LUSTERNIK–SCHNIRELMANN THEORY ON GENERAL SPACES

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ABSTRACT. We extend Lusternik–Schnirelmann theory to pairs (f, φ), where φ is a homotopy equivalence of a space X, f is a function on X which decreases along φ and (f, φ) satisfies a discrete analog of the Palais–Smale condition. The theory is carried out in an equivariant setting.

1. INTRODUCTION

The Lusternik–Schnirelmann category $\operatorname{cat}_X A$ of a subset A of a topological space X is the minimal number of open and in X contractible sets which cover A [14, 9]. We set $\operatorname{cat} X = \operatorname{cat}_X X$. Lusternik and Schnirelmann [14] proved that for any smooth function f on a closed manifold M

$$\text{(a)} \qquad \sum_{d \in \mathbb{R}} \operatorname{cat}_M \left(K \cap f^{-1}(d)\right) \geq \operatorname{cat} M$$

where K denotes the set of critical points of f. In particular, f has at least $\operatorname{cat} M$ critical points, and if M is connected, then

(b) $\qquad K$ is infinite or f has at least $\operatorname{cat} M$ critical levels.

For generic f, (a) can be written as

$$\text{(c)} \qquad \operatorname{cat} K \geq \operatorname{cat} M.$$

In [18], the estimates (a) and (b) have been generalized to a class of Banach manifolds M and functions $f\colon M \to \mathbb{R}$ which satisfy the Palais–Smale condition, and more recently, versions for continuous functions on certain metric spaces have been found (see [4]). In this paper, we extend (a), (b) and (c) to more general situations.

Consider a topological space X, a continuous map $\varphi\colon X \to X$ and a continuous function $f\colon X \to \mathbb{R}$. We say that f is a *Lyapunov function* for φ if $f(\varphi(x)) < f(x)$ whenever x is not a fixed point of φ.

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1.1. Definition. Let f and φ be as above and let Y be a subspace of X. We say that the pair (f, φ) satisfies condition (D) on Y if f is a Lyapunov function for φ and the following holds.

> If A is a subset of Y on which $|f|$ is bounded but on which $f(y) - f(\varphi(y))$ is not bounded away from zero, then there is a fixed point of φ in the closure of A.

Observe that if Y is compact or if f is proper then the pair (f, φ) satisfies condition (D) whenever f is a Lyapunov function for φ. Condition (D) is a discrete analog of the Palais–Smale condition for C^1 functions on Banach manifolds. In fact, we show in Proposition 9.1 that in the case of a Hilbert manifold, condition (D) generalizes the Palais–Smale condition.

A topological space X is called *weakly locally contractible* if each point of X has a neighborhood which is contractible in X. Observe that this property is equivalent to $\operatorname{cat} X$ being definite. A space X is called *binormal* if $X \times [0,1]$ (and hence X) is normal. In particular, every metric space and every compact space is binormal. A space X is called an *absolute neighborhood retract* (ANR) if for each closed subset Z of a normal space Y every map $\rho \colon Z \to X$ admits an extension to a neighborhood of Z. CW-complexes and topological manifolds are ANR's. Every binormal ANR is weakly locally contractible (see Lemma 4.2).

If some category $\operatorname{cat}_X A$ is not finite, i.e., infinite or indefinite, we write $\operatorname{cat}_X A = \infty$. We agree that $\infty \geq \infty$ and that $\infty \geq n$ for every $n \in \mathbb{Z}$. Given a real valued function f on a space X, we set

$$f^a = \{x \in X \mid f(x) \leq a\}.$$

1.2. Theorem. *Consider a space X and a homotopy equivalence φ of X. Let F be the set of fixed points of φ. Assume that the function $f \colon X \to \mathbb{R}$ is bounded below and that (f, φ) satisfies condition* (D) *on f^b for some $b \in \mathbb{R}$.*

(a) *We always have*

$$\sum_{d \leq b} \operatorname{cat}_X \left(F \cap f^{-1}(d)\right) \;\geq\; \operatorname{cat}_X f^b.$$

(b) *If X is connected and weakly locally contractible, then $F \cap f^b$ is infinite or $f(F) \cap]-\infty, b]$ contains at least $\operatorname{cat}_X f^b$ elements.*

(c) *If X is a binormal ANR and $f(F)$ is discrete, then*

$$\operatorname{cat}(F \cap f^b) \;\geq\; \operatorname{cat}_X f^b.$$

If φ is homotopic to the identity map, then these estimates also hold for $b = \infty$ (in which case $f^b = X$).

Our main results generalize Theorem 1.2 in two directions. Firstly, we overcome the assumption of f being bounded below: Given $-\infty < a < b < \infty$, Theorem 6.1 provides a lower bound for the number and the category of the rest points of a homotopy equivalence φ in the slice $\{x \in X \mid a < f(x) \leq b\}$, and in Theorem 7.1 we show that if φ is homotopic to the identity, then the categorical estimates in Theorem 6.1 can be replaced by various finer ones, which also hold for $b = \infty$. Secondly, all results readily extend to equivariant situations where a compact Lie group acts on X.

Finally, we observe in Section 8 that if φ is a homeomorphism or the time-1-map of a flow, then some of the previously established estimates can be improved by using different invariants of Lusternik–Schnirelmann type.

Notice that even if φ is homotopic to the identity, we in general do not require that there exists a homotopy between the identity and φ which decreases along f. Therefore, our theory should also prove useful in understanding the fixed point set of other than gradient-like systems such as those arising in simulated annealing.

The usefulness of extending Lusternik–Schnirelmann theory to spaces more general than manifolds was demonstrated by various proofs of the Arnold conjecture about the number of fixed points of Hamiltonian symplectomorphisms [10, 11, 19]. In these proofs different variants of the Lusternik–Schnirelmann category were considered. We thus develop an axiomatically defined version of Lusternik–Schnirelmann theory (Theorem 2.3) and derive Theorem 6.1 and Theorem 7.1 from Theorem 2.3. In [20] this axiomatic approach will be used to develop a Lusternik–Schnirelmann theory for certain symplectic and contact manifolds.

All spaces are assumed to be Hausdorff, and all maps and functions are assumed to be continuous.

2. An axiomatic version of LS-theory

2.1. Definition (cf. [14, 5, 6])**.** An *index function* on a topological space X is a function $\nu\colon 2^X \times 2^X \to \mathbb{N} \cup \{0\}$ satisfying the following axioms:

(monotonicity) If $A \subset B$, then $\nu(A, Y) \leq \nu(B, Y)$ for all $Y \subset X$;

(continuity) For every closed subset $A \subset X$ there exists a neighborhood U of A such that $\nu(A, Y) = \nu(U, Y)$ for all $Y \subset X$;

(mixed subadditivity) For all $A, B, Y \subset X$,

$$\nu(A \cup B, Y) \;\le\; \nu(A, Y) + \nu(B, \emptyset).$$

Given a map $\varphi : X \to X$ and a subset Z of X, an index function ν is (φ, Z)-*supervariant* if

((φ, Z)-supervariance) $\nu(\varphi(A), Z) \ge \nu(A, Z)$ for all $A \subset X$.

Below we write $\nu(A)$ instead of $\nu(A, \emptyset)$.

2.2. Lemma (cf. [14]). *Consider a self-map $\varphi \colon X \to X$ on the space X. Denote the set of fixed points of φ by F. Let $f \colon X \to \mathbb{R}$ be a function and let $a < b$ be real numbers such that (f, φ) satisfies condition* (D) *on $f^{-1}[a, b]$. Then the following assertions hold.*

(i) *Suppose that $f^{-1}[a, b[\cap F$ is empty. Then, given any neighborhood U of $f^{-1}(b) \cap F$, there exists $n \in \mathbb{N}$ such that $\varphi^n(f^b \setminus U) \subset f^a$. In particular, $\nu(f^b \setminus U, Z) \le \nu(f^a, Z)$ for every (φ, Z)-supervariant index function ν, and hence $\nu(f^b \setminus U, Z) = \nu(f^a, Z)$ whenever $f^a \cap U = \emptyset$.*

(ii) *Suppose that there is $\varepsilon > 0$ such that $f^{-1}]a, a + \varepsilon[\cap F$ is empty. Then, given any neighborhood U of f^a, there exists $\delta > 0$ such that $\varphi(f^{a+\delta}) \subset U$. In particular, for every (φ, Z)-supervariant index function ν, there exists $\delta > 0$ such that $\nu(f^{a+\delta}, Z) = \nu(f^a, Z)$.*

Proof. (i) Since (f, φ) satisfies condition (D) on $f^{-1}[a, b]$ and since φ has no fixed point on the closed set $f^{-1}[a, b] \setminus U$, there exists $\delta_1 > 0$ such that

$$f(x) - f(\varphi(x)) \ge \delta_1 \quad \text{for every } x \in f^{-1}[a, b] \setminus U.$$

If $b - \delta_1 \le a$, we are done. Otherwise, let $\delta_2 > 0$ be such that

$$f(x) - f(\varphi(x)) \ge \delta_2 \quad \text{for every } x \in f^{-1}[a, b - \delta_1].$$

Choose $n \in \mathbb{N}$ so large that $(n - 1)\delta_2 \ge b - \delta_1 - a$. Then $\varphi^n(f^b \setminus U) \subset f^a$. Indeed, assume that $f(\varphi^n(x)) > a$ for some $x \in f^b \setminus U$. Then $\varphi^k(x) \in f^{-1}[a, b - \delta_1]$ for $k = 1, \dots, n$. Hence,

$$\begin{aligned} f(\varphi^n(x)) \;&=\; f(\varphi^1(x)) + \sum_{k=1}^{n} f(\varphi^{k+1}(x)) - f(\varphi^k(x)) \\ &\le\; b - \delta_1 - (n - 1)\delta_2 \;\le\; a, \end{aligned}$$

a contradiction.

Assume that ν is a (φ, Z)-supervariant index function. Choose n such that $\varphi^n(f^b \setminus U) \subset f^a$. Then, by (φ, Z)-supervariance and monotonicity,

$$\nu(f^b \setminus U, Z) \le \nu(\varphi^n(f^b \setminus U), Z) \le \nu(f^a, Z).$$

(ii) Fix a neighborhood U of f^a and choose a sequence of real numbers

$$a+\varepsilon > a_1 > a_2 > \cdots > a_n > \cdots$$

which converges to a. We claim that $\varphi(f^{a_n}) \subset U$ for some n. Arguing by contradiction, suppose that $A_n = f^{a_n} \setminus \varphi^{-1}(U) \neq \emptyset$ for all n. Then $f(x)-f(\varphi(x)) \leq a_n - a$ for all $x \in A_n$, and so $\inf\{f(x)-f(\varphi(x)) \mid x \in A_1\} = 0$. Since A_1 is closed, condition (D) implies that $A_1 \cap F \neq \emptyset$. This contradicts $f^{-1}]a, a+\varepsilon[\cap F = \emptyset$. So, $\varphi(f^{a_n}) \subset U$ for some n. Set $\delta = a_n - a$.

Assume that ν is a (φ, Z)-supervariant index function. By continuity, there exists a neighborhood U of f^a such that $\nu(f^a, Z) = \nu(U, Z)$. According to what we said above, there exists $\delta > 0$ such that $\varphi(f^{a+\delta}) \subset U$. Then

$$\nu(f^a, Z) = \nu(U, Z) \geq \nu(\varphi(f^{a+\delta}), Z) \geq \nu(f^{a+\delta}, Z) \geq \nu(f^a, Z)$$

and so $\nu(f^{a+\delta}, Z) = \nu(f^a, Z)$. □

The Lusternik–Schnirelmann Theorem, from the modern standpoint, is

2.3. Theorem. *Consider a self-map $\varphi\colon X \to X$ on the topological space X. Let F be the set of fixed points of φ and let f be a Lyapunov function for φ. Let $a < b$ be real numbers such that (f, φ) satisfies condition* (D) *on $f^{-1}[a, b]$. Suppose that $f(F) \cap]a, b] = \{d_1, \dots, d_m\}$ is a finite set. Then*

$$\nu(f^a, f^a) \;+\; \sum_{i=1}^{m} \nu\left(F \cap f^{-1}(d_i)\right) \;\geq\; \nu(f^b, f^a).$$

for any (φ, f^a)-supervariant index function ν.

Proof. Denote $\nu(A, f^a)$ by $\mu(A)$. If $\mu(f^b) = \mu(f^a)$, there is nothing to prove. So assume that $\mu(f^a) < \mu(f^b)$. Set

$$c_k = \inf\{c \in \mathbb{R} \bigm| \mu(f^c) \geq k\}, \qquad k = \mu(f^a)+1, \dots, \mu(f^b).$$

By monotonicity of μ,

$$a \leq c_{\mu(f^a)+1} \leq \cdots \leq c_{\mu(f^b)} \leq b. \tag{2.1}$$

2.4. Claim. *For all $k = \mu(f^a)+1, \dots, \mu(f^b)$,*

$$\mu(f^{c_k-\varepsilon}) \;<\; k \quad \textit{for all } \varepsilon > 0, \tag{2.2}$$

$$\mu(f^{c_k}) \;\geq\; k. \tag{2.3}$$

Proof. Inequality (2.2) follows from the definition of c_k.

We prove (2.3). By (2.1), $c_k \le b$. Suppose first that $c_k < b$. Since $f(F) \cap]a,b]$ is finite, we find $\varepsilon > 0$ such that $f(F) \cap]c_k, c_k + \varepsilon[= \emptyset$, i.e., $f^{-1}]c_k, c_k+\varepsilon[\cap F = \emptyset$. By Lemma 2.2 (ii) there exists $\delta > 0$ such that $\mu(f^{c_k+\delta}) = \mu(f^{c_k})$. But, by monotonicity, $\mu(f^{c_k+\delta}) \ge k$, and so $\mu(f^{c_k}) \ge k$. Suppose now $c_k = b$. Then $\mu(f^{c_k}) = \mu(f^b) \ge k$ as well. $\diamond$

Plugging $k = \mu(f^a)+1$ into (2.3), we find $\mu(f^{c_{\mu(f^a)+1}}) \ge \mu(f^a)+1$. Hence,

$$a < c_{\mu(f^a)+1}. \tag{2.4}$$

2.5. Claim. *$f^{-1}(c_k)$ contains at least one point of F.*

Proof. Suppose the contrary. In view of (2.1), (2.4) and the finiteness of $f(F) \cap]a,b]$, we then find $\varepsilon > 0$ such that $f^{-1}[c_k - \varepsilon, c_k]$ does not contain points of F. By Lemma 2.2 (i), there exists n such that $\varphi^n(f^{c_k}) \subset f^{c_k-\varepsilon}$. Hence, by (φ, f^a)-supervariance, monotonicity and (2.2),

$$\mu(f^{c_k}) \le \mu(\varphi^n(f^{c_k})) \le \mu(f^{c_k-\varepsilon}) < k.$$

This contradicts (2.3). $\diamond$

2.6. Lemma (cf. [14, II, §4], [7, Lemma 19.12])**.** *If $c_k = c_{k+1} = \dots = c_{k+r}$, then*

$$\nu(F \cap f^{-1}(c_k)) \ge r+1.$$

Proof. Set $A = F \cap f^{-1}(c_k)$. By continuity, there is a neighborhood U of A with $\nu(U) = \nu(A)$. Arguing as above we find $\varepsilon > 0$ such that $f^{-1}[c_k - \varepsilon, c_k[\cap F$ is empty. By Lemma 2.2 (i) there exists n such that $\varphi^n(f^{c_k} \setminus U) \subset f^{c_k-\varepsilon}$. So, by (φ, f^a)-supervariance, monotonicity and (2.2), $\mu(f^{c_k} \setminus U) < k$. Now, by monotonicity, subadditivity and (2.3),

$$\begin{aligned}
\mu(f^{c_k} \setminus U) + \nu(U) &= \mu(f^{c_{k+r}} \setminus U) + \nu(U) \\
&\ge \mu(f^{c_{k+r}} \setminus U) + \nu(f^{c_{k+r}} \cap U) \\
&\ge \mu(f^{c_{k+r}}) \\
&\ge k + r,
\end{aligned}$$

and so $\nu(U) > r$. Thus, $\nu(A) \ge r+1$. $\diamond$

We continue the proof of the theorem. We have

$$\begin{aligned}
c_{\mu(f^a)+1} = \dots = c_{i_1} &< c_{i_1+1} = \dots = c_{i_2} \\
&< \dots < c_{i_n+1} = \dots = c_{i_{n+1}} = c_{\mu(f^b)}.
\end{aligned}$$

Set $F_k = F \cap f^{-1}(c_{i_k})$ and $i_0 = \mu(f^a)$. Then, by Lemma 2.6, $\nu(F_k) \geq i_k - i_{k-1}$. Thus

$$\sum_{k=1}^{n+1} \nu(F_k) \geq i_{n+1} - i_0 = \mu(f^b) - \mu(f^a).$$

In view of (2.4) and Claim 2.5, $\{c_{\mu(f^a)+1}, \dots, c_{\mu(f^b)}\} \subset \{d_1, \dots, d_m\}$, and so Theorem 2.3 follows. □

3. Relative equivariant category

Let G be a compact Lie group. A *G-space* is a space X together with a continuous action $G \times X \to X$, $(g, x) \mapsto gx$. A *subspace* of a G-space X is a G-invariant subspace. If X_1 and X_2 are G-spaces, then a *G-map* $\varphi\colon X_1 \to X_2$ is an equivariant map, i.e., $\varphi(gx) = g\varphi(x)$ for all $g \in G$ and $x \in X_1$, and a *G-homotopy* $H_t\colon X_1 \to X_2$ is a map $H\colon X_1 \times I \to X_2$ such that $H_t\colon X_1 \to X_2$ is a G-map for each t.

Let W, Y be subspaces of a G-space X. We say that W *is G-deformable to* Y if there is a G-homotopy $H_t\colon W \to X$ which starts with the inclusion and is such that $H_1(W) \subset Y$. If in addition $H_t(W \cap Y) \subset Y$ for all t, we say that W *is G-deformable to* $Y \bmod Y$.

Fix a class $\mathcal{G}$ of homogeneous G-spaces,

$$\mathcal{G} \subset \{G/H \mid H \subset G \text{ is a closed subgroup}\}.$$

A subspace A of a G-space X is called *$\mathcal{G}$-categorical* if there exist G-maps $\alpha\colon A \to G/H$ and $\beta\colon G/H \to X$ with $G/H \in \mathcal{G}$ such that the inclusion $A \hookrightarrow X$ is G-homotopic to the composition $\beta\alpha$.

3.1. Definition (cf. [8, 6])**.** Fix a subspace Y of the G-space X. If A is another subspace of X, we set $\mathcal{G}\text{-cat}_X(A, Y) = k$ if A can be covered by $k+1$ *open* subspaces $A_0, A_1, \dots, A_k$ of X such that

(i) A_0 is G-deformable to Y

(ii) $A_1, \dots, A_k$ are $\mathcal{G}$-categorical

and if k is minimal with this property. If there is no such number k, we set $\mathcal{G}\text{-cat}_X(A, Y) = \infty$.

The invariant $\mathcal{G}\text{-cat}_X(A \bmod Y)$ is defined by replacing (i) by

(i mod) A_0 is G-deformable to $Y \bmod Y$, and $A_0 \supset A \cap Y$.

We set $\mathcal{G}\text{-cat}_X A = \mathcal{G}\text{-cat}_X(A, \emptyset)$ and $\mathcal{G}\text{-cat}\, X = \mathcal{G}\text{-cat}_X X$.

3.2. Remarks. 1. If G acts trivially on X and $\mathcal{G}$ contains the point G/G, then A is $\mathcal{G}$-categorical iff it is contractible in X. Therefore, in this case $\mathcal{G}\text{-cat}_X A$ equals the classical open Lusternik–Schnirelmann category $\text{cat}_X A$ [14, 3, 9].

2. If $\mathcal{G}_1 \subset \mathcal{G}_2$, then $\mathcal{G}_1\text{-}\text{cat}_X(A,Y) \geq \mathcal{G}_2\text{-}\text{cat}_X(A,Y)$. If G has a fixed point x and $\mathcal{G}$ does not contain G/G, then $\mathcal{G}\text{-cat}_X A = \infty$ whenever $x \in A$.

3. If G acts freely on X and $\mathcal{G}$ is the full class of homogeneous G-spaces, then $\mathcal{G}\text{-cat}_X A = \text{cat}_{X/G} A/G$. In general, however, $\mathcal{G}\text{-cat}_X A \geq \text{cat}_{X/G} A/G$. E.g., if $\mathbb{Z}_2$ acts on $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ by complex conjugation, then $\mathcal{G}\text{-cat}\, S^1 \geq 2 > 1 = \text{cat}\, S^1/\mathbb{Z}_2$.

4. Agreeing that $\infty \geq \infty - \infty$ and $0 \geq n - \infty$ for every $n \in \mathbb{Z}$, we always have

$$\mathcal{G}\text{-cat}_X(A \bmod Y) \;\geq\; \mathcal{G}\text{-cat}_X(A,Y) \;\geq\; \mathcal{G}\text{-cat}_X A - \mathcal{G}\text{-cat}_X Y.$$

If $X = \{(x,y) \in S^1 \mid y \geq 0\}$ and $Y = X \cap \{y = 0\}$, then

$$\text{cat}_X(X \bmod Y) = 1 > 0 = \text{cat}_X(X,Y);$$

and if X is the union of the unit circle S centered at $(0,-1)$ and its translate centered at $(0,1)$, then $\text{cat}_X(X,S) = 1 > 2 - 2 = \text{cat}_X X - \text{cat}_X S$.

4. Some general topology

If A is a subspace of the G-space X, we set $\mathcal{G}\text{-}\overline{\text{cat}}_X A = k$ if A can be covered by k *closed* categorical subspaces $A_1, \dots, A_k$ of X and if k is minimal with this property. If there is no such k, we set $\mathcal{G}\text{-}\overline{\text{cat}}_X A = \infty$. We also set $\mathcal{G}\text{-}\overline{\text{cat}} X = \mathcal{G}\text{-}\overline{\text{cat}}_X X$, and, if G acts trivially and $\mathcal{G}$ contains G/G, $\overline{\text{cat}}_X A = \mathcal{G}\text{-}\overline{\text{cat}}_X A$.

4.1. Lemma. *Let A be a closed subspace of a normal G-space X. Then*

$$\mathcal{G}\text{-cat}_X A \;\geq\; \mathcal{G}\text{-}\overline{\text{cat}}_X A.$$

Proof. We may assume that $\mathcal{G}\text{-cat}_X A$ is finite. Let $A \subset U_1 \cup \cdots \cup U_k$ where each U_i is open and $\mathcal{G}$-categorical. Then $\{X \setminus A, U_1, \dots, U_k\}$ is an open covering of X. According to [1, § 2, Proposition 20] there is an open covering $\{V_0, V_1, \dots, V_k\}$ of X such that $\overline{V_0} \subset X \setminus A$ and $\overline{V_i} \subset U_i$, $i = 1, \dots, k$. Hence, $A \subset \overline{V_1} \cup \cdots \cup \overline{V_k}$. By Proposition 1.1.1 of [16], every set $G\overline{V_i} = \{gv \mid g \in G,\, v \in \overline{V_i}\}$ is closed. The inclusions $\overline{V_i} \subset G\overline{V_i} \subset GU_i \subset U_i$ imply that $A \subset \cup G\overline{V_i}$ and that each $G\overline{V_i}$ is a closed $\mathcal{G}$-categorical subspace. Thus, $\mathcal{G}\text{-}\overline{\text{cat}}_X A \leq k$. □

A G-space X is called a *G-ANR* if for each closed subspace Z of a normal G-space Y every G-map $\rho\colon Z \to X$ admits an equivariant extension to a G-invariant neighborhood of Z. Notice that G/H is a G-ANR for every closed subgroup H of G [16, Corollary 1.6.7].

4.2. Lemma. *Let A be a closed $\mathcal{G}$-categorical subset of a binormal G-ANR X. Then A is contained in an open $\mathcal{G}$-categorical subset of X.*

Proof (cf. [12, IV, Proposition 3.4] and [5, Appendix B]). Consider $\alpha\colon A \to G/H$ and $\beta\colon G/H \to X$ as in Definition 3.1 and let $H\colon A \times I \to X$ be a G-homotopy between $i\colon A \hookrightarrow X$ and $\beta\alpha$. Since G/H is a G-ANR, there exists a G-map $\gamma\colon W \to G/H$ where W is a G-neighborhood of A and $\gamma|_A = \alpha$. Since X is normal, we find a G-neighborhood V of A with $\overline{V} \subset W$. We convert $X \times I$ to a G-space by setting $g(x,t) = (g(x), t)$. Set

$$P = X \times \{0\} \cup A \times I \cup \overline{V} \times \{1\} \subset X \times I$$

and define $\varphi\colon P \to X$ by

$$\begin{aligned} \varphi(x,0) &= x \text{ for } x \in X, \\ \varphi(a,t) &= H(a,t) \text{ for } (a,t) \in A \times I, \\ \varphi(v,1) &= \beta(\gamma(v)) \text{ for } v \in \overline{V}. \end{aligned}$$

Then φ is well-defined and equivariant, and since $X \times \{0\}$, $A \times I$ and $\overline{V} \times \{1\}$ are closed subsets of $X \times I$, φ is continuous. Since X is a G-ANR and P is a closed subspace of the normal G-space $X \times I$, there exists a G-neighborhood Q of P in $X \times I$ and an equivariant extension $\psi\colon Q \to X$ of φ. For each $a \in A$ the set $\{a\} \times I$ is compact, and so there is a G-neighborhood U_a of a with $U_a \times I \subset Q$. Set $U = \bigcup_a U_a \cap W$. Clearly, U is a G-neighborhood of A with $U \subset W$ and $U \times I \subset Q$.

Now, $\psi|_{U\times I}\colon U \times I \to X$ yields a G-homotopy between the inclusion $U \hookrightarrow X$ and $\beta\gamma|_U$. □

4.3. Proposition. *If A is a closed subspace of a binormal G-ANR X, then*

$$\mathcal{G}\text{-cat}\, A \geq \mathcal{G}\text{-}\overline{\text{cat}}A \geq \mathcal{G}\text{-}\overline{\text{cat}}_X A = \mathcal{G}\text{-cat}_X\, A.$$

Proof. The first inequality follows from the simple fact that a closed subset of a normal space is normal and from Lemma 4.1, the second inequality is clear since A is closed, and the equality follows from Lemmata 4.1 and 4.2. □

4.4. Remarks. 1. The closedness condition on A in Lemma 4.1 is essential: If X is a circle, x is a point in X and $A = X \setminus \{x\}$, then $2 = \overline{\text{cat}}_X A > \text{cat}_X A = 1$.

2. If X fails to be an ANR, Proposition 4.3 might not hold: If

$$X = \bigcup_{n=1}^{\infty} \left\{ (x,y) \in \mathbb{R}^2 \mid \left(x - \tfrac{1}{n}\right)^2 + y^2 = \tfrac{1}{n^2} \right\}$$

and $A = (0,0)$, then $\operatorname{cat} A = 1$ while $\operatorname{cat}_X A = \infty$.

5. Equivariant categories as index functions

A G-map $\varphi\colon X_1 \to X_2$ between G-spaces is a *G-homotopy equivalence* if there exists a G-map $\psi\colon X_2 \to X_1$ such that $\psi\varphi$ and $\varphi\psi$ are G-homotopic to the identities.

5.1. Lemma. *Let $\varphi\colon X \to X$ be a G-homotopy equivalence. If U is an open categorical subspace of X, then so is $\varphi^{-1}(U)$.*

Proof. Consider the commutative square

$$\begin{array}{ccc} \varphi^{-1}(U) & \xrightarrow{\varphi} & U \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ X & \xrightarrow{\varphi} & X \end{array}$$

where i and j are the inclusions. Let $\psi\colon X \to X$ be a G-homotopy inverse of φ. If the composition

$$U \xrightarrow{\alpha} G/H \xrightarrow{\beta} X$$

is G-homotopic to $j\colon U \to X$, then the composition

$$\varphi^{-1}(U) \xrightarrow{\alpha\varphi} G/H \xrightarrow{\psi\beta} X$$

is G-homotopic to i, because $\psi\beta\alpha\varphi \simeq_G \psi j\varphi = \psi\varphi i \simeq_G i$. □

For any subset A of a G-space X we set $GA = \{\, ga \mid g \in G,\ a \in A \,\}$.

5.2. Lemma. *Let X be a G-space and let $\varphi\colon X \to X$ be a G-map. Then, for every $N \in \mathbb{N}$, each of the functions $2^X \times 2^X \to \mathbb{N} \cup \{0\}$,*

$$\begin{aligned} \nu_N^1(A,Y) &= \min\{\mathcal{G}\text{-}\mathrm{cat}_X\, GA, N\}, \\ \nu_N^2(A,Y) &= \min\{\mathcal{G}\text{-}\mathrm{cat}_X (GA, GY), N\}, \\ \nu_N^3(A,Y) &= \min\{\mathcal{G}\text{-}\mathrm{cat}_X (GA \operatorname{mod} GY), N\} \end{aligned}$$

is an index function. Furthermore, the following holds.

(i) *If φ is a G-homotopy equivalence, then the function ν_N^1 is (φ, Z)-supervariant for every $Z \subset X$.*

(ii) *If φ is G-homotopic to the identity, then the function ν_N^2 is (φ, Z)-supervariant for every $Z \subset X$.*

(iii) *If there is a G-homotopy Φ_t between the identity and φ with $\Phi_t(Z) \subset Z$ for all t, then the function ν_N^3 is (φ, Z)-supervariant.*

Proof. The first claim is readily verified.

(i) Let $\{A_1, \dots, A_k\}$ be a covering of $G\varphi(A)$ by open categorical subspaces. By Lemma 5.1, $\{\varphi^{-1}(A_1), \dots, \varphi^{-1}(A_k)\}$ is then a covering of $\varphi^{-1}(G\varphi(A))$ by open categorical subspaces. But $\varphi^{-1}(G\varphi(A)) \supset GA$.

(ii) Assume that $G\varphi(A) \subset A_0 \cup A_1 \cup \dots \cup A_k$ where $A_1, \dots, A_k$ are open $\mathcal{G}$-categorical subspaces and A_0 is an open subspace which is G-deformable to GZ. Since φ is G-homotopic to the identity, $\varphi^{-1}(A_0)$ is G-deformable to A_0 and hence to GZ. We conclude now as in (i).

(iii) Assume that $G\varphi(A) \subset A_0 \cup A_1 \cup \dots \cup A_k$ where $A_1, \dots, A_k$ are open $\mathcal{G}$-categorical subspaces and A_0 is an open subspace which contains $G\varphi(A) \cap GZ$ and is G-deformable to $GZ \operatorname{mod} GZ$. Set $A_i' = \varphi^{-1}(A_i)$. Then $GA \subset A_0' \cup A_1' \cup \dots \cup A_k'$, and $A_1', \dots, A_k'$ are open $\mathcal{G}$-categorical subspaces. Let Φ_t be a G-homotopy between the identity and φ with $\Phi_t(Z) \subset Z$. Then $GZ \subset \varphi^{-1}(GZ)$, and so A_0' is an open subspace containing $GA \cap GZ$. Moreover, the composition of Φ_t with a G-homotopy deforming A_0 to $GZ \operatorname{mod} GZ$ yields a G-homotopy deforming A_0' to $GZ \operatorname{mod} GZ$. □

6. LS theory for homotopy equivalences

We let G always act trivially on $\mathbb{R}$. Thus, a G-function $f\colon X \to \mathbb{R}$ is a G-invariant function. The *orbit type* of an orbit $Gx = G\{x\}$ is its G-homeomorphism type. Given a G-function $f\colon X \to \mathbb{R}$, two orbits in X are *equivalent* if f has the same value on them and if they have the same type and are G-deformable into each other [5, 2.9]. If x is a fixed point of the G-map $\varphi\colon X \to X$, then the whole orbit Gx is fixed by φ.

Recall that $\infty \geq \infty$ and $\infty \geq n$ for every $n \in \mathbb{Z}$. We also agree that $\infty \geq \infty - n$ for every $n \in \mathbb{Z}$.

6.1. Theorem. *Consider a G-space X and a class $\mathcal{G}$ of homogeneous G-spaces. Let $\varphi\colon X \to X$ be a G-homotopy equivalence, and let F be the set of fixed points of φ. Let $-\infty < a < b < \infty$, and let $f\colon X \to \mathbb{R}$ be a G-function such that (f, φ) satisfies condition* (D) *on $f^{-1}[a, b]$ and such that $\mathcal{G}\text{-cat}_X f^a$ is finite. Set $F_d = F \cap f^{-1}(d)$.*

(a) *We always have*

$$\sum_{d \in \,]a,b]} \mathcal{G}\text{-cat}_X F_d \geq \mathcal{G}\text{-cat}_X f^b - \mathcal{G}\text{-cat}_X f^a.$$

(b) *If X is a binormal G-ANR and $\mathcal{G}$ contains all orbit types in $F \cap f^{-1}]a,b]$, then $F \cap f^{-1}]a,b]$ contains infinitely many G-orbits, or the number of equivalence classes of G-orbits in $F \cap f^{-1}]a,b]$ is at least $\mathcal{G}\text{-cat}_X f^b - \mathcal{G}\text{-cat}_X f^a$.*

(c) *If X is a binormal G-ANR and $f(F) \cap]a,b]$ is discrete, then*

$$\mathcal{G}\text{-cat}\left(F \cap f^{-1}]a,b]\right) \ \geq\ \mathcal{G}\text{-cat}_X f^b - \mathcal{G}\text{-cat}_X f^a.$$

Proof. If $f(F) \cap]a,b]$ is infinite, there is nothing to prove. Suppose therefore that $f(F) \cap]a,b] = \{d_1, \dots, d_m\}$ is finite.

(a) If $\mathcal{G}\text{-cat}_X f^b$ is finite, Lemma 5.2 (i) shows that

$$\nu(A,Y) := \min\{\mathcal{G}\text{-cat}_X GA, \mathcal{G}\text{-cat}_X f^b\}$$

is a (φ, f^a)-supervariant index function. Since $\nu(A,Y) = \mathcal{G}\text{-cat}_X GA$ for every $A \subset f^b$, the claim then follows from Theorem 2.3. If $\mathcal{G}\text{-cat}_X f^b = \infty$, we have to show that

$$\sum_{i=1}^{m} \mathcal{G}\text{-cat}_X F_{d_i} = \infty. \tag{6.1}$$

Fix $N \geq \mathcal{G}\text{-cat}_X f^a$. By Lemma 5.2 (i), $\nu_N(A,Y) = \min\{\mathcal{G}\text{-cat}_X GA, N\}$ is a (φ, f^a)-supervariant index function. We have $\nu_N(f^b, f^a) = N$ and $\mathcal{G}\text{-cat}_X A \geq \nu_N(A)$ for every $A \subset X$. In view of Theorem 2.3 we therefore conclude that

$$\begin{aligned} \sum_{i=1}^{m} \mathcal{G}\text{-cat}_X F_{d_i} \ &\geq\ \sum_{i=1}^{m} \nu_N(F_{d_i}) \\ &\geq\ \nu_N(f^b, f^a) - \nu_N(f^a, f^a) \ =\ N - \mathcal{G}\text{-cat}_X f^a. \end{aligned}$$

Since $N \geq \mathcal{G}\text{-cat}_X f^a$ was arbitrary, (6.1) follows.

(b) Assume that all the sets F_{d_i} contain only finitely many orbits. Let $E_1, \dots, E_{e_i}$ be the equivalence classes of orbits in F_{d_i}. By assumption, all orbits in E_j are G-deformable to one of its elements, and $\mathcal{G}$ contains this element. Lemma 4.2 thus implies that $\mathcal{G}\text{-cat}_X E_j = 1$ for $j = 1, \dots, e_i$. Therefore, $e_i = \sum_{j=1}^{e_i} \mathcal{G}\text{-cat}_X E_j \geq \mathcal{G}\text{-cat}_X F_{d_i}$.

(c) By Proposition 4.3,

$$\mathcal{G}\text{-cat}\left(F \cap f^{-1}]a,b]\right) \ =\ \sum_{i=1}^{m} \mathcal{G}\text{-cat}\, F_{d_i} \ \geq\ \sum_{i=1}^{m} \mathcal{G}\text{-cat}_X F_{d_i}.$$

$\square$

6.2. Corollary (cf. [5, 2.5] and [6, 3.8 (1)]). *Suppose that under the assumptions of Theorem* 6.1 (b) *the set* $f(F)\cap\,]a,b]$ *contains less than* $\mathcal{G}\text{-cat}\,f^b - \mathcal{G}\text{-cat}\,f^a$ *elements. Then one of the sets* F_d *in* $f^{-1}]a,b]$ *is not* G*-deformable to a* G*-orbit in* $f^{-1}]a,b]$. □

6.3. Remarks. 1. Proposition 4.3 implies that for a binormal G-ANR, Theorem 6.1 holds with $\mathcal{G}$-cat replaced by $\mathcal{G}\text{-}\overline{\text{cat}}$.

2. We discuss the assumptions in Theorem 6.1.

(i) The condition that φ is a G-homotopy equivalence cannot be omitted: If $X = S^1 = \{x^2+y^2=1\}$, $\varphi(X) = (0,-1)$ and $f(x,y) = y$ is the height function, then $\sum_{d\in\mathbb{R}} \operatorname{cat}_X F_d = 1 < 2 = \operatorname{cat} X$.

(ii) The condition that b is finite cannot be omitted: Define X to be the telescope (homotopy direct limit) of the sequence

$$\cdots \xrightarrow{\ d\ } S^1 \xrightarrow{\ d\ } S^1 \xrightarrow{\ d\ } S^1 \xrightarrow{\ d\ } \cdots, \quad \deg d = 2.$$

In greater detail, X is the quotient space of the disjoint union

$$Y = \coprod_{k=-\infty}^{\infty} S^1 \times [k, k+1]$$

under the following equivalence relation: $(z,k) \in S^1 \times [k-1,k]$ is equivalent to $(z^2,k) \in S^1 \times [k,k+1]$.

We denote by $[z,k] \in X$ the image of $(z,k) \in S^1 \times [k-1,k]$ and by $[z,t] \in X$ the image of $(z,t) \in Y$ for $t \notin \mathbb{Z}$. Consider the map

$$\varphi\colon X \to X, \quad \varphi[z,t] = [z,t-1]$$

and the function $f\colon X \to \mathbb{R}$, $f[z,t] = t$. Then (f,φ) clearly satisfies condition (D), and it is easy to see that φ is a homotopy equivalence. (This can be seen directly or by observing that X is the Eilenberg–Mac Lane space $K(\mathbb{Z}[1/2],1)$ and that φ induces an isomorphism of fundamental groups.) Furthermore, $F = \emptyset$.

Notice that f^0 is homotopy equivalent to S^1, and so $\operatorname{cat}_X f^0 = 2$. On the other hand, $\operatorname{cat} X > 2$; indeed, $\pi_1(X) = \mathbb{Z}[1/2]$ is not a free group, and it is well known that the fundamental group of a space whose category is at most 2 is free [9]. Therefore, $0 = \sum_{d\in[0,\infty[} \operatorname{cat}_X F_d < \operatorname{cat} X - \operatorname{cat}_X f^0$.

(iii) Condition (D) cannot be omitted: If $X =]0,1[$, $\varphi(x) = \frac{x}{2}$ and $f(x) = x$, then $F = \emptyset$ but $\operatorname{cat} X = 1$.

(iv) The discreteness condition on $f(F)\cap\,]a,b]$ in (c) cannot be omitted: If $X = S^1$ and $\varphi\colon X \to X$, $(x,y) \mapsto (\varphi_1(x,y), \varphi_2(x,y))$ is such that $\varphi(x,y) = (x,y)$ for $x \geq 0$ and $\varphi_2(x,y) < y$ for $x < 0$, then φ is homotopic to the identity, and $f(x,y) = y$ is a Lyapunov function for φ. But $F = \{(x,y) \in X \mid x \geq 0\}$, and so $\operatorname{cat} F = 1 < 2 = \operatorname{cat} X$.

7. LS theory for maps homotopic to the identity

In this section we show that for maps homotopic to the identity, the estimates in Theorem 6.1 can often be strengthened, and that all estimates also hold for $b = \infty$.

7.1. Theorem. *Consider a G-space X and a class $\mathcal{G}$ of homogeneous G-spaces. Let $\varphi\colon X \to X$ be a G-map which is G-homotopic to the identity, and let F be the set of fixed points of φ. Let $-\infty < a < b \leq \infty$, and let $f\colon X \to \mathbb{R}$ be a G-function such that (f, φ) satisfies condition* (D) *on $f^{-1}[a,b]$. Set $F_d = F \cap f^{-1}(d)$.*

(I) *If $\mathcal{G}$-$\operatorname{cat}_X f^a$ is finite, then the statements* (a), (b), (c) *in Theorem 6.1 also hold for $b = \infty$.*

(II) *In case that $a \in f(F)$ and a is an isolated point in $f(F) \cap [a,b[$, assume that a G-neighborhood of f^a is G-deformable to f^a. Then* (a), (b), (c) *in Theorem 6.1 hold with $\mathcal{G}$-$\operatorname{cat}_X f^b - \mathcal{G}$-$\operatorname{cat}_X f^a$ replaced by $\mathcal{G}$-$\operatorname{cat}_X(f^b, f^a)$.*

(III) *If, in addition to the assumption in* (II)*, there is a G-homotopy Φ_t between the identity and φ with $\Phi_t(f^a) \subset f^a$ for all t, then* (a), (b), (c) *in Theorem 6.1 hold with $\mathcal{G}$-$\operatorname{cat}_X f^b - \mathcal{G}$-$\operatorname{cat}_X f^a$ replaced by $\mathcal{G}$-$\operatorname{cat}_X(f^b \operatorname{mod} f^a)$.*

Proof. We may again assume that $f(F) \cap \,]a,b] = \{d_1, \dots, d_m\}$.

7.2. Claim. *For any $c > d_m$ the space X is G-deformable to f^c.*

Proof. Choose a G-homotopy $\Phi\colon X \times [0,1] \to X$ between the identity and φ. For $x \in X$ and $t \geq 0$ set

$$\widetilde{\Phi}(x,t) = \varphi^{[t]}\left(\Phi(x, t - [t])\right)$$

where $[t] = \max\{n \in \mathbb{N} \cup \{0\} \mid n \leq t\}$. By Lemma 2.2 (i), for each $k \in \mathbb{Z}$ the number

$$n_k = \min\{n \in \mathbb{N} \cup \{0\} \mid \varphi^n(f^k) \subset f^c\}$$

is well-defined. Choose a non-decreasing continuous function $u\colon \mathbb{R} \to [0, \infty[$ such that $u(k) = n_{k+1}$ for all $k \in \mathbb{Z}$. Thus,

$$\varphi^{[u(r)]}(f^r) \subset f^c \tag{7.1}$$

for every $r \in \mathbb{R}$. Define a G-function $h\colon X \to \mathbb{R}_+$ by

$$h(x) = \max_{0 \leq t \leq 1} f(\Phi(x,t))$$

and define $\Psi\colon X \times [0,1] \to X$ by $\Psi(x,t) = \widetilde{\Phi}(x, u(h(x))\,t)$. Then Ψ is a G-homotopy between the identity and $\widetilde{\Phi}(x, u(h(x)))$. We verify that $\widetilde{\Phi}(x, u(h(x))) \in f^c$ for each $x \in X$: Set $y = \Phi(x, u(h(x)) - [u(h(x))])$. Then $y \in f^{h(x)}$. Therefore, by (7.1), $\widetilde{\Phi}(x, u(h(x))) = \varphi^{[u(h(x))]}(y) \in f^c$. $\diamond$

Claim 7.2 implies that $\mathcal{G}\text{-cat}\, X = \mathcal{G}\text{-cat}_X\, f^c$ and $\mathcal{G}\text{-cat}_X(X, f^a) = \mathcal{G}\text{-cat}_X(f^c, f^a)$. Moreover, if the homotopy Ψ above is constructed from a G-homotopy Φ_t with $\Phi_t(f^a) \subset f^a$, then $\Psi(f^a, t) \subset f^a$ for all t, and so X is G-deformable to $f^c \bmod f^a$, whence $\mathcal{G}\text{-cat}_X(X \bmod f^a) = \mathcal{G}\text{-cat}_X(f^c \bmod f^a)$. We may therefore assume that b is finite, and so, by the proof of Theorem 6.1, we are left with showing (a) in (II) and (III) for b finite.

(II)(a) If $a \in f(F)$, a G-neighborhood of f^a is G-deformable to f^a by assumption, and if $a \notin f(F)$, we conclude this by applying Lemma 2.2 (i) to some $b \in]a, d_1[$ and to $U = \emptyset$. So, $\mathcal{G}\text{-cat}_X(f^a, f^a) = 0$. If $\mathcal{G}\text{-cat}_X(f^b, f^a)$ is finite, we set

$$\nu(A, Y) = \min\left\{\mathcal{G}\text{-cat}_X(GA, GY), \mathcal{G}\text{-cat}_X(f^b, f^a)\right\}$$

and conclude from Lemma 5.2 (ii) and Theorem 2.3 that

$$\sum_{i=1}^{m} \mathcal{G}\text{-cat}_X\, F_{d_i} \;\geq\; \mathcal{G}\text{-cat}_X(f^b, f^a).$$

If $\mathcal{G}\text{-cat}_X(f^b, f^a) = \infty$, we fix $N \in \mathbb{N}$, set

$$\nu_N(A, Y) = \min\{\mathcal{G}\text{-cat}_X(GA, GY), N\}$$

and conclude from Lemma 5.2 (ii) and Theorem 2.3 that

$$\sum_{i=1}^{m} \mathcal{G}\text{-cat}_X\, F_{d_i} \geq N.$$

(III)(a) We use Lemma 5.2 (iii) to argue as for (II)(a). $\square$

7.3. Remarks. 1. Theorem 7.1 shows that if φ is G-homotopic to the identity, then Corollary 6.2 can be refined and extended to the case $b = \infty$.

2. (a) and (c) of Theorem 6.1 and Theorem 7.1 (I) imply (a) and (c) of Theorem 1.2; and since a connected and weakly locally contractible space is path-connected, Theorem 1.2 (b) follows from Theorem 1.2 (a).

3. For a compact metric space X, Theorem 1.2 (a) with cat_X replaced by $\overline{\text{cat}}_X$ and φ homotopic to the identity was stated by Mañé [15, Chapter II, Theorem 4.1].

Proposition 4.3 implies that for a binormal G-ANR, Theorem 7.1 (I) holds with $\mathcal{G}$-cat replaced by $\mathcal{G}$-$\overline{\text{cat}}$.

4. We discuss the assumptions in Theorem 7.1.

(i) Even if b is finite, (II) in general does not hold for homotopy equivalences: Let $S(k)$ be the unit circle in $\mathbb{R}^2$ centred at $(0,2k)$ and set $X = \bigcup_{k\in\mathbb{Z}} S(k)$. Define $\varphi\colon X \to X$ by $\varphi(x,y) = (x, y-2)$ and set $f(x,y) = y$. Then $F = \emptyset$ but $\operatorname{cat}_X(f^2, f^0) = 1$.

(ii) The assumptions in (II) and (III) cannot be omitted: Let

$$X = \{(x,0) \mid -1 \le x \le 1\} \cup \left\{ \left(\sin \tfrac{1}{y}, y\right) \,\Big|\, 0 < y \le 1 \right\} \subset \mathbb{R}^2,$$

$\varphi\left(\sin\frac{1}{y}, y\right) = \left(\sin\frac{1+2\pi y}{y}, \frac{y}{1+2\pi y}\right)$ and $\varphi(x,0) = (x,0)$, and $f(x,y) = y$. Then $F \cap f^{-1}]0,1] = \emptyset$ but $\operatorname{cat}_X(f^1, f^0) = \infty$. Moreover, if

$$X = \left\{ (x,y) \in S^1 \,\Big|\, x \ge -\tfrac{\sqrt{3}}{2},\ y \ge 0 \right\},$$

$\varphi(X) = (1,0)$ and $f(x,y) = y$, then $F \cap f^{-1}]\frac{1}{2}, 1] = \emptyset$ but

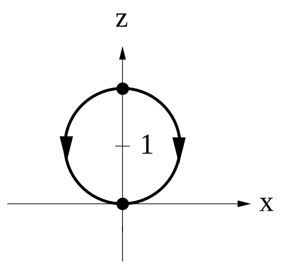
$$\operatorname{cat}_X \left(X \bmod f^{\frac{1}{2}} \right) = 1.$$

5. Replacing (i) in Definition 3.1 by

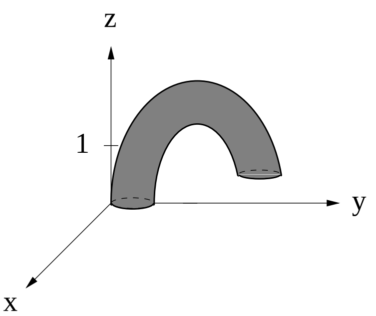
$$A_0 \text{ is } G\text{-deformable to } Y, \text{ and } A_0 \supset A \cap Y$$

we obtain a relative category $\mathcal{G}\text{-cat}_X(A;Y)$ which is at least as large as $\mathcal{G}\text{-cat}_X(A,Y)$. Since $\varphi(f^a) \subset f^a$, the proof of Theorem 7.1 (II) shows that if b is finite, then the lower bound $\mathcal{G}\text{-cat}_X(f^b, f^a)$ in (II) can be replaced by $\mathcal{G}\text{-cat}_X(f^b; f^a)$.

6. (II) is not covered by (I) and (III): Let $X \subset \mathbb{R}^3$ be the space obtained by gluing the circle X_1 and the cylinder X_2 depicted below in the origin. If f is the height function and φ restricts on X_1 to the time-1-map of the negative gradient flow of f and retracts X_2 onto its bottom circle, then $\operatorname{cat}_X(X, f^1) = 1 > 2-2 = \operatorname{cat} X - \operatorname{cat}_X f^1$, and (III) does not apply – indeed, $\sum_{d\,\in\,]1,2]} \operatorname{cat}_X F_d = 1 < 2 = \operatorname{cat}_X(X \bmod f^1)$.



X_1



X_2

(III) is not covered by (II): If $X = \{(x,y) \in S^1 \mid y \geq 0\}$, $\varphi\colon X \to X$, $(x,y) \mapsto (\varphi_1(x,y), \varphi_2(x,y))$ is given by $\varphi_2(x,y) = y^2$ and $x\,\varphi_1(x,y) \geq 0$, and f is the height function, then $\operatorname{cat}_X(X \bmod f^0) = 1 > 0 = \operatorname{cat}_X(X, f^0)$.

7. The main difficulty with (equivariant) categories is in their computation. A lower bound for $\mathcal{G}\text{-}\operatorname{cat}_X(f^b \bmod f^a)$ is given by an equivariant relative version of the cup-length, $(\mathcal{G}, h^*)\text{-cup-length}_X\left(f^b, f^a\right)$, which depends on a choice of a G-equivariant cohomology theory h^* [2]. Indeed, the sets $A_0, A_1, \ldots, A_k$ used in the definition of $\mathcal{G}\text{-}\operatorname{cat}_{(X,f^a)} f^b$ in [6] are required to be *relative open subsets* of f^b, whence

$$\mathcal{G}\text{-}\operatorname{cat}_X\left(f^b \bmod f^a\right) \;\geq\; \mathcal{G}\text{-}\operatorname{cat}_{(X,f^a)} f^b,$$

and the argument given in [2, p. 58] shows that

$$\mathcal{G}\text{-}\operatorname{cat}_{(X,f^a)} f^b \;\geq\; (\mathcal{G}, h^*)\text{-cup-length}_X\left(f^b, f^a\right).$$

A rougher but effectively computable lower bound for the category $\mathcal{G}\text{-}\operatorname{cat}_X(f^b \bmod f^a)$ is the $(\mathcal{G}, h^*)\text{-length}_X\left(f^b, f^a\right)$ [6, 2].

7.4. Definition. A *G-semiflow* on a G-space X is a family $\Phi = \{\varphi_t\}$, $t \geq 0$, of G-maps $\varphi_t\colon X \to X$ such that $\varphi_0 = id_X$ and $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for all $s, t \geq 0$. Moreover, the map $X \times [0, \infty[\to X$, $(x,t) \mapsto \varphi_t(x)$ is required to be continuous.

A point $x \in X$ is called a *rest point* of Φ if $\varphi_t(x) = x$ for all $t \geq 0$. The orbit of a rest point of Φ consists of rest points of Φ and is thus called a *rest orbit* of Φ.

A G-semiflow $\Phi = \{\varphi_t\}$ is called *gradient-like* if there exists a G-function $f\colon X \to \mathbb{R}$ such that $f(\varphi_t(x)) < f(\varphi_s(x))$ whenever $t > s$ and x is not a rest point of Φ. Any such function f is called a *Lyapunov function* for Φ.

7.5. Corollary. *Let R be the set of rest points of the gradient-like G-semiflow $\Phi = \{\varphi_t\}$ on a G-space X. Let f be a Lyapunov function for Φ which is bounded below. Assume that there exists $\tau > 0$ such that (f, φ_τ) satisfies condition* (D) *on X.*

(a) *We always have*

$$\sum_{d \in \mathbb{R}} \mathcal{G}\text{-}\operatorname{cat}_X\left(R \cap f^{-1}(d)\right) \;\geq\; \mathcal{G}\text{-}\operatorname{cat} X.$$

(b) *If X is a binormal G-ANR and $\mathcal{G}$ contains all orbit types in R, then Φ has at least $\mathcal{G}\text{-}\operatorname{cat} X$ rest orbits.*

(c) *If X is a binormal G-ANR and $f(R)$ is discrete, then*

$$\mathcal{G}\text{-}\operatorname{cat} R \;\geq\; \mathcal{G}\text{-}\operatorname{cat} X.$$

Proof. By the definition of a G-semiflow, φ_τ is G-homotopic to the identity. Denote the set of fixed points of φ_τ by F.

7.6. Claim. $F = R$.

Proof. The inclusion $R \subset F$ is obvious. Conversely, for $x \in F$ and $t \geq 0$, write $t = m\tau + r$ with $m \in \mathbb{N} \cup \{0\}$ and $r \in [0, \tau[$. Then $\varphi_{m\tau}(x) = \varphi_\tau^m(x) = x$, and since Φ is gradient-like, $\varphi_r(x) = x$. Thus, $\varphi_t(x) = \varphi_r(\varphi_{m\tau}(x)) = \varphi_r(x) = x$. ◇

The corollary now follows from applying Theorem 7.1 (III) to (f, φ_τ). □

We leave it to the reader to formulate a relative version of the above corollary.

8. LS theory for homeomorphisms and flows

8.1. Definition. Fix a class $\mathcal{B}$ of G-spaces. Given a subset A of a G-space X, we set $\mathcal{B}_X A = k$ if A can be covered by k *open* subspaces $A_1, \dots, A_k$ of X such that every A_i is G-homeomorphic to a space from $\mathcal{B}$, and if k is minimal with this property. If there is no such number k, we set $\mathcal{B}_X A = \infty$.

8.2. Theorem. *Consider a G-space X. Let $\varphi \colon X \to X$ be a G-homeomorphism, and let F be the set of fixed points of φ. Let $-\infty < a < b < \infty$, and let $f \colon X \to \mathbb{R}$ be a G-function such that (f, φ) satisfies condition* (D) *on $f^{-1}[a,b]$ and such that $\mathcal{B}_X f^a$ is finite. Then*

$$\sum_{d \in \,]a,b]} \mathcal{B}_X \left(F \cap f^{-1}(d)\right) \geq \mathcal{B}_X f^b - \mathcal{B}_X f^a.$$

Proof. It is easy to see that for every $N \in \mathbb{N}$ the function

$$2^X \times 2^X \to \mathbb{N} \cup \{0\}, \qquad (A, Z) \mapsto \min\{\mathcal{B}_X GA, N\}$$

is an index function which is (φ, Z)-supervariant for every $Z \subset X$. The proof can now be completed as the one of Theorem 6.1 (a). □

8.3. Definition. A semiflow $\{\varphi_t\}$ on a space X is called a *flow* if each map φ_t is a homeomorphism.

8.4. Corollary. *Let R be the set of rest points of the gradient-like G-flow $\Phi = \{\varphi_t\}$ on a G-space X. Let $-\infty < a < b < \infty$, and let f be a Lyapunov function for Φ such that there exists $\tau > 0$ for which (f, φ_τ) satisfies condition* (D) *on $f^{-1}[a,b]$ and such that $\mathcal{B}_X f^a$ is finite. Then*

$$\sum_{d \in \,]a,b]} \mathcal{B}_X \left(R \cap f^{-1}(d)\right) \geq \mathcal{B}_X f^b - \mathcal{B}_X f^a.$$

Proof. The corollary can be deduced from Theorem 8.2 in the same way as Corollary 7.5 (a) was deduced from Theorem 7.1 (III). □

8.5. Remark. Contrary to Corollary 7.5, the condition that b is finite cannot be omitted in Corollary 8.4: If $\mathcal{B}$ is the class consisting of the annulus $\{(x,y) \in \mathbb{R}^2 \mid 1 < x^2+y^2 < 4\}$, and if $X = \mathbb{R}^2$ and Φ is the negative gradient flow of $f(x,y) = x^2+y^2$, then $\sum_{d \in]1,\infty]} \mathcal{B}_X\left(R \cap f^{-1}(d)\right) = 0 < 2-1 = \mathcal{B}_X X - \mathcal{B}_X f^1$.

We can play a similar game in the category of smooth manifolds. We then consider a class $\mathcal{B}$ of diffeomorphism types. For example, let B be the class consisting of $\mathbb{R}^n$. Then $B_M A$ is the minimal number of open balls necessary to cover $A \subset M$. Of course, $B_M A \geq \operatorname{cat}_M A$. We set $B(M) = B_M M$, and denote the minimal number of critical points of a smooth function on M by $\operatorname{Crit} M$. Following the proof of Theorem 8.2, we obtain

8.6. Corollary. *Let K be the set of critical points of a smooth function on a closed smooth manifold M. Then*

$$\sum_{d \in]a,b]} B_M\left(K \cap f^{-1}(d)\right) \;\geq\; B_M f^b - B_M f^a.$$

In particular, $\operatorname{Crit} M \geq B(M)$. □

We leave it to the reader to define a relative invariant $\mathcal{B}_X(A \bmod Y)$ which can be used to refine Corollaries 8.4 and 8.6.

8.7. Remark. Singhof [22] proved that $B(M) = \operatorname{cat} M$ for every closed smooth p-connected manifold M with

$$\operatorname{cat} M \;\geq\; \frac{n+p+4}{2(p+1)}$$

provided $\operatorname{cat} M \geq 3$ and $\dim M \geq 4$. Moreover, it is easy to see that if $B(M) = \operatorname{cat} M$, then $\operatorname{cat}(M \setminus \{x\}) = \operatorname{cat} M - 1$ for every $x \in M$ [22, p. 29]. On the other hand, there is an example of a closed manifold Q with $\operatorname{cat}(Q \setminus \{x\}) = \operatorname{cat} Q$, [13]. In particular, $B(Q) > \operatorname{cat} Q$. It is, however, still unknown whether there are closed manifolds M with $\operatorname{Crit} M > B(M)$.

9. The Palais–Smale condition (C) and condition (D)

Consider a connected and complete Riemannian manifold M without boundary modelled on a separable Hilbert space. For $m \in M$ we denote by $\langle\ ,\ \rangle_m$ the inner product on $T_m M$ and by $\|\ \|_m$ the norm induced by $\langle\ ,\ \rangle_m$. We say that a map is $C^{r,1}$, $r \geq 0$, if all its derivatives up to order r exist and are locally Lipschitz continuous. We assume that M is of class $C^{1,1}$ and that the Riemannian metric is $C^{0,1}$. Let $f\colon M \to \mathbb{R}$ be a $C^{1,1}$ function and let $Df(m)$ be its derivative at m. The $C^{0,1}$ vector field ∇f, defined at m by

$$Df(m)(v) = \langle \nabla f(m), v \rangle_m \quad \text{for all } v \in T_m M,$$

is called the gradient vector field of f. A point $m \in M$ is called a critical point of f if $\nabla f(m)$ vanishes. The function f is said to satisfy the Palais–Smale condition (C) if the following holds:

> If A is a subset of M on which $|f|$ is bounded but on which $\|\nabla f\|$ is not bounded away from zero, then there is a critical point of f in the closure of A.

Choose a smooth, monotone function $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$g(x) = 1 \text{ for } x \leq 1, \quad g(x) \leq x \text{ for } 1 \leq x \leq 2, \quad g(x) = x \text{ for } x \geq 2$$

and set $h(m) = 1/g(\|\nabla f(m)\|)$, $m \in M$. Then $V = -h\,\nabla f$ is $C^{0,1}$ and bounded, and so, M being complete, V integrates to a flow $\{\varphi_t\}$, $t \in \mathbb{R}$, on M.

9.1. Proposition. *Let M, f and φ_τ, $\tau > 0$, be as above. If f satisfies the Palais–Smale condition* (C)*, then the pair (f, φ_τ) satisfies condition* (D)*.*

Proof. Clearly, f is a Lyapunov function for φ_τ. Let $A \subset M$ and assume that $|f(a)| \leq c < \infty$ for all $a \in A$ and that $\inf_{a\in A}\{f(a) - f(\varphi_\tau(a))\} = 0$. We compute that for each $m \in M$,

$$\begin{aligned} \frac{d}{dt} f\left(\varphi_t(m)\right) &= Df\left(\varphi_t(m)\right)\left(\frac{d}{dt}\varphi_t(m)\right) \\ &= Df\left(\varphi_t(m)\right)\left(V\left(\varphi_t(m)\right)\right) \\ &= -h(\varphi_t(m))\left\langle \nabla f\left(\varphi_t(m)\right), \nabla f\left(\varphi_t(m)\right)\right\rangle, \end{aligned}$$

and so

$$\begin{aligned} f(m) - f(\varphi_\tau(m)) &= -\int_0^\tau \frac{d}{dt} f\left(\varphi_t(m)\right)\, dt \\ &= \int_0^\tau h(\varphi_t(m))\, \|\nabla f\left(\varphi_t(m)\right)\|^2\, dt. \end{aligned} \tag{9.1}$$

By assumption, there is a sequence $(a_n)_{n\geq 1} \subset A$ such that

$$f(a_n) - f(\varphi_\tau(a_n)) < \frac{\tau}{n}. \tag{9.2}$$

Observe that $h(\varphi_t(m))\,\|\nabla f\,(\varphi_t(m))\,\|^2 < 1$ only if $\|\nabla f\,(\varphi_t(m))\,\| < 1$. Therefore, (9.1) and (9.2) imply that there exists a sequence $(t_n)_{n\geq 1} \subset [0,\tau]$ such that

$$\|\nabla f\,(\varphi_{t_n}(a_n))\,\|^2 < \frac{1}{n}. \tag{9.3}$$

Set $b_n = \varphi_{t_n}(a_n)$ and $B = \{b_n\}_{n\geq 1}$. Then, by assumption and (9.2),

$$\begin{aligned} |f(b_n)| &\leq |f(a_n)| + |f(b_n) - f(a_n)| \\ &\leq c + |f(\varphi_{t_n}(a_n)) - f(a_n)| \;<\; c + \tfrac{\tau}{n} \;\leq\; c + \tau. \end{aligned} \tag{9.4}$$

In view of (9.3), (9.4) and condition (C), we conclude that there exists $b^* \in \overline{B}$ with $\varphi_\tau(b^*) = b^*$. After passing to a subsequence, if necessary, we may assume that $b^* = \lim_{n\to\infty} b_n$.

For any C^1 path $\sigma\colon [a,b] \to M$ the length of σ is defined by

$$\int_a^b \left\| \frac{d}{dt}\sigma(t) \right\| \, dt.$$

For $m, m' \in M$ let $d(m,m')$ be the infimum of the length of all C^1 paths joining m and m'. The function d thus defined is a metric on M which is consistent with the topology of M [17, §9]. For each $n \geq 1$ the path $[0,t_n] \to M$, $t \mapsto \varphi_t(a_n)$ is of class C^1. Therefore, by Schwartz's inequality, (9.1) and (9.2),

$$\begin{aligned} d(b_n,a_n) \;=\; d(\varphi_{t_n}(a_n),a_n) &\leq \int_0^{t_n} \left\| \frac{d}{dt}\varphi_t(a_n) \right\| dt \\ &\leq \int_0^{\tau} \left\| \frac{d}{dt}\varphi_t(a_n) \right\| dt \\ &= \int_0^{\tau} h(\varphi_t(a_n))\ \|\nabla f(\varphi_t(a_n))\|\ dt \\ &\leq \tau^{\frac{1}{2}} \left(\int_0^{\tau} h(\varphi_t(a_n))^2\ \|\nabla f\,(\varphi_t(a_n))\|^2\ dt \right)^{\frac{1}{2}} \\ &\leq \tau^{\frac{1}{2}} \left(\int_0^{\tau} h(\varphi_t(a_n))\ \|\nabla f\,(\varphi_t(a_n))\|^2\ dt \right)^{\frac{1}{2}} \\ &\leq \frac{\tau}{\sqrt{n}}. \end{aligned}$$

We conclude that

$$\lim_{n\to\infty} d(b^*,a_n) \leq \lim_{n\to\infty} d(b^*,b_n) + \lim_{n\to\infty} d(b_n,a_n) = 0,$$

i.e., $b^* \in \overline{A}$. □

A metric G-space X is called a *metric G-ANR* if for each closed G-subspace Z of a *metric* G-space Y every G-map $\rho\colon Z \to X$ admits an equivariant extension to a G-neighborhood of Z.

9.2. Corollary. *Let M be a Riemannian Hilbert manifold as above. Suppose that M is a G-space and that the Riemannian metric on M is G-invariant. Let K be the set of critical points of a $C^{1,1}$ G-function $f\colon M \to \mathbb{R}$ which satisfies condition* (C) *on M and is bounded below.*

(a) *We always have*

$$\sum_{d\in\mathbb{R}} \mathcal{G}\text{-cat}_M\, K \cap f^{-1}(d) \;\geq\; \mathcal{G}\text{-cat}\, M.$$

(b) *If M is a metric G-ANR and $\mathcal{G}$ contains all orbit types in K, then f has at least $\mathcal{G}$-cat M critical orbits.*

(c) *If M is a metric G-ANR and $f(K)$ is discrete, then*

$$\mathcal{G}\text{-cat}\, K \;\geq\; \mathcal{G}\text{-cat}\, M.$$

Proof. Going once more through Section 4, we see that Proposition 4.3 holds for metric G-ANR's X. Therefore, Corollary 7.5 holds for metric G-ANR's. Since the Riemannian metric on M is G-invariant, ∇f and V are G-equivariant, and so is φ_t, $t \in \mathbb{R}$. Moreover, the critical points of f are the rest points of φ_1. The corollary now follows in view of Proposition 9.1. □

Localizing Proposition 9.1 and applying the relative version of Corollary 7.5, we obtain a relative version of the above corollary.

9.3. Remarks. 1. Let M be as in the basic assumptions of Corollary 9.2. Since M is metrizable, it is paracompact. Appendix B of [6] thus provides sufficient conditions for M being a metric G-ANR. They are, e.g., fulfilled if G acts trivially, or if M and the action $G \times M \to M$ are $C^{2,1}$.

2. In view of the above remark, Corollary 9.2 (b) recovers the basic Lusternik–Schnirelmann theorem for Hilbert manifolds first obtained in [21].

3. Versions of (a) and (b) in Corollary 9.2 have been proved for $C^{1,1}$ functions on complete $C^{1,1}$ Finsler manifolds ([18, Theorems 7.1 and 7.2] and [6, § 3]) and for continuous functions on weakly locally contractible complete metric spaces (see [4]). We notice that in these situations also the analogs of (c) hold.

References

[1] A. V. Arkhangel'skiĭ, V. V. Fedorchuk. *General Topology I.* The Basic Concepts and Constructions of General Topology. In: Encyclopaedia of Mathematical Sciences **17**. Springer-Verlag, Berlin, 1990.

[2] T. Bartsch. *Topological methods for variational problems with symmetries.* Lecture Notes in Mathematics **1560**. Springer-Verlag, Berlin, 1993.

[3] K. Borsuk. Über den Lusternik–Schnirelmannschen Begriff der Kategorie. *Fund. Math.* **26** (1936) 123–136.

[4] A. Canino, M. Degiovanni. Nonsmooth critical point theory and quasilinear elliptic equations. *Topological methods in differential equations and inclusions.* Montreal, 1994, 1–50, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **472**, Kluwer Acad. Publ., Dordrecht, 1995.

[5] M. Clapp, D. Puppe. Invariants of the Lusternik–Schnirelmann type and the topology of critical sets. *Trans. Amer. Math. Soc.* **298** (1986), 603–620.

[6] M. Clapp, D. Puppe. Critical point theory with symmetries. *J. Reine Angew. Math.* **418** (1991), 1–29.

[7] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov. *Modern Geometry - Methods and Applications, Part III. Introduction to Homology Theory.* Springer-Verlag, Berlin Heidelberg New York, 1990.

[8] E. Fadell. The equivariant Ljusternik–Schnirelmann method for invariant functionals and relative cohomological index theories. *Topological methods in nonlinear analysis*, 41–70, Sm. Math. Sup., 95, Presses Univ. Montreal, Montreal, Que., 1985.

[9] R. Fox. On the Lusternik–Schnirelmann category. *Ann. of Math.* **42** (1941) 333–370.

[10] H. Hofer. Lusternik–Schnirelman-theory for Lagrangian intersections. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **5** (1988), 465–499.

[11] H. Hofer, E. Zehnder. *Symplectic Invariants and Hamiltonian Dynamics.* Birkhäuser-Verlag, Basel, 1994.

[12] S. Hu. *Theory of retracts.* Wayne State University Press, Detroit 1965.

[13] P. Lambrecht, D. Stanley, L. Vandembroucq. *Embeddings up to homotopy of two-cones into Euclidean space.* Preprint, 2000.

[14] L. A. Lusternik, L. G. Schnirelmann. *Méthodes topologiques dans les problèmes variationnels.* Hermann, Paris, 1934.

[15] R. Mañé. *Global Variational Methods in Conservative Dynamics.* 18º Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro, 1989–1991.

[16] R. S. Palais. The classification of G-spaces. *Memoirs of the Amer. Math. Soc.* **36**, Providence, Rhode Island, 1960.

[17] R. S. Palais. Morse theory on Hilbert manifolds. *Topology* **2** (1963), 299–340.

[18] R. S. Palais. Lusternik–Schnirelman theory on Banach manifolds. *Topology* **5** (1966), 115–132.

[19] Yu. B. Rudyak. On analytical applications of stable homotopy (the Arnold conjecture, critical points). *Math. Z.* **230** (1999), 659–672.

[20] Yu. B. Rudyak, F. Schlenk. Lusternik–Schnirelmann theory for symplectic and contact manifolds. *In preparation.*

[21] J. T. Schwartz. Generalizing the Lusternik-Schnirelman Theory of Critical Points. *Comm. Pure Appl. Math.* **17** (1964), 307–315.

[22] W. Singhof. Minimal coverings of manifolds with balls. *Manuscripta Math.* **29** (1979), 385–415.

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