ON THE SPECIAL LAGRANGIAN GEOMETRY ON HERMITIAN MANIFOLDS

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§ 1. Introduction

The special Lagrangian geometry on the complex space $\mathbb{C}^n$ was constructed and studied in depth by R. Harvey and H.B. Lawson in the foundational paper $[HL_1]$. The submanifolds in this geometry are usual Lagrangian submanifolds of "constant phase". It turns out that such submanifolds are associated to so – called special Lagrangian calibrations and therefore absolutely volume minimizing. This deep relationship of Lagrangian geometry to the theory of minimal surfaces leads to a large new class of volume minimizing submanifolds in $\mathbb{C}^n$.

It is natural to define and study such a geometry on Hermitian manifolds. The present paper is devoted to this problem.

One of the important points of the problem is to introduce a natural notion of a Lagrangian calibration on an arbitrary Hermitian manifold. For the case $\mathbb{C}^n$ the special Lagrangian calibrations can be defined explicitly by means of the standard coordinates to be $\mathrm{Re}(e^{i\varphi}dz_1 \wedge dz_2 \wedge \dots \wedge dz_n)$, $0 \leq \varphi < 2\pi$, as in $[HL_1]$. In section 2 a Lagrangian form on an 2n–dimensional Hermitian manifold M is presented as the real part $\mathrm{Re}(\overline{\omega})$ of any complex differential n–form $\overline{\omega}$ that has unit comass and characterizes the tangent Lagrangian n–planes at each point. Closed Lagrangian forms will be called Langrangian calibrations.

The research was done when the author was staying at Max–Planck–Institut in Bonn.

The main tool of the paper is <u>the Lagrangian calibration equation</u>, derived in Section 3. Suppose M is a Hermitian manifold with the complex structure J and the metric tensor g and let $(z_1,z_2, \dots ,z_n)$ be local complex coordinates. Recall that for $M = \mathbb{C}^n$ the coordinate forms $\mathrm{Re}(e^{i\varphi}dz_1 \wedge dz_2 \wedge \dots \wedge dz_n)$ are closed and of comass one simultaneously. However, in the general case even the local picture is rather complicated. Namely, the comass of $e^{i\varphi}dz_1 \wedge dz_2 \wedge \dots \wedge dz_n$ depends on g and generally changes from point to point. More precisely, in local coordinates any Lagrangian form can be expressed as $\omega = \mathrm{Re}(e^{G-iH}dz_1 \wedge dz_2 \wedge \dots \wedge dz_n)$, where $G = \ln\sqrt{\det(g)}$ and H is a real valued function. The main idea for overcoming the difficulty above can be described as follows: in order for ω to be a calibration, i.e. $d\omega = 0$, the change of the comass of $dz_1 \wedge dz_2 \wedge \dots \wedge dz_n$ that is equal to e^G must be annihilated by a change of the phase H in the corresponding imaginary direction. It turns out that this idea can be realized globally and allows to establish a necessary and sufficient condition for a Lagrangian form ω to be a calibration.

<u>THEOREM</u> I (<u>Theorems</u> 3.3 <u>and</u> 3.5). <u>The differential</u> 1–<u>form</u> $dG-JdH$, <u>given locally as above, is correctly defined on the whole manifold</u> M . <u>A Lagrangian form</u> ω <u>on</u> M <u>is a Lagrangian calibration if and only if the</u> 1–<u>form</u> $dG-JdH$, <u>corresponding to</u> ω, <u>vanishes on</u> M .

The existence of Lagrangian calibrations and finding them are studied in Section 5. It turns out that the question can be explained completely.

<u>THEOREM</u> II (<u>Theorem</u> 5.2). <u>A simply connected Hermitian manifold</u> M <u>is</u> L–<u>calibrated</u> (<u>i.e. has a Lagrangian calibration</u>) <u>if and only if</u> $dJdG = 0$, <u>where differential</u> 2–<u>form</u> $dJdG$ <u>is correctly defined on the whole</u> M .

In the local coordinates the condition $dJdG = 0$ has a simple form: $\frac{\partial G}{\partial z_i \partial \bar{z}_j} = 0$ for any $i,j = 1,2, \dots ,n$. For a Kahler manifold M this means simply that M has the trivial Ricci tensor. In [HL_1] R. Harvey and H.B. Lawson noted that the special Lagrangian geometry is naturally defined on any Ricci–flat Kahler manifold (the existence of such manifolds is established by S.T. Yau [Y]). Theorem II justifies this comment. Moreover, in fact, the special Lagrangian geometry can be constructed only on such manifolds.

The property of being a Lagrangian calibration seems to be so strong that every such one is determined completely by its behaviour at an arbitrary fixed point. This important observation leads to a complete classification of Lagrangian calibrations.

THEOREM III (Theorem 5.8). *Suppose* M *is a* L–*calibrated Hermitian manifold. Then the set of all the Lagrangian calibrations on* M *is* $\{\mathrm{Re}(e^{i\varphi}\bar{\omega}), 0 \le \varphi < 2\pi\}$, *where* $\mathrm{Re}(\bar{\omega})$ *is a Lagrangian calibration.*

In particular, for the case $\mathbb{C}^n$ Theorem III shows that the special Lagrangian calibrations $\mathrm{Re}(e^{i\varphi}dz_1 \wedge dz_2 \wedge \dots \wedge dz_n)$ cover, in fact, all the Lagrangian calibrations on $\mathbb{C}^n$.

Section 6 is devoted to investigating special Lagrangian submanifolds of L–calibrated manifolds. An n–dimensional oriented submanifold N in M is called *a special Lagrangian submanifold* if $\omega(T_zN) = 1$ at each point $z \in N$ for a Lagrangian calibration ω on M . By virtue of Theorem III for a special Lagrangian submanifold N the corresponding Lagrangian calibration ω is determined uniquely. This remarkable fact allows a localization of the property of being special Lagrangian. Namely we prove the following result.

THEOREM IV (Theorem 6.1). A submanifold N in M is special Lagrangian if and only if each point of N has a special Lagrangian neighborhood.

By using Theorem IV and the criterion for the local minimality of Lagrangian submanifolds discovered by Le Hong Van and A. Formenko [LF] and R. Bryant [B] we can establish the equivalence between the properties of being stationary and special Lagrangian for Lagrangian submanifolds.

THEOREM V (Theorem 6.3). Suppose M is a L–calibrated manifold. Then every connected stationary Lagrangian submanifold is special Lagrangian.

For the case $M = \mathbb{C}^n$ this result was proved in [HL_1].

Sections 4, 7, 8 present some methods and constructions for finding special Lagrangian submanifolds. In Section 4 we introduce special Lagrangian sections as the first step in integrating the special Lagrangian equation. A section $p : M \longrightarrow \mathrm{Lag}(M)$ of the Lagrangian bundle is said to be special Lagrangian if $\omega(p(z)) = 1$ everywhere for a Lagrangian calibration ω. Clearly, any integral submanifold of a special Lagrangian section is special Lagrangian. For each Lagrangian section p there exists an unique Lagrangian form ω_p such that $\omega_p(p) = 1$ everywhere. This form is called the characteristic form of p.

THEOREM VI (Theorem 4.3). A Lagrangian section $p : M \longrightarrow \mathrm{Lag}(M)$ is special Lagrangian if and only if $dG - JdH = 0$, where $dG - JdH$ is the 1–form associated to the characteristic form ω_p of p.

By virtue of Theorem IV we can investigate special Lagrangian submanifolds in each coordinate neighborhood separately. This means that we deal with a complex space

$\mathbb{C}^n = \{(z_1, z_2, \dots, z_n)\}$ equipped with a generally non–standard Hermitian metric $g = (g_{ij})$. In Section 7 criteria for a submanifold X to be special Lagrangian are established for the cases when X is given implicitly as zeros of n smooth real valued functions or when X is described explicitly by parametric equations. Each of these cases are accompanied by calculating illustrations and concrete examples. In Section 8 we consider the case when $\mathbb{C}^n$ can be expressed as a unitary sum of subspaces and construct special Lagrangian submanifolds of $\mathbb{C}^n$ as the sums of special Lagrangian submanifolds of each terms. In particular, if G is a linear function of the real coordinates $x_1, \dots, x_n$, $y_1, \dots, y_n$ then any curvelinear cylinder parallel a special Lagrangian curve in $\mathrm{Span}\{\mathrm{grad}\, G,\ J\, \mathrm{grad}\, G\}$ and through a special Lagrangian submanifold of $\mathbb{C}^{n-1} = \{K = K(0),\ G = G\{0\}\}$ is special Lagrangian, where K is a function determined by the formula $dK = JdG$. Note that $\mathbb{C}^{n-1}$ can be equipped with coordinates such that the corresponding metric tensor g' is of standard type, that is $\sqrt{\det(g')} = \mathrm{const}$. Example 8.3 generalizes the construction of special Lagrangian normal bundles in $[HL_1]$. Suppose $T_*\mathbb{R}^n$ is the tangent bundle of $\mathbb{R}^n$ with a metric g such that $G = \ln\sqrt{\det(g)}$ is linear function. Let X be a cylinder parallel $\mathrm{grad}\, G$ such that all the invariants of odd order of the second fundamental form at each normal vector to X vanish. Then the normal bundle $N(X)$ of X is special Lagrangian.

Finally, we point out that the special Lagrangian geometry can be extended to n–currents of locally finite mass on M, including, in particular, submanifolds with singularities. All the main results above remain true. For details of the method of calibrations one can refer to $[HL_1]$, $[D_1]$ and [DF]. The basic facts and results of the current theory can be found in [F] and [FF].

§ 2. Lagrangian submanifolds and Lagrangian forms.

Let M be a 2n–dimensional Hermitian manifold with the complex structure J and the metric tensor g. The fundamental 2–form Ω and the Hermitian complex valued product h on M are given by setting

$$\Omega(X,Y) = g(X,JY) \tag{2.1}$$

$$h(X,Y) = g(X,Y) + ig(X,JY) \tag{2.2}$$

for any vector fields X and Y on M. If the form Ω is closed, then the manifold M is called a Kähler manifold. Clearly, the form Ω is never degenerate (i.e. Ω^n never vanishes). Therefore, the manifold M equipped with the closed 2–form Ω is a symplectic manifold.

An oriented real tangent n–plane $\xi \subset T_zM$ is said to be Lagrangian if the restriction of Ω to ξ vanishes, that is

$$J\,u \perp \xi \tag{2.3}$$

for any $u \in \xi$. Obviously, ξ is maximally isotropic with respect to Ω on T_zM, and the last is equivalent to the following condition.

$$J\,\xi \perp \xi \quad \text{and} \quad J\,\xi \oplus \xi = T_zM\,. \tag{2.4}$$

We denote by $\mathrm{Lag}(M)$ the bundle of oriented Lagrangian planes on M. Each oriented n–plane of T_zM can be identified naturally with an unit n–vector in $\Lambda_n(T_zM)$. In this way the bundle $\Gamma_n(M)$ of oriented tangent n–planes is embedded into the unit sphere of the Grassmann bundle $\Lambda_n(M)$.

PROPOSITION 2.1. Let ξ be a Lagrangian plane in T_zM . Then any orthonormal basis of ξ forms a unitary basis of T_zM . Conversely, if a orthonormal basis of a n–plane $\xi \subset T_zM$ is also a unitary basis of T_zM then ξ is a Lagrangian plane.

PROOF. Suppose that ξ is a Lagrangian plane of T_zM and $\{e_1, e_2, \dots, e_n\}$ is an orthonormal basis of ξ . By definition $J\, e_j \perp e_i$ for any $i,j\ (1 \le i,j \le n)$. Hence

$$h(e_i,e_j) = g(e_i,e_j) + ig(e_i,Je_j) = g(e_i,e_j) = \delta_{ij}$$

for any $i,j\ (1 \le i,j \le n)$, that is $\{e_1, e_2, \dots, e_n\}$ is a unitary basis of T_zM .

Conversely, let ξ be an n–plane of T_zM and suppose that an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of ξ is also a unitary basis of T_zM . For any $i,j\ (1 \le i,j \le n)$ we have

$$g(Je_i,e_j) = h(Je_i,e_j) - ig(Je_i,Je_j) = ih(e_i,e_j) - ig(e_i,e_j) = 0 .$$

Hence, $J\,u \perp \xi$ for any $u \in \xi$. Consequently, ξ is a Lagrangian plane.

An n–dimensional oriented submanifold N in a Hermitian manifold M is called a Lagrangian submanifold if its tangent space at each point is Lagrangian, that is the restriction of the form Ω to N vanishes.

The Hermitian metric on M induces a Hermitian structure on cotangent spaces T_z^*M .

DEFINITION 2.2 A real exterior n–form ω on $T_z M$ is called a Lagrangian exterior form if there exists an unitary basis $\{e_1^*, e_2^*, \dots, e_n^*\}$ of $T_z M$ such that $\omega = \mathrm{Re}\{e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*\}$.

We define the comass of a complex exterior k–form ω to be the number

$$||\omega|| = \sup_{\xi} |\omega(\xi)| ,$$

where ξ runs through all unit simple k–vectors on $T_z M$.

PROPOSITION 2.3. For any complex basis $\{f_1^*, f_2^*, \dots, f_n^*\}$ of $T_z^* M$ a Lagrangian exterior form ω on $T_z M$ admit the representation

$$\omega = \mathrm{Re}(\lambda f_1^* \wedge f_2^* \wedge \dots \wedge f_n^*) \tag{2.5}$$

where λ is a complex number such that $|\lambda| = ||f_1^* \wedge f_2^* \wedge \dots \wedge f_n^*||^{-1}$. Conversely, any n–form of the form (2.5) is Lagrangian.

PROOF. Let $\omega = \mathrm{Re}(e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*)$ be a Lagrangian form, where $\{e_1^*, e_2^*, \dots, e_n^*\}$ is a unitary basis of $T_z^* M$. Suppose that $\{f_1^*, f_2^*, \dots, f_n^*\}$ is a complex basis of $T_z^* M$ and $\{e_1^*, e_2^*, \dots, e_n^*\} = A\,\{f_1^*, f_2^*, \dots, f_n^*\}$, that is

$$e_i^* = \sum_{j=1}^{n} a_{ij} f_j^* \ , \quad i = 1, \dots, n \ .$$

We have

$$e_1^* \wedge e_2^* \wedge \dots \wedge e_n^* = \det A \, f_1^* \wedge f_2^* \wedge \dots \wedge f_n^* ,$$

where

$$|\det A| = \frac{||\overset{*}{e}_1 \wedge \overset{*}{e}_2 \wedge \dots \wedge \overset{*}{e}_n||}{||\overset{*}{f}_1 \wedge \overset{*}{f}_2 \wedge \dots \wedge \overset{*}{f}_n||} = ||\overset{*}{f}_1 \wedge \overset{*}{f}_2 \wedge \dots \wedge \overset{*}{f}_n||^{-1}.$$

Hence

$$\omega = \mathrm{Re}(\overset{*}{e}_1 \wedge \overset{*}{e}_2 \wedge \dots \wedge \overset{*}{e}_n) = \mathrm{Re}(\lambda \overset{*}{f}_1 \wedge \overset{*}{f}_2 \wedge \dots \wedge \overset{*}{f}_n)$$

with $\lambda = \det A$, satisfying the condition $|\lambda| = ||\overset{*}{f}_1 \wedge \overset{*}{f}_2 \wedge \dots \wedge \overset{*}{f}_n||^{-1}$. Conversely, suppose that $\omega = \mathrm{Re}(\lambda \overset{*}{f}_1 \wedge \overset{*}{f}_2 \wedge \dots \wedge \overset{*}{f}_n)$ for a complex basis $\{\overset{*}{f}_1,\overset{*}{f}_2,\dots,\overset{*}{f}_n\}$ of $T_z\overset{*}{M}$ and a complex number λ such that $|\lambda| = ||\overset{*}{f}_1 \wedge \overset{*}{f}_2 \wedge \dots \wedge \overset{*}{f}_n||^{-1}$. Consider a unitary basis $\{\overset{*}{e}_1,\overset{*}{e}_2,\dots,\overset{*}{e}_n\}$ of $\overset{*}{T_z}M$ and let A be the complex linear transformation of $\overset{*}{T_z}M$, mapping $\{\overset{*}{e}_1,\overset{*}{e}_2,\dots,\overset{*}{e}_n\}$ into $\{\overset{*}{f}_1,\overset{*}{f}_2,\dots,\overset{*}{f}_n\}$. We have
$\overset{*}{f}_1 \wedge \overset{*}{f}_2 \wedge \dots \wedge \overset{*}{f}_n = \det A\, \overset{*}{e}_1 \wedge \overset{*}{e}_2 \wedge \dots \wedge \overset{*}{e}_n$. Therefore,
$\omega = \mathrm{Re}(\lambda \overset{*}{f}_1 \wedge \overset{*}{f}_2 \wedge \dots \wedge \overset{*}{f}_n) = \mathrm{Re}(\lambda \det A\, \overset{*}{e}_1 \wedge \overset{*}{e}_2 \wedge \dots \wedge \overset{*}{e}_n)$. Moreover,
$1 = ||\lambda \overset{*}{f}_1 \wedge \overset{*}{f}_2 \wedge \dots \wedge \overset{*}{f}_n|| = ||\lambda \det A\, \overset{*}{e}_1 \wedge \overset{*}{e}_2 \wedge \dots \wedge \overset{*}{e}_n|| =$
$= |\lambda \det A|\; ||\overset{*}{e}_1 \wedge \overset{*}{e}_2 \wedge \dots \wedge \overset{*}{e}_n|| = |\lambda \det A|$. Consequently, $\lambda \det A = e^{i\varphi}$ for a real number φ. Consider the new unitary basis $\{\tilde{\overset{*}{e}}_1,\tilde{\overset{*}{e}}_2,\dots,\tilde{\overset{*}{e}}_n\} = \{e^{-i\varphi}\overset{*}{e}_1,\overset{*}{e}_2,\dots,\overset{*}{e}_n)$. Clearly, $\omega = \mathrm{Re}(\tilde{\overset{*}{e}}_1 \wedge \tilde{\overset{*}{e}}_2 \wedge \dots \wedge \tilde{\overset{*}{e}}_n)$, that is ω is a Lagrangian form. The proof is complete.

<u>PROPOSITION</u> 2.4. <u>Suppose</u> $\{e_1,e_2\dots,e_n\}$ <u>is a real basis of a Lagrangian plane</u> $\xi \subset T_zM$ <u>and that</u> $\{\overset{*}{e}_1,\overset{*}{e}_2,\dots,\overset{*}{e}_n\}$ <u>is the dual basis to</u> $\{e_1,e_2,\dots,e_n\}$, that is $\overset{*}{e}_i(e_j) = \delta_{ij}$ <u>for any</u> $i,j (1 \le i,j \le n)$. <u>Then</u>

$$\mathrm{Re}(e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*)\,(e_1 \wedge e_2 \wedge \dots \wedge e_n) = 1\,.$$

PROOF. The statement of Proposition 2.4 is obvious for the case when $\{e_1,e_2, \dots ,e_n\}$ is orthonormal. We consider the general case. Let $\{f_1,f_2, \dots ,f_n\}$ be an orthonormal basis of ξ. By Proposition 2.1 $\{f_1,f_2, \dots ,f_n\}$ is also a unitary basis of T_zM. Denote by $\{f_1^*,f_2^*, \dots ,f_n^*\}$ the unitary basis of T_z^*M dual to $\{f_1,f_2, \dots ,f_n\}$. Suppose, that the real linear transformation A sends the basis $\{e_1,e_2, \dots ,e_n\}$ into the basis $\{f_1,f_2, \dots ,f_n\}$, i.e. $\{f_1,f_2, \dots ,f_n\} = A\{e_1,e_2, \dots ,e_n\}$. Then $\{e_1^*,e_2^*, \dots ,e_n^*\} = A^t\{f_1^*,f_2^*, \dots ,f_n^*\}$. Therefore,

$$f_1 \wedge f_2 \wedge \dots \wedge f_n = (\det A)\, e_1 \wedge e_2 \wedge \dots \wedge e_n \tag{2.6}$$

$$\begin{aligned} e_1^* \wedge e_2^* \wedge \dots \wedge e_n^* &= (\det A^t)\, f_1^* \wedge f_2^* \wedge \dots \wedge f_n^* \\ &= (\det A)\, f_1^* \wedge f_2^* \wedge \dots \wedge f_n^*\,. \end{aligned} \tag{2.7}$$

Since $\mathrm{Re}(f_1^* \wedge f_2^* \wedge \dots \wedge f_n^*)(f_1 \wedge f_2 \wedge \dots \wedge f_n) = 1$ because of the above remark and (2.6), (2.7) we have

$$\det A = (\det A^t)\mathrm{Re}(f_1^* \wedge f_2^* \wedge \dots \wedge f_n^*)(f_1 \wedge f_2 \wedge \dots \wedge f_n) = \tag{2.8}$$

$$= (e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*)(\det A\; e_1 \wedge e_2 \wedge \dots \wedge e_n) = (\det A) e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*\,(e_1 \wedge e_2 \wedge \dots \wedge e_n)$$

From (2.8) it follows that

$$e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*\,(e_1 \wedge e_2 \wedge \dots \wedge e_n) = 1$$

completing the proof.

Propositions 2.3 and 2.4 yield the following result.

COROLLARY 2.5. Let $\omega = \mathrm{Re}\,\overline{\omega}$ be a Lagrangian form on T_zM. Then

$$|\overline{\omega}(\xi)| \le |\xi| \tag{2.9}$$

for any n–plane ξ in T_zM. Moreover, the equality holds if and only if ξ is a Lagrangian plane. In particular, $\|\overline{\omega}\| = 1$ and $\|\omega\| = \|\mathrm{Re}\,\overline{\omega}\| = 1$.

Fix a unitary basis $\{e_1^*, e_2^*, \dots, e_n^*\}$ of T_z^*M. Then $\|e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*\| = 1$. According to Proposition 2.3 any Lagrangian exterior form in T_zM can be expressed in the form $\mathrm{Re}(e^{i\varphi}e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*)$. Further, suppose ξ is a given Lagrangian plane of T_zM. By force of Proposition 2.1 one can choose a unitary basis $\{e_1^*, e_2^*, \dots, e_n^*\}$ of T_z^*M so that the dual basis $\{e_1, e_2, \dots, e_n\}$ to $\{e_1^*, e_2^*, \dots, e_n^*\}$ is an orthonormal basis of ξ. Clearly, $\mathrm{Re}(e^{i\varphi}e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*)(\xi) = \cos\varphi$. In particular, it implies that there exists an unique Lagrangian exterior form ω on T_zM satisfying the condition : $\omega(\xi) = 1$. Thus we have proved the following

COROLLARY 2.6. There exists precisely a S^1 – family of Lagrangian exterior forms on each tangent space T_zM. Moreover, for a given Lagrangian plane ξ there exists an unique one that has the largest value 1 at ξ.

DEFINITION 2.7. A differential n–form ω on M is called a Lagrangian form if the restriction of ω to the tangent space T_zM, at each point, is Lagrangian.

Given complex local coordinates $z = (z_1, z_2, \dots, z_n)$ on M. Set $G = \ln\sqrt{\det(g)} = \ln\sqrt{\det(g_{i\bar{j}})}$, where $g = \Sigma\, g_{i\bar{j}}dz_i d\bar{z}_j$ is the Hermitian metric on M.

PROPOSITION 2.8. *Suppose* ω *be a Lagrangian form on* M . *Then in local coordinates* ω *can be expressed in the form*

$$\omega = \mathrm{Re}(E^{G-iH} dz_1 \wedge dz_2 \wedge \dots dz_n) \tag{2.10}$$

where H *is a real valued function on* M . *Conversely, any* n–*form given by the formula* (2.10) *is Lagrangian.*

Proof. The complex linear forms $dz_1, dz_2, \dots, dz_n$ form a complex basis of T_z^*M at each point z. Choose a unitary basis $\{e_1^*, e_2^*, \dots, e_n^*\}$ of T_z^*M and let $\{dz_1, dz_2, \dots, dz_n\} = A\{e_1^*, e_2^*, \dots, e_n^*\}$. Then $(g_{ij}) = (A^{-1})^t(\bar{A}^{-1})$. Consequently, $\sqrt{\det(g)} = |\det A^{-1}| = |\det A|^{-1}$. On the other hand,
$||dz_1 \wedge dz_2 \wedge \dots dz_n|| = ||\det A\, e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*|| =$
$= |\det A|\ ||e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*|| = |\det A|$. Now, using Proposition 2.3 proves both statements of the proposition.

§ 3. The Lagrangian calibration equation

DEFINITION 3.1. *A Lagrangian form* ω *on a Hermitian manifold* M *is called a Lagrangian calibration if* $d\omega = 0$.

DEFINITION 3.2. *An* n–*dimensional oriented submanifold* N *in* M *is called a special Lagrangian submanifold if* N *is a* φ – *submanifold for a Lagrangian calibration* φ *on* M , *that is* $\varphi(T_zN) = 1$ *at each point* $z \in N$.

By force of Corollary 2.5 $T_z N$ must be Lagrangian for every $z \in N$. Therefore, N is, particularly, a Lagrangian submanifold.

From Definitions 3.1 and 3.2 it follows immediately that any special Lagrangian submanifold is volume–minimizing (see, for example, $[HL_1]$ or $[D_1]$).

Consider a Lagrangian form ω on M. Let given complex local coordinates $z = (z_1, z_2, \dots, z_n)$ on M. By virtue of Proposition 2.8 ω can be represented in the form

$$\omega = \mathrm{Re}(e^{G-iH} dz_1 \wedge dz_2 \wedge \dots \wedge dz_n) . \tag{3.1}$$

The real valued functions G and H are defined on the domain of the local coordinates $z = (z_1, z_2, \dots, z_n)$. They depend on the local coordinates.

<u>THEOREM</u> 3.3. <u>The differential 1–form $dG - JdH$, where G and H are given locally by the formula (3.1), is correctly defined on the whole manifold M, that is it is independent of the choice of the local coordinates.</u>

<u>PROOF</u>. We show that $dG - JdH$, although defined by means of the coordinates, does not depend on them in fact. Suppose $(z_1', z_2', \dots, z_n')$ is another complex local coordinates on M and let $A = \left[\frac{\partial z_i'}{\partial z_j}\right]$ be the Jacobian matrix. Let $\det A = e^{\alpha + i\beta}$ (α, β are real valued functions). Then we have

$$dz_1' \wedge dz_2' \wedge \dots \wedge dz_n' = \det A \, dz_1 \wedge dz_2 \wedge \dots \wedge dz_n = e^{\alpha+i\beta} dz_1 \wedge dz_2 \dots \wedge z_n . \tag{3.2}$$

Suppose that in the new coordinates ω has the form

$$\omega = \mathrm{Re}(e^{G'-iH'} dz_1' \wedge dz_2' \wedge \ldots \wedge dz_n') \,. \tag{3.3}$$

From (3.1), (3.2) and (3.3) it follows that

$$e^{G-iH} dz_1 \wedge dz_n \wedge \ldots \wedge dz_2 = e^{G'-iH'} dz_1' \wedge dz_2' \wedge \ldots \wedge dz_n' =$$

$$= e^{G'-iH'} e^{\alpha+i\beta} dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n = e^{(G'+\alpha)-i(H'-\beta)} dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n \,.$$

Hence

$$G = G' + \alpha \quad \text{and} \quad H = H' - \beta \,. \tag{3.4}$$

Since the function $\det \Lambda$ is holomorphic the function $\alpha + i\beta$ is holomorphic. By using the Cauchy–Riemann condition we obtain.

$$\frac{\partial \alpha}{\partial x_i} = \frac{\partial \beta}{\partial y_i}\,, \quad \frac{\partial \alpha}{\partial y_i} = -\frac{\partial \beta}{\partial x_i} \quad (i = 1,2, \ldots ,n)\,, \tag{3.5}$$

where $z_i = x_i + iy_i$. Obviously, (3.5) is equivalent to the following equality

$$J d\alpha = d\,\beta \,. \tag{3.6}$$

Taking (3.4) and (3.6) into account we have

$$dG - JdH = d(G' + \alpha) - Jd(H' - \beta) = dG' + d\alpha - JdH' + Jd\beta =$$
$$= (dG' - JdH') - J(Jd\alpha - d\beta) = dG' - JdH' \,,$$

completing the proof.

<u>COROLLARY</u> 3.4. <u>The differential</u> 2–<u>forms</u> $dJdG$ <u>and</u> $dJdH$, <u>where</u> G <u>and</u> H <u>are defined locally by</u> (3.1)., <u>are correctly defined on the whole manifold</u> M .

<u>PROOF</u>. By using (3.4) and (3.6) we have

$$dJdG = dJ(dG' + d\alpha) = dJdG' + d(Jd\alpha) = dJdG' + d(d\beta) = dJdG'$$

$$dJdH = dJd(H' - \beta) = dJ(dH' - d\beta) = dJdH' - dJd\beta = dJdH' - dJ(Jd\alpha) =$$
$$= dJdH' + d(d\alpha) = dJdH' .$$

Thus the proof is complete.

<u>THEOREM</u> 3.5. <u>A Lagrangian form</u> ω <u>on</u> M <u>is a Lagrangian calibration if and only if the</u> 1–<u>form</u> $dG - JdH$, <u>associated to</u> ω <u>by the formula</u> (3.1), <u>vanishes on</u> M .

Fix an arbitrary point $p \in M$. Since the form $dG - JdH$ is independent of coordinates we can choose local complex coordinates $z = (z_1, z_2, \dots, z_n)$ on M so that $\{dz_1, dz_2, \dots, dz_n\}$ is a unitary basis of T_z^*M . To prove Theorem 3.5 we need the following lemmas.

<u>LEMMA</u> 3.6. <u>Set</u> $\overline{\omega} = e^{G-iH} dz_1 \wedge dz_2 \wedge \dots \wedge dz_n$. <u>Let</u> $\mathrm{Re}\,\overline{\omega}$ <u>and</u> $\mathrm{Im}\,\overline{\omega}$ <u>denote the real and imaginary parts of</u> $\overline{\omega}$ <u>respectively</u>. <u>Then the equality</u>

$$dH \wedge \mathrm{Im}\,\overline{\omega} = - JdH \wedge \mathrm{Re}\,\overline{\omega} \tag{3.7}$$

holds at the point p.

PROOF. Let $z_i = x_i + iy_i$, $i = 1,2, \dots ,n$. Then the real linear forms $dx_1, dy_1, dx_2, dy_2, \dots , dx_n, dy_n$ constitute an orthomormal basis of T_z^*M. The complex structure operator J acts on this basis as follows

$$\text{(3.8)} \qquad Jdx_i = dy_i\,,\ \ Jy_i = -dx_i\,,\ \ i = 1,2, \dots ,n\,.$$

Put $e_k^+ = \left.\dfrac{\partial}{\partial x_k}\right|_p$, $e_k^- = \left.\dfrac{\partial}{\partial y_k}\right|_p$ $(k = 1,2, \dots ,n)$. By a straight forward calculation of the values of the forms $dx_i \wedge \operatorname{Re}\overline{\omega}$, $dy_i \wedge \operatorname{Re}\overline{\omega}$, $dx_i \wedge \operatorname{Im}\overline{\omega}$ and $dy_i \wedge \operatorname{Im}\overline{\omega}$ $(i = 1,2, \dots ,n)$ at the basic real $(n + 1)$ – vectors $\xi = e_{i_0}^{\varepsilon_0} \wedge e_{i_1}^{\varepsilon_1} \wedge \dots \wedge e_{i_n}^{\varepsilon_n}$, $\varepsilon_i = \pm$, $i = 0,1, \dots ,n$, we obtain

(3.9)

$$dx_i \wedge \operatorname{Re}\overline{\omega}(\xi) = \begin{cases} (-1)^i & \text{if } \xi = e_i^- \wedge e_i^+ \wedge e_1^{\varepsilon_1} \wedge \dots \wedge e_{i-1}^{\varepsilon_{i-1}} \wedge e_{i+1}^{\varepsilon_{i+1}} \wedge \dots \wedge e_n^{\varepsilon_n} \\ & \text{and the number of minuses amomg } \varepsilon_1, \dots , \varepsilon_{i-1}, \varepsilon_{i+1}, \dots , \varepsilon_n \\ & \text{is equal to } 1 (\mathrm{mod}\ 4) \\ (-1)^{i+1} & \text{if } \xi = e_i^- \wedge e_i^+ \wedge e_1^{\varepsilon_1} \wedge \dots \wedge e_{i-1}^{\varepsilon_{i-1}} \wedge e_{i+1}^{\varepsilon_{i+1}} \wedge \dots \wedge e_n^{\varepsilon_n} \\ & \text{and the number of minuses among } \varepsilon_1, \dots , \varepsilon_{i-1}, \varepsilon_{i+1}, \dots , \varepsilon_n \\ & \text{is equal to } 3 (\mathrm{mod}\ 4) \\ 0 & \text{for others}. \end{cases}$$

(3.10)

$$dy_i \wedge \operatorname{Re}\overline{\omega}(\xi)=\begin{cases}(-1)^i & \text{if } \xi = e_i^- \wedge e_i^+ \wedge e_1^{\varepsilon_1} \wedge \dots \wedge e_{i-1}^{\varepsilon_{i-1}} \wedge e_{i+1}^{\varepsilon_{i+1}} \wedge \dots \wedge e_n^{\varepsilon_n} \\ & \text{and the number of minuses amomg } \varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}, \dots, \varepsilon_n \\ & \text{is equal to } 0 \pmod 4 \\ (-1)^{i+1} & \text{if } \xi = e_i^- \wedge e_i^+ \wedge e_1^{\varepsilon_1} \wedge \dots \wedge e_{i-1}^{\varepsilon_{i-1}} \wedge e_{i+1}^{\varepsilon_{i+1}} \wedge \dots \wedge e_n^{\varepsilon_n} \\ & \text{and the number of minuses among } \varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}, \dots, \varepsilon_n \\ & \text{is equal to } 2 \pmod 4 \\ 0 & \text{for others.}\end{cases}$$

(3.11)

$$dx_i \wedge \operatorname{Im}\overline{\omega}(\xi)=\begin{cases}(-1)^i & \text{if } \xi = e_i^+ \wedge e_i^- \wedge e_1^{\varepsilon_1} \wedge \dots \wedge e_{i-1}^{\varepsilon_{i-1}} \wedge e_{i+1}^{\varepsilon_{i+1}} \wedge \dots \wedge e_n^{\varepsilon_n} \\ & \text{and the number of minuses amomg } \varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}, \dots, \varepsilon_n \\ & \text{is equal to } 0 \pmod 4 \\ (-1)^{i+1} & \text{if } \xi = e_i^+ \wedge e_i^- \wedge e_1^{\varepsilon_1} \wedge \dots \wedge e_{i-1}^{\varepsilon_{i-1}} \wedge e_{i+1}^{\varepsilon_{i+1}} \wedge \dots \wedge e_n^{\varepsilon_n} \\ & \text{and the number of minuses among } \varepsilon_1, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}, \dots, \varepsilon_n \\ & \text{is equal to } 2 \pmod 4 \\ 0 & \text{for others.}\end{cases}$$

(3.12)

$$dy_i \wedge \operatorname{Im} \overline{\omega}(\xi) = \begin{cases} (-1)^i & \text{if } \xi = e_i^- \wedge e_i^+ \wedge e_1^{\varepsilon_1} \wedge \ldots \wedge e_{i-1}^{\varepsilon_{i-1}} \wedge e_{i+1}^{\varepsilon_{i+1}} \wedge \ldots \wedge e_n^{\varepsilon_n} \\ & \text{and the number of minuses amomg } \varepsilon_1, \ldots, \varepsilon_{i-1}, \varepsilon_{i+1}, \ldots, \varepsilon_n \\ & \text{is equal to } 1 (\mathrm{mod}\ 4) \\ (-1)^{i+1} & \text{if } \xi = e_i^- \wedge e_i^+ \wedge e_1^{\varepsilon_1} \wedge \ldots \wedge e_{i-1}^{\varepsilon_{i-1}} \wedge e_{i+1}^{\varepsilon_{i+1}} \wedge \ldots \wedge e_n^{\varepsilon_n} \\ & \text{and the number of minuses among } \varepsilon_1, \ldots, \varepsilon_{i-1}, \varepsilon_{i+1}, \ldots, \varepsilon_n \\ & \text{is equal to } 3 (\mathrm{mod}\ 4) \\ 0 & \text{for others.} \end{cases}$$

Comparing (3.9), (3.10), (3.11) and (3.12) we can conclude that

(3.13) $$dx_i \wedge \operatorname{Im} \overline{\omega} = - dy_i \wedge \operatorname{Re} \overline{\omega}$$

(3.14) $$dy_i \wedge \operatorname{Im} \overline{\omega} = dx_i \wedge \operatorname{Re} \overline{\omega}.$$

Assume that $dH = \sum_{i=1}^{n} (\alpha_i\, dx_i + \beta_i dy_i)$. Taking (3.8), (3.13) and (3.14) into account we have

$$dH \wedge \operatorname{Im} \overline{\omega} = \sum_{i=1}^{n} (\alpha_i dx_i \wedge \operatorname{Im} \overline{\omega} + \beta_i dy_i \wedge \operatorname{Im} \overline{\omega}) = \sum_{i=1}^{n} (-\alpha_i dy_i \wedge \operatorname{Re} \overline{\omega} + \beta_i dx_i \wedge \operatorname{Re} \overline{\omega})$$

$$= \sum_{i=1}^{n} (-\alpha_i J dx_i \wedge \operatorname{Re} \overline{\omega} - \beta_i J dy_i \wedge \operatorname{Re} \overline{\omega}) = -(J \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i dy_i)) \wedge \operatorname{Re} \overline{\omega} = - J dH \wedge \operatorname{Re} \overline{\omega}.$$

Thus, the lemma is proved.

LEMMA 3.7. Let the notations be as in Lemma 3.6. Then the equality

$$d(\mathrm{Re}\,\overline{\omega}) = (dG - JdH) \wedge \mathrm{Re}\,\overline{\omega} \tag{3.15}$$

holds at the point p .

Proof. Really,

$$d\overline{\omega} = d\,[e^{G-iH}dz_1 \wedge dz_2 \wedge \dots \wedge dz_n] = d(e^{G-iH}) \wedge dz_1 \wedge dz_2 \wedge \dots \wedge dz_n =$$
$$= d(G - iH) \wedge (e^{G-iH}\, dz_1 \wedge dz_2 \wedge \dots \wedge dz_n) = d(G - iH) \wedge \overline{\omega}\,.$$

Hence

$$d(\mathrm{Re}\,\overline{\omega}) = \mathrm{Re}(d\,\overline{\omega}) = \mathrm{Re}(d(G - iH) \wedge \overline{\omega}) = \mathrm{Re}((dG - idH) \wedge \overline{\omega}) = \tag{3.16}$$
$$= dG \wedge \mathrm{Re}\,\overline{\omega} + dH \wedge \mathrm{Im}\,\overline{\omega}\,.$$

From (3.7) and (3.16) it follows that

$$d(\mathrm{Re}\,\overline{\omega}) = dG \wedge \mathrm{Re}\,\overline{\omega} - JdH \wedge \mathrm{Re}\,\overline{\omega} = (dG - JdH) \wedge \mathrm{Re}\,\overline{\omega}\,,$$

completing the proof.

PROOF OF THEOREM 3.5. Assume that $dG - JdH = 0$. Then by virtue of Lemma 3.7

$$d(\mathrm{Re}\,\overline{\omega}) = (dG - JdH) \wedge \mathrm{Re}\,\overline{\omega} = 0\,,$$

that is $\omega = \mathrm{Re}\,\overline{\omega}$ is a Lagrangian calibration. Conversely, if $\omega = \mathrm{Re}\,\overline{\omega}$ is a Lagrangian calibration, then $d(\mathrm{Re}\,\overline{\omega}) = 0$. Consequently, by Lemma 3.7

$$(dG - JdH) \wedge \mathrm{Re}\,\overline{\omega} = 0 . \tag{3.17}$$

For any nontrivial vector $e \in T_pM$ one can choose in the orthogonal supplement to e a unitary basis $\{e_1, e_2, \dots, e_n\}$ of T_pM so that $e_1 = -Je$. Then from (3.17) it follows that

$$0 = (dG - JdH) \wedge \mathrm{Re}\,\overline{\omega}(e \wedge e_1 \wedge e_2 \wedge \dots \wedge e_n) = (dG - JdH)(e)\mathrm{Re}\,\overline{\omega}(e_1 \wedge e_2 \wedge \dots \wedge e_n) .$$

By using Corollary 2.5 we have $|\mathrm{Re}\,\overline{\omega}(e_1 \wedge e_2 \wedge \dots \wedge e_n)| = 1$ because $e_1, e_2, \dots, e_n$ form an orthonormal basis of a Lagrangian plane. Hence $(dG - JdH)(e) = 0$. Consequently $dG - JdH = 0$. Thus the theorem is completely proved.

<u>THEOREM</u> 3.5 establishes that a Lagrangian form ω is a calibration if and only if

$$dG - JdH = 0 \tag{3.18}$$

on M . In what follows for notational convenience we will use the form $JdG + dH = J(dG - JdH)$ together with $dG - JdH$. Clearly, the equality $dG - JdH = 0$ is equivalent to

$$JdH + dH = 0 \tag{3.19}$$

We call (3.18) or (3.19) *the Lagrangian calibration equation*.

§ 4. Special Lagrangian sections

A Lagrangian section on a Hermitian manifold M is defined to be any section $p : M \longrightarrow \mathrm{Lag}(M)$ of the Lagrangian bundle on M.

DEFINITION 4.1. *A Lagrangian section p on M is said to be special Lagrangian if there exists a Lagrangian calibration ω on M such that $\omega(p(z)) = 1$ for every $z \in M$.*

Obviously, any integral submanifold of a special Lagrangian section (i.e. such a submanifold that has $p(z)$ as the tangent space at each its point z) is special Lagrangian.

EXAMPLE. Suppose F is an oriented n–dimensional foliation of M such that tangent planes to the leaves are Lagrangian. Then the mapping $p : M \longrightarrow \mathrm{Lag}(M)$, sending each point z to the tangent plane to the leaf at z, gives a Lagrangian section. Note that a connected integral submanifold without boundary of this section is just a closed leaf of the foliation.

Let $p : M \longrightarrow \mathrm{Lag}(M)$ be a Lagrangian section on M. Given local complex coordinates $z = (z_1, z_2, \dots, z_n)$ on M. Set

$$\bar{\omega} = dz_1 \wedge dz_2 \wedge \dots \wedge dz_n\,, \quad \bar{\omega}_p = \frac{\bar{\omega}}{\bar{\omega}(p)} \tag{4.1}$$

THEOREM 4.1. The differential n–form $\bar{\omega}_p$, given locally by the formula (4.1), is correctly defined on the whole manifold M. Moreover, the real part $\mathrm{Re}(\bar{\omega}_p)$ of $\bar{\omega}_p$ is a Lagrangian form.

PROOF. First we show that, in fact, the form $\bar{\omega}_p$ is independent of the choice of the complex coordinates. Really, suppose $(z_1', z_2', \dots, z_n')$ is another local complex coordinates on M and denote by $A = \left[\frac{\partial z_i'}{\partial z_j}\right]$ the Jacobian matrix. Let

$$(4.2) \qquad \bar{\omega}' = dz_1' \wedge dz_2' \wedge \dots \wedge dz_n' , \quad \bar{\omega}_p' = \frac{\bar{\omega}'}{\bar{\omega}(p)}$$

Then we have

$$(4.3) \qquad \bar{\omega}' = dz_1' \wedge dz_2' \wedge \dots \wedge dz_n' = \det A \; dz_1 \wedge dz_2 \wedge \dots \wedge dz_n = \det A \; \bar{\omega} .$$

In particular,

$$(4.4) \qquad \bar{\omega}'(p) = \det A \; \bar{\omega}(p) .$$

By substituting (4.3) and (4.4) into (4.1) and (4.2) we obtain

$$\bar{\omega}_p' = \frac{\bar{\omega}'}{\bar{\omega}'(p)} = \bar{\omega}_p' = \frac{\det A \;\; \bar{\omega}}{\det A \;\; \bar{\omega}(p)} = \frac{\bar{\omega}}{\bar{\omega}(p)} = \bar{\omega}_p .$$

This proves the first statement of the theorem. It remains to prove that $\mathrm{Re}(\bar{\omega}_p)$ is a Lagrangian form. Set $\tilde{\omega} = e^G \, \bar{\omega} = e^G dz_1 \wedge dz_2 \wedge \dots \wedge dz_n$. By Proposition 2.8 $\mathrm{Re}\, \tilde{\omega}$ is Lagrangian. Then $|\tilde{\omega}(p)| = 1$ by force of Corollary 2.5, because p is a Lagrangian plane. Hence, $|\overline{\omega}(p)| = |e^{-G}\cdot\tilde{\omega}(p)| = e^{-G}|\tilde{\omega}(p)| = e^{-G}$, i.e. $\overline{\omega}(p) = e^{-G+iH}$ for some real valued function H. Applying Proposition 2.8 again we see that the form $\mathrm{Re}\,\overline{\omega}_p$, where

$$\overline{\omega}_p = \frac{\overline{\omega}}{\overline{\omega}(p)} = e^{G-iH} dz_1 \wedge dz_2 \wedge \dots \wedge dz_n ,$$

is Lagrangian. The proof is complete.

Given the Lagrangian section $p : M \longrightarrow \mathrm{Lag}(M)$ on M. By virtue of Corollary 2.6 there exists an unique Lagrangian form ω on M such that $\omega(p) = 1$ everywhere. This form is called <u>the characteristic form</u> of the section p and denoted by ω_p.

<u>THEOREM</u> 4.2. <u>The Lagrangian form</u> $\mathrm{Re}\,\overline{\omega}_p$, <u>constructed in Theorem</u> 4.1 <u>is nothing but the characteristic form</u> ω_p <u>of the Lagrangian section</u> p.

<u>PROOF</u>. Since $\mathrm{Re}\,\overline{\omega}_p$ is Lagrangian by Theorem 4.1 it remains to show that $\mathrm{Re}\,\overline{\omega}_p(p) = 1$ at each point. Really, because $\overline{\omega}_p$ is independent of the choice of the coordinates we can choose local complex coordinates $z = (z_1, z_2, \dots, z_n)$ such that $\{dz_1, dz_2, \dots, dz_n\}$ is a unitary basis of T_z^*M dual to an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of $p(z)$. (Note that $\{e_1, e_2, \dots, e_n\}$ is also a unitary basis of T_zM). Then $\overline{\omega}(p) = dz_1 \wedge dz_2 \wedge \dots \wedge dz_n(p) = 1$. Hence, $\mathrm{Re}\,\overline{\omega}_p(p) = \overline{\omega}_p(p) = 1$ by Proposition 2.4. This proves the theorem.

<u>THEOREM</u> 4.3. <u>A Lagrangian section</u> $p : M \longrightarrow \mathrm{Lag}(M)$ <u>is special Lagrangian if and only if</u>

$$dG - JdH = 0$$

where $dG - JdH$ is the 1–form associated to the characteristic form ω_p of the section p .

PROOF. Suppose that $dG - JdH = 0$. By Theorem 3.5 ω_p is a Lagrangian calibration. That means that p is a special Lagrangian section. Conversely, if p is a special Lagrangian section, then there exists a Lagrangian calibration ω on M such that $\omega(p) = 1$ everywhere. But this property provides ω to be the characteristic form of the section p . Thus, $\omega = \omega_p$. Applying Theorem 3.5 again we have $dG - JdH = 0$, completing the proof.

§ 5. The existence and classification of Lagrangian calibrations

DEFINITION 5.1. A Hermitian manifold M is called a L–calibrated manifold if there exists a Lagrangian calibration on M .

THEOREM 5.2. A simply connected Hermitian manifold M is L–calibrated if and only if $d(JdG) = 0$.

Let given local complex coordinates $z = (z_1, z_2, \dots, z_n)$. Assume that the 1–form JdG is closed. Then $-JdG = dH$, where H is a function determined uniquely up to a constant. We consider the family of the following Lagrangian forms

$$\omega_H = \mathrm{Re}(e^{G-iH} dz_1 \wedge dz_2 \wedge \dots \wedge dz_n) , \tag{5.1}$$

where $G = \ln\sqrt{\det(g)}$ and H is any real valued function satisfying the condition: $dH = -JdG$.

REMARK. By Proposition 2.8 and Theorem 3.5 ω_H are closed Lagrangian forms on the domain of the local coordinates $z = (z_1, z_2, \dots, z_n)$, i.e. ω_H are local Lagrangian calibrations.

LEMMA 5.3. The family $\{\omega_H\}$ given by (5.1) is independent of the choice of the local complex coordinates.

PROOF. Suppose $z' = (z'_1, z'_2, \dots, z'_n)$ are another local complex coordinates and let $A = \left[\dfrac{\partial z'_i}{\partial z_j}\right]$ be the Jacobian matrix, $\det A = e^{\alpha + i\beta}$. The functions α and β are related by the equality (3.6): $Jd\alpha = d\beta$. Consider the family of Lagrangian forms

$$\omega'_{H'} = \mathrm{Re}(e^{G' - iH'} dz'_1 \wedge dz'_2 \wedge \dots \wedge dz'_n), \tag{5.2}$$

where $G' = \ln\sqrt{\det(g')}$ and H' is any real valued function such that $dH' = -JdG'$. Representing $\omega'_{H'}$, in the coordinates $z = (z_1, z_2, \dots, z_n)$ we have

$$\begin{aligned}\omega'_{H'} &= \mathrm{Re}(e^{(G' - iH'} e^{\alpha + i\beta} dz_1 \wedge dz_2 \wedge \dots \wedge dz_n) = \\ &= \mathrm{Re}(e^{(G' + \alpha) - i(H' - \beta)} dz_1 \wedge dz_2 \wedge \dots \wedge dz_n).\end{aligned}$$

According to Proposition 2.8

$$G' + \alpha = G. \tag{5.3}$$

Taking (3.6) and (5.3) into account we have

$$d(H' - \beta) = dH' - d\beta = JdG' - Jd\alpha = -Jd(G' + \alpha) = -JdG\,.$$

Hence, $\{\omega_{H'}\} \subset \{\omega_H\}$. Since the local coordinates $z = (z_1, z_2, \dots, z_n)$ and $z' = (z_1', z_2', \dots, z_n')$ are equivalent in our above argument the converse implication $\{\omega_H\} \subset \{\omega_{H'}\}$ is true as well. Consequently, $\{\omega_H\} \equiv \{\omega_{H'}\}$. This proves the lemma.

LEMMA 5.4. Each form of the family $\{\omega_H\}$ is determined completely by its value at a fixed point $z = p$.

PROOF. Fix a point $z = p$. Suppose

$$\omega_H = \mathrm{Re}(e^{G-iH} dz_1 \wedge dz_2 \wedge \dots \wedge dz_n)$$

$$\omega_{H'} = \mathrm{Re}(e^{G-iH'} dz_1 \wedge dz_2 \wedge \dots \wedge dz_n)$$

are two forms of the family $\{\omega_H\}$. By virtue of Corollary 2.6 $\omega_H = \omega_{H'}$ if and only if $H \equiv H'$. However, $H \equiv H'$ if and only if $H(p) = H'(p)$ because H and H' differ by only a constant. This proves the statement of the lemma.

Let $\gamma : [0,1] \longrightarrow M$ be a path, joining fixed points $p = \gamma(0)$ and $q = \gamma(1)$. Given a Lagrangian exterior form ω_0 on T_pM . Assume that $\{U_1, U_2, \dots, U_k\}$ is a chain of neighborhood in M such that,

(i) $U = \bigcup_{i=1}^{k} U_i \supset \gamma[0,1]$;

(ii) $p \in U_1$, $q \in U_k$, $U_i \cap U_j \neq \phi \Leftrightarrow j = i \pm 1 (1 \leq i < j \leq k)$

(iii) There exist complex coordinates on U_i for each $i = 1,2, \ldots,k$.

Clearly, there always exist such chains of neighborhoods on M for every path γ. Now, by force of Lemma 5.3 and Lemma 5.4 there exists an unique set of local Lagrangian calibrations $\{\varphi_1,\varphi_2, \ldots ,\varphi_k\}$ such that φ_i is defined on $U_i (i = 1,2, \ldots ,k)$, $\varphi_1(p) = \omega_0$ and $\varphi_i \equiv \varphi_j$ on $U_i \cap U_j (1 \leq i,j \leq k)$. Then we can get a local calibration φ defined on $U = \bigcup_{i=1}^{k} U_i$ by setting $\varphi = \varphi_i$ on $U_i (i = 1,2, \ldots ,k)$. Put $\omega_1^\gamma = \varphi(q) (= \varphi_k(q))$.

<u>LEMMA</u> 5.5. <u>The Lagrangian exterior form</u> ω_1^γ <u>on</u> T_qM <u>constructed as above does not depend on the choice of the chain</u> $\{U_1,U_2, \ldots ,U_k\}$.

<u>PROOF</u>. Suppose $\{U'_1,U'_2, \ldots ,U'_s\}$ is another chain of neighborhoods satisfying the conditions (i), (ii) and (iii). Denote the corresponding local calibrations on $U'_1,U'_2, \ldots ,U'_s$ and U' by $\varphi'_1,\varphi'_2, \ldots ,\varphi'_s$, and φ' respectively. Set $\omega_t = \varphi(\gamma(t))$, $\omega'_t = \varphi'(\gamma(t))$. First we note that ω_t and ω'_t depend on t continuously. Let $K = \{t \in [0,1] : \omega_t = \omega'_t\}$. Obviously, $0 \in K$, i.e. $K \neq \phi$. The fact that ω_t and ω'_t are continuous implies that K is closed. On the other hand, assume that $t_0 \in K$, i.e. $\omega_{t_0} = \omega'_{t_0}$. Suppose that $\gamma(t_0) \in U_i \cap U'_j$. There exists a neighborhood $(t_0 - \varepsilon, t_0 + \varepsilon)$ in $[0,1]$ such that $\gamma(t_0 - \varepsilon, t_0 + \varepsilon) \subset U_i \cap U'_j$. By Lemma 5.4 $\varphi_i \equiv \varphi'_j$ on $U_i \cap U'_j$. Consequently, $\omega_t = \varphi(\gamma(t)) \equiv \varphi'(\gamma(t)) = \omega'_t$ on $(t_0 - \varepsilon, t_0 + \varepsilon)$. Hence,

$(t_0 - \varepsilon\ ,\ t_0 + \varepsilon) \subset K$, i.e. K is open. Thus, $K = [0,1]$, therefore $\omega_t = \omega_t'$ for any $t \in [0,1]$. In particular, $\omega_1 = \omega_1'$. The proof is complete.

Let $\Omega_{pq}(M)$ denote the space of paths joining points p and q on M . $\Omega_{pq}(M)$ can be equipped with the topology of uniform convergence. In this topology each homotopy class of paths is a connected component of $\Omega_{pq}(M)$. In particular, if M is simply connected, then $\Omega_{pq}(M)$ is connected.

By force of Lemma 5.5 we can construct correctly a map:
$\psi : \Omega_{pq}(M) \longrightarrow \Lambda_n(T_q^*M)$, corresponding each path γ to the Lagrangian exterior form ω_1^γ constructed as above.

LEMMA 5.6. The map $\psi : \Omega_{pq}(M) \longrightarrow \Lambda_n(T_q^*M)$ is locally constant.

PROOF. Really, assume that ω_1^γ is constructed by using a chain of neighborhoods $\{U_1, U_2, \dots , U_k\}$ satisfying the conditions (i), (ii), and (iii). Then every path γ' in $\Omega_{pq}(M)$ near γ enough is contained in $U = \bigcup_{i=1}^{k} U_i$. Consequently, $\omega_1^{\gamma'}$ can be constructed by using the chain $\{U_1, U_2, \dots , U_k\}$ as well. Hence, $\omega_1^{\gamma'} = \varphi(q) = \omega_1^\gamma$, where φ is the local calibration on U . This completes the proof.

From Lemma 5.6 it follows immediately

COROLLARY 5.7. The map $\psi : \Omega_{pq}(M) \longrightarrow \Lambda_n(T_q^*M)$ is constant on each homotopy class of paths. In particular, if M is simply connected, then ψ is constant.

PROOF OF THEOREM 5.2. Assume that $dJdG = 0$. Then we may apply Lemmas 5.3 – 5.6 and Corollary 5.7. Since M is simply connected Corollary 5.7 means that, in fact, the Lagrangian exterior form $\omega_1^\gamma = \psi(\gamma)$ does not depend on γ . So, fixing a point $p \in M$ and a Lagrangian exterior form ω_0 on T_pM we can construct a map $\omega : M \longrightarrow \Lambda_n(T^*M)$, sending each point $q \in M$ to n–form ω_1^γ for a path γ joining p and q. In particular, $\omega(p) = \omega_0$. Clearly, the differential n–form ω constructed in this way is Lagrangian and the 1–form associated to ω satisfies the condition $JdG + dH = 0$ locally. Consequently, the equality $JdG + dH = 0$ holds everywhere. Therefore, ω is a Lagrangian calibration and M is a L–calibrated manifold.

Conversely, suppose that M is L–calibrated and ω is a Lagrangian calibration on M. Then for ω the equality $JdG + dH = 0$ holds. Hence
$dJdG = d(JdG + dH) = 0$. Thus, the proof is complete.

THEOREM 5.8. Suppose that M is a L–calibrated Hermitian manifold. Then each Lagrangian calibration on M is determined completely by its value at a point. If $\mathrm{Re}(\overline{\omega})$ is a Lagrangian calibration, then the set of all Lagrangian calibrations on M is $\{\mathrm{Re}(e^{i\varphi}\,\overline{\omega}) \ , \ 0 \le \varphi < 2\pi\}$. In other words, there exists precisely a S^1–family of Lagrangian calibrations on M.

PROOF. In fact, the statement of Theorem 5.8 can be obtained by looking more carefully at the proof of Theorem 5.2 (and of Lemmas 5.3 – 5.6). However, here we will present a direct proof. Since $\mathrm{Re}(\overline{\omega})$ is a Lagrangian form, it follows from Proposition 2.8 that every Lagrangian form on M is of the form $\mathrm{Re}(e^{i\varphi}\,\overline{\omega})$, where φ is a real valued function on M. In local complex coordinates $z = (z_1, z_2, \dots, z_n)$ we have

$$\mathrm{Re}(\overline{\omega}) = \mathrm{Re}(e^{G-iH}dz_1 \wedge dz_2 \wedge \dots \wedge dz_n)$$

$$\mathrm{Re}(e^{i\varphi}\,\overline{\omega}) = \mathrm{Re}(e^{G-i(H-\varphi)}dz_1 \wedge dz_2 \wedge \dots \wedge dz_n) .$$

Since $\mathrm{Re}(\overline{\omega})$ and $\mathrm{Re}(e^{i\varphi}\,\overline{\omega})$ are both Lagrangian calibrations, by Theorem 3.5 we have: $JdG + dH = 0 = JdG + d(H-\varphi)$. Hence $d\varphi = 0$; that means that φ is locally constant. Consequently, φ is constant on M.

REMARK. Theorem 5.2 states that the necessary and sufficient condition for a Hermitian manifold M to be L–calibrated is $dJdG = 0$, where $G = \ln\sqrt{\det(g)}$. Given local complex coordinates $z = (z_1, z_2, \ldots, z_n)$ on M. We have

$$dG = \sum_{\alpha}\left[\frac{\partial G}{\partial z_\alpha}\,dz_\alpha + \frac{\partial G}{\partial \overline{z}_\alpha}\,d\overline{z}_\alpha\right]$$

$$JdG = \sum_{\alpha} i\left[\frac{\partial G}{\partial z_\alpha}\,dz_\alpha - \frac{\partial G}{\partial \overline{z}_\alpha}\,d\overline{z}_\alpha\right]$$

$$dJdG = 2i\sum_{\alpha,\beta}\frac{\partial^2 G}{\partial z_\alpha\,\partial \overline{z}_\beta}\,dz_\alpha\wedge d\overline{z}_\beta\ .$$

Hence, the condition $dJdG = 0$ is equivalent to

$$\frac{\partial^2 G}{\partial z_\alpha\,\partial \overline{z}_\beta} = 0 \quad \text{for any } \alpha,\beta = 1,2,\ldots,n\ . \tag{5.4}$$

If M is a Kähler manifold, then the Ricci tensor on M is given as follows:
$K_{\alpha\overline{\beta}} = 2\cdot\dfrac{\partial^2 G}{\partial z_\alpha\,\partial \overline{z}_\beta}$ (cf. [H]). Thus we have proved the following Corollary

COROLLARY 5.9. A Kähler manifold M is L–calibrated if and only if it is Ricci flat.

§ 6. Special Lagrangian submanifolds on L–calibrated manifolds

Let M be a connected 2n–dimensional Hermitian manifold, N an n–dimensional oriented submanifold in M .

THEOREM 6.1. A submanifold N in M is special Lagrangian if and only if each point of N has a special Lagrangian neighborhood.

PROOF. Of course, if N is a special Lagrangian submanifold then each its neighborhood is special Lagrangian. Thus, it will suffice to prove the converse statement. Assume that each point $p \in N$ has a special Lagrangian neighborhood $U_p \subset N$ and let φ_p denote the corresponding local Lagrangian calibration. By definition $\varphi_p(T_pM) = 1$. From Theorem 5.8 and Corollary 2.6 it follows that φ_p must be the restriction of a (globally defined) Lagrangian calibration ω_p. Moreover, for any points $p,q \in N$, if $U_p \cap U_q \neq \phi$ then $\omega_p(T_zN) = 1 = \omega_q(T_zN)$ for all $z \in U_p \cap U_q$. Consequently, $\omega_p = \omega_q$. This means that all local Lagrangian calibrations φ_p are restrictions of the same Lagrangian calibration ω on M . Thus, N is special Lagrangian (with respect to ω) and the proof is complete.

REMARK. Theorem 6.1 establishes the equivalence between the properties of being special Lagrangian and locally special Lagrangian. Below we will prove that these properties are equivalent to the property of the local minimality.

Let given local complex coordinates $z = (z_1,z_2, \dots ,z_n)$ on M . Set $\overline{\omega} = e^G dz_1 \wedge dz_2 \wedge \dots \wedge dz_n (G = \ln\sqrt{\det(g)})$ .Denote by π the projection of the

Lagrangian bundle Lag(M) . We can consider the functions f and θ on Lag(M) by setting $f(\xi) = G \circ \pi(\xi)$ and $\theta(\xi) = -i \ln \overline{\omega}(\xi)$ for any $\xi \in \mathrm{Lag}(M)$. As is known (cf. [LF], [B]) the differential 1–form $Jdf + d\theta$, where J is the operator of the complex structure, is independent of the choice of the local coordinates; so that it is defined correctly on the whole bundle Lag(M) .

Suppose N is a Lagrangian submanifold in M , $p : N \longrightarrow \mathrm{Lag}(M)$ is the map, sending each point $z \in N$ to the tangent space to N at z .

PROPOSITION 6.2. ([LF], [B]). A Lagrangian submanifold N in M is (stationary) minimal if and only if the induced form $p^*(Jdf + d\theta)$ vanishes on N .

THEOREM 6.3. Suppose M is a L–calibrated manifold. Then every connected (stationary) minimal Lagrangian submanifold is special Lagrangian. In particular, minimal Lagrangian submanifolds seem to be volume minimizing.

REMARK. For the case $M = \mathbb{C}^n$ this result was proved by R . Harvey and H.B. Lawson [HL_1].

PROOF. Assume that N is a stationary Lagrangian submanifold. According to Proposition 6.2 $p^*(Jdf + d\theta) = 0$. Consider local complex coordinates $z = (z_1, z_2, \dots, z_n)$ on M and set $\overline{\omega} = e^G dz_1 \wedge dz_2 \wedge \dots \wedge dz_n$. Let j denote the embedding of N into M . We have: $p^*(Jdf) = j^*(JdG)$, $p^*(d\theta) = -id \ln \overline{\omega}(p)$. Fix a point $z_0 \in N$ and choose the Lagrangian calibration φ on M such that $\varphi(p(z_0)) = 1$. In the local coordinates φ has the form

$$\varphi = \mathrm{Re}(e^{G-iH} dz_1 \wedge dz_2 \wedge \dots \wedge dz_n) = \mathrm{Re}(e^{-iH}\overline{\omega}) .$$

Set $\alpha = -i \ln \overline{\omega}(p)$. We have $e^{i\alpha} = \overline{\omega}(p)$ or $e^{-i\alpha}\,\overline{\omega}(p) = 1$. Hence, $\varphi(p(z)) = 1$ if and only if $H = \alpha = -i \ln \overline{\omega}(p)$. Now, since $\varphi(p(z_0)) = 1$ we have $H(z_0) = -i \ln \overline{\omega}(p(z_0))$ or $H(z_0) + i \ln \omega(p(z_0)) = 0$. On the other hand, $d[H \circ j + i \ln \overline{\omega}(p)] = j^*dH + d(i \ln \overline{\omega}(p)) = j^*(dH + JdG) - (j^*(JdG) + p^*(d\theta)) =$ $= j^*(JdG + dH) - p^*(Jdf + d\theta) = 0$ by using Theorem 3.5 and Proposition 6.2. That means that $H(z) + i \ln \overline{\omega}(p(z))$ is locally constant on N . Since N is connected and $H(z_0) + i \ln \overline{\omega}(p(z_0)) = 0$ we have $H + i \ln \overline{\omega}(p) = 0$ or $H = -i \ln \overline{\omega}(p)$ on N . By virtue of the above remark $\varphi(p(z)) = 1$ on N . Hence, N is a φ – submanifold, and consequently, a special Lagrangian submanifold,

REMARK. In [LF] it was proved that $dJdG = 0$ if and only if $Jdf + d\theta$ is integrable. Combining that with the statement of Theorem 5.2 one can conclude that a necessary and sufficient condition for a simply connected Hermitian manifold M to be L–calibrated is the integrability of the form $Jdf + d\theta$. The integrability of $Jdf + d\theta$ means that it determines a foliation of codimension 1 of Lag(M) . From the proof of Theorem 5.2 it is easy to see that the image of each leaf of this foliation under the projection π is the whole M . Now, using Theorems 4.3, 6.3 and Proposition 6.2 we can obtain the following result

COROLLARY 6.4. Suppose M is a simply connected L–calibrated Hermitian manifold

(i) A connected Lagrangian submanifold N in M is special Lagrangian if and only if p(N) is contained in a leaf of the foliation determined by the form $Jdf + d\theta$, where $p : N \longrightarrow \mathrm{Lag}(M)$ is the map, sending every point z to the tangent plane to N at z.

(ii) A Lagrangian section $p : M \longrightarrow Lag(M)$ is special Lagrangian if and only if $p(M)$ is contained in a leaf of the foliation determined by the form $Jdf + d\theta$.

§ 7. Special Lagrangian condition in coordinates

THEOREM 6.1 shows that the study of the property of being special Lagrangian is reduced to that of the property of being locally special Lagrangian, so we can use a fix system of local complex coordinates. In other words, we will deal with a complex space $M = \mathbb{C}^n = \{(z_1, z_2, \dots, z_n)\}$ equipped with a generally non–standard Hermitian metric $g = \Sigma\, g_{ij} dz_i d\bar{z}_j$. In this case the condition for $(\mathbb{C}^n; g)$ to be L–calibrated is the following

$$(7.1) \qquad \frac{\partial^2 G}{\partial z_i \partial \bar{z}_j} = 0 \text{ for any } i,j = 1,2, \dots, n,$$

where as always $G = \ln\sqrt{\det(g)}$. The equalities (7.1) mean that G is the real part of a holomorphic function $G + iK$. Consider the real coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$, where $z_i = x_i + iy_i$ $(i = 1, \dots, n)$.

Then the Cauchy–Riemann condition is written as follows

$$(7.2) \qquad \frac{\partial G}{\partial x_i} = \frac{\partial K}{\partial y_i}, \quad \frac{\partial G}{\partial y_i} = -\frac{\partial K}{\partial x_i}, \quad i = 1,2, \dots, n.$$

We have

$$dG = \sum_{i=1}^{n} \left[\frac{\partial G}{\partial x_i} dx_i + \frac{\partial G}{\partial y_i} dy_i\right]$$

$$JdG = \sum_{i=1}^{n} \left[\frac{\partial G}{\partial x_i} dy_i + \frac{\partial G}{\partial y_i} dx_i\right] = \sum_{i=1}^{n} \left[\frac{\partial K}{\partial y_i} dy_i + \frac{\partial K}{\partial x_i} dx_i\right] = dK$$

Thus, the equality $JdG + dH = 0$ just means, that $-H$ and K differ by only a constant. By the way we note, that $JdK = -dG$. Hence $dJdK = 0$, that is

(7.3) $$\frac{\partial^2 K}{\partial z_i \partial \bar{z}_j} = 0 \text{ for any } i,j = 1,2, \dots ,n .$$

First we consider the case when an n–dimensional submanifold X is described implicity as the set $X = \{z \in \Omega : f_1(z) = f_2(z) = \dots = f_n(z) = 0\}$, where $f_1, f_2, \dots ,f_n$ are smooth real valued functions on an open set Ω of M such that $df_1, df_2, \dots ,df_n$ are linearly independent at each point of X . Then normal n–plane $N_z X$ to X at point z is spanned (over $\mathbb{R}$) by $\operatorname{grad} f_1(z)$, $\operatorname{grad} f_2(z), \dots ,\operatorname{grad} f_n(z)$. Obviously, the tangent n–plane $T_z M$ is Lagrangian if and only if the normal n–plane $N_z X$ is Lagrangian. In this case $T_z X = J(N_z X)$.

Let $g_{ij} = g'_{ij} + ig''_{ij}$ $(1 \le i,j \le n)$. Then in real coordinates $(x_1, \dots ,x_n, y_1, \dots ,y_n)$ the metric on M can be expressed as $g = \Sigma\, g'_{ij}(dx_i \otimes dx_j + dy_i \otimes dy_j) + g''_{ij}(dx_i \otimes dy_j - dy_i \otimes dx_j)$. Besides that, $g'_{ij} = g'_{ji}$ and $g''_{ij} = -\, g''_{ji}$ for any i , j . Suppose that $\operatorname{grad} f = (u_1, \dots ,u_n, v_1, \dots ,v_n)$. By definition $g(\operatorname{grad} f, \xi) = df(\xi)$ for any vector $\xi = (\alpha_1, \dots ,\alpha_n, \beta_1, \dots ,\beta_n) \in T_z M \ (= \mathbb{C}^n)$, that is

$$(7.4)\qquad \sum_{i,j=1}^{n} g'_{ij}(u_i\alpha_j + v_i\beta_j) + g''_{ij}(u_i\beta_j - v_i\alpha_j) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\alpha_i + \frac{\partial f}{\partial y_i}\beta_i .$$

From (7.4) it follows that

$$(7.5)\qquad \frac{\partial f}{\partial x_j} = \sum_{i=1}^{n} g'_{ij}u_i\,,\quad \frac{\partial f}{\partial y_j} = -\sum_{i=1}^{n} g''_{ij}\,v_i\,.$$

Hence

$$(7.6)\qquad 2\frac{\partial f}{\partial \bar{z}_j} = \frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j} = \sum_{i=1}^{n} g_{ij}\,\xi_i = \sum_{i=1}^{n} \bar{g}_{ji}\,\xi_i$$

where $\xi_j = u_j + iv_j$, $j = 1,2,\dots,n$. Thus, applying (7.6) for $f_1, f_2, \dots, f_n$ we see that the matrix $(\bar{g}_{ij})$ maps vectors $\operatorname{grad} f_1$, $\operatorname{grad} f_2, \dots, \operatorname{grad} f_n$ to vectors $\frac{2\,\partial f_1}{\partial \bar{z}}$, $\frac{2\,\partial f_2}{\partial \bar{z}}, \dots, \frac{2\,\partial f_n}{\partial \bar{z}}$ respectively.

<u>THEOREM</u> 7.1. <u>Suppose</u> X <u>is a Lagrangian submanifold, described implicitly as above</u>. <u>Then</u> X <u>is special Lagrangian if and only if</u>.

$$\operatorname{Arg}\left[\det\left[\frac{\partial f_i}{\partial \bar{z}_j}\right]\right] + K = \text{const. } \underline{\text{on}}\ X\,,$$

<u>where</u> Arg(z) <u>denotes the argument of the complex number</u> z .

PROOF. Since X is Lagrangian $T_zX = J(N_zX)$ by the force of the above remark, i.e. T_zX is spanned by $i \operatorname{grad} f_1$, $i \operatorname{grad} f_2, \dots, i \operatorname{grad} f_n$. In the other hand

$$(2^n)\frac{\partial f_1}{\partial \bar{z}} \wedge \frac{\partial f_2}{\partial \bar{z}} \wedge \dots \wedge \frac{\partial f_n}{\partial \bar{z}} = e^{2G} \operatorname{grad} f_1 \wedge \operatorname{grad} f_2 \wedge \dots \wedge \operatorname{grad} f_n .$$

Hence

$$T_zX = (i)^n \lambda \frac{\partial f_1}{\partial \bar{z}} \wedge \frac{\partial f_2}{\partial z} \wedge \dots \wedge \frac{\partial f_n}{\partial \bar{z}} , \tag{7.7}$$

where λ is a real number. Letting

$$\bar{\omega} = e^G dz_1 \wedge dz_2 \wedge \dots \wedge dz_n$$

$$\xi_0 = \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \wedge \dots \wedge \frac{\partial}{\partial z_n} ,\ p(z) = T_zX ,$$

we have

$$p = \lambda(i)^n \det\left[\frac{\partial f_i}{\partial \bar{z}_j}\right] \xi_0 .$$

Consequently,

$$\alpha = -i \ln \bar{\omega}(p) = \operatorname{Arg}(\bar{\omega}(p)) = \operatorname{Arg}\left[\lambda e^G (i)^n \det\left[\frac{\partial f_i}{\partial \bar{z}_j}\right]\right] . \tag{7.8}$$

Theorem 6.3 and Proposition 6.2 state that X is special Lagrangian if and only if $\alpha + K = \text{const}$ on X. Combining that with (7.8) completes the proof.

We note that if the metric tensor (g_{ij}) is real then $\left\{\frac{\partial f_1}{\partial \bar z}, \frac{\partial f_2}{\partial \bar z}, \dots \frac{\partial f_n}{\partial \bar z}\right\}$ is another real basis of the normal n–plane $T_z X$. We have

$$2\frac{\partial f_k}{\partial \bar z} = \left[\frac{\partial f_k}{\partial x_1} + i\frac{\partial f_k}{\partial y_1}, \dots, \frac{\partial f_k}{\partial x_n} + i\frac{\partial f_k}{\partial y_n}\right]$$

$$2J\left[\frac{\partial f_k}{\partial \bar z}\right] = \left[-\frac{\partial f_k}{\partial y_1} + i\frac{\partial f_k}{\partial x_1}, \dots, -\frac{\partial f_k}{\partial y_n} + i\frac{\partial f_k}{\partial x_n}\right]$$

$$g\left[\frac{\partial f_k}{\partial \bar z}, J\frac{\partial f_h}{\partial \bar z}\right] = -\sum_{i,j=1}^{n} g_{ij}\left[\frac{\partial f_k}{\partial x_i}\frac{\partial f_h}{\partial y_j} - \frac{\partial f_k}{\partial y_i}\frac{\partial f_h}{\partial x_j}\right].$$

By that we obtain the following criterion for a submanifold to be Lagrangian.

<u>PROPOSITION</u> 7.2. <u>Assume that the metric tensor</u> (g_{ij}) <u>on</u> $\mathbb{C}^n$ <u>is real and let</u> X <u>is given implicitly as in Theorem</u> 7.1. <u>Then</u> X <u>is Lagrangian if and only if</u>

$$\text{(7.9)}\qquad \sum_{i,j=1}^{n} g_{ij}\left[\frac{\partial f_k}{\partial x_i}\frac{\partial f_h}{\partial y_j} - \frac{\partial f_k}{\partial y_i}\frac{\partial f_h}{\partial x_j}\right] \equiv$$

$$\equiv 2i\sum_{i,j=1}^{n} g_{ij}\left[\frac{\partial f_k}{\partial z_i}\frac{\partial f_h}{\partial \bar z_j} - \frac{\partial f_k}{\partial \bar z_i}\frac{\partial f_h}{\partial z_j}\right] = 0$$

for any $k, h = 1,2, \dots, n$.

Note that if grad K never vanishes on an open set Ω then in Theorem 7.1 $K + c$ $(c = \text{const})$ may be chosen as one of the functions $f_1, f_2, \dots, f_n$. In this case $K = -\text{const}$ on X and from Theorem 7.1 it immediately follows

<u>COROLLARY</u> 7.3. <u>Let</u> X <u>be given as in Theorem</u> 7.1; <u>moreover</u> $f_1 = K$. <u>Then</u> X <u>is special Lagrangian if only if</u>

$$\operatorname{Arg}\left[\det\left[\frac{\partial f_i}{\partial \bar z_j}\right]\right] = \text{const on } X .$$

<u>EXAMPLE</u> 7.4. Now we present a class of special Lagrangian submanifolds delivered by Corollary 7.3. For any real number $\theta \in [0,2\pi)$ we consider the real valued function

$$\ell(\theta,z) = 2(x\cos\theta + y\sin\theta) ,$$

where $z = x + iy$. A straightforward calculation shows that

$$\frac{\partial \ell(\theta,z)}{\partial z} = e^{-i\theta} , \quad \frac{\partial \ell(\theta,z)}{\partial \bar z} = e^{i\theta} . \tag{7.10}$$

Suppose $\bar h(t_1,t_2, \dots ,t_n)$ is an arbitrary real function of variables $t_1,t_2, \dots ,t_n$ and let

$$h(z_1,z_2, \dots ,z_n) = \bar h(\ell(\theta_1,z_1),\ell(\theta_2,z_2), \dots ,\ell(\theta_n,z_n)) . \tag{7.11}$$

It is clear that

(7.12) $$\frac{\partial h}{\partial z_i} = \frac{\partial \bar{h}}{\partial t_i} e^{-i\theta_i}, \quad \frac{\partial h}{\partial \bar{z}_i} = \frac{\partial \bar{h}}{\partial t_i} e^{i\theta_i} .$$

In particular, if $\bar{h} = \lambda_1 t_1 + \lambda_2 t_2 + \dots + \lambda_n t_n$ (λ_i are real numbers) then h satisfies the condition (7.3), i.e. h is the imaginary part of a holomorphic function.

Now in Corollary 7.3 we choose

$$f_i = h_i = \bar{h}_i(\ell(\theta_{i1},z_1)\ ,\ \ell(\theta_{i2},z_2), \dots ,\ell(\theta_{in},z_n))\ ,\ i = 1,2, \dots ,n\ ,$$

where h_i are of the form (7.11) with $\bar{h}_i (1 \leq i \leq n)$ and $\theta_{ij} (1 \leq i,j \leq n)$ chosen arbitrarily so that $\bar{h}_1$ is linear and the assumption of Theorem 7.1 is satisfied. Letting $h_{ij} = \dfrac{\partial \bar{h}_i}{\partial t_j} (1 \leq i,j \leq n)$ we have

$$\left[\frac{\partial h_i}{\partial \bar{z}_j}\right] = \begin{bmatrix} h_{11}e^{i\theta_{11}} & h_{12}e^{i\theta_{12}} & \dots & h_{1n}e^{i\theta_{1n}} \\ h_{21}e^{i\theta_{21}} & h_{22}e^{i\theta_{22}} & \dots & h_{2n}e^{i\theta_{2n}} \\ \dots & \dots & \dots & \dots \\ h_{n1}e^{i\theta_{n1}} & h_{n2}e^{i\theta_{n2}} & \dots & h_{nn}e^{i\theta_{nn}} \end{bmatrix}$$

$\operatorname{Arg}\left[\dfrac{\partial h_i}{\partial \bar{z}_j}\right]$ is constant, for example, in each of the following cases:

(1) $\theta_{i1} = \theta_{i2} = \dots = \theta_{in}$, $i = 1,2, \dots ,n$

(2) $\theta_{1i} = \theta_{2i} = \dots = \theta_{ni}$, $i = 1,2, \dots ,n$

(3) $h_{ij} = c_{ij} h_j$, where $c_{ij} \in \mathbb{R}$, h_j are real valued functions $(1 \le i,j \le n)$.

(4) $h_{ij} = c_{ij} h_i$, where $c_{ij} \in \mathbb{R}$, h_i are real valued functions $(1 \le i,j \le n)$.

Assume that the metric tensor (g_{ij}) is real.

Substituting (7.12) into (7.9) we obtain

$$(7.13) \qquad (2i) \sum_{i,j=1}^{n} g_{ij} \left[\frac{\partial h_k}{\partial z_i} \frac{\partial h_q}{\partial \bar{z}_j} - \frac{\partial h_k}{\partial \bar{z}_i} \frac{\partial h_q}{\partial z_j} \right] =$$

$$= 2i \sum_{i,j=1}^{n} g_{ij} h_{ki} h_{qj} \left[\left(e^{i(\theta_{qj} - \theta_{ki})} - e^{i(\theta_{ki} - \theta_{qj})} \right) \right] .$$

The sum in (7.13) vanishes, for example, in each of the following cases:

(a) $\theta_{ij} = \theta$ for any $i,j = 1,2, \dots ,n$

(b) $\theta_{1i} = \theta_{2i} = \dots = \theta_{ni}$ for any $i = 1,2, \dots ,n$ and $h_{ki} h_{qj} = h_{kj} h_{qi}$ for any $i,j,k,q = 1,2, \dots ,n$.

(c) $\theta_{i1} = \theta_{i2} = \dots = \theta_{in}$ for any $i = 1,2, \dots ,n$ and vectors $(h_{k1}, h_{k2}, \dots ,h_{kn})$, $k = 1,2, \dots ,n$, are orthogonal

(d) $\theta_{ii} = \theta$ for any $i = 1,2, \ldots ,n$ and $h_{kj} = 0$ for any $k \neq j$

(e) $\theta_{1i} = \theta_{2i} = \ldots = \theta_{ni}$ for any $i = 1,2, \ldots ,n$ and $g_{ij} = 0$ for any $i \neq j$.

According to Proposition 7.2 each of the conditions (a) – (e) guarantees that the submanifold $X = \{h_1 = h_2 = \ldots = h_n = 0\}$ is Lagrangian. On the other hand, clearly, each of (a) – (b) implies one of the conditions (1) – (4). Thus, by applying now Theorem 7.1 we can conclude that the above construction provided by one of (a) – (e) gives a special Lagrangian submanifold.

EXAMPLE 7.5. Consider $\mathbb{C}^2 = \{(z_1,z_2)\}$ with $K = x_1x_2 - y_1y_2$. Let in Corollary 7.3 choose $f_1 = K$, $f_2 = z_1\bar{z}_1 - z_2\bar{z}_2$. A straightforward calculation shows that the submanifold $X = \{z \in \mathbb{C}^2 : f_1(z) = f_2(z) = 0\}$ is Lagrangian provided $g_{11} = g_{22}$, $g_{12} = g_{21} = 0$. On the other hand

$$\det \begin{bmatrix} \dfrac{\partial f_1}{\partial \bar{z}_1} & \dfrac{\partial f_1}{\partial \bar{z}_2} \\ \dfrac{\partial f_2}{\partial \bar{z}_1} & \dfrac{\partial f_2}{\partial \bar{z}_2} \end{bmatrix} = \det \begin{bmatrix} \bar{z}_2 & \bar{z}_1 \\ z_1 & -z_2 \end{bmatrix} = -|z_2|^2 - |z_1|^2 .$$

Consequently, $\operatorname{Arg}\left[\det\left[\dfrac{\partial f_i}{\partial \bar{z}_j}\right]\right] = 0$ on X. Hence, X is special Lagrangian.

Now we consider the case when submanifolds are given by parametric equations. Suppose, an n–dimensional submanifold X in M is described as the set $X = \{(f_1(t),f_2(t), \ldots ,f_n(t)) \in M \mid t \in \Omega\}$, where $f_1,f_2, \ldots ,f_n$ are smooth complex valued functions on an open set Ω of $\mathbb{R}^n$ with the standard coordinates $t = (t_1, t_2, \dots, t_n)$. $\operatorname{rank}_{\mathbb{R}} \left[\frac{\partial f_i}{\partial t_j} \right]$ is assumed to be equal to n everywhere on Ω . Vectors

$$\frac{\partial f(t)}{\partial t_i} = \left[\frac{\partial f_1(t)}{\partial t_i}, \frac{\partial f_2(t)}{\partial t_i}, \dots, \frac{\partial f_n(t)}{\partial t_i} \right], \quad i = 1,2, \dots, n ,$$

form a real basis of the tangent plane to X at each point $z = (f_1(t), f_2(t), \dots, f_n(t))$.

<u>THEOREM</u> 7.6. <u>Let</u> X <u>be a Lagrangian submanifold described parametriccally as above</u>. <u>Then</u> X <u>is special Lagrangian if and only if</u>

$$\operatorname{Arg}\left[\det \left[\frac{\partial f_i}{\partial t_j} \right] \right] + K = \text{const} \ \underline{\text{on}} \ X . \tag{7.14}$$

<u>Proof</u>. Since $\left\{ \frac{\partial f}{\partial t_1}, \dots, \frac{\partial f}{\partial t_n} \right\}$ is a real basis of $T_z X$,

$$p(z) = T_z X = \lambda \frac{\partial f}{\partial t_1} \wedge \frac{\partial f}{\partial t_2} \wedge \dots \wedge \frac{\partial f}{\partial t_n} ,$$

where λ is a real number. Therefore

$$p = \lambda \det \left[\frac{\partial f_i}{\partial t_j} \right] \xi_0 ,$$

where

$$\xi_0 = \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \wedge \dots \wedge \frac{\partial}{\partial z_n} .$$

Consequently, letting $\overline{\omega} = e^G dz_1 \wedge dz_2 \wedge \dots \wedge dz_n$ we have

(7.15) $$\alpha = -i \ln \overline{\omega}(p) = \mathrm{Arg}(\overline{\omega}(p)) = \mathrm{Arg}\left[\lambda\, e^G \det\left[\frac{\partial f_i}{\partial t_j}\right]\right] .$$

Using Theorem 6.3, Proposition 6.2 and (7.15) completes the proof.

Since $\left\{\frac{\partial f}{\partial t_1}, \frac{\partial f}{\partial t_2}, \dots, \frac{\partial f}{\partial t_n}\right\}$ is a basis of $T_z X$, the submanifold X is Lagrangian if and only if

(7.16) $$\frac{\partial f}{\partial t_i} \perp J\left[\frac{\partial f}{\partial t_j}\right]$$

for any $i,j = 1,2, \dots ,n$ and at any point $t \in \Omega$. Let

$$f = f' + if'' , \quad f_k = f_k' + if_k'' \quad (k = 1,2, \dots ,n) .$$

Then

$$\frac{\partial f}{\partial t_k} = \frac{\partial f'}{\partial t_k} + \frac{\partial f''}{\partial t_k} = \left[\frac{\partial f_1'}{\partial t_k} + i\frac{\partial f_1''}{\partial t_k}, \dots, \frac{\partial f_n'}{\partial t_k} + i\frac{\partial f_n''}{\partial t_k}\right]$$

$$J\left[\frac{\partial f}{\partial t_q}\right] = -\frac{\partial f''}{\partial t_q} + i\frac{\partial f'}{\partial t_q} = \left[-\frac{\partial f_1''}{\partial t_q} + i\frac{\partial f_1'}{\partial t_q}, \dots, -\frac{\partial f_n''}{\partial t_q} + i\frac{\partial f_n'}{\partial t_q}\right] .$$

Assuming the metric tensor (g_{ij}) to be real the condition (7.16) is equivalent to the following equalities

$$(7.17) \qquad \sum_{i,j=1}^{n} g_{ij}\left[\frac{\partial f_i'}{\partial t_k}\frac{\partial f_j''}{\partial t_q} - \frac{\partial f_n''}{\partial t_k}\frac{\partial f_j'}{\partial t_q}\right] = 0$$

for any $k,q = 1,2, \ldots ,n$.

EXAMPLE 7.7. It is easy to make sure that the condition (7.14) in Theorem 7.6 holds if $f_1,f_2, \ldots ,f_n$ satisfy the following equation

$$(7.18) \qquad \det\left[\frac{\partial f_i}{\partial t_j}\right] = \varphi(f(t),t)e^{-iK(f(t))+ic} ,$$

where $\varphi(f(t),t)$ is a real function of t and c is a real constant. In particular, if $f_i = f_i(t_i)$, $K = h_1(z_1) + h_2(z_2) + \ldots + h_n(z_n)$,
$\varphi = \varphi_1(f_1(t_1),t_1)\varphi_2(f_2(t_2),t_2) \ldots \varphi_n(f_n(t_n),t_n)$ then (7.18) decomposed to n equations, separate for each variable z_k :

$$(7.19) \qquad \frac{df_k(t_k)}{dt_k} = \varphi_k(f_k(t_k),t_k)e^{-ih_k(f_k(t_k))+ic_k} .$$

All the equations (7.19) are of the same form

$$(7.20) \qquad \frac{d\lambda}{d\theta} = a(\lambda,\theta)e^{-ih(\lambda(\theta))+ic} .$$

Letting $\lambda(\theta) = p(\theta) + iq(\theta)$ we can rewrite (7.20) in the form

$$\begin{cases} p' = a(p,q,\theta)\cos(h(p,q) - c) \\ q' = -a(p,q,\theta)\sin(h(p,q) - c)\ . \end{cases} \tag{7.21}$$

Consider, for example, some special cases :

1) $a \equiv 1$, $h(p,q) = \alpha p + \beta q$, where α and β are real numbers that are not trivial simultaneously. Note that $h(p,q) = 0$ if $p/q = -\beta/\alpha$. Choose $c = \text{arc tg}(-\alpha/\beta)$. A straightforward calculation shows that

$$p = \frac{\beta\theta}{\sqrt{\alpha^2 + \beta^2}}\ , \quad p = \frac{-\alpha\theta}{\sqrt{\alpha^2 + \beta^2}} \tag{7.22}$$

are solutions of (7.21).

2) Suppose $h(p,q)$ is linear as above. Without lost of generality one can assume that $h(p,q) = q$. Choose $a = -[\sin(h(p,q) - c]^{-1} = -[\sin(q - c)]^{-1}$. Then (7.21) is of the form

$$\begin{cases} p' = -\text{ctg}(q - c) \\ q' = 1 \end{cases} . \tag{7.23}$$

The solutions of (7.23) are

$$p = -\int \text{ctg}(\theta + b - c)d\theta + d\ , \quad q = \theta + b \tag{7.24}$$

where b , d are arbitrary real constants.

REMARK. Assume that $g_{ij} = 0$ for any $i \neq j$. Then (7.17) is satisfied for functions $f_i = f_i(t_i)$ $(i = 1,2, \dots ,n)$. Thus, in this case solutions of (7.19) give special Lagrangian submanifolds

EXAMPLE 7.8. Suppose $M = \mathbb{C}^2$ with
$K(z_1,z_2) = x_1 \sin a + y_1 \cos a + x_2 \sin b + y_2 \cos b$ $(a,b \in \mathbb{R})$. Let X be given by the parametric equations

$$(7.25) \qquad z_1 = f_1(r,\theta_1) = re^{i\theta_1}, \; z_2 = f_2(r,\theta_2) = re^{i\theta_2}, \; \theta_1 + \theta_2 = -(a+b).$$

For $z = (z_1,z_2) \in X$ we have

$$K(z) = r(\cos\theta_1 \sin a + \sin\theta_1 \cos a + \cos\theta_2 \sin b + \sin\theta_2 \cos b) =$$

$$= r(\sin(\theta_1 + a) + \sin(\theta_2 + b)) =$$

$$= 2r \sin\frac{\theta_1 + \theta_2 + a + b}{2} \cos\frac{\theta_1 + a - \theta_2 - b}{2} = 0 .$$

Now we calculate the tangent plane to X.

$$\frac{\partial f}{\partial r} = \left[e^{i\theta_1}, e^{i\theta_2}\right], \; \frac{\partial f}{\partial \theta_1} = \left[ire^{i\theta_1}, -ire^{i\theta_2}\right]$$

$$\det \begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_2}{\partial r} \\ \frac{\partial f_1}{\partial \theta_1} & \frac{\partial F_2}{\partial \theta_1} \end{bmatrix} = 2ire^{i(\theta_1+\theta_2)} .$$

Therefore f satisfies (7.14). A straightforward calculation shows that (7.17) holds if $g_{11} = g_{22}$(g_{12} and g_{21} are not necessarily trivial). Thus in this case the construction above gives a special Lagrangian cone.

§ 8. Special Lagrangian submanifolds of unitary sums

In this section we present constructions giving special Lagrangian submanifolds in the form of unitary sums. We recall that as above $G = \ln\sqrt{\det(g)}$ and K denotes the function related to G by the equality $dK = JdG$. Suppose that grad K never vanishes on an open set $\Omega \subset \mathbb{C}^n$. By definition, for every vector ξ we have

$$g(J \text{ grad } K,\xi) = g(\text{grad } K - J\xi) = dK(-J\xi) = JdG(-J\xi) = dG(-\xi) .$$

That means, that $J \text{ grad } K = -\text{grad } G$. Assume that $\mathbb{C}^n$ can be represented as a direct sum $\mathbb{C}^n = P \oplus Q$ of complex linear subspaces P and Q with metric g' and g'' respectively. Moreover, the metric g is expressed through g' and g'' as follows. For any vectors $u,v \in T_{p+q}\mathbb{C}^n$ $g(u,v) = g'(u',v') + g''(u'',v'')$, where $u = u' + u''$, $v = v' + v''$, u' , $v' \in T_pP$, $U'',V'' \in T_qQ$. By a complex linear transformation we can get new complex coordinates $z'_1,z'_2, \dots ,z'_n$ on $\mathbb{C}^n$ such that $z'_1,z'_2, \dots ,z'_k$ are coordinates on P and $z'_{k+1}, \dots ,z'_n$ are coordinates on Q . Letting $(\tilde{g}_{ij})$, $(\tilde{g}'_{ij})$ $(\tilde{g}''_{ij})$ denote the corresponding metric tensors on $\mathbb{C}^n$, P and Q respectively we have

$$\tilde{g}_{ij} = \tilde{g}_{ji} = 0 \text{ if } i \le k\,,\ j > k$$

$$\tilde{g}_{ij}(p+q) = \tilde{g}'_{ij}(p) \text{ if } i \le k\,,\ j \le k\,,\ p \in P\,,\ q \in Q$$

$$\tilde{g}_{ij}(p+q) = \tilde{g}''_{ij}(q) \text{ if } i \ge k\,,\ j \ge k\,,\ p \in P\,,\ q \in Q\,.$$

Hence $\tilde{G} = \tilde{G}' + \tilde{G}''$ and $\tilde{K} = \tilde{K}' + \tilde{K}''$, where $\tilde{G} = \ln\sqrt{\det(\tilde{g}_{ij})}$, $\tilde{G}' = \ln\sqrt{\det(\tilde{g}'_{ij})}$, $\tilde{G}'' = \ln\sqrt{\det(\tilde{g}''_{ij})}$ and $d\tilde{K} = Jd\tilde{G}$, $d\tilde{K}'' = Jd\tilde{G}'$, $d\tilde{K}' = Jd\tilde{G}''$. Suppose $X = X' \oplus X''$, where X', X'' are submanifolds of P and Q respectively. Obviously, in order for X to be Lagrangian (in $\mathbb{C}^n$) it is necessary and sufficient that X' and X'' are Lagrangian in P and Q respectively. Set

$$\overline{\omega} = e^{\tilde{G}} dz'_1 \wedge dz'_2 \wedge \ldots \wedge dz'_n$$

$$\overline{\omega}' = e^{\tilde{G}'} dz'_1 \wedge dz'_2 \wedge \ldots \wedge dz'_k$$

$$\overline{\omega}'' = e^{\tilde{G}''} dz'_{k+1} \wedge dz'_{k+2} \wedge \ldots \wedge dz'_n$$

$$\tilde{\alpha}(p+q) = -i \ln \overline{\omega}(T_{p+q}X) \quad (p \in P, q \in Q)$$

$$\tilde{\alpha}'(p) = -i \ln \overline{\omega}'(T_pX')$$

$$\tilde{\alpha}''(p) = -i \ln \overline{\omega}''(T_qX'')\,.$$

Since the sum q $P \oplus Q$ is orthogonal and $\tilde{G} = \tilde{G}' + \tilde{G}''$, it is easy to see that $\overline{\omega} = \overline{\omega}' \wedge \overline{\omega}''$, $\overline{\omega}(T_{p+q}X) = \overline{\omega}'(T_pX')\overline{\omega}''(T_qX')$. Hence $\tilde{\alpha}(p+q) = \tilde{\alpha}'(p) + \tilde{\alpha}''(q)$. Because $\tilde{K}'$, $\tilde{\alpha}'$ are functions on P and $\tilde{K}''$, $\tilde{\alpha}''$ are functions on Q, $\tilde{K} + \tilde{\alpha} = \text{const}$ if and only if $\tilde{K}' + \tilde{\alpha}' = \text{const}$ and $\tilde{K}'' + \tilde{\alpha}'' = \text{const}$. Now, applying Theorem 6.3 and Proposition 6.2 we can conclude that X is special Lagrangian if and only if X' and X'' are special Lagrangian. Finally we note that since the coordinate systems $(z_1, z_2, \dots, z_n)$ and $(z_1', z_2', \dots, z_n')$ differ by only a linear transformation, $\tilde{K} = K + \text{const}$. Thus we have proved the following

PROPOSITION 8.1. Suppose $\mathbb{C}^n = P \oplus Q$ is a unitary sum and K, $\tilde{K}$, $\tilde{K}'$, $\tilde{K}''$ are described as above. Then

(a) $\tilde{K} = K + \text{const}$, $\tilde{K} = \tilde{K}' + \tilde{K}''$.

(b) The sum $X = X' \oplus X''$ of submanifolds $X' \subset P$ and $X'' \subset Q$ is special Lagrangian if and only if X' and X'' are special Lagrangian in P and Q respectively.

EXAMPLE 8.2. Assume that the metric g on $\mathbb{C}^n$ is given so that K is a linear function of $x_1, y_1, \dots, x_n, y_n$. Then $\operatorname{grad} K$ and $J \operatorname{grad} K = -\operatorname{grad} H$ are fixed vectors. Set $P = \{z \in \mathbb{C}^n : K = K(0),\ G = G(0)\}$, $Q = \operatorname{Span}\{\operatorname{grad} K, \operatorname{grad} G\}$. It is easy to verify that P is a complex linear subspace and $\mathbb{C}^n = P \oplus Q$ is a unitary sum.

Choosing complex coordinates $(z_1', z_2', \dots, z_n')$ as in Proposition 8.1 we have $\tilde{K} = \text{const}$ on P because $K = \text{const}$ on P. Consequently, $\tilde{K}' = \text{const}$ and $\tilde{G}' = \text{const}$. Thus, P is a complex space with a metric g' like the standard metric (that is $\det(g') = \text{const}$) and special Lagrangian submanifolds in P may be found by methods of $[HL_1]$. Further, the function $\tilde{K}''$ is linear on Q and special Lagrangian lines on Q are determined as in Example 7.7. In particular, choosing $X''//\text{grad}\, G$ (see(7.22)) we obtain that the cylinder parallel $\text{grad}\, G$ through any special Lagrangian submanifold of P is special Lagrangian in $\mathbb{C}^n$. Similarly, the lines determined by (7.24) give special Lagrangian "curvelinear cylinders" in $\mathbb{C}^n$.

<u>EXAMPLE</u> 8.3. Consider the real space $\mathbb{R}^n$ with a (non–standard) metric g chosen so that $G = \ln\sqrt{\det(g)}$ is a linear function. By using the metric g one can identify the tangent bundle $T\mathbb{R}^n$ with the cotangent bundle $T^*\mathbb{R}^n$ which is equipped with the natural complex (and symplectic) structure of $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$ (the metric g is expended naturally to the metric of $\mathbb{C}^n$). Assume that $\mathbb{C}^n$ with the metric g is L–calibrated. Then the function K related to G by the equality $dK = JdG$ is also linear and depends only on the second term in the sum $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$. In this case the space $P = \{z \in \mathbb{C}^n : K = K(0),\ G = G(0)\}$ is nothing but the tangent subbundle $T\mathbb{R}^{n-1}$ of the space $\mathbb{R}^{n-1} = \{x \in \mathbb{R}^n : G = G(0)\}$. Suppose X' is a submanifold in $\mathbb{R}^{n-1}$ and $X = X' \oplus \{\text{grad}\, G\}$ is the cylinder parallel $\text{grad}\, G$ through X'. Let $N(X')$ and $N(X)$ denote the normal bundles of X' in $T\mathbb{R}^{n-1}$ and X in $T\mathbb{R}^n$ respectively. Note that $N(X)$ is the cylinder parallel $\text{grad}\, G$ through $N(X')$. By virtue of Example 8.2 $N(X)$ is special Lagrangian if and only if $N(X')$ is special Lagrangian (in $T\mathbb{R}^{n-1}$). $[HL_1]$ proved that $N(X')$ is special Lagrangian if all the invariants of odd order of the second fundamental form at each normal vector to X' vanish, i.e. the set of eigenvalues of the second fundamental form is invariant under

multiplication by -1 . On the other hand, one can prove that the last condition is equivalent to the same one with replacing X' by X . Thus, the normal bundle $N(X)$ of any cylinder X parallel grad G such that all the invariants of odd order of the second fundamental form at each normal vector to X vanish is special Lagrangian.

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