^\infty(\overline{\mathbb{R}}_+)$ for $\mathbf{g} = (\gamma, \gamma-\mu, \Theta)$.

Proof: By homogeneity $\phi(r[\tau,\eta]_l)(m(t,y,\tau,\eta)r^\beta - r^\beta m_\beta(t,y,\tau,\eta))\psi(r[\tau,\eta]_l)$ is a classical operator–valued symbol with respect to the subspaces with asymptotics. Moreover, the point–wise GREEN property is known by the cone calculus (cf. [SCH1], [SCH2]), which yields the assertion. □

Proof: (of Proposition 5) Choosing a cut-off function $\tilde{\omega}(r)$ with $\tilde{\omega}\phi = \phi$ we obtain $\phi(r[\tau,\eta]_l)m(t,y,\tau,\eta)\omega_0(r[\tau,\eta]_l) = \phi(r[\tau,\eta]_l)\tilde{\omega}(r[\tau,\eta]_l)m(t,y,\tau,\eta)\omega_0(r[\tau,\eta]_l)$. The operator valued function $\phi(r[\tau,\eta]_l)\tilde{\omega}(r[\tau,\eta]_l)m(t,y,\tau,\eta)\omega_0(r[\tau,\eta]_l)$ is because of its homogeneity in operator–valued sense for large $|\tau,\eta|_l$ a classical symbol with respect to the subspaces with asymptotics, such that it remains to show the point–wise GREEN property of the symbols. For that reason we omit the fixed variables (t,y,τ,η) from the notation.

We have to check that there are asymptotic types $P \in \mathrm{As}(\gamma-\mu,\Theta)$ and $Q \in \mathrm{As}(-\gamma,\Theta)$, such that $(gu)(r) = \phi(r[\tau,\eta]_l)r^{-\nu+j}\mathrm{op}_M^{\delta-\frac{n}{2}}(h)r^{|\alpha|_l}\omega_0(r[\tau,\eta]_l)u(r) \in \mathcal{S}_P^{\gamma-\mu}$ for all $u \in \bigcup_{s\in\mathbb{R}} \mathcal{K}^{s,\gamma}$ and $(g^*u)(r) = \omega_0(r[\tau,\eta]_l)r^{|\alpha|_l}\mathrm{op}_M^{-\delta-\frac{n}{2}}(h^*)r^{-\nu+j}\phi(r[\tau,\eta]_l)u(r) \in \mathcal{S}_Q^{-\gamma}$ for every $u \in \bigcup_{s\in\mathbb{R}} \mathcal{K}^{s,-\gamma+\mu}$. Here $h \in M_R^{-\infty}$ and h^* is given in the proof of 1.4 Proposition 4.

But $h \in M_R^{-\infty}$ implies for the above cut–off function $\tilde{\omega}$ that $(gu)(r)$ belongs to $[\tilde{\omega}]\mathcal{K}^{\infty,\gamma-\mu}$. Note that supp $\tilde{\omega}(r[\tau,\eta]_l) \subseteq$ supp $\tilde{\omega}(r)$ for every $(\tau,\eta) \in \mathbb{R}^{1+q}$ in view of $[\tau,\eta]_l \geq 1$, which makes an (τ,η)–independent choice of $[\tilde{\omega}]\mathcal{K}^{\infty,\gamma-\mu}$ possible.

From $\phi \in C_0^\infty(\mathbb{R}_+)$ it follows, that $(gu)(r)$ has a zero of infinite order at $r = 0$, that leads to trivial asymptotics such that we get $(gu)(r) \in [\tilde{\omega}]\mathcal{K}_O^{\infty,\gamma-\mu} \subset \mathcal{S}_O^{\gamma-\mu}$.

The same argument gives $\phi(r)u(r) \in \mathcal{K}_O^{s,-\gamma+\mu}$ for all $u \in \mathcal{K}^{s,-\gamma+\mu}$; and as in Proposition 2 we have $(g^*u)(r) \in [\omega_0]\mathcal{K}_Q^{\infty,-\gamma} \subset \mathcal{S}_Q^{-\gamma}$ for some asymptotic type Q.

The proof for $\omega(r[\tau,\eta]_l)m(t,y,\tau,\eta)\phi(r[\tau,\eta]_l)$ follows from the above case by applying Lemma 6 that allows commuting the r–powers through the MELLIN action. □

Corollary 7 *Let ω, ω_0 and $\tilde{\omega}, \tilde{\omega}_0$ be arbitrary cut-off functions, then for the function $m = m(t,y,\tau,\eta)$ given as in Proposition 5 we have*

$$\omega(r[\tau,\eta]_l)m\omega_0(r[\tau,\eta]_l) - \tilde{\omega}(r[\tau,\eta]_l)m\tilde{\omega}_0(r[\tau,\eta]_l) \in R_G^{\nu,l}(U \times \mathbb{R}^{1+q}, \mathbf{g}) \tag{4}$$

for $\mathbf{g} = (\gamma, \gamma-\mu, \Theta)$.

Proof: Let us omit for abbreviation the independent variables in the notation. Then $\omega m\omega_0 - \bar{\omega}m\bar{\omega}_0 = \omega m\omega_0 - \omega m\bar{\omega}_0 - (\bar{\omega}m\bar{\omega}_0 - \omega m\bar{\omega}_0) = \omega m(\omega_0 - \bar{\omega}_0) - (\bar{\omega} - \omega)m\bar{\omega}_0 \in R_G^{\nu,l}$ by Proposition 5, because of $\omega_0 - \bar{\omega}_0, \bar{\omega} - \omega \in C_0^\infty(\mathbb{R}_+)$. □

Proposition 8 *Let $a_\delta(t,y,\tau,\eta) = \omega(r[\tau,\eta]_l)r^{-\nu+j}\mathrm{op}_M^{\delta-\frac{n}{2}}h(t,y)(r^l\tau, r\eta)^\alpha\omega_0(r[\tau,\eta]_l)$ with h and δ satisfying the conditions of Definition 1. Then, if δ' is another allowed choice for the given h, it follows*

$$a_\delta(t,y,\tau,\eta) - a_{\delta'}(t,y,\tau,\eta) \in R_G^{\nu,l}(U \times \mathbb{R}^{1+q}, \mathbf{g}) \tag{5}$$

for $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$.

Proof: In view of the homogeneity the a_δ and $a_{\delta'}$ are classical operator–valued symbols with respect to the subspaces with asymptotics. Moreover, $a_\delta - a_{\delta'}$ are point–wise GREEN operators, which is known by the cone calculus. Thus the difference is necessarily of GREEN–type. □

Lemma 9 *Let $\nu, \tilde{\nu}, \mu, \tilde{\mu}$ be arbitrary real numbers . Then for every smoothing* MELLIN *symbol $a(t,y,\tau,\eta)$ (that belongs to $\bigcap_{s\in\mathbb{R}} S_{cl}^{\nu,l}(U \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}, \mathcal{K}^{\infty,\gamma-\mu})$, cf. Proposition 2), and every* GREEN *symbol $g(t,y,\tau,\eta) \in R_G^{\tilde{\nu},l}(U \times \mathbb{R}^{1+q}, \tilde{\mathbf{g}})$ with $\tilde{\mathbf{g}} = (\gamma-\mu, \gamma-(\mu+\tilde{\mu}), \Theta)$, $\Theta = (\theta, 0]$, $-\infty < \theta < 0$, we have $(ga)(t,y,\tau,\eta) \in R_G^{\nu+\tilde{\nu},l}(U \times \mathbb{R}^{1+q}, \mathbf{g})$ for $\mathbf{g} = (\gamma, \gamma-(\mu+\tilde{\mu}), \Theta)$. Moreover, for $\tilde{\mathbf{g}} = (\gamma+\tilde{\mu}, \gamma, \Theta)$ we find $(ag)(t,y,\tau,\eta) \in R_G^{\nu+\tilde{\nu},l}(U \times \mathbb{R}^{1+q}, \mathbf{g})$ for $\mathbf{g} = (\gamma+\tilde{\mu}, \gamma-\mu, \Theta)$.*

Proof: We will only consider ga. For ag the proof is analogous.

In view of the anisotropic ν–homogeneity in the operator–valued sense for large $|\tau,\eta|_l$ the smoothing MELLIN symbol $a(t,y,\tau,\eta)$ is a classical operator–valued symbol of order ν. Since also every GREEN symbol is a classical one, we get that ag has this property, too, because of the general fact that the composition of classical symbols remains classical.

By definition we have

$$R_G^{\tilde{\nu},l}(U \times \mathbb{R}^{1+q}, \tilde{\mathbf{g}})_{\tilde{P},\tilde{Q}} \ni g(t,y,\tau,\eta) \sim \sum_{j=0}^{\infty} \chi(\tau,\eta) g_{(\tilde{\nu}-j)}(t,y,\tau,\eta)$$

with uniquely determined $(\tilde{\nu}-j)$–homogeneous operator–valued functions $g_{(\tilde{\nu}-j)}(t,y,\tau,\eta)$ such that $\chi(\tau,\eta)g_{(\tilde{\nu}-j)}(t,y,\tau,\eta)$ belongs to $R_G^{\tilde{\nu},l}(U \times \mathbb{R}^{1+q}, \tilde{\mathbf{g}})_{\tilde{P}_j,\tilde{Q}_j}$ with asymptotic types $\tilde{P}_j \subseteq \tilde{P}$ and $\tilde{Q}_j \subseteq \tilde{Q}$. For every $j \in \mathbb{N}$ we then get via point–wise composition the function

$$\chi(\tau,\eta)g_{(\tilde{\nu}-j)}a(t,y,\tau,\eta) = \chi(\tau,\eta)g_{(\tilde{\nu}-j)}(t,y,\tau,\eta)a(t,y,\tau,\eta) \tag{6}$$

which is anisotropic $(\nu+\tilde{\nu}-j)$–homogeneous in the operator–valued sense for large $|\tau,\eta|_l$. Moreover, the cone calculus gives point–wise GREEN property of (6) with $(\chi g_{(\tilde{\nu}-j)}a)u \in \mathcal{S}_{\tilde{P}_j}^{\gamma-\mu+\tilde{\mu}}$ and $(\chi g_{(\tilde{\nu}-j)}a)^*u = (a^*(\chi g_{(\tilde{\nu}-j)})^*)u \in \mathcal{S}_{Q_j}^{-\gamma}$ where $Q_j \subseteq Q$ depends on $\tilde{Q}_j$ and the MELLIN symbol a. Note that Q follows in the same way from $\tilde{Q}$ and the asymptotic behaviour of $a(t,y,\tau,\eta)$. But then for any further excision function $\chi'(\tau,\eta)$ we have $\chi g_{(\tilde{\nu}-j)}a =$

$\chi'\chi g_{(\tilde{\nu}-j)}a+(1-\chi')\chi g_{(\tilde{\nu}-j)}a$, where $\chi'\chi g_{(\tilde{\nu}-j)}a \in R_G^{\nu+\tilde{\nu}-j,l}(U\times\mathbb{R}^{1+q},\mathbf{g})_{\widetilde{P},Q}$ and $(1-\chi')\chi g_{(\tilde{\nu}-j)}a$ belongs to $R_G^{-\infty}(U\times\mathbb{R}^{1+q},\mathbf{g})_{\widetilde{P},Q}$ because of compact (τ,η)–support. Finally using 1.3 Lemma 6 we find

$$ag \sim \sum_{j=0}^{\infty} \chi'\chi g_{(\tilde{\nu}-j)}a \in R_G^{\nu+\tilde{\nu},l}(U\times\mathbb{R}^{1+q},\mathbf{g})_{\widetilde{P},Q}$$

that was to be proved. □

1.5 The algebra of smoothing MELLIN and GREEN symbols

We now turn to the announced ideal of smoothing MELLIN and GREEN symbols in the anisotropic edge symbol algebra.

Definition 1 *$R_{M+G}^{\nu,l}(U\times\mathbb{R}^{1+q},\mathbf{g})$ for $\mathbf{g}=(\gamma,\gamma-\mu,\Theta)$, $\Theta=(-k,0]$, $k\in\mathbb{N}\setminus\{0\}$ and $\mu-\nu\in\mathbb{N}$, is the space of all operator families*

$$(m+g)(t,y,\tau,\eta) = m(t,y,\tau,\eta)+g(t,y,\tau,\eta)$$

with $g\in R_G^{\nu,l}(U\times\mathbb{R}^{1+q},\mathbf{g})$ and $m(t,y,\tau,\eta)$ being a finite sum of operator-valued symbols of the form

$$\omega(r[\tau,\eta]_l)r^{-\nu+j}\mathrm{op}_M^{\delta_{j\alpha}-\frac{n}{2}}(h_{j\alpha})(t,y)(r^l\tau,r\eta)^\alpha\omega_0(r[\tau,\eta]_l)$$

with varying $j=0,\ldots,k-1$ and multi-indices α, where $h_{j\alpha}\in C^\infty(U,M_{R_{j\alpha}}^{-\infty}(X))$ satisfy the conditions of 1.4 Definition 1 with respect to $\delta_{j\alpha}$.

Definition 2 *$\mathfrak{R}_{M+G}^{\nu,l}(U\times\mathbb{R}^{1+q},\mathbf{g};N_-,N_+)$ for $\mathbf{g}=(\gamma,\gamma-\mu,\Theta)$, $\Theta=(-k,0]$, $k\in\mathbb{N}\setminus\{0\}$ and $\mu-\nu\in\mathbb{N}$, is the space of all matrices*

$$(\mathfrak{m}+\mathfrak{g})(t,y,\tau,\eta) = \begin{pmatrix} m(t,y,\tau,\eta) & 0 \\ 0 & 0 \end{pmatrix} + \mathfrak{g}(t,y,\tau,\eta)$$

with $\mathfrak{g}\in\mathfrak{R}_G^{\nu,l}(U\times\mathbb{R}^{1+q},\mathbf{g};N_-,N_+)$ and $m(t,y,\tau,\eta)$ being as in Definition 1.

As in the previous section we define the modified smoothing MELLIN and GREEN symbols indicated by subscripts $M+(G)$.

Corollary 3 *Let $\mathfrak{b}_j(t,y,\tau,\eta)\in\mathfrak{R}_{M+G}^{\nu-j,l}(U\times\mathbb{R}^{1+q},\mathbf{g};N_-,N_+)$, $j\in\mathbb{N}$, be any sequence of smoothing MELLIN and GREEN symbols, where the asymptotic types P and Q appearing in the GREEN part of the symbol are the same for all $j\in\mathbb{N}$. Then there exists a $\mathfrak{b}(t,y,\tau,\eta)\in\mathfrak{R}_{M+G}^{\nu,l}(U\times\mathbb{R}^{1+q},\mathbf{g};N_-,N_+)$, which is uniquely determined* mod $\mathfrak{R}_G^{-\infty}(U\times\mathbb{R}^{1+q},\mathbf{g};N_-,N_+)_{P,Q}$, *such that*

$$\mathfrak{b}(t,y,\tau,\eta)\sim\sum_{j=0}^{\infty}\mathfrak{b}_j(t,y,\tau,\eta).$$

The same is true if we insert everywhere (G) instead of G and replace $\sim$ by $\sim_G$.

Proof: The assertion follows immediately from the general property that the subspace $R^{\nu-j,l}_{M+G}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g})$ with fixed asymptotic types P and Q in the GREEN part of the symbols is a subspace of $R^{\nu-j,l}_{G}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g})_{P,Q}$ for $j>k-\mu+\nu-1$ (cf. [SCH3]) and 1.3 Lemma 6 after subtracting a finite sum. □

Proposition 4 *Let $\omega(r)$, $\omega_0(r)$ and $\tilde\omega(r)$, $\tilde\omega_0(r)$ be arbitrary cut-off functions and $h(t,y,w)\in C^\infty(U,M^{-\infty}_R(X))$ with some asymptotic type R for* MELLIN *symbols and $\tilde h(t,y,w)\in C^\infty(U,M^{-\infty}_{\tilde R}(X))$ with another asymptotic type $\tilde R$ for* MELLIN *symbols. Assume further $\nu,\tilde\nu,\gamma,\tilde\gamma\in\mathbb{R}$, $j,\tilde j\in\mathbb{N}$ and $\alpha,\tilde\alpha\in\mathbb{N}^{1+q}$. Then with*

$$a(t,y,\tau,\eta)=\omega(r[\tau,\eta]_l)r^{-\nu+j}\mathrm{op}_M^{\delta-\frac{n}{2}}(h)(t,y)(r^l\tau,r\eta)^\alpha\omega_0(r[\tau,\eta]_l)$$

for $\delta\in\mathbb{R}$ with $\mathrm{sg}(h)(t,y)\cap\Gamma_{\frac{n+1}{2}-\delta}=\emptyset$ and

$$b(t,y,\tau,\eta)=\tilde\omega(r[\tau,\eta]_l)r^{-\tilde\nu+\tilde j}\mathrm{op}_M^{\tilde\delta-\frac{n}{2}}(\tilde h)(t,y)(r^l\tau,r\eta)^{\tilde\alpha}\tilde\omega_0(r[\tau,\eta]_l)$$

for $\tilde\delta\in\mathbb{R}$ with $\mathrm{sg}(\tilde h)(t,y)\cap\Gamma_{\frac{n+1}{2}-\tilde\delta}=\emptyset$ we have $b(t,y,\tau,\eta)a(t,y,\tau,\eta)\in R^{\nu+\tilde\nu,l}_{M+G}(U\times\mathbb{R}^{1+q},\mathbf{g})$ for $\mathbf{g}=(\gamma,\gamma-\mu,\Theta)$ for every μ satisfying $\delta'+j+\tilde j+\mu-\nu-\tilde\nu\ge\gamma\ge\delta'$ with some $\mathbb{R}\ni\delta'\ge\delta+\tilde\delta$.

Proof: First using 1.4 Lemma 6 we get

$$\begin{aligned} ba(t,y,\tau,\eta) &= \tilde\omega r^{-\tilde\nu+\tilde j}\mathrm{op}_M^{\tilde\delta-\frac{n}{2}}(\tilde h)(t,y)(r^l\tau,r\eta)^{\tilde\alpha}\tilde\omega_0\omega r^{-\nu+j}\mathrm{op}_M^{\delta-\frac{n}{2}}(h)(t,y)(r^l\tau,r\eta)^\alpha\omega_0\\ &= \tilde\omega r^{-\tilde\nu+\tilde j-\nu+j}\mathrm{op}_M^{\tilde\delta-\frac{n}{2}}(T^{\nu-j}\tilde h)(t,y)\tilde\omega_0\omega\mathrm{op}_M^{\delta-\frac{n}{2}}(T^{|\tilde\alpha|_l}h)(t,y)(r^l\tau,r\eta)^{\alpha+\tilde\alpha}\omega_0+g_1\end{aligned}$$

with a GREEN edge symbol g_1.

Let us set $H:=\mathrm{op}_M^{\delta-\frac{n}{2}}(T^{|\tilde\alpha|_l}h)(t,y)$ and $\tilde H:=\mathrm{op}_M^{\tilde\delta-\frac{n}{2}}(T^{\nu-j}\tilde h)(t,y)$. Then we have for some cut-off function ω' satisfying $\omega'\tilde\omega=\omega'$ and $\omega'\omega_0=\omega'$

$$\begin{aligned} ba &= \tilde\omega r^{-\tilde\nu+\tilde j-\nu+j}\tilde H\tilde\omega_0\omega H(r^l\tau,r\eta)^{\alpha+\tilde\alpha}\omega_0\\ &= \omega' r^{-\tilde\nu+\tilde j-\nu+j}\tilde H\tilde\omega_0\omega H(r^l\tau,r\eta)^{\alpha+\tilde\alpha}\omega_0+(\tilde\omega-\omega')r^{-\tilde\nu+\tilde j-\nu+j}\tilde H\tilde\omega_0\omega H(r^l\tau,r\eta)^{\alpha+\tilde\alpha}\omega_0\\ &= \omega' r^{-\tilde\nu+\tilde j-\nu+j}\tilde H H(r^l\tau,r\eta)^{\alpha+\tilde\alpha}\omega_0-\omega' r^{-\tilde\nu+\tilde j-\nu+j}\tilde H(1-\tilde\omega_0\omega)H(r^l\tau,r\eta)^{\alpha+\tilde\alpha}\omega_0 \quad (1)\\ &\quad +(\tilde\omega-\omega')r^{-\tilde\nu+\tilde j-\nu+j}\tilde H\tilde\omega_0\omega H(r^l\tau,r\eta)^{\alpha+\tilde\alpha}\omega_0. \quad (2)\end{aligned}$$

From 1.4 Lemma 9 and 1.4 Proposition 5 it follows the GREEN property of (2). Writing the second item of (1) in the form

$$\begin{aligned}\omega' r^{-\tilde\nu+\tilde j-\nu+j}\tilde H(1-\tilde\omega_0\omega)H(r^l\tau,r\eta)^{\alpha+\tilde\alpha}\omega_0 &= \omega' r^{-\tilde\nu+\tilde j-\nu+j}\tilde H(1-\tilde\omega_0)(1-\omega)H(r^l\tau,r\eta)^{\alpha+\tilde\alpha}\omega_0\\ &\quad+\omega' r^{-\tilde\nu+\tilde j-\nu+j}\tilde H(1-\tilde\omega_0)\omega H(r^l\tau,r\eta)^{\alpha+\tilde\alpha}\omega_0\\ &\quad+\omega' r^{-\tilde\nu+\tilde j-\nu+j}\tilde H\tilde\omega_0(1-\omega)H(r^l\tau,r\eta)^{\alpha+\tilde\alpha}\omega_0\end{aligned}$$

we immediately find the GREEN property using 1.4 Lemma 9 and 1.4 Proposition 5 once again.

So it remains to deal with $c(t,y,\tau,\eta) := \omega' r^{-\tilde{\nu}+\tilde{j}-\nu+j}\tilde{H}H(r^l\tau,r\eta)^{\alpha+\tilde{\alpha}}\omega_0$. But 1.4 Proposition 8 allows to write

$$\begin{aligned} c(t,y,\tau,\eta) &= \omega' r^{-\tilde{\nu}+\tilde{j}-\nu+j}\mathrm{op}_M^{\tilde{\delta}-\frac{n}{2}}(T^{\nu-j}\tilde{h})(t,y)\mathrm{op}_M^{\delta-\frac{n}{2}}(T^{|\tilde{\alpha}|_l}h)(t,y)(r^l\tau,r\eta)^{\alpha+\tilde{\alpha}}\omega_0 \\ &= \omega' r^{-\tilde{\nu}+\tilde{j}-\nu+j}\mathrm{op}_M^{\delta'-\frac{n}{2}}(T^{\nu-j}\tilde{h}\circ T^{|\tilde{\alpha}|_l}h)(t,y)(r^l\tau,r\eta)^{\alpha+\tilde{\alpha}}\omega_0 + g_2 \end{aligned}$$

with a GREEN edge symbol g_2 and $\delta' \geq \delta + \tilde{\delta}$ such that $\mathrm{sg}(T^{\nu-j}\tilde{h}\circ T^{|\tilde{\alpha}|_l}h)(t,y)\cap\Gamma_{\frac{n+1}{2}-\delta'} = \emptyset$.

Because of $h(t,y,w)\in C^\infty(U,M_R^{-\infty}(X))$ as well as $\tilde{h}(t,y,w)\in C^\infty(U,M_{\tilde{R}}^{-\infty}(X))$ the function $(T^{\nu-j}\tilde{h}\circ T^{|\tilde{\alpha}|_l}h)(t,y)$ belongs to $C^\infty(U,M_{R'}^{-\infty}(X))$ with some asymptotic type for MELLIN symbols R'. □

Theorem 5 *Let $a(t,y,\tau,\eta)\in R_{M+G}^{\nu,l}(U\times\mathbb{R}^{1+q},\mathbf{g}_1)$ and $b(t,y,\tau,\eta)\in R_{M+G}^{\tilde{\nu},l}(U\times\mathbb{R}^{1+q},\mathbf{g}_2)$ for $\mathbf{g}_1=(\gamma,\gamma-\mu,\Theta)$ and $\mathbf{g}_2=(\gamma-\mu,\gamma-(\mu+\tilde{\mu}),\Theta)$ be given.*

Then we have $(ba)(t,y,\tau,\eta)\in R_{M+G}^{\nu+\tilde{\nu},l}(U\times\mathbb{R}^{1+q},\mathbf{g})$ for $\mathbf{g}=(\gamma,\gamma-(\mu+\tilde{\mu}),\Theta)$ and $a^*(t,y,\tau,\eta)\in R_{M+G}^{\nu,l}(U\times\mathbb{R}^{1+q},\mathbf{g}^*)$ for $\mathbf{g}^*=(-\gamma+\mu,-\gamma,\Theta)$.

Proof: From $a = g_a + m_a$ and $b = g_b + m_b$ it follows

$$ba = (g_b+m_b)(g_a+m_a) = g_bg_a + g_bm_a + m_bg_a + m_bm_a. \tag{3}$$

1.4 Lemma 9 and 1.3 Remark 4 imply that $g_bg_a+g_bm_a+m_bg_a$ belongs to $R_{M+G}^{\nu+\tilde{\nu},l}(U\times\mathbb{R}^{1+q},\mathbf{g})$. The last item of (3) is a finite sum of operator-valued functions we dealed with in Proposition 4, such that we also have $m_bm_a\in R_{M+G}^{\nu+\tilde{\nu},l}(U\times\mathbb{R}^{1+q},\mathbf{g})$.

The second assertion follows from the fact, that the set of GREEN edge operators and smoothing MELLIN operators remains preserved under the *–operations. □

The properties stated in Theorem 5 give rise to an algebra of smoothing MELLIN and GREEN symbols. With smoothing MELLIN and GREEN symbols we also associate pseudo–differential operators.

Definition 6 *Let $\nu,\mu\in\mathbb{R}$, an open set $U\subseteq\mathbb{R}^{1+q}$ and weight data $\mathbf{g}=(\gamma,\gamma-\mu,\Theta)$ with $\Theta=(-k,0]$, $k\in\mathbb{N}\setminus\{0\}$ be given, where $\mu-\nu\in\mathbb{N}$. Then we define the space*

$$Y_{M+G}^{\nu,l}(U,\mathbf{g}) := \mathrm{Op}(R_{M+G}^{\nu,l}(U\times\mathbb{R}^{1+q},\mathbf{g})) + Y_G^{-\infty}(U,\mathbf{g}). \tag{4}$$

In an analogous manner we define $\mathfrak{Y}_{M+G}^{\nu,l}(U,\mathbf{g};N_-,N_+)$.

Corollary 7 *Every $A\in Y_{M+G}^{\nu,l}(U,\mathbf{g})$ extends for arbitrary $s\in\mathbf{R}$ to a continuous operator*

$$A\ :\ \mathcal{W}_{comp}^{s,l}(U,\mathcal{K}^{s,\gamma})\to\mathcal{W}_{loc}^{s-\nu,l}(U,\mathcal{K}^{\infty,\gamma-\mu})$$

which restricts for every $P\in\mathrm{As}(\gamma,\Theta)$ to a continuous mapping

$$A\ :\ \mathcal{W}_{comp}^{s,l}(U,\mathcal{K}_P^{s,\gamma})\to\mathcal{W}_{loc}^{s-\nu,l}(U,\mathcal{K}_Q^{\infty,\gamma-\mu})$$

with some $Q \in \mathrm{As}(\gamma - \mu, \Theta)$ *that depends on* P *and* A.

Furthermore, every $A \in Y_G^{\nu,l}(U, g)$ *induces for all* $s \in \mathbb{R}$ *an operator*

$$A \; : \; \mathcal{W}^{s,l}_{comp}(U, \mathcal{K}^{s,\gamma}) \to \mathcal{W}^{s-\nu,l}_{loc}(U, \mathcal{K}_Q^{\infty,\gamma-\mu})$$

with some asymptotic type $Q \in \mathrm{As}(\gamma - \mu, \Theta)$.

Proof: Corollary 7 is a consequence of 1.2 Theorem 11 and of the fact that every smoothing MELLIN and GREEN symbol is an anisotropic operator–valued symbol between the cone SOBOLEV spaces as well as the subspaces with asymptotics. □

2 Anisotropic edge pseudo–differential operators

2.1 Interior symbols and MELLIN operator convention

In this section we describe the local interior symbols of the anisotropic edge pseudo–differential operators and formulate a MELLIN operator convention.

Let us first remind once again of the singular configuration of the wedge. We study operators on $\mathbb{R} \times W$, where $\mathbb{R} \ni t$ describes the time axis and W is any compact manifold with edge Y, which is itself a closed compact C^∞–manifold of dimension q. Furthermore, we assume that for every $y \in Y$ there exists an open neighbourhood $\Lambda \subset W$ of y, such that we have a homeomorphism $\Lambda \to (\overline{\mathbb{R}}_+ \times X \times U')/(\{0\} \times X \times U')$ with an open set $U' \subset \mathbb{R}^q$ that restricts to a diffeomorphism $\Lambda \setminus Y \to X^\wedge \times U'$ Here the base X of the model cone is itself a closed compact n–dimensional C^∞–manifold.

If $V = \chi(\widetilde{V})$ is the image of a chart on X in $\mathbb{R}^n$ we have local interior symbols on $(\mathbb{R}_+)_r \times V_x \times \mathbb{R}_t \times U'_y$ of the form $r^{-\nu} p(r, x, t, y, r\rho, \xi, r^l \tau, r\eta)$ where $p(r, x, t, y, \tilde{\rho}, \xi, \tilde{\tau}, \tilde{\eta})$ belongs to $S^{\nu,l}(\overline{\mathbb{R}}_+ \times V \times \mathbb{R} \times U' \times \mathbb{R}^{1+n+1+q})$. Note that the anisotropy of this space only refers to the time covariable τ. Here $S^{\nu,l}(\overline{\mathbb{R}}_+ \times V \times \mathbb{R} \times U' \times \mathbb{R}^{1+n+1+q}_{\tilde{\rho},\xi,\tilde{\tau},\tilde{\eta}})$ denotes the subspace of anisotropic symbols in $\mathbb{R}_+ \times V \times \mathbb{R} \times U' \times \mathbb{R}^{1+n+1+q}_{\tilde{\rho},\xi,\tilde{\tau},\tilde{\eta}}$ that are C^∞ in r up to $r = 0$ and for that the constants in the symbol estimates are uniform in $0 \le r \le r_1$ for arbitrary $r_1 > 0$.

We denote by $\tilde{S}^{\nu,l}(\overline{\mathbb{R}}_+ \times V \times \mathbb{R} \times U' \times \mathbb{R}^{1+n+1+q})$ the space of all $\tilde{p} = \tilde{p}(r, x, t, y, \rho, \xi, \tau, \eta)$ such that there is a $p(r, x, t, y, \tilde{\rho}, \xi, \tilde{\tau}, \tilde{\eta}) \in S^{\nu,l}(\overline{\mathbb{R}}_+ \times V \times \mathbb{R} \times U' \times \mathbb{R}^{1+n+1+q})$ with

$$\tilde{p}(r, x, t, y, \rho, \xi, \tau, \eta) = p(r, x, t, y, r\rho, \xi, r^l \tau, r\eta).$$

With the symbols $\tilde{p}$ we associate the (t, y, τ, η)–dependent operator families

$$\mathrm{op}_{\psi,(r,x)}(\tilde{p})(t, y, \tau, \eta) = \mathcal{F}^{-1}_{(\rho,\xi)\to(r,x)} \tilde{p}(r, x, t, y, \rho, \xi, \tau, \eta) \mathcal{F}_{(r',x')\to(\rho,\xi)}, \tag{1}$$

which acts for fixed (t, y, τ, η) first on $C_0^\infty(\mathbb{R}_+ \times V)$.

Fixing any finite atlas $\{(\tilde{V}_j, \chi_j)\}_{j=1}^N$ of X and a subordinate partition of unity $\{\phi_j\}_{j=1}^N$ as well as an N-tuple of functions $\{\psi_j \in C_0^\infty(\tilde{V}_j)\}_{j=1}^N$ with $\phi_j\psi_j = \phi_j$ for all j we form to an arbitrary system $\{\tilde{p}_j \in \tilde{S}^{\nu,l}(\overline{\mathbb{R}}_+ \times V_j \times \mathbb{R} \times U' \times \mathbb{R}^{1+n+1+q})\}_{j=1}^N$, $V_j = \chi_j(\tilde{V}_j)$, the operator family

$$P(t,y,\tau,\eta) = \sum_{j=1}^N \phi_j\{\chi_j^* \mathrm{op}_{\psi,(r,x)}(\tilde{p}_j)(t,y,\tau,\eta)\}\psi_j, \tag{2}$$

where χ_j^* denotes the operator pull-back under the map χ_j. For every fixed (t,y,τ,η) we therefore have $P(t,y,\tau,\eta) \in \Psi^\nu(\mathbb{R}_+ \times X)$. Analogous considerations make sense (and will be employed below) also for classical symbols.

We first interpret $P(t,y,\tau,\eta)$ as an operator family $P(t,y,\tau,\eta) : C_0^\infty(X^\wedge) \to C^\infty(X^\wedge)$ and write $P(t,y,\tau,\eta) \in C^\infty(\mathbb{R} \times U', \Psi^{\nu,l}(X^\wedge; \mathbb{R}^{1+q}))$ and $C^\infty(\mathbb{R} \times U', \Psi_{cl}^{\nu,l}(X^\wedge; \mathbb{R}^{1+q}))$, respectively.

Proposition 1 *For all $P(t,y,\tau,\eta) \in C^\infty(\mathbb{R} \times U', \Psi^{\nu,l}(X^\wedge; \mathbb{R}^{1+q}))$ and arbitrary cut-off functions ω, ω_1 and σ, σ_0 the operator family*

$$\sigma(r)a_1(t,y,\tau,\eta)\sigma_0(r) := \sigma(r)(1-\omega(r[\tau,\eta]_l))r^{-\nu}P(t,y,\tau,\eta)(1-\omega_1(r[\tau,\eta]_l))\sigma_0(r) \tag{3}$$

belongs to $S^{\nu,l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\nu,\gamma-\nu}(X^\wedge))$ for all $s, \gamma \in \mathbb{R}$.

Proof: We have by construction $\sigma(r)r^{-\nu}P(t,y,\tau,\eta)\sigma_0(r) = r^{-\nu}\sum_{j=0}^N \phi_j\chi_j^*\mathrm{op}_{\psi,(r,x)}(\sigma\tilde{p}_j\sigma_0)\psi_j$. The symbol $\tilde{q}_j(r,x,t,y,\rho,\xi,\tau,\eta) := \sigma(r)\tilde{p}_j(r,x,t,y,\rho,\xi,\tau,\eta)\sigma_0(r)$ belongs to $\tilde{S}^{\nu,l}(\overline{\mathbb{R}}_+ \times V_j \times \mathbb{R} \times U' \times \mathbb{R}^{1+n+1+q})$ and does not depend on r for $r > c$ for some constant $c > 0$. But then it follows

$$\tilde{q}_j \in \tilde{S}^{\nu,l}(V_j \times \mathbb{R} \times U' \times \mathbb{R}^{1+n+1+q}) + (C_0^\infty(\overline{\mathbb{R}}_+)\hat{\otimes}_\pi \tilde{S}^{\nu,l}(V_j \times \mathbb{R} \times U' \times \mathbb{R}^{1+n+1+q})),$$

such that $\tilde{q}_j$ has a representation

$$\tilde{q}_j(r,x,t,y,\rho,\xi,\tau,\eta) = \tilde{q}_j^0(x,t,y,\rho,\xi,\tau,\eta) + \sum_{k=1}^\infty \lambda_{jk}\varphi_j^k(r)\tilde{q}_j^k(x,t,y,\rho,\xi,\tau,\eta) \tag{4}$$

where $\{\lambda_{jk}\}_{k=1}^\infty \in l_1$, φ_j^k tends to zero in $C_0^\infty(\overline{\mathbb{R}}_+)$ and $\tilde{q}_j^k$ tends to zero in $\tilde{S}^{\nu,l}(V_j \times \mathbb{R} \times U' \times \mathbb{R}^{1+n+1+q})$ for $k \to \infty$.

Moreover, for $Q_j(t,y,\tau,\eta) := \mathrm{op}_{\psi,(r,x)}(\tilde{q}_j)$ as well as for $Q_j^k(t,y,\tau,\eta) := \mathrm{op}_{\psi,(r,x)}(\tilde{q}_j^k)$, $k = 0,1,\ldots$, we have

$$\begin{aligned}
\sigma(r)r^{-\nu}P(t,y,\tau,\eta)\sigma_0(r) &= r^{-\nu}\sum_{j=0}^N \phi_j\chi_j^* Q_j(t,y,\tau,\eta)\psi_j \\
&= \sum_{j=0}^N \phi_j\chi_j^*\left(r^{-\nu}Q_j^0(t,y,\tau,\eta) + \sum_{k=1}^\infty \lambda_{jk}M_{\varphi_j^k}r^{-\nu}Q_j^k(t,y,\tau,\eta)\right)\psi_j \\
&= \sum_{j=0}^N \phi_j\chi_j^* r^{-\nu}Q_j^0(t,y,\tau,\eta)\psi_j \\
&\quad + \sum_{k=1}^\infty\sum_{j=0}^N \phi_j\chi_j^*\lambda_{jk}M_{\varphi_j^k}r^{-\nu}Q_j^k(t,y,\tau,\eta)\psi_j.
\end{aligned} \tag{5}$$

An easy computation shows that all the $r^\nu Q_j^k(t,y,\tau,\eta)$ are anisotropic ν-homogeneous in the operator-valued sense (for $\kappa_\lambda u(r,x) = \lambda^{\frac{n+1}{2}} u(\lambda r, x)$) such that $\sum_{j=0}^N \phi_j \chi_j^* r^{-\nu} Q_j^k(t,y,\tau,\eta)\psi_j$ belongs to $S^{\nu,l}(\mathbb{R}\times U' \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}, \mathcal{K}^{s-\nu,\gamma-\nu})$ for all $s,\gamma \in \mathbb{R}$.

Furthermore, for every j the sequence of operator families $\{r^{-\nu}Q_j^k(t,y,\tau,\eta)\}_{k=1}^\infty$ tends to zero for $k \to \infty$. Using 1.1 Lemma 3 we get that $\sum_{j=0}^N \phi_j \chi_j^* M_{\varphi_j^k} \psi_j$ belongs to $S^{0,l}(\mathbb{R}\times U' \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}, \mathcal{K}^{s,\gamma})$ for all $s,\gamma \in \mathbb{R}$ and $\{M_{\varphi_j^k}\}_{k=1}^\infty$ tends to zero for $k \to \infty$.

Therefore (5) converges in $S^{\nu,l}(\mathbb{R}\times U' \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}, \mathcal{K}^{s-\nu,\gamma-\nu})$ and hence it has the property to be an operator-valued symbol of order ν.

Moreover, $(1-\omega(r[\tau,\eta]_l))$ as well as $(1-\omega_1(r[\tau,\eta]_l))$ are anisotropic 0-homogeneous in the operator-valued sense for large $|\tau,\eta|_l$, such that the assertion follows from 1.1 Lemma 9 and the composition rule for operator-valued symbols. □

Definition 2 *For any $\nu \in \mathbb{R}$ we denote by $N_O^{\nu,l}(X;\mathbb{R}^{1+q}_{\tau,\eta})$ the space of all functions $h(w,\tau,\eta) \in \mathcal{A}(\mathbb{C}_w, \Psi^{\nu,l}(X;\mathbb{R}^{1+q}))$ with*

$$h(w,\tau,\eta)|_{\Gamma_\beta} \in \Psi^{\nu,l}(X;\Gamma_\beta \times \mathbb{R}^{1+q}) \tag{6}$$

for all $\beta \in \mathbb{R}$, uniformly in $c_0 \le \beta \le c_1$ for $c_0 < c_1$. The anisotropy in (6) only refers to τ such that Im w *is formally treated as a further η-variable.*

Remark 3 We have in $N_O^{\nu,l}(X;\mathbb{R}^{1+q})$ a canonical FRÉCHETtopology (cf. analogously [SCH4]). Furthermore, there are the subspaces $M_O^{\nu,l}(X;\mathbb{R}^{1+q})$ of classical symbols, which are analogously defined in terms of classical operator families in $\Psi_{cl}^{\nu,l}(X;\mathbb{R}^{1+q}))$ and $\Psi_{cl}^{\nu,l}(X;\Gamma_\beta\times\mathbb{R}^{1+q})$, respectively.

Proposition 4 *Let a function $f(r,t,y,w,\tau,\eta) = h(r,t,y,w,r^l\tau, r\eta)$ be given with some $h(r,t,y,w,\tilde\tau,\tilde\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}_t \times U'_y, N_O^{\nu,l}(X;\mathbb{R}^{1+q}))$. Then $a_0(t,y,\tau,\eta)$, defined by*

$$a_0(t,y,\tau,\eta) := \omega(r[\tau,\eta]_l) r^{-\nu} \mathrm{op}_M^{\gamma-n/2}(f)(t,y,\tau,\eta)\omega_0(r[\tau,\eta]_l) \tag{7}$$

with arbitrary cut-off functions $\omega,\omega_0 \in C_0^\infty(\overline{\mathbb{R}}_+)$, belongs to the symbol space $S^{\nu,l}(\mathbb{R}\times U' \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\nu,\gamma-\nu}(X^\wedge))$ for all $s \in \mathbb{R}$.

If h, in particular, does not depend on r we obtain that $a_0(t,y,\tau,\eta)$ belongs to the subspace of classical operator-valued symbols.

Proof: Because of $[\tau,\eta]_l \ge 1$ there exist cut-off functions σ,σ_0 such that the relations $\sigma(r)\omega(r[\tau,\eta]_l) = \omega(r[\tau,\eta]_l)$ and $\sigma_0(r)\omega_0(r[\tau,\eta]_l) = \omega_0(r[\tau,\eta]_l)$ are fulfilled for all $[\tau,\eta]_l$. Therefore we may assume, that the function $f(r,t,y,w,\tau,\eta)$ does not depend on r for $r > c$ with some constant $c>0$, such that f comes from some h with

$$h(r,t,y,w,\tilde\tau,\tilde\eta) \in C^\infty(\mathbb{R}\times U', N_O^{\nu,l}(X;\mathbb{R}^{1+q})) + (C_0^\infty(\overline{\mathbb{R}}_+)\hat\otimes_\pi C^\infty(\mathbb{R}\times U', N_O^{\nu,l}(X;\mathbb{R}^{1+q})))$$

and h has a representation

$$h(r,t,y,w,\tilde{\tau},\tilde{\eta}) = h^0(t,y,w,\tilde{\tau},\tilde{\eta}) + \sum_{k=1}^{\infty} \lambda_k \varphi^k(r) h^k(t,y,w,\tilde{\tau},\tilde{\eta})$$

where $\{\lambda_k\}_{k=1}^{\infty} \in l_1$, φ^k tends to zero in $C_0^{\infty}(\overline{\mathbb{R}}_+)$ and h^k tends to zero in $C^{\infty}(\mathbb{R} \times U', N_O^{\nu,l}(X;\mathbb{R}^{1+q})$ for $k \to \infty$.

On the other hand from the edge degeneracy it follows that $r^{-\nu}\mathrm{op}_M^{\gamma-n/2}(f^k)(t,y,\tau,\eta)$ for $f^k(t,y,w,\tau,\eta) = h^k(t,y,w,r^l\tau,r\eta)$ is ν-homogeneous in the operator-valued sense, which implies especially the second part of the proposition. But then it follows the assertion in the same manner as in Proposition 1. □

Let us now state the MELLIN operator convention. It says that one can express the pseudo-differential action of $P(t,y,\tau,\eta)$ along the cone axis variable $r \in \mathbb{R}_+$ as a family of MELLIN pseudo-differential operators. We will construct the associated MELLIN symbol, which may be chosen as an holomorphic function in the MELLIN covariable $w \in \mathbb{C}$.

Theorem 5 *For every operator family $P(t,y,\tau,\eta) \in C^{\infty}(\mathbb{R} \times U', \Psi^{\nu,l}(X^{\wedge};\mathbb{R}^{1+q}))$ of the form (2) there exists a function $h(r,t,y,w,\tilde{\tau},\tilde{\eta}) \in C^{\infty}(\overline{\mathbb{R}}_+ \times \mathbb{R}_t \times U', N_O^{\nu,l}(X;\mathbb{R}^{1+q}))$ such that for*

$$f(r,t,y,w,\tau,\eta) = h(r,t,y,w,r^l\tau,r\eta) \tag{8}$$

we have

$$\mathrm{op}_M^{\delta}(f)(t,y,\tau,\eta) - P(t,y,\tau,\eta) \in C^{\infty}(\mathbb{R} \times U', \mathcal{S}(\mathbb{R}^{1+q}_{\tau,\eta}, \Psi^{-\infty}(X^{\wedge}))) \tag{9}$$

for all $\delta \in \mathbb{R}$. Conversely for every such f there is an operator family $P(t,y,\tau,\eta)$ such that (9) holds also for all $\delta \in \mathbb{R}$. In particular, for $P(t,y,\tau,\eta) \in C^{\infty}(\mathbb{R} \times U', \Psi_{cl}^{\nu,l}(X^{\wedge};\mathbb{R}^{1+q}))$ it follows $h(r,t,y,w,\tilde{\tau},\tilde{\eta}) \in C^{\infty}(\overline{\mathbb{R}}_+ \times \mathbb{R}_t \times U', M_O^{\nu,l}(X;\mathbb{R}^{1+q}))$ and vice versa.

For proving Theorem 5 we first study a local version of the MELLIN operator convention. Let us denote by $S^{\nu,l}(\overline{\mathbb{R}}_+ \times V \times \mathbb{R} \times U' \times \mathbb{C}_w \times \mathbb{R}^{n+1+q}_{\xi,\tau,\eta})$ the subspace of all $h(r,x,t,y,w,\xi,\tau,\eta) \in \mathcal{A}(\mathbb{C}_w, S^{\nu,l}(\overline{\mathbb{R}}_+ \times V \times \mathbb{R} \times U' \times \mathbb{R}^{n+1+q}_{\xi,\tau,\eta}))$ such that $h(r,x,t,y,\beta+i\rho,\xi,\tau,\eta)$ belongs to $S^{\nu,l}(\overline{\mathbb{R}}_+ \times V \times \mathbb{R} \times U' \times \Gamma_{\beta} \times \mathbb{R}^{n+1+q}_{\xi,\tau,\eta})$ for all $\beta \in \mathbb{R}$, uniformly in $c_0 \leq \beta \leq c_1$ for arbitrary reals $c_0 < c_1$. The space $S^{\nu,l}(\overline{\mathbb{R}}_+ \times V \times \mathbb{R} \times U' \times \mathbb{C}_w \times \mathbb{R}^{n+1+q}_{\xi,\tau,\eta})$ has a natural FRÉCHET topology.

Lemma 6 *The kernel cut-off operator H_{ψ} defined by $H_{\psi} = \mathcal{M}_{\tilde{r}\to w}\psi(\tilde{r})\mathcal{M}^{-1}_{\tilde{w}\to\tilde{r}}$ for a function $\psi(\tilde{r}) \in C_0^{\infty}(\mathbb{R}_+)$ with $\psi(\tilde{r}) \equiv 1$ in some open neighbourhood of $\tilde{r} = 1$ induces a continuous mapping*

$$H_{\psi} \;:\; S^{\nu,l}(\overline{\mathbb{R}}_+ \times V \times \mathbb{R} \times U' \times \Gamma_0 \times \mathbb{R}^{n+1+q}) \to S^{\nu,l}(\overline{\mathbb{R}}_+ \times V \times \mathbb{R} \times U' \times \mathbb{C} \times \mathbb{R}^{n+1+q})$$

for every $\nu \in \mathbb{R}$, where we have $(H_{\psi}f)|_{w=i\rho} - f \in S^{-\infty}(\overline{\mathbb{R}}_+ \times V \times \mathbb{R} \times U' \times \Gamma_0 \times \mathbb{R}^{n+1+q})$, such that for $h = (H_{\psi}f)|_{w=i\rho}$ the operator family $\mathrm{op}_{M,r}^{\frac{1}{2}}\mathrm{op}_{\psi,x}(h-f)(t,y,\tau,\eta)$ belongs to $C^{\infty}(\mathbb{R} \times U', \Psi^{-\infty}(\mathbb{R}_+ \times V;\mathbb{R}^{1+q}))$.

We omit the proof of Lemma 6 ; the anisotropy causes no additional difficulties compared with the case treated in [SCH1] or [DOR1]. Now we are ready to state the mentioned local version of the MELLIN operator convention.

Lemma 7 *Let $\tilde{p}(r,x,t,y,\rho,\xi,\tau,\eta) = p(r,x,t,y,r\rho,\xi,r^l\tau,r\eta)$ be given, where the function $p(r,x,t,y,\tilde{\rho},\xi,\tilde{\tau},\tilde{\eta})$ belongs to $S^{\nu,l}(\overline{\mathbb{R}}_+ \times V \times \mathbb{R} \times U' \times \mathbb{R}^{1+n+1+q}_{\tilde{\rho},\xi,\tilde{\tau},\tilde{\eta}})$. Then there exists an $h(r,x,t,y,w,\xi,\tilde{\tau},\tilde{\eta}) \in S^{\nu,l}(\overline{\mathbb{R}}_+ \times V \times \mathbb{R} \times U' \times \mathbb{C}_w \times \mathbb{R}^{n+1+q}_{\xi,\tilde{\tau},\tilde{\eta}})$ such that for $\tilde{h}(r,x,t,y,w,\xi,\tau,\eta) = h(r,x,t,y,w,\xi,r^l\tau,r\eta)$ the relations*

$$\mathrm{op}_{\psi,(r,x)}(\tilde{p})(t,x,\tau,\eta) - \mathrm{op}^{\delta}_{M,r}\mathrm{op}_{\psi,x}(\tilde{h})(t,x,\tau,\eta) \in C^\infty(\mathbb{R} \times U', \Psi^{-\infty}(\mathbb{R}_+ \times V; \mathbb{R}^{1+q}_{\tau,\eta})) \tag{10}$$

and

$$\mathrm{op}_{\psi,(r,x,t,y)}(\tilde{p}) - \mathrm{op}^{\delta}_{M,r}\mathrm{op}_{\psi,(x,t,y)}(\tilde{h}) \in \Psi^{-\infty}(\mathbb{R}_+ \times V \times \mathbb{R} \times U') \tag{11}$$

hold for all $\delta \in \mathbb{R}$.

Proof: First we have $\mathrm{op}^{\delta}_M(h) = \mathrm{op}^{\delta'}_M(h)$ for every $\delta, \delta' \in \mathbb{R}$. This follows by CAUCHY's integral formula from the holomorphy of h with respect to w and the holomorphy of the MELLIN image of a function with compact support with respect to r. Thus we may restrict ourselves to the weight $\delta = \frac{1}{2}$ such that the associated weight line is $\Gamma_0 = \{\mathrm{Re}\ w = 0\}$.

We will prove for any symbol $p(r,x,t,y,\tilde{\rho},\xi,\tilde{\tau},\tilde{\eta})$ the existence of an

$$h_0(r,x,t,y,w,\xi,\tilde{\tau},\tilde{\eta}) \in S^{\nu,l}(\overline{\mathbb{R}}_+ \times V \times \mathbb{R} \times U' \times \Gamma_0 \times \mathbb{R}^{n+1+q}_{\xi,\tilde{\tau},\tilde{\eta}})$$

and a symbol

$$p_1(r,x,t,y,\tilde{\rho},\xi,\tilde{\tau},\tilde{\eta}) \in S^{\nu-1,l}(\overline{\mathbb{R}}_+ \times V \times \mathbb{R} \times U' \times \mathbb{R}^{1+n+1+q}_{\tilde{\rho},\xi,\tilde{\tau},\tilde{\eta}})$$

such that for $p_0 := p$ the relation

$$\mathrm{op}_{\psi,(r,x)}(p_0|_{\tilde{\rho}=r\rho})(t,x,\tau,\eta) \simeq (\mathrm{op}^{\frac{1}{2}}_{M,r}\mathrm{op}_{\psi,x}(h_0) + \mathrm{op}_{\psi,(r,x)}(p_1|_{\tilde{\rho}=r\rho}))(t,x,\tau,\eta) \tag{12}$$

holds, where $\simeq$ means equality mod $C^\infty(\mathbb{R} \times U', \Psi^{-\infty}(\mathbb{R}_+ \times V; \mathbb{R}^{1+q}_{\tau,\eta}))$.

Then doing the same with p_1 leads to h_1 and p_2 satisfying the analogous relation. Therefore we get inductively for the already constructed p_k of order $\nu - k$ a symbol h_k of order $\nu - k$ and a p_{k+1} of order $\nu - (k+1)$ such that

$$\mathrm{op}_{\psi,(r,x)}(p_k|_{\tilde{\rho}=r\rho})(t,x,\tau,\eta) \simeq (\mathrm{op}^{\frac{1}{2}}_{M,r}\mathrm{op}_{\psi,x}(h_k) + \mathrm{op}_{\psi,(r,x)}(p_{k+1}|_{\tilde{\rho}=r\rho}))(t,x,\tau,\eta) \tag{13}$$

holds. Summing up we have

$$\mathrm{op}_{\psi,(r,x)}(p|_{\tilde{\rho}=r\rho})(t,x,\tau,\eta) \simeq (\mathrm{op}^{\frac{1}{2}}_{M,r}\mathrm{op}_{\psi,x}(\sum_{j=0}^{k} h_j) + \mathrm{op}_{\psi,(r,x)}(p_{k+1}|_{\tilde{\rho}=r\rho}))(t,x,\tau,\eta). \tag{14}$$

In $S^{\nu,l}(\overline{\mathbb{R}}_+ \times V \times \mathbb{R} \times U' \times \Gamma_0 \times \mathbb{R}^{n+1+q}_{\xi,\tilde{\tau},\tilde{\eta}})$ we then form the asymptotic sum

$$\underline{h}(r,x,t,y,w,\xi,\tilde{\tau},\tilde{\eta}) \sim \sum_{j=0}^{\infty} h_j(r,x,t,y,w,\xi,\tilde{\tau},\tilde{\eta}),$$

which implies

$$\mathrm{op}_{\psi,(r,x)}(p|_{\tilde{\rho}=r\rho})(t,x,\tau,\eta) \simeq \mathrm{op}^{\frac{1}{2}}_{M,r}\mathrm{op}_{\psi,x}(\underline{h})(t,x,\tau,\eta). \tag{15}$$

Like in Lemma 6 we can set $h := H_\psi \underline{h}$ and obtain

$$\mathrm{op}^{\frac{1}{2}}_{M,r}\mathrm{op}_{\psi,x}(h-\underline{h})(t,y,\tau,\eta) \in C^\infty(\mathbb{R}\times U', \Psi^{-\infty}(\mathbb{R}_+ \times V; \mathbb{R}^{1+q}_{\tau,\eta}))$$

as desired. In h it is allowed to set $\tilde{\tau} = r^l\tau$ and $\tilde{\eta} = r\eta$ because it acts like an r–dependence of coefficients, that means it is involved as an action from the left hand side in the operator.

Now (11) follows immediately from (10) such that it remains to show the first step, namenly (12). The way does not depend on $(x,t,y,\xi,\tilde{\tau},\tilde{\eta})$ such that we omit this variables for abbreviation.

We have to choose for $p(r,\tilde{\rho}) \in S^\nu(\overline{\mathbb{R}}_+ \times \mathbb{R})$ elements $h_0(r,w) \in S^\nu(\overline{\mathbb{R}}_+ \times \Gamma_0)$ and $p_1(r,\tilde{\rho}) \in S^{\nu-1}(\overline{\mathbb{R}}_+ \times \mathbb{R})$ such that

$$\mathrm{op}_{\psi,r}(p|_{\tilde{\rho}=r\rho}) \simeq \mathrm{op}^{\frac{1}{2}}_{M,r}(h_0) + \mathrm{op}_{\psi,r}(p_1|_{\tilde{\rho}=r\rho}) \tag{16}$$

holds, where $\simeq$ now means equality mod $\Psi^{-\infty}(\mathbb{R}_+)$. Let us set $h(r,i\rho) = p(r,-\rho)$ and look for p_1. For $\chi : \mathbb{R} \to \mathbb{R}_+$ defined by $\chi(s) = r = e^{-s}$ we have

$$\begin{aligned} \mathrm{op}^{\frac{1}{2}}_{M}(h_0)u(r) &= \int_{-\infty}^{\infty}\int_0^\infty \left(\frac{r}{r'}\right)^{-i\rho} h_0(r,i\rho)u(r')\frac{dr'}{r'}đ\rho \\ &= (\chi^*)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(s-s')\rho}h_0(e^{-s},i\rho)\chi^*u(s')ds'đ\rho, \end{aligned}$$

where χ^* denote the pull–back of functions with respect to χ. For $q(s,\rho) = h_0(e^{-s},i\rho)$, that means

$$q(s,\rho)|_{s=-\ln(r)} = p(r,-\rho), \tag{17}$$

which implies

$$\mathrm{op}^{\frac{1}{2}}_{M}(h_0) = \chi_*\mathrm{op}_{\psi,r}(q), \tag{18}$$

where χ_* denotes the operator push–forward with respect to χ. Thus there exists a symbol $\tilde{b}$ such that

$$\mathrm{op}_{\psi,r}(\tilde{b}) \simeq \chi_*\mathrm{op}_{\psi,r}(q), \tag{19}$$

where we get $\tilde{b}$ mod symbols of order $-\infty$ by

$$\tilde{b}(r,\rho)|_{r=\chi(s)} \sim \sum_{k=0}^{\infty}\frac{1}{k!}(\partial_\rho^k q)(s,\chi'(s)\rho)\phi_k(s,\rho)$$

with $\phi_k(s,\rho) = D^k_{\tilde{s}}e^{i\Delta(s,\tilde{s},\rho)}|_{\tilde{s}=s}$ and $\Delta(s,\tilde{s},\rho) = (\chi(\tilde{s}) - \chi(s) - \chi'(s)(\tilde{s}-s))\rho$.

From the definition of χ we get $\chi'(s) = -r$. Therefore $\tilde{b}(r,\rho)$ is detemined by $q(s,-r\rho)|_{s=-\ln(r)}$ modulo symbols of lower order. Furthermore, $\phi_k(s,\rho)|_{s=-\ln(r)}$ has the form $\phi_k(r,r\rho)$ where $\phi_k(r,\rho) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R})$ is a polynomial with respect to ρ of order $\leq k/2$. It follows $\tilde{b}(r,\rho) = b(r,r\rho)$ with $b(r,\tilde{\rho})$ of order ν which is C^∞ in r up to $r=0$. Note that $b(r,\rho) - q(s,-\rho)|_{s=\ln(r)}$ belongs to $S^{\nu-1}(\overline{\mathbb{R}}_+ \times \mathbb{R})$. Using (17) we obtain

$$b(r,\rho) - p(r,\rho) \in S^{\nu-1}(\overline{\mathbb{R}}_+ \times \mathbb{R}). \tag{20}$$

The formulas (18), (19) and (20) give with $\tilde{p} = p|_{\rho=r\rho}$ and $\tilde{b} = b|_{\tilde{\rho}=r\rho}$

$$\mathrm{op}_{M,r}^{\frac{1}{2}}(h_0) \simeq \mathrm{op}_{\psi,r}(\tilde{p}) + \mathrm{op}_{\psi,r}(\tilde{b}-\tilde{p}). \tag{21}$$

This implies (16) for $p_1 = \tilde{b} - \tilde{p}$. In the same manner one can treat the general case of $(x,t,y,\xi,\tilde{\tau},\tilde{\eta})$–dependent symbols; the obvious details will be dropped. □

Proof: (of Theorem 5) By assumption $P(t,y,\tau,\eta)$ has the form (2). Now we use Lemma 7 for every $\tilde{p}_j$ and set with the resulting $h_j(r,x,t,y,w,\xi,\tilde{\tau},\tilde{\eta}) \in S^{\nu,l}(\overline{\mathbb{R}}_+ \times V_j \times \mathbb{R} \times U' \times \mathbb{C}_w \times \mathbb{R}^{n+1+q}_{\xi,\tilde{\tau},\tilde{\eta}})$

$$h(r,t,y,w,\tilde{\tau},\tilde{\eta}) = \sum_{j=1}^{N} \phi_j \mathrm{op}_{\psi,x}(h_j)(r,t,y,w,\tilde{\tau},\tilde{\eta})\psi_j,$$

which belongs to $C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R} \times U'; N_O^{\nu,l}(X;\mathbb{R}^{1+q}))$. Then with $f(r,t,y,w,\tau,\eta) = h(r,t,y,w,r^l\tau,r\eta)$ we obtain (9).

If we suppose in addition that all $\tilde{p}_j$ in (2) are classical symbols we get even an $h(r,t,y,w,\tilde{\tau},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}_t \times U'_y, M_O^{\nu,l}(X;\mathbb{R}^{1+q}))$ such that for $f(r,t,y,w,\tau,\eta) = h(r,t,y,w,r^l\tau,r\eta)$ we have (9) for all $\delta \in \mathbb{R}$. If on the other hand such a function f is given, then we find an operator family $P(t,y,\tau,\eta)$, which turns out to be classical, such that (9) is satisfied for all $\delta \in \mathbb{R}$. □

2.2 Pseudo–differential operators with exit behaviour

The anisotropic operator–valued edge symbols of the following section will contain operators on the infinite cone $X^\wedge \ni (r,x)$ for $r \to \infty$ where the specific properties for large r play a role. $r \to \infty$ will be regarded as an exit of the underlying manifold to infinity. So we will talk about operators with corresponding exit behaviour. This was studied, in particular, by CORDES [COR1] and SCHROHE [SCR1]. Here we will develop a variant with additional covariables in the anisotropic set–up.

The main point is to introduce the symbols and operators in local coordinates (r,x,t,y) with the covariables (ρ,ξ,τ,η). In order to simplify notations we shall first neglect (t,y). The final symbol and operator families will depend as C^∞–functions on (t,y) and this can easily accepted afterwards. The variable x will run over an open set $V \subseteq \mathbb{R}^n$, r over $\mathbb{R}_+$. We shall first allow $r \in \mathbb{R}$ and then cut–off the objects for $r > c$ with some $c > 0$.

The exit behaviour of a symbol $q(r,x,\rho,\xi,\tau,\eta)$ for $|r| \to \infty$ is defined by the symbol estimates

$$|D_r^k D_x^\alpha D_{\tau,\eta}^\beta D_{\rho,\xi}^\gamma q(r,x,\rho,\xi,\tau,\eta)| \le c[\rho,\xi,\tau,\eta]_l^{\nu-|\beta|_l-|\gamma|}(1+|r|)^{\delta-k} \tag{1}$$

for all $r \in \mathbb{R}$, $(\rho,\xi,\tau,\eta) \in \mathbb{R}^{1+n+1+q}$, $x \in K \subset\subset V$ and all k,α,β,γ, with constants $c = c(k,\alpha,\beta,\gamma,K) > 0$. Here we denote by $[\rho,\xi,\tau,\eta]_l$ the smoothed norm function associated with the anisotropic norm $|\rho,\xi,\tau,\eta|_l := (|\tau|^2+|(\rho,\xi,\eta)|^{2l})^{\frac{1}{2l}}$. The anisotropic order $\nu \in \mathbb{R}$ is arbitrary as well as the order $\delta \in \mathbb{R}$ in r. For our purposes it suffices to restrict the consideration to $\delta = 0$.

Let us denote the class of all symbols p satisfying the estimates (1) for $\delta = 0$ by $S^{\nu,l}(\mathbb{R} \times V \times \mathbb{R}^{1+n+1+q})_e$, endowed with the FRÉCHET topology, defined by the best constants in (1) as semi-norms.

Let us now define the space $\Psi^{\nu,l}(\mathbb{R} \times X; \mathbb{R}^{1+q})_e$ of parameter dependent pseudo-differential operators on $\mathbb{R} \times X$ of order ν, depending anisotropically on $(\tau,\eta) \in \mathbb{R}^{1+q}$ and satisfying the exit condition for $|r| \to \infty$.

Definition 1 *The space $\Psi^{\nu,l}(\mathbb{R} \times X; \mathbb{R}^{1+q})_e$ consists of all*

$$Q(\tau,\eta) = \sum_{j=1}^{N} \phi_j\{\chi_j^* \mathrm{op}_{\psi,(r,x)}(q_j)(\tau,\eta)\}\psi_j + C(\tau,\eta) \tag{2}$$

for arbitrary $q_j(r,x,\rho,\xi,\tau,\eta) \in S^{\nu,l}(\mathbb{R} \times V_j \times \mathbb{R}^{1+n+1+q})_e$ for $j = 1,\dots,N$, (cf. analogously 2.1 formula (2)) and for an arbitrary family of smoothing operators $C(\tau,\eta)$ on $\mathbb{R} \times X$ with kernels in $\mathcal{S}(\mathbb{R}^{1+q}_{\tau,\eta}, C^\infty(X \times X, \mathcal{S}(\mathbb{R} \times \mathbb{R})))$.

Note that if $C(\tau,\eta)$ has a kernel $c(\tau,\eta,x,x',r,r') \in \mathcal{S}(\mathbb{R}^{1+q}_{\tau,\eta}, C^\infty(X \times X, \mathcal{S}(\mathbb{R} \times \mathbb{R})))$ then the operator acts as $(C(\tau,\eta)u)(r,x) = \int_X \int_{\mathbb{R}} c(\tau,\eta,x,x',r,r')u(r',x')dr'dx'$.

The space $\Psi^{\nu,l}(\mathbb{R} \times X; \mathbb{R}^{1+q})_e$ has a natural FRÉCHET topology. The same is true of $\Psi^{\nu,l}(X^\wedge; \mathbb{R}^{1+q})_e$ obtained from $\Psi^{\nu,l}(\mathbb{R} \times X; \mathbb{R}^{1+q})_e$ by restriction to $r > 0$. Then it makes sense to talk about $C^\infty(\mathbb{R}_t \times U'_y, \Psi^{\nu,l}(X^\wedge; \mathbb{R}^{1+q}_{\tau,\eta})_e)$ for an open set $U' \subseteq \mathbb{R}^q$.

The following results are easy consequences of CORDES [COR1], SCHROHE [SCR1] (cf. also the material in EGOROV, SCHULZE [EGO1]), that will tacitly be used below.

Proposition 2 *Let arbitrary cut-off functions $\sigma(r), \sigma_1(r)$ and an operator family $Q(\tau,\eta) \in \Psi^{\nu,l}(X^\wedge; \mathbb{R}^{1+q})_e$ be given. Then for every $(\tau,\eta) \in \mathbb{R}^{1+q}$ the operator*

$$(1-\sigma)Q(\tau,\eta)(1-\sigma_1) \ : \ \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\nu,\gamma'}(X^\wedge)$$

is continuous for all $s \in \mathbb{R}$ and all weights $\gamma, \gamma' \in \mathbb{R}$.

Proposition 3 *Let arbitrary cut-off functions $\sigma(r), \sigma_1(r)$ be given. Then for any operator family $Q(t,y,\tau,\eta) \in C^\infty(\mathbb{R} \times U', \Psi^{\nu,l}(X^\wedge; \mathbb{R}^{1+q})_e)$ the operator-valued function $(1 -$

$\sigma)Q(t,y,\tau,\eta)(1-\sigma_1)$ belongs to $S^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q};\mathcal{K}^{s,\gamma},\mathcal{K}^{s-\nu,\gamma'})$ for every $s\in\mathbb{R}$ and all weights $\gamma,\gamma'\in\mathbb{R}$.

Remark 4 Because of the finite support of a $\phi(r)\in C_0^\infty(\mathbb{R}_+)$ we have that M_ϕ regarded as a constant function of (τ,η) belongs to $\Psi^{0,l}(X^\wedge;\mathbb{R}^{1+q})_e$.

Proposition 5 *Every $Q(t,y,\tau,\eta)\in C^\infty(\mathbf{R}_t\times U'_y,\Psi^{\nu,l}(X^\wedge;\mathbb{R}^{1+q})_e)$ induces for fixed $(t,y,\tau,\eta)\in\mathbf{R}\times U'\times\mathbb{R}^{1+q}$ with some cut-off functions $\sigma(r),\sigma_1(r)\in C_0^\infty(\overline{\mathbb{R}}_+)$ a continuous operator*

$$(1-\sigma(r))Q(t,y,\tau,\eta)(1-\sigma_1(r))\ :\ \mathcal{S}(X^\wedge)\to\mathcal{S}(X^\wedge).$$

Here we set as before $\mathcal{S}(X^\wedge)=\mathcal{S}(\mathbb{R},C^\infty(X))|_{\mathbb{R}_+}$.

Theorem 6 *$A(\tau,\eta)\in\Psi^{\nu,l}(X^\wedge;\mathbf{R}^{1+q})_e$ and $B(\tau,\eta)\in\Psi^{\tilde\nu,l}(X^\wedge;\mathbb{R}^{1+q})_e$, $\nu,\tilde\nu\in\mathbb{R}$, implies $(A(1-\sigma(r))B)(\tau,\eta)\in\Psi^{\nu+\tilde\nu,l}(X^\wedge;\mathbb{R}^{1+q})_e$ for every cut-off function $\sigma(r)$. Moreover, we have $A^*(\tau,\eta)\in\Psi^{\nu,l}(X^\wedge;\mathbb{R}^{1+q})_e$, where $A^*(\tau,\eta)$ denotes the point-wise formal adjoint of $A(\tau,\eta)$.*

2.3 The edge symbol algebra

In this section we describe the properties of the anisotropic edge pseudo–differential operators on symbolic level. We will introduce an algebra of matrices of anisotropic operator–valued symbols, which also contain trace and potential conditions as in BOUTET DE MONVEL's algebra of boundary value problems for elliptic operators.

The algebra of smoothing MELLIN and GREEN symbols as well as the algebra of non–classical GREEN symbols form ideals in the mentioned edge symbol algebra. Furthermore, we will get an analogue of the symbolic calculus from SCHULZE [SCH2].

Definition 1 *Let $\mu,\nu\in\mathbb{R}$, $\mu-\nu\in\mathbb{N}$, and weight data $\mathbf{g}=(\gamma,\gamma-\mu,\Theta)$ with $\gamma\in\mathbb{R}$ and $\Theta=(-k,0]$, $k\in\mathbb{N}\setminus\{0\}$, be given. Moreover, let $U'\subseteq\mathbb{R}^q$ be an open set. Then we denote by $R^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g})$ the set of all operator families*

$$\begin{aligned} a(t,y,\tau,\eta) &= \sigma(r)\{a_0(t,y,\tau,\eta)+a_1(t,y,\tau,\eta)\}\sigma_0(r)\\ &\quad +(1-\sigma(r))a_\infty(t,y,\tau,\eta)(1-\sigma_1(r))\\ &\quad +(m+g)(t,y,\tau,\eta)\end{aligned}$$

with

$$a_0(t,y,\tau,\eta)=\omega(r[\tau,\eta]_l)r^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(f)(t,y,\tau,\eta)\omega_0(r[\tau,\eta]_l)$$

for $f(r,t,y,w,\tau,\eta)=h(r,t,y,w,r^l\tau,r\eta)$ where $h(r,t,y,w,\tilde\tau,\tilde\eta)$ belongs to $C^\infty(\overline{\mathbb{R}}_+\times\mathbb{R}\times U',N_O^{\nu,l}(X;\mathbb{R}^{1+q}))$ and

$$a_1(t,y,\tau,\eta)=(1-\omega(r[\tau,\eta]_l))P(t,y,\tau,\eta)(1-\omega_1(r[\tau,\eta]_l))$$

for $P(t,y,\tau,\eta) \in C^\infty(\mathbb{R} \times U', \Psi^{\nu,l}(X^\wedge;\mathbf{R}^{1+q})_e)$.

Moreover, we demand that $a_\infty(t,y,\tau,\eta) \in C^\infty(\mathbb{R} \times U', \Psi^{\nu,l}(X^\wedge;\mathbf{R}^{1+q})_e)$ *as well as* $(m+g)(t,y,\tau,\eta) \in R^{\nu,l}_{M+G}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g})$, *and we suppose that* $\omega, \omega_0, \omega_1$ *and* $\sigma, \sigma_0, \sigma_1$ *are cut-off functions satisfying* $\omega\omega_0 = \omega$ *and* $\omega\omega_1 = \omega_1$ *as well as* $\sigma\sigma_0 = \sigma$ *and* $\sigma\sigma_1 = \sigma_1$.

Furthermore, we assume that f *and* P *are related one to another via* MELLIN *operator convention (cf. 2.1 Theorem 5).*

The elements of $R^{\nu,l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g})$ *are called anisotropic edge symbols.*

We get by definition the space $R^{\nu,l}_{cl}(\mathbf{R} \times U' \times \mathbf{R}^{1+q}, \mathbf{g})$ of classical anisotropic edge symbols if we demand the condition $P(t,y,\tau,\eta) \in C^\infty(\mathbf{R} \times U', \Psi^{\nu,l}_{cl}(X^\wedge;\mathbb{R}^{1+q})_e)$ where we set $C^\infty(\mathbb{R} \times U', \Psi^{\nu,l}_{cl}(X^\wedge;\mathbf{R}^{1+q})_e) := C^\infty(\mathbf{R} \times U', \Psi^{\nu,l}_{cl}(X^\wedge;\mathbf{R}^{1+q})) \cap C^\infty(\mathbf{R} \times U', \Psi^{\nu,l}(X^\wedge;\mathbb{R}^{1+q})_e)$ which has the consequence that $h(r,t,y,w,\tilde{\tau},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R} \times U', M^{\nu,l}_O(X;\mathbb{R}^{1+q}))$ holds.

Remark 2 Using the notation of 2.2 we get for the anisotropic edge symbols

$$R^{\nu,l}(\mathbb{R} \times U' \times \mathbf{R}^{1+q}, \mathbf{g}) \subset C^\infty(\mathbb{R} \times U', \Psi^{\nu,l}(X^\wedge;\mathbf{R}^{1+q})_e) \tag{1}$$

and for the smoothing elements

$$R^{\nu,l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g}) \cap C^\infty(\mathbf{R} \times U', \Psi^{-\infty}(X^\wedge;\mathbf{R}^{1+q})_e) = R^{\nu,l}_{M+G}(\mathbf{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g}). \tag{2}$$

In view of Remark 2 the elements in $R^{\nu,l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g})$ are uniquely determined by the systems of symbols

$$r^{-\nu}\tilde{p}_j(r,x,t,y,\rho,\xi,\tau,\eta) \text{ for } x \in V_j \text{ and } 0 < r < c \text{ (cf. (2.1,(2))}$$

and

$$q_j(r,x,t,y,\rho,\xi,\tau,\eta) \text{ for } x \in V_j \text{ and } \tilde{c} < r \leq \infty \text{ (cf. (2.1,(2))}$$

with $0 < c < \tilde{c}$, $j = 1,\ldots,N$.

Recall that the mapping which assigns to the tuple $\{r^{-\nu}\tilde{p}_j\}_{j=1,\ldots,N} \cup \{q_j\}_{j=1,\ldots,N}$ an element of $R^{\nu,l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g})$ was obtained by operator conventions. They contain

(i) some finite covering $\{\widetilde{V}_j\}$, a subordinate partition of unity $\{\phi_j\}$ on X and an associated system of functions $\{\psi_j\}$ with $\psi_j \in C_0^\infty(\widetilde{V}_j)$, $\phi_j\psi_j = \phi_j$ for all j;

(ii) the global definition of operators along X;

(iii) the MELLIN convention for the $\{\tilde{p}_j\}$ near $r = 0$ and construction of the concrete operator–valued symbols near $r = 0$ by using, in particular, the cut–off functions $\omega, \omega_0, \omega_1$;

(iv) adding the operator family associated with the symbols $\{q_j\}$ with exit behaviour by using, in particular, $\sigma, \sigma_0, \sigma_1$.

Without loss of generality

(a) we may assume the symbols $\tilde{p}_j$ and $\tilde{p}_k$ are compatible over the subsets corresponding to $\widetilde{V}_j \cap \widetilde{V}_k$ in the sense of the push-forward rule with respect to the coordinate diffeomorphisms on symbolic level;

(b) it suffices to start the discussion with the symbols

$$\sigma(r)r^{-\nu}\tilde{p}_j(r,x,t,y,\rho,\xi,\tau,\eta) + (1-\sigma(r))q_j(r,x,t,y,\rho,\xi,\tau,\eta), \tag{3}$$

$j = 1, \ldots, N$.

From now on for convenience we will write $(3) = q_j(r,x,t,y,\rho,\xi,\tau,\eta)$. Then, in the above points (i)–(iv), we may argue in terms of q_j both near $r = 0$ and $r = \infty$.

This gives us altogether operator conventions

$$\text{op} \ : \ \{q_j\}_{j=1,\ldots,N} \mapsto a(t,y,\tau,\eta) \tag{4}$$

which map the symbol tuples to the operator-valued symbols in $R^{\nu,l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g})$.

Definition 3 *Let $\mu, \nu \in \mathbb{R}$, $\mu - \nu \in \mathbb{N}$, and weight data $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$ with $\gamma \in \mathbb{R}$ and $\Theta = (-k, 0]$, $k \in \mathbb{N} \setminus \{0\}$, be given. Moreover, let $U' \subseteq \mathbb{R}^q$ be an open set.*

Then we denote by $\mathfrak{R}^{\nu,l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g}; N_-, N_+)$ the set of all matrices

$$\mathfrak{a}(t,y,\tau,\eta) = \begin{pmatrix} a(t,y,\tau,\eta) & 0 \\ 0 & 0 \end{pmatrix} + (\mathfrak{m}+\mathfrak{g})(t,y,\tau,\eta)$$

with $a(t,y,\tau,\eta) = \sigma(r)\{a_0(t,y,\tau,\eta) + a_1(t,y,\tau,\eta)\}\sigma_0(r) + (1-\sigma(r))a_\infty(t,y,\tau,\eta)(1-\sigma_1(r))$ with a_0, a_1, a_∞ and $\sigma, \sigma_0, \sigma_1$ being given as in Definition 1 and a smoothing MELLIN *and* GREEN *symbol $(\mathfrak{m}+\mathfrak{g})(t,y,\tau,\eta) \in \mathfrak{R}^{\nu,l}_{M+G}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g}; N_-, N_+)$. Analogously we define $\mathfrak{R}^{\nu,l}_{cl}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g}; N_-, N_+)$ by demanding $a(t,y,\tau,\eta) \in R^{\nu,l}_{cl}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g})$.*

As before, we investigate at first the left upper corners of the matrices in $\mathfrak{R}^{\nu,l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g}; N_-, N_+)$ which belongs to $R^{\nu,l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g})$. This part corresponds to the pseudo–differential equation while the other items represent the additional trace and potential conditions of the problem.

Remark 4 Definition 3 can immediately be generalized to symbols $\mathfrak{a}(t,y,t',y',\tau,\eta)$ with $(t,y,t',y') \in (\mathbb{R} \times U') \times (\mathbb{R} \times U')$. This variant will be tacitly used below in the edge operator algebra generated by $\text{Op}(\mathfrak{a})$, $\mathfrak{a} \in \mathfrak{R}^{\nu,l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g}; N_-, N_+)$, modulo the smoothing operators in $\mathfrak{Y}^{-\infty}_G(\mathbb{R} \times U', \mathbf{g}; N_-, N_+)$.

Proposition 5 *Let $\mu, \nu \in \mathbb{R}$, $\mu - \nu \in \mathbb{N}$, and fix weight data $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$ with $\gamma \in \mathbb{R}$ and $\Theta = (-k, 0]$, $k \in \mathbb{N} \setminus \{0\}$. Moreover, let $U' \subseteq \mathbb{R}^q$ be an open set. Then we have $\mathfrak{R}^{\nu,l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g}; N_-, N_+) \subseteq \bigcap_{s \in \mathbb{R}} S^{\nu,l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}; \mathcal{K}^{s,\gamma} \oplus \mathbb{C}^{N_-}, \mathcal{K}^{s-\nu,\gamma-\mu} \oplus \mathbb{C}^{N_+})$ and for every asymptotic type $P \in \mathrm{As}(\gamma, \Theta)$ there is some $Q \in \mathrm{As}(\gamma - \mu, \Theta)$ such that $\mathfrak{R}^{\nu,l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g}; N_-, N_+) \subseteq \bigcap_{s \in \mathbb{R}} S^{\nu,l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}; \mathcal{K}_P^{s,\gamma} \oplus \mathbb{C}^{N_-}, \mathcal{K}_Q^{s-\nu,\gamma-\mu} \oplus \mathbb{C}^{N_+})$.*

Proof: Because of 2.1 Proposition 1, 2.1 Proposition 4, 2.2 Proposition 3 and the definition of smoothing MELLIN and GREEN symbols all the ingredients of an edge symbol are anisotropic operator-valued symbols between the cone SOBOLEV spaces as well as the subspaces with asymptotics which yields the assertion. □

Lemma 6 *For every anisotropic edge symbol $a(t, y, \tau, \eta) \in R^{\nu,l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g})$, an arbitrary cut-off function ω and some $\phi(r) \in C_0^\infty(\mathbb{R}_+)$ with $\omega(r)\phi(r) = 0$ the functions $\omega(r[\tau,\eta]_l) a(t, y, \tau, \eta) \phi(r[\tau,\eta]_l)$ and $\phi(r[\tau,\eta]_l) a(t, y, \tau, \eta) \omega(r[\tau,\eta]_l)$ belong to $R_G^{\nu,l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathbf{g})$.*

Proof: The point-wise GREEN property follows from the cone calculus (cf. [SCH1]). Furthermore, using Proposition 5, from the standard homogeneity argument and the composition law for operator-valued symbols we get immediately that $\omega(r[\tau,\eta]_l) a(t, y, \tau, \eta) \phi(r[\tau,\eta]_l)$ and $\phi(r[\tau,\eta]_l) a(t, y, \tau, \eta) \omega(r[\tau,\eta]_l)$ are operator-valued symbols with respect to the subspaces with asymptotics such that it remains to prove that the treated functions are classical symbols. We can prove this by TAYLOR expansion arguments with respect to r. The method is the same for all items and we are going to show it only for the MELLIN part of the edge symbol. Then we have to look at $\phi(r[\tau,\eta]_l) a_0(t, y, \tau, \eta) \omega(r[\tau,\eta]_l)$ with $a_0(t, y, \tau, \eta) = \tilde{\omega}(r[\tau,\eta]_l) r^{-\nu} \mathrm{op}_M^{\gamma - \frac{n}{2}}(f)(t, y, \tau, \eta) \tilde{\omega}_0(r[\tau,\eta]_l)$, where $f(r, t, y, w, \tau, \eta) = h(r, t, y, w, r^l \tau, r\eta)$ with some $h(r, t, y, w, \tilde{\tau}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R} \times U', N_O^{\nu,l}(X; \mathbb{R}^{1+q}))$ (cf. Section 2.1).

In 2.1 Proposition 4 we stated that $a_0(t, y, \tau, \eta)$ is anisotropic ν-homogeneous in the operator-valued sense for large $|\tau, \eta|_l$ if $h(r, t, y, w, \tilde{\tau}, \tilde{\eta})$ does not depend on r.

For given $N \in \mathbb{N}$ we get by TAYLOR's formula

$$h(r, t, y, w, \tilde{\tau}, \tilde{\eta}) = \sum_{j=0}^{N-1} r^j h_j(t, y, w, \tilde{\tau}, \tilde{\eta}) + r^N h_{(N)}(r, t, y, w, \tilde{\tau}, \tilde{\eta}).$$

Then setting $a_0^{(j)}(t, y, \tau, \eta) = \tilde{\omega}(r[\tau,\eta]_l) r^{-\nu-N+j} \mathrm{op}_M^{\gamma - \frac{n}{2}}(f_j)(t, y, \tau, \eta) \tilde{\omega}_0(r[\tau,\eta]_l)$ for $j = 0, \ldots, N$ with $f_j(t, y, w, \tau, \eta) = h_j(t, y, w, r^l \tau, r\eta)$ for $j = 0, \ldots, N-1$ and $f_N(r, t, y, w, \tau, \eta) = h_{(N)}(r, t, y, w, r^l \tau, r\eta)$, we have

$$\begin{aligned} a_0(t, y, \tau, \eta) &= r^N \left(\sum_{j=0}^{N-1} a_0^{(j)}(t, y, \tau, \eta) + a_0^{(N)}(t, y, \tau, \eta) \right) \\ &= (r[\tau,\eta]_l)^N \left(\sum_{j=0}^{N-1} [\tau,\eta]_l^{-N} a_0^{(j)}(t, y, \tau, \eta) + [\tau,\eta]_l^{-N} a_0^{(N)}(t, y, \tau, \eta) \right). \end{aligned}$$

But here for $j = 0, \dots, N-1$ we observe that $[\tau,\eta]_l^{-N} a_0^{(j)}(t,y,\tau,\eta)$ is anisotropic $(-N+\nu-N+j=\nu-j)$–homogeneous in the operator–valued sense for large $|\tau,\eta|_l$ and $[\tau,\eta]_l^{-N} a_0^{(N)}(t,y,\tau,\eta)$ is an anisotropic operator–valued symbol of order $\nu - N$.

Finally multiplication by $(r[\tau,\eta]_l)^N$ which is anisotropic 0–homogeneous in the operator–valued sense for large $|\tau,\eta|_l$ preserves the desired structure because of the compact support of ϕ such that the proof is complete. □

Proposition 7 *Let $\tilde\omega, \tilde\omega_0, \tilde\omega_1$ be cut-off functions satisfying the conditions $\tilde\omega\tilde\omega_0 = \tilde\omega$ and $\tilde\omega\tilde\omega_1 = \tilde\omega_1$. Furthermore, we assume $a(t,y,\tau,\eta) = a_0(t,y,\tau,\eta) + a_1(t,y,\tau,\eta)$, where a_0 and a_1 are defined as in Definition 1. Then for $\tilde a(t,y,\tau,\eta) = \tilde a_0(t,y,\tau,\eta) + \tilde a_1(t,y,\tau,\eta)$ with*

$$\tilde a_0(t,y,\tau,\eta) = \tilde\omega(r[\tau,\eta]_l) r^{-\nu} \mathrm{op}_M^{\gamma-\frac{n}{2}}(f)(t,y,\tau,\eta)\tilde\omega_0(r[\tau,\eta]_l)$$

and

$$\tilde a_1(t,y,\tau,\eta) = (1-\tilde\omega(r[\tau,\eta]_l))P(t,y,\tau,\eta)(1-\tilde\omega_1(r[\tau,\eta]_l))$$

we have $\sigma(r)\{a(t,y,\tau,\eta) - \tilde a(t,y,\tau,\eta)\}\sigma_0(r) \in R_G^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q}, \mathbf{g})$ where σ, σ_0 are cut-off functions as in Definition 1.

Proof: We omit for abbreviation the variables (t,y,τ,η) and r, but keep in mind that cut-off functions are denoted by ω if the argument r is multiplied by $[\tau,\eta]_l$ and by σ if not. Furthermore, we set $F := r^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(f)$. Then we have $a = \omega F\omega_0 + (1-\omega)P(1-\omega_1)$ and $\tilde a = \tilde\omega F\tilde\omega_0 + (1-\tilde\omega)P(1-\tilde\omega_1)$. Now we choose cut-off functions $\omega', \omega_0', \omega_1'$ satisfying

$$\omega'\omega_0' = \omega', \quad \omega'\omega_1' = \omega_1', \quad \omega'\omega = \omega', \quad \omega'\tilde\omega = \omega'. \tag{5}$$

Then we have

$$\begin{aligned} \sigma\{a-\tilde a\}\sigma_0 &= \sigma\{\omega'(a-\tilde a)\omega_0' + \omega'(a-\tilde a)(1-\omega_0') \\ &\quad +(1-\omega')(a-\tilde a)\omega_1' + (1-\omega')(a-\tilde a)(1-\omega_1')\}\sigma_0, \end{aligned}$$

where the second and the third item have the Green property by Lemma 6. For the first item we get

$$\begin{aligned} \sigma\omega'(a-\tilde a)\omega_0'\sigma_0 &= \sigma\omega'\{\omega F\omega_0 - \tilde\omega F\tilde\omega_0 + (1-\omega)P(1-\omega_1) - (1-\tilde\omega)P(1-\tilde\omega_1)\}\omega_0'\sigma_0 \\ &= \sigma\omega' F(\omega_0 - \tilde\omega_0)\omega_0'\sigma_0, \end{aligned}$$

cf. (5), which has also the Green property by Lemma 6, because of $\omega'(\omega_0 - \tilde\omega_0)\omega_0' = 0$.

It remains to deal with $\sigma(1-\omega')(a-\tilde a)(1-\omega_1')\sigma_0$. Using the Mellin operator convention we get for all $\phi, \psi \in C_0^\infty(\mathbb{R}_+)$ that $\phi F\psi = \phi(P+g)\psi$ with some $g \in R_G^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q}, \mathbf{g})$.

Then we have

$$\begin{aligned}
&\sigma(1-\omega')(a-\tilde{a})(1-\omega_1')\sigma_0 \\
=\ &\sigma(1-\omega')\{\omega F\omega_0+(1-\omega)P(1-\omega_1)-\tilde{\omega}F\tilde{\omega}_0-(1-\tilde{\omega})P(1-\tilde{\omega}_1)\}(1-\omega_1')\sigma_0 \\
=\ &\sigma(1-\omega')\{\omega P\omega_0+(1-\omega)P(1-\omega_1)- \\
&-\tilde{\omega}P\tilde{\omega}_0-(1-\tilde{\omega})P(1-\tilde{\omega}_1)+\omega g\omega_0-\tilde{\omega}g\tilde{\omega}_0\}(1-\omega_1')\sigma_0.
\end{aligned}$$

The supposed conditions on the cut–off functions yield $\omega(1-\omega_0)=\tilde{\omega}(1-\tilde{\omega}_0)=(1-\omega)\omega_1=(1-\tilde{\omega})\tilde{\omega}_1=0$ such that from Lemma 6 it follows that the symbol

$$b:=(1-\omega')\{\omega P(1-\omega_0)+(1-\omega)P\omega_1-\tilde{\omega}P(1-\tilde{\omega}_0)-(1-\tilde{\omega})P\tilde{\omega}_1\}(1-\omega_1')$$

belongs to $R_G^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathrm{g})$. But then

$$\begin{aligned}
\sigma(1-\omega')(a-\tilde{a})(1-\omega_1')\sigma_0 &= \sigma\{(1-\omega')(a-\tilde{a})(1-\omega_1')+b-b\}\sigma_0 \\
&= \sigma(1-\omega')\{(P-P)-\omega P(1-\omega_0)-(1-\omega)P\omega_1+ \\
&\quad +\tilde{\omega}P(1-\tilde{\omega}_0)+(1-\tilde{\omega})P\tilde{\omega}_1+\omega g\omega_0-\tilde{\omega}g\tilde{\omega}_0\}(1-\omega_1')\sigma_0
\end{aligned}$$

also belongs to $R_G^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathrm{g})$. □

Remark 8 Without loss of generality we may assume in Definition 1 that the operator family $P(t,y,\tau,\eta)$ coincides with $a_\infty(t,y,\tau,\eta)$. Under this condition we obtain analogously to Proposition 7 that the concrete choice of the cut–off functions σ,σ_0,σ_1 with $\sigma\sigma_0=\sigma$ and $\sigma\sigma_1=\sigma_1$ only affects $a(t,y,\tau,\eta)$ in Definition 1 mod $R_G^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q};\mathrm{g})$.

Proposition 9 *Let* $\mathfrak{a}_j(t,y,\tau,\eta)\in\mathfrak{R}^{\nu-j,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathrm{g};N_-,N_+)$, $j\in\mathbb{N}$, *be any sequence of symbols, where we assume that the asymptotic types* P *and* Q *of the involved* Green *symbols are independent of* $j\in\mathbb{N}$.

Then there exists an $\mathfrak{a}(t,y,\tau,\eta)\in\mathfrak{R}^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathrm{g};N_-,N_+)$, *which is uniquely determined modulo* $R_{(G)}^{-\infty}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathrm{g};N_-,N_+)_{P,Q}$, *such that*

$$\mathfrak{a}(t,y,\tau,\eta)\sim_{(G)}\sum_{j=0}^{\infty}\mathfrak{a}_j(t,y,\tau,\eta),$$

where " $\sim_{(G)}$*" means, that for every* $M\in\mathbb{N}$ *there is an* $N\in\mathbb{N}$ *with* $\mathfrak{a}(t,y,\tau,\eta)-\sum_{j=0}^{N}\mathfrak{a}_j(t,y,\tau,\eta)\in\mathfrak{R}_{(G)}^{\nu-M,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathrm{g};N_-,N_+)_{P,Q}$.

An analogous result holds for classical symbols $\mathfrak{a}_j$ *with a resulting* $\mathfrak{a}(t,y,\tau,\eta)\in\mathfrak{R}_{cl}^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathrm{g};N_-,N_+)$

Proof: For the sake of simplicity we will only prove the local version of the proposition and drop all arguments with respect to localization and globalization on the manifold X. Moreover, we will only deal with the left upper corners of the matrices, the case of the whole matrix is then an obvious generalization. Furthermore, we will keep in mind that the varius cut-off functions ω in the argument are multiplied by $[\tau,\eta]_l$ whereas cut-off functions σ only depend on r. We will often drop the variables (t,y,τ,η) and r in the arguments, Moreover, note that $\nu_j = \nu - j$.

Then we have $a_j = \sigma(\omega r^{-\nu_j}\mathrm{op}_M^{\gamma-\frac{n}{2}}(f_j)\omega_0 + (1-\omega)P_j(1-\omega_1))\sigma_0 + (1-\sigma)Q_j(1-\sigma_1) + m_j + g_j$. Note that in view of Proposition 7 and Remark 8 it is possible to pass to j-independent cut-off functions.

For proving Proposition 9 we will construct a sequence of symbols $f_j^0(r,t,y,\tau,\eta) = h_j^0(r,t,y,r^l\tau,r\eta)$ with $h_j^0(r,t,y,\tilde\tau,\tilde\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}_t \times U'_y, N_O^{\nu_j,l}(V;\mathbb{R}^{1+q}_{\tilde\tau,\tilde\eta}))$ and a sequence of symbols with exit behaviour $Q_j^0 \in C^\infty(\mathbb{R}\times U', \Psi^{\nu_j,l}(\mathbb{R}_+\times V;\mathbb{R}^{1+q})_e)$ such that

$$\omega r^{-\nu_j}\mathrm{op}_M^{\gamma-\frac{n}{2}}(f_j - f_j^0)\omega_0 \in R^{\nu_j,l}_{M+(G)}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathrm{g})_{P,Q} \tag{6}$$

and

$$(1-\sigma)(Q_j - Q_j^0)(1-\sigma_1) \in R^{\nu_j,l}_{M+(G)}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathrm{g})_{P,Q} \tag{7}$$

and for which $f := \sum_{j=0}^{\infty} r^j f_j^0$ as well as $Q := \sum_{j=0}^{\infty} Q_j^0$ are absolutely convergent sums in the corresponding symbol spaces.

Then setting $b_j = \sigma(\omega r^{-\nu_j}\mathrm{op}_M^{\gamma-\frac{n}{2}}(f_j - f_j^0)\omega_0 + (1-\omega)(P_j - P_j^0)(1-\omega_1))\sigma_0 + (1-\sigma)(Q_j - Q_j^0)(1-\sigma_1) + m_j + g_j$, where P_j^0 is associated with f_j^0 via the MELLIN operator convention, we obtain $b_j \in R^{\nu_j,l}_{M+(G)}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathrm{g})_{P,Q}$. Thus using 1.5 Corollary 3 we get a $b \in R^{\nu,l}_{M+(G)}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathrm{g})_{P,Q}$ with $b \sim_{(G)} \sum b_j$ and b is unique mod $R^{-\infty}_{(G)}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathrm{g})_{P,Q}$.

But now $a = \sigma(\omega r^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(f)\omega_0 + (1-\omega)P(1-\omega_1))\sigma_0 + (1-\sigma)Q(1-\sigma_1) + b$ with $P := \sum_{j=0}^{\infty} P_j^0$ has the asserted properties.

Finally it remains to construct the sequences Q_j^0 and f_j^0. Locally for some chart $\tilde V \to V$ on X the operator family Q_j is defined by $Q_j = \mathrm{op}_{\psi,(r,x)}\{q_j(r,x,t,y,\rho,\xi,\tau,\eta)\}$ with some $q_j \in S^{\nu_j,l}(\mathbb{R}\times V\times\mathbb{R}\times U'\times\mathbb{R}^{1+n+1+q})_e$. But then for

$$q_j^0(r,x,t,y,\rho,\xi,\tau,\eta) = \chi\left(\frac{\rho,\xi,\tau,\eta}{c_j}\right) q_j(r,x,t,y,\rho,\xi,\tau,\eta),$$

where $\chi(\rho,\xi,\tau,\eta)$ is some excision function, we get the desired operator family $Q_j^0 = \mathrm{op}_{\psi,(r,x)}\{q_j^0(r,x,t,y,\rho,\xi,\tau,\eta)\}$ satisfying condition (7). If further c_j tends to ∞ sufficiently fast we obtain an absolutely convergent sum $Q = \sum Q_j^0$.

For constructing f_j^0 we first restrict the symbol $h_j(r,t,y,w,\tilde\tau,\tilde\eta)$ that is associated with f_j to the weight line $\Gamma_{\frac{n+1}{2}-\gamma}$. Now we get by 2.1 Lemma 6 (analogously applied to $\Gamma_{\frac{n+1}{2}-\gamma}$ istead of Γ_0) with the kernel cut-off operator H_ψ from

$$h_j^0(r,t,y,i\rho,\tilde\tau,\tilde\eta) = \chi\left(\frac{\rho,\tilde\tau,\tilde\eta}{c_j}\right) h_j|_{\Gamma_{\frac{n+1}{2}-\gamma}}(r,t,y,i\rho,\tilde\tau,\tilde\eta)$$

with some excision function $\chi(\rho,\tilde{\tau},\tilde{\eta})$ the symbol $\tilde{h}_j^0 = H_\psi h_j^0$. Then $f_j^0(r,t,y,w,\tau,\eta) = \tilde{h}_j^0(r,t,y,w,r^l\tau,r\eta)$ fulfills the condition (6). The sum $\sum_{n=0}^{\infty} r^j f_j^0$ now is absolutely convergent if the constants c_j tend to ∞ sufficiently fast for $j \to \infty$. □

Theorem 10 *From* $\mathfrak{a}(t,y,\tau,\eta) \in \mathfrak{R}^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g}_1;N,N_+)$ *and* $\mathfrak{b}(t,y,\tau,\eta) \in \mathfrak{R}^{\tilde{\nu},l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g}_2;N_-,N)$ *for weight data* $\mathbf{g}_1 = (\gamma-\tilde{\mu},\gamma-(\tilde{\mu}+\mu),\Theta)$ *and* $\mathbf{g}_2 = (\gamma,\gamma-\tilde{\mu},\Theta)$, *respectively, it follows* $(\mathfrak{a}\mathfrak{b})(t,y,\tau,\eta) \in \mathfrak{R}^{\nu+\tilde{\nu},l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g};N_-,N_+)$ *for the weight data* $\mathbf{g} = (\gamma,\gamma-(\mu+\tilde{\mu}),\Theta)$.

Moreover, $\mathfrak{a}(t,y,\tau,\eta) \in \mathfrak{R}^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g};N_-,N_+)$ *for* $\mathbf{g} = (\gamma,\gamma-\mu,\Theta)$ *implies for the point-wise taken formal adjoint* $\mathfrak{a}^*(t,y,\tau,\eta) \in \mathfrak{R}^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g}^*;N_+,N_-)$ *for* $\mathbf{g}^* = (-\gamma+\mu,-\gamma,\Theta)$. *Analogous relations hold for the symbol classes with subscript cl.*

For proving Theorem 10 we need the following lemma.

Lemma 11 *Let the symbols* $g(t,y,\tau,\eta) \in R_G^{\tilde{\nu},l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g}_1)$ *and* $(m+g)(t,y,\tau,\eta) \in R_{M+G}^{\tilde{\nu},l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g}_1)$ *for weight data* $\mathbf{g}_1 = (\gamma,\gamma-\tilde{\mu},\Theta)$ *be given.*

Then for every $a(t,y,\tau,\eta) \in R^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g}_2)$ *for* $\mathbf{g}_2 = (\gamma-\tilde{\mu},\gamma-(\tilde{\mu}+\mu),\Theta)$ *we have* $ag \in R_G^{\nu+\tilde{\nu},l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g})$ *and* $a(m+g) \in R_{M+G}^{\nu+\tilde{\nu},l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g})$ *for* $\mathbf{g} = (\gamma,\gamma-(\tilde{\mu}+\mu),\Theta)$. *An analogous result holds if we compose a symbol a from the right hand side to an* GREEN *or smoothing* MELLIN *and* GREEN *symbol.*

Proof: In view of Definition 1 we have $a = \sigma(a_0-a_1)\sigma_0 + (1-\sigma)a_\infty(1-\sigma_1) + \tilde{m} + \tilde{g}$ with $\tilde{m}+\tilde{g} \in R_{M+G}^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g}_2)$ and the non-smoothing MELLIN, the FOURIER pseudo-differential and the exit parts a_0, a_1 and a_∞, respectively. Then we have to look at the compositions

$$ag = \sigma(a_0+a_1)\sigma_0 g + (1-\sigma)a_\infty(1-\sigma_1)g + (\tilde{m}+\tilde{g})g \tag{8}$$

and

$$a(m+g) = \sigma(a_0+a_1)\sigma_0 m + (1-\sigma)a_\infty(1-\sigma_1)m + (\tilde{m}+\tilde{g})m + ag. \tag{9}$$

Using the composition rule for operator-valued symbols we obtain that ag belongs to $\bigcap_{s\in\mathbb{R}} S^{\nu+\tilde{\nu},l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q};\mathcal{K}^{s,\gamma},\mathcal{K}^{\infty,\gamma-(\mu+\tilde{\mu})})$. So it remains to prove that (8) is a classical symbol with point-wise GREEN property. In virtue of 1.4 Lemma 9 and the definition of GREEN edge symbols we get $(\tilde{m}+\tilde{g})g \in R_G^{\nu+\tilde{\nu},l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g})$.

Moreover, for fixed $(t,y,\tau,\eta) \in \mathbb{R}\times U'\times\mathbb{R}^{1+q}$ it follows

$$\mathcal{K}^{s,\gamma} \xrightarrow{\sigma g} \mathcal{K}_P^{\infty,\gamma-\mu} \xrightarrow{a_1} \mathcal{K}_\Theta^{\infty,\gamma-(\mu+\tilde{\mu})} \xrightarrow{\sigma_0} \mathcal{S}_\Theta^{\gamma-(\mu+\tilde{\mu})}$$

and

$$\mathcal{K}^{s,-\gamma+(\mu+\tilde{\mu})} \xrightarrow{a_1^*\sigma_0} \mathcal{K}^{s-\nu,-\gamma+\mu} \xrightarrow{g^*\sigma} \mathcal{S}_Q^{-\gamma}$$

for every $s \in \mathbb{R}$, which implies the point-wise GREEN property of $\sigma a_1 \sigma_0 g$.

The point-wise GREEN property of $\sigma a_0 g \sigma_0$ is a consequence of the cone calculus, cf. [SCH1] or [SCH3].

For the exit part we observe from 2.2 Proposition 5 and 2.2 Proposition 2 for fixed (t, y, τ, η) and arbitrary $s \in \mathbb{R}$ the following mapping properties –

$$\mathcal{K}^{s,\gamma} \stackrel{(1-\sigma)g}{\longrightarrow} \mathcal{S}_P^{\gamma-\mu} \stackrel{a_\infty}{\longrightarrow} \mathcal{S}_P^{\gamma-(\mu+\tilde{\mu})} \stackrel{1-\sigma_1}{\longrightarrow} \mathcal{S}_\Theta^{\gamma-(\mu+\tilde{\mu})}$$

and

$$\mathcal{K}^{s,-\gamma+(\mu+\tilde{\mu})} \stackrel{a_\infty^*(1-\sigma_1)}{\longrightarrow} \mathcal{K}^{s-\nu,-\gamma+\mu} \stackrel{1-\sigma_1}{\longrightarrow} \mathcal{K}_\Theta^{s-\nu,-\gamma+\mu} \stackrel{g^*}{\longrightarrow} \mathcal{S}_Q^{-\gamma}$$

such that (8) has point-wise GREEN property.

The proof of the fact that the composition of an edge and a GREEN symbol is classical in the operator-valued sense is essentially the same as in Lemma 6 using that the appearing r-powers will be compensated by the GREEN symbol. In virtue of 1.5 Theorem 5 and the first part of this proof we obtain $(\tilde{m} + \tilde{g})m + ag \in R_{M+G}^{\nu+\tilde{\nu},l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathrm{g})$.

By definition $m(t, y, \tau, \eta)$ is a finite sum of operator-valued symbols of the form $\omega H \omega_0$, where $H = r^{-\tilde{\nu}+j} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(t, y)(r^l \tau, r\eta)^\alpha$. But with $a_1 = (1 - \tilde{\omega})P(1 - \tilde{\omega}_1)$ we have

$$\sigma a_1 \sigma_0 m = \sigma(1 - \tilde{\omega})P(1 - \tilde{\omega}_1)\sigma_0 \omega H \omega_0 = \sigma(1 - \tilde{\omega})P\omega_1' \sigma_0 \varphi H \omega_0$$

where $\varphi = (1 - \tilde{\omega}_1)\omega \in C_0^\infty(\mathbb{R}_+)$ and ω_1' satisfies $\omega_1' \varphi = \varphi$. Now $\varphi H \omega_0$ is GREEN by 1.4 Proposition 5 such that $\sigma a_1 \sigma_0 m$ has the GREEN property by the first part of this proof.

The composition of a smoothing MELLIN symbol and the exit part of an edge symbol is GREEN. This is an easy consequence of the freedom to choose the cut-off functions σ, σ_1 modulo GREEN remainders.

Finally the MELLIN pseudo-differential calculus, as it may be found in EGOROV, SCHULZE [EGO1], ensures that $\sigma a_0 \sigma_0 m$ also belongs to $R_{M+G}^{\nu+\tilde{\nu},l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}, \mathrm{g})$ which completes the proof. □

Remark 12 In order to treat the composition of the operator-valued symbols $\mathfrak{a} = (a_{ij})_{i,j=1,2}$ and $\mathfrak{b} = (b_{jk})_{j,k=1,2}$ given as in Theorem 10 we have to look at compositions $a_{ij} b_{jk}$, where in case of $(i, j) \neq (1, 1)$ or $(j, k) \neq (1, 1)$ always one of the elements has the GREEN property. Thus all compositions, except of $a_{11} b_{11}$, can be treated by the scheme of proving Lemma 11.

Proof: (of Theorem 10) For proving the anisotropic version of this theorem we need no new idea compared with the isotropic one which is elaborated in [EGO1]. Therefore we will only sketch the proof. Using Remark 12 we have only to deal with the left upper corners of the block matrices $\mathfrak{a}$ and $\mathfrak{b}$ denoted here for convenience by a and b, respectively. In view of Remark 2 it suffices to know for the operator convention (4)

$$ab = \mathrm{op}(\{q_j \#_{r,x} \tilde{q}_j\}) \bmod R_{M+G}^{\nu+\tilde{\nu},l}(\mathbb{R} \times U' \times \mathbb{R}^{1+q}; \mathrm{g}). \tag{10}$$

Here we associate $\{q_j\}$ with a and $\{\tilde{q}_j\}$ with b, respectively, in the sense of the above operator convention op, cf. (4), and by $\sharp_{r,x}$ we denote the LEIBNIZ product with respect to the variables r and x.

But (10) follows immediately from (2) in Remark 2 and from the fact that the (t,y,τ,η)–wise LEIBNIZ compositions are compatible with the composition rules from the cone theory. The property that the (t,y,τ,η)–dependent $C^{\nu+\tilde{\nu}}_{M+G}(X^\wedge,\mathbf{g})$–remainders which constitute the remainders in (10) are classical follows by TAYLOR expansion arguments with respect to r, similarly as above. □

2.4 Edge pseudo–differential operators

In this section we establish the local anisotropic pseudo–differential algebra of edge problens. As mentioned before the local model of the anisotropic edge is $\mathbb{R}\times\mathbf{W} := \mathbb{R}\times U'\times X^\wedge$. Then we are looking for a suitable subalgebra of

$$\bigcap_{s\in\mathbb{R}} \Psi^{\nu,l}(\mathbb{R}\times U';\mathcal{K}^{s,\gamma}\oplus\mathbb{C}^{N_-},\mathcal{K}^{s-\nu,\gamma-\mu}\oplus\mathbb{C}^{N_+}).$$

Definition 1 *Let $\mu,\nu\in\mathbb{R}$, $\mu-\nu\in\mathbb{N}$, and weight data $\mathbf{g}=(\gamma,\gamma-\mu,\Theta)$ with $\gamma\in\mathbb{R}$ and $\Theta=(-k,0]$, $k\in\mathbb{N}\setminus\{0\}$, be given. Then we define the space $Y^{\nu,l}(\mathbb{R}\times\mathbf{W},\mathbf{g}) := \mathrm{Op}(R^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g})) + Y_G^{-\infty}(\mathbb{R}\times\mathbf{W},\mathbf{g})$ and analogously $\mathfrak{Y}^{\nu,l}(\mathbb{R}\times\mathbf{W},\mathbf{g};N_-,N_+) := \mathrm{Op}(\mathfrak{R}^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g};N_-,N_+)) + \mathfrak{Y}_G^{-\infty}(\mathbb{R}\times\mathbf{W},\mathbf{g};N_-,N_+)$; moreover, the subspaces of classical operators $Y^{\nu,l}_{cl}(\ldots)$ and $\mathfrak{Y}^{\nu,l}_{cl}(\ldots)$ in terms of $R^{\nu,l}_{cl}(\ldots)$ and $\mathfrak{R}^{\nu,l}_{cl}(\ldots)$, respectively.*

Concerning the nature of the smoothing GREEN operators compare 1.3 Definition 7 and Definition 8. Remember that in view of 2.3 Proposition 5 the space $\mathfrak{R}^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g};N_-,N_+)$ is a subset of $S^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q};\mathcal{K}^{s,\gamma}\oplus\mathbb{C}^{N_-},\mathcal{K}^{s-\nu,\gamma-\mu}\oplus\mathbb{C}^{N_+})$ for every $s\in\mathbb{R}$. Here $\mathrm{Op}(\mathfrak{a})$ for $\mathfrak{a}\in\mathfrak{R}^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g};N_-,N_+)$ has the form $\mathrm{Op}(\mathfrak{a}) = (\mathrm{Op}(a_{jk}))_{j,k=1,2}$ with the entries $\mathrm{Op}(a_{11})\in\Psi^{\nu,l}(\mathbb{R}\times U';\mathcal{K}^{s,\gamma},\mathcal{K}^{s-\nu,\gamma-\mu})$, $\mathrm{Op}(a_{12})\in\Psi^{\nu,l}(\mathbb{R}\times U';\mathbb{C}^{N_-},\mathcal{K}^{s-\nu,\gamma-\mu})$, $\mathrm{Op}(a_{21})\in\Psi^{\nu,l}(\mathbb{R}\times U';\mathcal{K}^{s,\gamma},\mathbb{C}^{N_+})$ and $\mathrm{Op}(a_{11})\in\Psi^{\nu,l}(\mathbb{R}\times U';\mathbb{C}^{N_-},\mathbb{C}^{N_+})$.

The operators $\mathfrak{A}\in\mathfrak{Y}^{\nu,l}(\mathbb{R}\times\mathbf{W},\mathbf{g};N_-,N_+)$ will also be written as block matrices $\mathfrak{A} = (A_{jk})_{j,k=1,2}$. Occasionally we set l.u.c.$\mathfrak{A} := A_{11}$ (left upper corner). A_{12} has the interpretation as a potential and A_{21} as a trace operator, respectively, associated with the anisotropic edge $\mathbb{R}\times U'$.

Theorem 2 *Let $\Phi = \mathrm{diag}(\phi_1,\phi_2)$ and $\Psi = \mathrm{diag}(\psi_1,\psi_2)$ with abitrary functions $\phi_1,\phi_2,\psi_1,\psi_2\in C_0^\infty(\mathbb{R}^{1+q})$ be given. Then every $\mathfrak{A}\in\mathfrak{Y}^{\nu,l}(\mathbb{R}\times\mathbb{R}^q\times X^\wedge,\mathbf{g};N_-,N_+)$ induces for all $s\in\mathbb{R}$ continuous operators*

$$\Phi\mathfrak{A}\Psi : \begin{array}{c}\mathcal{W}^{s,\gamma;l}(\mathbb{R}\times\mathbb{R}^q\times X^\wedge)\\ \oplus\\ H^{s,l}(\mathbb{R}^{1+q},\mathbb{C}^{N_-})\end{array} \longrightarrow \begin{array}{c}\mathcal{W}^{s-\nu,\gamma-\mu;l}(\mathbb{R}\times\mathbb{R}^q\times X^\wedge)\\ \oplus\\ H^{s-\nu,l}(\mathbb{R}^{1+q},\mathbb{C}^{N_+}).\end{array}$$

Furthermore, for every $P \in \mathrm{As}(\gamma,\Theta)$ there is some $Q \in \mathrm{As}(\gamma-\mu,\Theta)$ depending on P and $\mathfrak{A}$, such that $\mathfrak{A}$ induces for all $s \in \mathbb{R}$ continuous operators

$$\Phi\mathfrak{A}\Psi \; : \; \begin{matrix} \mathcal{W}_P^{s,\gamma;l}(\mathbb{R}\times\mathbb{R}^q\times X^\wedge) \\ \oplus \\ H^{s,l}(\mathbb{R}^{1+q},\mathbb{C}^{N_-}) \end{matrix} \longrightarrow \begin{matrix} \mathcal{W}_Q^{s-\nu,\gamma-\mu;l}(\mathbb{R}\times\mathbb{R}^q\times X^\wedge) \\ \oplus \\ H^{s-\nu,l}(\mathbb{R}^{1+q},\mathbb{C}^{N_+}). \end{matrix}$$

Proof: The proof is an easy consequence of 2.3 Proposition 5 and 1.2 Theorem 11 and 1.2 Corollary 14 for the subspaces with asymptotics. □

Lemma 3 *Let $\mathfrak{b}(t,y,t',y',\tau,\eta)$ be in $\mathfrak{R}^{\nu,l}((\mathbb{R}\times U')\times(\mathbb{R}\times U')\times\mathbb{R}^{1+q},\mathbf{g};N_-,N_+)$ (cf. 2.3 Remark 4). Then there exists a $\underline{\mathfrak{b}}(t,y,\tau,\eta) \in \mathfrak{R}^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g};N_-,N_+)$ such that $\mathrm{Op}(\mathfrak{b})-\mathrm{Op}(\underline{\mathfrak{b}})$ belongs to $\mathfrak{Y}^{-\infty}(\mathbb{R}\times U',\mathbf{g};N_-,N_+)$. For $\underline{\mathfrak{b}}$ we have the asymptotic expansion*

$$\underline{\mathfrak{b}}(t,y,\tau,\eta) \sim_{(G)} \sum_{\alpha\in\mathbb{N}^{1+q}} \frac{1}{\alpha!} D^\alpha_{t',y'}\partial^\alpha_{\tau,\eta}\mathfrak{b}(t,y,t',y',\tau,\eta))|_{(t',y')=(t,y)}.$$

The proof of Lemma 3 is completely analogous to the isotropic edge pseudo-differential calculus.

Corollary 4 *Let $\mathfrak{b}(t',y',\tau,\eta)$ be in $\mathfrak{R}^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g};N_-,N_+)$ (cf. 2.3 Remark 4). Then there exists a $\tilde{\mathfrak{b}}(t,y,\tau,\eta) \in \mathfrak{R}^{\nu,l}(\mathbb{R}\times U'\times\mathbb{R}^{1+q},\mathbf{g};N_-,N_+)$ such that $\mathrm{Op}(\mathfrak{b})-\mathrm{Op}(\tilde{\mathfrak{b}})$ belongs to $\mathfrak{Y}^{-\infty}(\mathbb{R}\times U',\mathbf{g};N_-,N_+)$. For $\tilde{\mathfrak{b}}$ we have the asymptotic expansion*

$$\tilde{\mathfrak{b}}(t,y,\tau,\eta) \sim_{(G)} \sum_{\alpha\in\mathbb{N}^{1+q}} \frac{1}{\alpha!} D^\alpha_{t',y'}(-\partial)^\alpha_{\tau,\eta}\mathfrak{b}(t',y',\tau,\eta))|_{(t',y')=(t,y)}.$$

Remark 5 Note that $\tilde{\mathfrak{b}}$ is the dual symbol of $\mathfrak{b}$, which means that

$$(\mathcal{F}\mathrm{Op}(\mathfrak{b}))u)(\tau,\eta) = \mathcal{F}(\tilde{\mathfrak{b}}u)(\tau,\eta) \bmod \mathfrak{Y}^{-\infty}(\mathbb{R}\times U',\mathbf{g};N_-,N_+) \tag{1}$$

is fulfilled for all $u(t,y) \in \mathcal{W}^{s,\gamma;l}_{comp}(\mathbb{R}\times\mathbf{W}) \oplus H^{s,l}_{comp}(\mathbb{R}^{1+q},\mathbb{C}^{N_-})$.

Theorem 6 *Let for $\nu,\tilde{\nu},\mu,\tilde{\mu} \in \mathbb{R}$ and weight data $\mathbf{g}_1 = (\gamma-\tilde{\mu},\gamma-(\tilde{\mu}+\mu),\Theta)$ and $\mathbf{g}_2 = (\gamma,\gamma-\tilde{\mu},\Theta)$ with $\gamma\in\mathbb{R}$ and $\Theta = (-k,0]$, $k\in\mathbb{N}\setminus\{0\}$ the operators $\mathfrak{A} \in \mathfrak{Y}^{\nu,l}(\mathbb{R}\times\mathbf{W},\mathbf{g}_1;N,N_+)$ and $\mathfrak{B} \in \mathfrak{Y}^{\tilde{\nu},l}(\mathbb{R}\times\mathbf{W},\mathbf{g}_2;N_-,N)$ be given. Then we have $\mathfrak{A}\Phi\mathfrak{B} \in \mathfrak{Y}^{\nu+\tilde{\nu},l}(\mathbb{R}\times\mathbf{W},\mathbf{g};N_-,N_+)$ with $\mathbf{g} = (\gamma,\gamma-(\tilde{\mu}+\mu),\Theta)$ for every $\Phi = \mathrm{diag}(\phi_1,\phi_2)$ with abitrary functions $\phi_1,\phi_2 \in C_0^\infty(\mathbb{R}^{1+q})$.*

Moreover, $\mathfrak{A} \in \mathfrak{Y}^{\nu,l}(\mathbb{R}\times\mathbf{W},\mathbf{g};N_-,N_+)$ with $\mathbf{g} = (\gamma,\gamma-\mu,\Theta)$ implies $\mathfrak{A}^* \in \mathfrak{Y}^{\nu,l}(\mathbb{R}\times\mathbf{W},\mathbf{g}^*;N_+,N_-)$ with $\mathbf{g}^* = (-\gamma+\mu,-\gamma,\Theta)$. Analogous relations hold within the corresponding spaces of classical symbols.

Proof: From 2.3 Lemma 11 it follows immediately that $\mathfrak{A}\Phi\mathfrak{B} = \mathrm{Op}(\mathfrak{a})\Phi\mathrm{Op}(\mathfrak{b})+\mathfrak{G}$ holds with some $\mathfrak{G} \in \mathfrak{Y}_G^{-\infty}(\mathbb{R}\times\mathbf{W},\mathbf{g};N_-,N_+)$.

But in view of (1) we can write

$$\begin{aligned}(\mathrm{Op}(\mathfrak{a})\Phi\mathrm{Op}(\mathfrak{b}))u &= (\mathcal{F}^{-1}\mathfrak{a}\mathcal{F}\Phi\mathcal{F}^{-1}\mathcal{F}\mathrm{Op}(\mathfrak{b}))u = \mathcal{F}^{-1}\mathfrak{a}\mathcal{F}\Phi\mathcal{F}^{-1}\mathcal{F}(\tilde{\mathfrak{b}}u) \\ &= \mathcal{F}^{-1}\mathfrak{a}\mathcal{F}(\Phi\tilde{\mathfrak{b}}u) = \mathrm{Op}(\mathfrak{c})u\end{aligned}$$

with $\mathfrak{c}(t,y,t',y',\tau,\eta) = \mathfrak{a}(t,y,\tau,\eta)\Phi(t',y')\tilde{\mathfrak{b}}(t',y',\tau,\eta)$. Applying Lemma 3 for $\mathfrak{c}$ we get $\mathfrak{A}\Phi\mathfrak{B} = \mathrm{Op}(\underline{\mathfrak{c}}) + \mathfrak{G}$ with

$$\underline{\mathfrak{c}}(t,y,\tau,\eta) \sim_{(G)} \sum_{\alpha\in\mathbb{N}^{1+q}} \frac{1}{\alpha!} D^{\alpha}_{t',y'}\partial^{\alpha}_{\tau,\eta}\mathfrak{c}(t,y,t',y',\tau,\eta)|_{(t',y')=(t,y)}.$$

Finally using the shadow condition of the involved asymptotic types of the GREEN parts of the operators we can apply 2.3 Proposition 9 to obtain that $\underline{\mathfrak{c}}(t,y,\tau,\eta)$ belongs to $\mathfrak{R}^{\nu+\tilde{\nu},l}(\mathbb{R}\times\mathbb{R}^q\times\mathbb{R}^{1+q},\mathrm{g};N_-,N_+)$. The second part of the theorem is an easy consequence of 2.3 Theorem 10. For classical operators it suffices to note that the asymptotic procedures preserve the property of being classical. □

Remark 7 The mapping properties between the anisotropic wedge SOBOLEV spaces in Theorem 2 were formulated on $\mathbb{R}\times\mathbb{R}^q\times X^\wedge$ in avoid indicating the *comp*– and *loc*–subscripts with respect to $y\in U'$. Of course, if we drop the cut-off factors Φ,Ψ in Theorem 2 we can obtain corresponding operators between the *comp*- and *loc*-wedge SOBOLEV spaces over $\mathbb{R}\times\mathbf{W}$.

Remark 8 The present calculus of anisotropic wedge pseudo-differential operators was here formulated in the framework of the discrete asymptotics; it is also possible in the version of continuous asymptotics, with vector-valued functionals in the complex w-plane. Such a theory is necessary for understanding the effects with variable branching asymptotics, cf. [SCH1], [SCH3]. Details will be published elsewhere.

Remark 9 The scheme of our anisotropic wedge theories can also be extended to more general model cones of "hedgehog type", cf [SCH5]. The corresponding parabolicity to be developed in a future paper then covers the problem of heat conduction in singular bodies which have components of different dimensions.

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