AUTOMORPHISMS OF NONDEGENERATE CR QUADRICS AND SIEGEL DOMAINS. EXPLICIT DESCRIPTION

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ABSTRACT. In this paper we give the complete explicit description of the holomorphic automorphisms of any nondegenerate CR-quadric Q of arbitrary CR-dimension and codimension. In particular, the obtained formula describes the automorphisms of Siegel domains of second kind with Levi-nondegenerate Shilov-boundary.

We introduce a family of k-dimensional chains ($k = \operatorname{codim} Q$), the analogues of one-dimensional Chern-Moser chains for hyperquadrics.

We also analyse some different types of rigid quadrics and give a simple proof of Beloshapka's theorem on the description of the infinitesimal automorphisms of nondegenerate quadrics.

1. INTRODUCTION

Let $z = (z^1, \dots, z^n)$, $w = (w^1, \dots, w^k)$ be coordinates in $\mathbb{C}^{n+k}$, $k \geq 1$, and

$$\langle z, z\rangle = \begin{pmatrix} \langle z, z\rangle^1 \\ \vdots \\ \langle z, z\rangle^k \end{pmatrix}$$

be a $\mathbb{C}^k$-valued Hermitian form on $\mathbb{C}^n$.

Consider the cone $C = \text{convex hull}\{\langle z, z\rangle : z \in \mathbb{C}^n\}$. Suppose C is an acute cone, i.e., C does not contain any entire line. This property takes place if and only if the form $\langle z, z\rangle$ is positive definite, i.e., in appropriate coordinates all the forms $\langle z, z\rangle^\varkappa$ are positive definite.

Let $V \supset C$ be an open acute cone in $\mathbb{R}^k$. The domain

$$\Omega_V = \{(z, w) \in \mathbb{C}^{n+k} : \operatorname{Im} w - \langle z, z\rangle \in V\}$$

is called Siegel domain of the second kind, associated with the cone V. (For simplicity we shall call them Siegel domains.)

Siegel domains were introduced by Pyatetskii-Shapiro [12] for the study of automorphic forms in several variables, homogeneous and symmetric domains. In particular,

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Pyatetskii-Shapiro constructed an example of a Siegel domain which is homogeneous but not symmetric. In general, a Siegel domain Ω_V is not necessarily homogeneous.

Kaup, Matsushima and Ochiai [8] proved that the infinitesimal automorphisms of Siegel domains are quadratic vector fields and that the automorphisms of Ω extend to birational maps of $\mathbb{C}^{n+k}$.

Henkin and Tumanov [7] established a natural correspondence between $\operatorname{Aut}\Omega_C$ and the group of CR automorphisms of its Shilov boundary, the quadric

$$Q = \{(z, w) \in \mathbb{C}^{n+k} : \operatorname{Im} w = \langle z, z\rangle\}.$$

Under the assumption that the forms $\langle z, z\rangle^\varkappa$, $\varkappa = 1, \dots, k$, are linearly independent they proved that any $\phi \in \operatorname{Aut}\Omega_C$ extends to a biholomorphic automorphism of Q and, conversely, any locally defined CR automorphism of Q extends to an automorphism of the entire domain Ω and, in particular to a global automorphism of Q.

Considering the group $\operatorname{Aut} Q$ of an arbitrary Hermitian quadric Q, Beloshapka [2] found a necessary and sufficient condition for $\langle z, z\rangle$ (not necessarily positive definite) so that $\operatorname{Aut} Q$ is a finite dimensional Lie group:

i.) The forms $\langle\cdot,\cdot\rangle^\varkappa$, $\varkappa = 1, \dots, k$ are linearly independent. Geometrically this condition means that C has nonempty interior.

ii.) The form $\langle z, z\rangle$ does not have an annihilator, i.e., the condition $\langle a, z\rangle = 0$ for all $z \in \mathbb{C}^n$ implies that $a = 0$.

Quadrics Q which satisfy these conditions are called nondegenerate.

The nondegenerate quadrics which represent Shilov boundaries of Siegel domains should just satisfy condition *i.)* because no positive definite form $\langle z, z\rangle$ has an annihilator.

For nondegenerate quadrics Tumanov [14] proved that their automorphisms are rational and extend to birational automorphisms of $\mathbb{C}^{n+k}$.

In this paper we obtain an explicit formula for the automorphisms of arbitrary nondegenerate quadrics, in particular, for the automorphisms of Siegel domains of second kind with nondegenerate Shilov boundary.

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2. Infinitesimal automorphisms of CR quadrics

The quadric $Q : \operatorname{Im} w = \langle z, z\rangle$ is a homogeneous manifold. The group H of Heisenberg translations $(z, w) \mapsto (z + p, w + q + 2i\langle z, p\rangle)$, $(p, q) \in Q$ acts transitively on Q. Thus, $\operatorname{Aut} Q$ splits into the semidirect product

$$\operatorname{Aut} Q = H \ltimes \operatorname{Aut}_0 Q,$$

where $\operatorname{Aut}_0 Q = \{\phi \in \operatorname{Aut} Q : \phi(0) = 0\}$ is the isotropy group of the origin. $\operatorname{Aut}_0 Q$ also splits:

$$\operatorname{Aut}_0 Q = L \ltimes \operatorname{Aut}_{0,\mathrm{id}} Q,$$

where $\mathrm{Aut}_{0,\mathrm{id}}\, Q = \{\phi \in \mathrm{Aut}_0\, Q : d\phi(0)|_{T_0^{\mathbb{C}}Q} = \mathrm{id}\}$, and L is the group of linear transformations $(z,w) \mapsto (Cz, \rho w)$ $(C \in \mathrm{GL}(n,\mathbb{C}), \rho \in \mathrm{GL}(k,\mathbb{R}))$ such that $\langle Cz, Cz\rangle = \rho\langle z, z\rangle$.

Hence,

$$\mathrm{Aut}\, Q = H \ltimes L \ltimes \mathrm{Aut}_{0,\mathrm{id}}\, Q. \tag{1}$$

Beloshapka [1], [3] showed that any $\phi \in \mathrm{Aut}_{0,\mathrm{id}}\, Q$ lies in a 1-parametric subgroup and explicitly described the Lie algebra $\mathfrak{g}$ of $\mathrm{Aut}\, Q$. Below, we suggest a simple proof of this result which is based on elementary properties of the Fourier transformation.

The splitting (1) implies that $\mathfrak{g}$ can be represented as sum

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+,$$

where $\mathfrak{g}_-, \mathfrak{g}_0, \mathfrak{g}_+$ are the Lie algebras of $H, L, \mathrm{Aut}_{0,\mathrm{id}}$, respectively.

Let $\Phi_t = (F_t(z,w), G_t(z,w))$ be a 1-parametric subgroup of $\mathrm{Aut}\, Q$, where F_t and G_t are the z and w components of Φ_t.

The correspondent element χ of $\mathfrak{g}$ can be represented by a holomorphic vector field

$$\chi = \sum_{j=1}^{n} f^j \frac{\partial}{\partial z^j} + \sum_{j=1}^{k} g^j \frac{\partial}{\partial w^j},$$

where

$$f^j = \left.\frac{dF_t^j}{dt}\right|_{t=0}, \quad g^j = \left.\frac{dG_t^j}{dt}\right|_{t=0}.$$

The condition $\chi \in \mathfrak{g}$ is equivalent to the identity

$$\mathrm{Re}\, \chi(\mathrm{Im}\, w - \langle z, z\rangle)|_{\mathrm{Im}\, w = \langle z,z\rangle} = 0. \tag{2}$$

Thus, to describe $\mathfrak{g}$ one has to solve (2).

Theorem 1. *(Beloshapka) The algebra $\mathfrak{g}_-$ consists of the vector fields*

$$\chi_- = \sum_{j=1}^{n} p^j \frac{\partial}{\partial z^j} + \sum_{j=1}^{k} (q^j + 2i\langle z, p\rangle^j) \frac{\partial}{\partial w^j},$$

with $p \in \mathbb{C}^n$, $q \in \mathbb{R}^k$.

The algebra $\mathfrak{g}_0$ consists of the vector fields

$$\chi_0 = \sum_{j=1}^{n} (Xz)^j \frac{\partial}{\partial z^j} + \sum_{j=1}^{k} (sw)^j \frac{\partial}{\partial w^j},$$

where $X \in \mathfrak{gl}(n,\mathbb{C})$, $s \in \mathfrak{gl}(k,\mathbb{R})$ satisfy the condition $2\,\mathrm{Re}\langle Xz, z\rangle = s\langle z, z\rangle$.

The algebra $\mathfrak{g}_+$ consists of the vector fields

$$\chi_+ = \sum_{j=1}^{n}(aw + A(z,z) + B(w,z))^j \frac{\partial}{\partial z^j} + \sum_{j=1}^{k}(2i\langle z, a\bar{w}\rangle + r(w,w))^j \frac{\partial}{\partial w^j},$$

where $a : \mathbb{C}^k \longrightarrow \mathbb{C}^n$ is a linear operator, A is a $\mathbb{C}^n$-valued symmetric bilinear form on $\mathbb{C}^n \otimes \mathbb{C}^n$, r is an $\mathbb{R}^k$-valued symmetric bilinear form on $\mathbb{R}^k$, and B is a $\mathbb{C}^n$-valued bilinear form on $\mathbb{C}^k \otimes \mathbb{C}^n$ satisfying

$$\begin{aligned} (3)\qquad \langle A(z,z), z\rangle &= 2i\langle z, a\langle z,z\rangle\rangle, \\ (4)\qquad \operatorname{Re}\langle B(u,z), z\rangle &= r(u, \langle z,z\rangle), \\ \operatorname{Im}\langle B(\langle z,z\rangle, z), z\rangle &= 0, \end{aligned}$$

for all $z \in \mathbb{C}^n$ and $u \in \mathbb{R}^k$.

Proof. Let $\chi = f\frac{\partial}{\partial z} + g\frac{\partial}{\partial w}$ be a solution of (2) and let $f = \sum_{i=0}^{\infty} f_i$ and $g = \sum_{i=0}^{\infty} g_i$ with $f_i(tz, t^2w) = t^i f_i(z,w)$, resp., $g_i(tz,t^2w) = t^i g_i(z,w)$ the decomposition into weighted homogeneous components. It is easy to verify that then all weighted components $f_i\frac{\partial}{\partial z} + g_{i+1}\frac{\partial}{\partial w}$ are also solutions of (2). Thus, we may restrict ourselves to look for polynomial solutions only.

Collecting in (2) the components of degree p with respect to z and of degree q with respect to $\bar{z}$ and performing elementary transformations one obtains

$$\begin{aligned} (5)\qquad g_p &= 0 \text{ for } p \geq 2, \\ (6)\qquad f_p &= 0 \text{ for } p \geq 3, \\ (7)\qquad \operatorname{Im} g_0 &= 0, \\ (8)\qquad g_1 &= 2i\langle z, f_0\rangle, \\ (9)\qquad 2\operatorname{Re}\langle f_1, z\rangle &= \operatorname{Re}\Delta g_0, \\ (10)\qquad \langle f_2, z\rangle &= 2i\langle z, \Delta f_0\rangle, \\ (11)\qquad \operatorname{Im}\langle \Delta f_1, z\rangle &= 0, \\ (12)\qquad \langle z, \Delta^2 f_0\rangle &= 0, \\ (13)\qquad \operatorname{Re}\Delta^3 g_0 &= 0, \end{aligned}$$

where $\Delta = \sum_{\varkappa=1}^{k}\langle z,z\rangle^{\varkappa}\frac{\partial}{\partial u^{\varkappa}}$ (cp. [2]).

To solve this system of partial differential equations Beloshapka used Palamodov's theorem on exponential representation of the solutions of systems of PDE with constant coefficients ([9]). Here, we suggest a selfcontained reasoning.

From (13) immediately follows that $\operatorname{Re} g_0$, and, therefore, g_0 as well, are polynomials whose degree with respect to u does not exceed 2. From (9) and (11) one obtains

$$\langle \Delta^2 f_1, z\rangle = 0. \tag{14}$$

We show that this implies that f_1 is linear with respect to u. Since we are looking for polynomial solutions we may suppose that f_1 is a polynomial in u and linear in z. The Fourier transform with respect to u of (14) equals

$$\sum_{|m|=0}^{M} \left\langle \sum_{\nu=1}^{n} \alpha_m^\nu z^\nu, z \right\rangle (\langle z, z\rangle, \xi)^2 D^m \delta = 0, \tag{15}$$

where ξ is the dual variable to u, δ is the delta-functional, $(\cdot,\cdot)$ is the standard scalar product in $\mathbb{R}^k$, $m = (m_1, \dots, m_k)$ are multiindices with $|m| = m_1 + \cdots + m_k$, $D^m = \frac{\partial^{|m|}}{(\partial u^1)^{m_1} \dots (\partial u^k)^{m_k}}$, and α_m^ν are constant $\mathbb{C}^n$-vectors.

Without loss of generality we may assume that M is the biggest number such that there exists some $\alpha_m^\nu \neq 0$ with $|m| = M$. Then M equals to the degree of f_1 with respect to u. Among all nonvanishing α_m^ν with $|m| = M$ we choose these with maximal m_1, among the latter these with maximal m_2 and so on. This way we come to some uniquely determined nonvanishing matrix-valued coefficient $\alpha_{\tilde{m}} = (\alpha_{\tilde{m}}^\nu)$. Assume $M \geq 2$. Apply the functional from the right hand side of (15) on the following $\mathbb{R}^k$-valued test function: If the maximal number $r \leq k$ with $\tilde{m}_r \neq 0$ is not smaller than 2 then set

$$\psi = \psi_0 \xi_1^{\tilde{m}_1} \dots \xi_r^{\tilde{m}_r - 2}.$$

Otherwise, if $\tilde{m}_r = 1$, let s be the maximal number with $\tilde{m}_s \neq 0$ and $s < r$. Then set

$$\psi = \psi_0 \xi_1^{\tilde{m}_1} \dots \xi_s^{\tilde{m}_s - 1}.$$

The vector ψ_0 will be determined later.

Because of the choice of ψ we obtain

$$\left(\left\langle \sum_{\nu=1}^{n} \alpha_{\tilde{m}}^\nu z^\nu, \zeta \right\rangle (\langle z, \zeta\rangle^r)^2, \psi_0 \right) = 0, \tag{16}$$

resp.,

$$\left(\left\langle \sum_{\nu=1}^{n} \alpha_{\tilde{m}}^\nu z^\nu, \zeta \right\rangle \langle z, \zeta\rangle^r \langle z, \zeta\rangle^s, \psi_0 \right) = 0. \tag{17}$$

(We substituted the antiholomorphic z-variables by ζ.)

Now, choose $z_0 \in \mathbb{C}^n$ such that $\langle \alpha_{tildem} z_0 \rangle = \sum_{\nu=1}^{n} \alpha_{\tilde{m}}^{\nu} z_0^{\nu} \neq 0$. Then, according to *i.)*, there exists a ζ_0 such that $\langle \alpha_{\tilde{m}}^{\nu} z_0, \zeta_0 \rangle \neq 0$. By continuity the latter inequality remains true for z, ζ in sufficiently small neighbourhoods of z_0, ζ_0. According to *ii.)* there exist z_1, ζ_1, from these neighbourhoods such that $\langle z, \zeta \rangle^r$ and $\langle z, \zeta \rangle^s$ do not vanish. Hence, for suitable ψ_0, the left hand side of (16) (resp., (17)) does not vanish. Contradiction. It follows that the assumption $M \geq 2$ was false. Consequently, f_1 is linear with respect to u.

In exactly the same way one deduces from (12) that f_0 depends linearly from u. Taking into account (8) this implies that g_1 is also linear with respect to u.

We obtain

$$\begin{aligned} f &= f_0 + f_1 + f_2, \\ g &= g_0 + g_1, \end{aligned}$$

with

$$\begin{aligned} f_0 &= p + aw, \\ f_1 &= Xz + B(w, z), \\ f_2 &= A(z, z), \\ g_0 &= q + sw + r(w, w), \\ g_1 &= 2i\langle z, p \rangle + 2i\langle z, a\bar{w} \rangle, \end{aligned}$$

where p, q, X, s, a, A, r, B satisfy the indicated conditions. □

Remark. It is an easy consequence of the equations (3) resp. (4) and the nondegeneracy of Q that the tensor A resp. r are uniquely determined by a, resp. B. By the same arguments as in the proof above it follows that a is uniquely determined by A and B is uniquely determined by r. In order to prove this, we have to show that the homogeneous equations

$$\langle z, a\langle z, z \rangle \rangle = \langle z, \Delta a u \rangle = 0$$

and

$$\langle B(\langle z, z \rangle, z), z \rangle = \langle \Delta B(u, z), z \rangle = 0$$

have trivial solution only. Considering the Fourier transforms of these equations one obtains that a, resp. B must vanish.

For any a and r as above there exists a unique automorphism $\phi = (f, g) \in \mathrm{Aut}_{0,\mathrm{id}}\, Q$ with

$$\left.\frac{\partial f}{\partial w}\right|_0 = a,$$
$$\operatorname{Re}\left.\frac{\partial^2 g}{(\partial w)^2}\right|_0 = 2r.$$

Inserting ϕ into the equation of Q and taking into account that the image of Q is Q itself one obtains the following second order derivatives of ϕ which we will need below:

$$(18)\qquad \begin{aligned} f &= z + aw + A(z,z) + \tilde{B}(w,z) + K(w,w) + o(|z|^2 + |w|^2) \\ g &= w + 2i\langle z, a\bar{w}\rangle + r(w,w) + i\langle aw, a\bar{w}\rangle + o(|z|^2 + |w|^2), \end{aligned}$$

where A is the tensor which is determined by (3) and $\tilde{B}$ and K will be determined later.

The automorphism ϕ coincides with Φ_1 where Φ_t is the 1-parametric subgroup corresponding to χ_+.

If $\mathcal{A}$ is the space of all tensors a and $\mathcal{R}$ the space of all tensors r as above then $\hat{Q} = \{r + i\langle a\cdot, a\cdot\rangle : a \in \mathcal{A}, r \in \mathcal{R}\}$ form a (not necessarily nondegenerate) quadric in $\mathcal{A} \times (\mathcal{R} \otimes \mathbb{C})$ and the Heisenberg group of $\hat{Q}$ is isomorphic to $\operatorname{Aut}_{0,\mathrm{id}} Q$. Thus, we have

Theorem 2. *(see [6]) For any nondegenerate quadric Q $\operatorname{Aut}_{0,\mathrm{id}} Q$ is isomorphic to the Heisenberg group of some CR quadric and therefore has a canonical CR structure.*

It is still an open question whether the dimension of $\hat{Q}$ can be estimated by $2n + k$. For strictly pseudoconvex quadrics, i.e., the Shilov boundaries of Siegel domains this sharp estimate was proved by Kaup, Matsushima and Ochiai [8] (see also [13]).

3. Results

Let Q be a nondegenerate quadric and $\phi = (f, g) \in \operatorname{Aut}_{0,\mathrm{id}} Q$ be the automorphism which corresponds to the parameters (a, r). Furthermore, let $f = \sum_{l=0}^{\infty} f_l$, $g = \sum_{l=0}^{\infty} g_l$ be the expansion into homogeneous polynomials then we prove

Theorem 3. *The polynomials f_l, g_l are determined by the recursive relations*

$$(19)\quad (l-1)\begin{pmatrix} f_l \\ g_l \end{pmatrix} = \begin{pmatrix} \frac{\partial f_{l-1}}{\partial z} & \frac{\partial f_{l-1}}{\partial w} \\ \frac{\partial g_{l-1}}{\partial z} & \frac{\partial g_{l-1}}{\partial w} \end{pmatrix} \begin{pmatrix} A(z,z) + B(w,z) + A(aw,z) - ia\langle z, a\bar{w}\rangle \\ 2i\langle z, a\bar{w}\rangle + r(w,w) + i\langle aw, a\bar{w}\rangle \end{pmatrix},$$

for $l > 1$ and the initial conditions $f_0 = 0, g_0 = 0, f_1 = z + aw, g_1 = w$.

Consider the real k-plane $\Gamma_0 = \{z = 0, \operatorname{Im} w = 0\}$ which is contained in Q. The orbit of Γ_0 under the action of $\operatorname{Aut}_0 Q$ composes a biholomorphically invariant family of real k-manifolds on Q passing through the origin. These k-manifolds are called chains as the analogous objects on hypersurfaces. The following theorem generalizes the fact that the chains on hyperquadrics are the intersections of the hyperquadric with complex lines passing through the origin and being transversal to the complex tangent space.

Theorem 4. *Any chain $\Gamma \subset Q$ is the intersection of Q with the complex k-plane $\{z = aw\}$, with $a \in \mathcal{A}$.*

The main result of this paper is the following explicit description of the automorphisms from $\operatorname{Aut}_{0,\mathrm{id}} Q$.

Theorem 5. *Let $(z^*, w^*) = \phi(z, w)$ be from $\operatorname{Aut}_{0,\mathrm{id}} Q$, then*

$$\begin{pmatrix} z^* \\ w^* \end{pmatrix} = \left(\mathrm{id} - \begin{pmatrix} \mathfrak{P}_p & \mathfrak{P}_q \\ \mathfrak{Q}_p & \mathfrak{Q}_q \end{pmatrix}\right)^{-1} \begin{pmatrix} z - aw - A(z,z) - 2B(w,z) \\ w - i\langle z, a\bar{w}\rangle - r(w,w) \end{pmatrix},$$

where $\mathfrak{P}_p$, $\mathfrak{Q}_p$, $\mathfrak{P}_q$, $\mathfrak{Q}_q$ are the following polynomial matrices:

$$\begin{aligned}
\mathfrak{P}_p \;=\;& 2A(z,\cdot) + B(w,\cdot) + A(aw,\cdot) - ia\langle\cdot, a\bar{w}\rangle - 2A(A(z,\cdot),z) + \\
&+ A(A(z,z),\cdot) + A(B(w,z),\cdot) - A(B(w,\cdot),z) - iB(\langle\cdot, a\bar{w}\rangle, z) + \\
&+ iB(\langle z, a\bar{w}\rangle,\cdot) - B(w, A(z,\cdot)) + A(A(z,aw),\cdot) - A(A(z,\cdot),aw) - \\
&- A(A(\cdot,aw),z) - 2a\langle z, a\langle a\bar{w},\cdot\rangle\rangle, \\
\mathfrak{Q}_p \;=\;& i\langle\cdot, a\bar{w}\rangle + 2\langle z, a\langle a\bar{w},\cdot\rangle\rangle - \frac{i}{2}\langle B(w,\cdot), a\bar{w}\rangle + \frac{i}{2}\langle\cdot, ar(\bar{w},\bar{w})\rangle - \\
&- ir(\langle\cdot, a\bar{w}\rangle, w) + \langle aw, a\langle a\bar{w},\cdot\rangle\rangle, \\
\mathfrak{P}_q \;=\;& 2a + 2B(\cdot,z) - 2A(a\cdot,z) - 2ia\langle z, a\overline{\cdot}\rangle - 2B(w,a\cdot) - 2ia\langle aw, a\overline{\cdot}\rangle - \\
&- 4iB(\langle z, a\overline{\cdot}\rangle, z) + 2A(A(z,a\cdot),z) - A(A(z,z),a\cdot) - B(w, B(\cdot,z)) + \\
&+ B(\cdot, B(w,z)) - 2B(r(\cdot,w),z) + B(w, A(a\cdot,z)) + iB(\langle a\cdot, a\bar{w}\rangle, z) + \\
&+ iB(\cdot, a\langle z, a\bar{w}\rangle) - iB(w, a\langle z, a\overline{\cdot}\rangle) + B(\cdot, A(aw,z)) - 2iB(\langle z, a\overline{\cdot}\rangle, aw) - \\
&- iB(\langle aw, a\overline{\cdot}\rangle, z) + 4ia\langle z, B(\bar{w}, a\overline{\cdot})\rangle - 2ia\langle z, ar(\bar{w},\overline{\cdot})\rangle - A(B(w,z),a\cdot) + \\
&+ A(B(w,a\cdot),z) - A(B(\cdot,z),aw) - A(B(\cdot,aw),z) + ia\langle B(w,z), a\overline{\cdot}\rangle - \\
&- ia\langle B(\cdot,z), a\bar{w}\rangle + ia\langle z, A(a\bar{w}, a\overline{\cdot})\rangle + A(A(aw,a\cdot),z) - A(A(aw,z),a\cdot) + \\
&+ A(A(a\cdot,z),aw) + ia\langle A(z,aw), a\overline{\cdot}\rangle - ia\langle A(z,a\cdot), a\bar{w}\rangle + a\langle aw, a\langle a\overline{\cdot}, z\rangle\rangle - \\
&- a\langle a\cdot, a\langle a\bar{w}, z\rangle\rangle,
\end{aligned}$$

$$\begin{aligned}\mathfrak{Q}_q &= 2i\langle z, a\bar{\cdot}\rangle + 2r(\cdot, w) - i\langle a\cdot, a\bar{w}\rangle + i\langle aw, a\bar{\cdot}\rangle - 2i\langle z, B(\bar{w}, a\bar{\cdot})\rangle - \\ &\quad -2\langle z, a\langle a\bar{w}, a\cdot\rangle\rangle - 2r(r(w,\cdot), w) + r(r(w,w),\cdot) + i\langle B(w, a\cdot), a\bar{w}\rangle - \\ &\quad -i\langle ar(w,\cdot), a\bar{w}\rangle + ir(\langle a\cdot, a\bar{w}\rangle, w) - ir(\langle aw, a\bar{\cdot}\rangle, w) + ir(\langle aw, a\bar{w}\rangle, \cdot) - \\ &\quad -\frac{i}{2}\langle a\cdot, ar(\bar{w},\bar{w})\rangle + \frac{i}{2}\langle ar(w,w), a\bar{\cdot}\rangle - i\langle aw, ar(\bar{w},\bar{\cdot})\rangle - \langle aw, a\langle a\bar{w}, a\cdot\rangle\rangle.\end{aligned}$$

In $\mathfrak{P}_p$ and $\mathfrak{Q}_p$ the dot stands instead of a complex n-dimensional vector argument and in $\mathfrak{P}_q$ and $\mathfrak{Q}_q$ instead of a complex k-dimensional vector argument.

4. Recursive formulas for the automorphisms

For shortness of the notations we introduce the following abbreviations: in the given fixed coordinates we will denote the vector field $\chi = \sum_{\nu=1}^{n} C^\nu \frac{\partial}{\partial z^\nu} + \sum_{\kappa=1}^{k} D^\kappa \frac{\partial}{\partial w^\kappa}$ by $\chi = (C, D)$ as well. If f is an n-vector and E is an $n \times m$ matrix with columns E_μ then by $\langle f, E\rangle$ we denote the $k \times m$-matrix with columns $\langle f, E_\mu\rangle$.

We consider the canonical action of $\mathrm{Aut}_{0,\mathrm{id}}\, Q$ on the Lie algebra $\mathfrak{g}$: Let $\chi \in \mathfrak{g}$ and $\phi = (f, g) \in \mathrm{Aut}_{0,\mathrm{id}}\, Q$, then

$$\phi^*(\chi)(z, w) = (d\phi)^{-1}(\chi(f, g)).$$

Hence, if $\chi = (C, D) = \sum_{j=1}^{n} C^j \frac{\partial}{\partial z^j} + \sum_{m=1}^{k} D^k \frac{\partial}{\partial w^k}$, then $\phi^*(C, D) = (P, Q)$ and

$$\begin{pmatrix} P(z,w) \\ Q(z,w) \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix}^{-1} \begin{pmatrix} C(f,g) \\ D(f,g) \end{pmatrix} \tag{20}$$

is also from $\mathfrak{g}$.

Since the polynomials P and Q are of second degree they are uniquely determined by the values of their derivatives up to second order in the origin. Restricting (20) and its derivatives to the origin and taking into account that

$$\left.\begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix}\right|_0 = \begin{pmatrix} \mathrm{id} & a \\ 0 & \mathrm{id} \end{pmatrix}$$

one can obtain the values of (P, Q) and its derivatives in 0 for given $\chi = (C, D)$ if one knows the derivatives of ϕ in 0 up to third order.

For any quadric Q $\mathfrak{g}_0$ contains a vector field $\chi_e = (z, 2w)$. This infinitesimal automorphism corresponds to the 1-parametric subgroup

$$\begin{aligned} z^* &= e^t z \\ w^* &= e^{2t} w. \end{aligned}$$

Let now $\Phi \in \mathrm{Aut}_{0,\mathrm{id}}\, Q$ be the automorphism corresponding to (a, r). Then one can compute $\phi^*(\chi_e) = (P_e, Q_e)$ using (18):

$$\begin{aligned} P_e &= z - aw - A(z,z) - 2B(w,z) \\ Q_e &= 2w - 2i\langle z, a\bar{w}\rangle - 2r(w,w). \end{aligned}$$

Moreover, one obtains

$$\begin{aligned} \tilde{B}(w,z) &= B(w,z) + A(aw,z) + i\langle z, a\bar{w}\rangle \\ K(w,w) &= \frac{1}{3}B(w,aw) + \frac{2}{3}ar(w,w) + \frac{1}{3}A(aw,aw) + \frac{i}{3}a\langle aw, a\bar{w}\rangle, \end{aligned} \tag{21}$$

where B is the tensor from (4) which is determined by r.

For $(C, D) = (z, 2w)$ the identity (20) takes the form

$$\begin{pmatrix} f \\ 2g \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix} \begin{pmatrix} z - aw - A(z,z) - 2B(w,z) \\ 2w - 2i\langle z, a\bar{w}\rangle - 2r(w,w) \end{pmatrix}. \tag{22}$$

Before studying this system, we will consider the action of ϕ on the vector field $\chi_i = (iz, 0)$. This infinitesimal automorphism corresponds to the 1-parametric subgroup

$$\begin{aligned} z^* &= e^{it}z \\ w^* &= w. \end{aligned}$$

One obtains $\phi^*(\xi_i) = (P_i, Q_i)$ with

$$\begin{aligned} P_i &= iz + iaw - iA(z,z) - 2iA(aw,z) - 2a\langle z, a\bar{w}\rangle \\ Q_i &= 2\langle z, a\bar{w}\rangle + 2\langle aw, a\bar{w}\rangle. \end{aligned}$$

It follows

$$\begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix} \begin{pmatrix} z + aw - A(z,z) - 2A(aw,z) + 2ia\langle z, a\bar{w}\rangle \\ -2i\langle z, a\bar{w}\rangle - 2i\langle aw, a\bar{w}\rangle \end{pmatrix}. \tag{23}$$

Combining (22) and (23) leads to

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix} \begin{pmatrix} z - A(z,z) - B(w,z) - A(aw,z) + ia\langle z, a\bar{w}\rangle \\ w - 2i\langle z, a\bar{w}\rangle - r(w,w) - i\langle aw, a\bar{w}\rangle \end{pmatrix}. \tag{24}$$

Let f_l, g_l be the homogeneous components of f and g with respect to z and w. Then

$$\begin{aligned} \frac{\partial f_l}{\partial z}z + \frac{\partial f_l}{\partial w}w &= lf_l \\ \frac{\partial g_l}{\partial z}z + \frac{\partial g_l}{\partial w}w &= lg_l. \end{aligned}$$

Isolating in (24) the component of degree l, one obtains a recursive formula which determines f_l, g_l for $l > 1$:

$$(l-1)\begin{pmatrix} f_l \\ g_l \end{pmatrix} = \begin{pmatrix} \frac{\partial f_{l-1}}{\partial z} & \frac{\partial f_{l-1}}{\partial w} \\ \frac{\partial g_{l-1}}{\partial z} & \frac{\partial g_{l-1}}{\partial w} \end{pmatrix} \begin{pmatrix} A(z,z) + B(w,z) + A(aw,z) - ia\langle z, a\bar{w}\rangle \\ 2i\langle z, a\bar{w}\rangle + r(w,w) + i\langle aw, a\bar{w}\rangle \end{pmatrix},$$

with initial conditions $f_0 = 0$, $g_0 = 0$, $f_1 = z + aw$, $g_1 = w$. Thus, we have proved Theorem 3.

5. Geometric description of k-dimensional chains

The description of the chains formulated in Theorem 4 is a direct consequence of the formula (19):

The image of Γ_0 under $\phi = (f, g)$ is $\{f(0,u), g(0,u) : u \in \mathbb{R}^k\}$. From (19) follows

$$\begin{aligned} (l-1)f_l(0,u) &= \frac{\partial f_{l-1}(0,u)}{\partial u}(r(u,u) + i\langle au, au\rangle) \\ f_0(u) &= 0,\ f_1(u) = au \\ (l-1)g_l(0,u) &= \frac{\partial g_{l-1}(0,u)}{\partial u}(r(u,u) + i\langle au, au\rangle) \\ g_0(u) &= 0,\ g_1(u) = u. \end{aligned}$$

For any solution $g(0,u) = \sum_{l=0}^{\infty} g_l(0,u)$, evidently, $f(0,u) = ag(0,u)$ is the uniquely determined solution for $f(0,u)$. This finishes the proof.

Any automorphism $\phi \in \mathrm{Aut}_{0,\mathrm{id}}\, Q$ with parameters (a, r) can be uniquely decomposed into $\phi_a \circ \phi_r$ corresponding to $(a,r) = (a,0) \circ (0,r)$. Then ϕ_a maps the standard chain Γ_0 onto the chain $\{z = aw\} \cap Q$, ϕ_r leaves the standard chain invariant, but changes the parameter.

6. Explicit formula for the automorphisms

We consider now the action of ϕ on the infinitesimal Heisenberg automorphisms:

$$\begin{aligned} \chi_p &= (p, 2i\langle z, p\rangle) \text{ with } p \in \mathbb{C}^n \\ \chi_q &= (0, q) \text{ with } q \in \mathbb{R}^k. \end{aligned}$$

Let (P_p, Q_p) and (P_q, Q_q) the images of χ_p and χ_q under ϕ^*. If p resp. q runs over the standard basis in $\mathbb{C}^n$ resp. $\mathbb{R}^k$, one can collect the resulting equations (20) into a matrix equation:

$$\begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{id} & 0 \\ 2i\langle f, \mathrm{id}\rangle & \mathrm{id} \end{pmatrix} = \begin{pmatrix} \Pi_p & \Pi_q \\ \Psi_p & \Psi_q \end{pmatrix},$$

which is equivalent to

$$\begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix}^{-1} = \begin{pmatrix} \Pi_p & \Pi_q \\ \Psi_p & \Psi_q \end{pmatrix} \begin{pmatrix} \mathrm{id} & 0 \\ -2i\langle f, \mathrm{id}\rangle & \mathrm{id} \end{pmatrix}. \tag{25}$$

Before determining the matrix blocks $\Pi_p, \Psi_p, \Pi_q, \Psi_q$ we simplify (25) and obtain an expression for the Jacobian matrix of ϕ which does not depend on f. Inserting this expression into (22) one gets an explicit formula for ϕ.

Let $\phi_{a,r} \in \mathrm{Aut}_{0,\mathrm{id}}\, Q$ be the automorphism corresponding to (a, r). Furthermore, set $\Phi_c(z, w) = (cz, |c|^2 w)$ with $c \in \mathbb{C}^*$. Then $\Phi_c^{-1} \circ \phi_{a,r} \circ \Phi_c \in \mathrm{Aut}_{0,\mathrm{id}}\, Q$ is the automorphism corresponding to $(\bar{c}a, |c|^2 r)$. Hence, if we substitute z, w, a, r, z^*, w^* by $cz, |c|^2 w, \frac{a}{\bar{c}}, \frac{r}{|c|^2}, cz^*, |c|^2 w^*$ in $\phi_{a,r}$ we obtain again $\phi_{a,r}$. This can be formulated as follows: If we associate z, w, a, r with the weights $(1,0), (1,1), (0,-1), (-1,-1)$, respectively, then f is homogeneous with weight $(1,0)$ and g is homogeneous with weight $(1,1)$. It follows

$$\begin{aligned} weight\left(\frac{\partial f}{\partial z}\right) &= (0,0) \\ weight\left(\frac{\partial f}{\partial w}\right) &= (0,-1) \\ weight\left(\frac{\partial g}{\partial z}\right) &= (0,1) \\ weight\left(\frac{\partial g}{\partial w}\right) &= (0,0). \end{aligned}$$

Set

$$H = \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix}^{-1} = \begin{pmatrix} H_{I,I} & H_{I,II} \\ H_{II,I} & H_{II,II} \end{pmatrix},$$

where $H_{I,I}, H_{I,II}, H_{II,I}, H_{II,II}$ are blocks of dimensions $(n,n), (n,k), (k,n)$ and (k,k). Then we have

Lemma 1.

$$\begin{aligned} weight(H_{I,I}) &= (0,0) \\ weight(H_{I,II}) &= (0,-1) \\ weight(H_{II,I}) &= (0,1) \\ weight(H_{II,II}) &= (0,0). \end{aligned}$$

Proof. Set

$$J = \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix}.$$

Then

$$H_{ij} = (-1)^{\epsilon(i,j)} \frac{\det \hat{J}_{ji}}{\det J},$$

where $\hat{J}_{ji}$ is the $(n+k-1) \times (n+k-1)$-matrix which is obtained by omitting the j-th line and the i-th column in J and

$$\epsilon(i,j) = \begin{cases} 0, & \text{if } |i-j| \text{ even} \\ 1, & \text{if } |i-j| \text{ odd} \end{cases}$$

It is easy to see that $weight(\det J) = (0,0)$, since $\det J$ is a sum of products containing as many factors from $\frac{\partial f}{\partial w}$ as from $\frac{\partial g}{\partial z}$. By the same reason, $weight(\det \hat{J}_{ji}) = (0,0)$ for $i,j \le n$ and $i,j > n$.

In the products of $\det \hat{J}_{ji}$ with $i \le n, j > n$ there will be one factor from $\frac{\partial f}{\partial w}$ more than factors from $\frac{\partial g}{\partial z}$. Hence, $\det \hat{J}_{ji}$ has the weight $(0,-1)$. Analogously, for $i < n, j \ge n$ $weight(\det \hat{J}_{ji})$ equals $(0,1)$. □

Now we are going to compute the weights of Π_p, Π_q, Ψ_p, Ψ_q: Let (P_p, Q_p) be the image of $(p, 2i\langle z,p\rangle)$. If p was associated with the weight $(1,0)$, then P_p would have the weight $(1,0)$ and Q_p would have the weight $(1,1)$. Passing to Π_p resp. Ψ_p we substitute p by constants of weight $(0,0)$. Consequently, the components which depend holomorphically on p get the weight $(0,0)$, resp. $(0,1)$, at the same time those components which depend antiholomorphically on p get the weight $(1,-1)$ resp. $(1,0)$.

Analogously one obtains $weight(\Pi_q) = (0,-1)$ and $weight(\Psi_q) = (0,0)$. Finally, the weight of $\langle f, \mathrm{id}\rangle$ is $(1,0)$.

From (25) follows $H_{I,I} = \Pi_p - 2i\Pi_q\langle f, \mathrm{id}\rangle$. Since $weight(\Pi_q\langle f, \mathrm{id}\rangle) = (1,-1)$ and $weight(H_{I,I}) = (0,0)$ then $H_{I,I} = (\Pi_p)_{(0,0)}$, where $(\Pi_p)_{(0,0)}$ is the $(0,0)$-component of Π_p.

In the same manner from $H_{II,I} = \Psi_p - 2i\Psi_q\langle f, \mathrm{id}\rangle$ follows $H_{II,I} = (\Psi_p)_{(0,1)}$.

Thus, the desired expression for the Jacobian matrix is

$$\begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix} = \begin{pmatrix} (\Pi_p)_{(0,0)} & \Pi_q \\ (\Psi_p)_{(0,0)} & \Psi_q \end{pmatrix}^{-1},$$

where $(\Pi_p)_{(0,0)}, (\Psi_p)_{(0,1)}$ can be obtained from (P_p, Q_p) by omitting the antiholomorphic terms with respect to p and by substituting p in the holomorphic terms by a free argument. To get (Π_q, Ψ_q) one inserts in (P_q, Q_q) q by a free holomorphic complex argument.

Now we go to compute (P_p, Q_p) and (P_q, Q_q). Therefore we need the derivatives of ϕ in 0 up to third order. They can be easily obtained by means of the recursive formula.

The recursive formula gives at once a simpler expression for f_2:

$$\begin{aligned} f_2 &= A(z,z) + B(w,z) + A(aw,z) + ia\langle z, a\bar{w}\rangle + \\ &\quad + ar(w,w) + ia\langle aw, a\bar{w}\rangle. \end{aligned}$$

Comparing with (21) leads to the following identities:

$$B(w, az) = ar(w,w) \tag{26}$$

$$A(aw, aw) = 2ia\langle aw, a\bar{w}\rangle. \tag{27}$$

Identity (27) is evidently equivalent to

$$A(aw, a\omega) = ia\langle aw, a\bar{\omega}\rangle + ia\langle a\omega, a\bar{w}\rangle.$$

Set now $f_3 = f_{zzz} + f_{zzw} + f_{zww} + f_{www}$, where the indices show the distribution of z and w variables. By means of (19) one obtaines

$$\begin{aligned} f_{zzz} &= A(A(z,z),z) \\ f_{zzw} &= A(B(w,z),z) + \frac{1}{2}B(w, A(z,z)) + iB(\langle z, a\bar{w}\rangle, z) + \\ &\quad + A(A(aw,z),z) + \frac{1}{2}A(A(z,z), aw) + ia\langle A(z,z), a\bar{w}\rangle \end{aligned}$$

$$\begin{aligned} f_{zww} &= \frac{1}{2}B(w,B(w,z)) + \frac{1}{2}B(r(w,w),z) + \frac{1}{2}B(w,A(aw,z)) - \\ &-\frac{i}{2}B(w,a\langle z,a\bar{w}\rangle) + \frac{1}{2}A(B(w,z),aw) + 2iar(\langle z,a\bar{w}\rangle,w) + \\ &+\frac{i}{2}a\langle B(w,z),a\bar{w}\rangle + \frac{i}{2}a\langle z,ar(\bar{w},\bar{w})\rangle + \frac{i}{2}B(\langle aw,a\bar{w}\rangle,z) + \\ &+\frac{1}{2}A(ar(w,w),z) + \frac{1}{2}A(A(aw,z),aw) + \frac{i}{2}a\langle A(aw,z),a\bar{w}\rangle + \\ &+\frac{i}{2}A(a\langle aw,a\bar{w}\rangle,z) - \frac{1}{2}a\langle z,a\langle a\bar{w},aw\rangle\rangle - \frac{1}{2}a\langle aw,a\langle a\bar{w},z\rangle\rangle. \end{aligned}$$

We do not need the expression for f_{www}.

For $g_3 = g_{zzz} + g_{zzw} + g_{zww} + g_{www}$ one gets

$$\begin{aligned} g_{zzz} &= 0 \\ g_{zzw} &= 2i\langle A(z,z),a\bar{w}\rangle,z) \\ g_{zww} &= i\langle B(w,z),a\bar{w}\rangle + 2ir(\langle z,a\bar{w}\rangle,w) + i\langle z,ar(\bar{w},\bar{w})\rangle + \\ &\quad +2i\langle A(aw,z),a\bar{w}\rangle \\ g_{www} &= r(r(w,w),w) + ir(\langle aw,a\bar{w}\rangle,w) + \frac{i}{2}\langle aw,ar(\bar{w},\bar{w})\rangle + \\ &\quad +\frac{i}{2}\langle ar(w,w),a\bar{w}\rangle + \frac{i}{2}\langle A(aw,aw),a\bar{w}\rangle, \end{aligned}$$

As in the case of (P_e, Q_e) we can now determine the vector fields (P_p, Q_p) as well as (P_q, Q_q). Let $P_p = P_0^p + P_z^p + P_w^p + P_{zz}^p + P_{zw}^p$ and $Q_p = Q_0^p + Q_z^p + Q_w^p + Q_{zw}^p + Q_{ww}^p$ be the expansion into homogeneous components with respect to z and w. Then

$$
\begin{aligned}
(28)\quad P^p_0 &= \underline{p} \\
P^p_z &= \underline{-2A(z,p)} - 2ia\langle z,p\rangle \\
P^p_w &= \underline{-B(w,p) - A(aw,p) + ia\langle p, a\bar{w}\rangle} - 2ia\langle aw,p\rangle \\
P^p_{zz} &= \underline{2A(A(z,p),z) - A(A(z,z),p)} + 2iA(z,a\langle z,p\rangle) + 2a\langle z, a\langle p,z\rangle\rangle - \\
&\quad -2iB(\langle z,p\rangle, z) \\
P^p_{zw} &= \underline{A(B(w,p),z) - A(B(w,z),p) + iB(\langle p,a\bar{w}\rangle,z) - iB(\langle z,a\bar{w}\rangle,p)} + \\
&\quad \underline{+B(w,A(z,p)) - A(A(z,aw),p) + A(A(z,p),aw) +} \\
&\quad \underline{+A(A(p,aw),z) + 2a\langle z, a\langle a\bar{w},p\rangle\rangle} - 2iB(\langle aw,p\rangle,z) + \\
&\quad +2iA(a\langle aw,p\rangle,z) + 2i\langle z, A(aw,p)\rangle - 2a\langle a\langle z,p\rangle, a\bar{w}\rangle \\
Q^p_0 &= 0 \\
Q^p_z &= 2i\langle z,p\rangle \\
Q^p_w &= \underline{-2i\langle p,a\bar{w}\rangle} + 2i\langle aw,p\rangle \\
Q^p_{zw} &= \underline{-4\langle z, a\langle a\bar{w},p\rangle\rangle} - 2i\langle z, B(\bar{w},p)\rangle - 2i\langle z, A(a\bar{w},p)\rangle + 2\langle z, a\langle p,aw\rangle\rangle \\
Q^p_{ww} &= \underline{i\langle B(w,p),a\bar{w}\rangle - i\langle p, ar(\bar{w},\bar{w})\rangle + 2ir(\langle p,a\bar{w}\rangle,w) -} \\
&\quad \underline{-2\langle aw, a\langle a\bar{w},p\rangle\rangle} + 2i\langle ar(w,w),p\rangle - 4ir(\langle aw,p\rangle,w) - \\
&\quad -2\langle a\langle aw,a\bar{w}\rangle,p\rangle - 2\langle a\langle aw,p\rangle,a\bar{w}\rangle + 2\langle aw, a\langle p,aw\rangle\rangle
\end{aligned}
$$

The terms which depend holomorphically on p and, therefore, contribute to the formula of the Jacobian are underlined.

The computation of (P_q, Q_q) leads to

$$
\begin{aligned}
P^q_0 &= -aq \\
P^q_z &= -B(q,z) + A(aq,z) + ia\langle z,aq\rangle \\
P^q_w &= B(w,aq) + ia\langle aw,aq\rangle \\
P^q_{zz} &= 2iB(\langle z,aq\rangle,z) - A(A(z,aq),z) + \frac{1}{2}A(A(z,z),aq)
\end{aligned}
$$

$$
\begin{aligned}
P^q_{zw} &= \frac{1}{2}B(w,B(q,z)) - \frac{1}{2}B(q,B(w,z)) + B(r(q,w),z) - \frac{1}{2}B(w,A(aq,z)) - \\
&- \frac{i}{2}B(\langle aq,a\bar{w}\rangle,z) - \frac{i}{2}B(q,a\langle z,a\bar{w}\rangle) + \frac{i}{2}B(w,a\langle z,aq\rangle) - \\
&- \frac{1}{2}B(q,A(aw,z)) + iB(\langle z,aq\rangle,aw) + \frac{i}{2}B(\langle aw,aq\rangle,z) - \\
&- 2ia\langle z,B(\bar{w},aq)\rangle + ia\langle z,ar(\bar{w},q)\rangle + \frac{1}{2}A(B(w,z),aq) - \\
&- \frac{1}{2}A(B(w,aq),z) + \frac{1}{2}A(B(q,z),aw) + \frac{1}{2}A(B(q,aw),z) - \\
&- \frac{i}{2}a\langle B(w,z),aq\rangle + \frac{i}{2}a\langle B(q,z),a\bar{w}\rangle - \frac{i}{2}a\langle z,A(a\bar{w},aq)\rangle - \\
&- \frac{1}{2}A(A(aw,aq),z) + \frac{1}{2}A(A(aw,z),aq) - \frac{1}{2}A(A(aq,z),aw) - \\
&- \frac{i}{2}a\langle A(z,aw),aq\rangle + \frac{i}{2}a\langle A(z,aq),a\bar{w}\rangle - \frac{1}{2}a\langle aw,a\langle aq,z\rangle\rangle + \\
&+ \frac{1}{2}a\langle aq,a\langle a\bar{w},z\rangle\rangle, \\
Q^q_0 &= q \\
Q^q_z &= -2i\langle z,aq\rangle \\
Q^q_w &= -2r(q,w) + i\langle aq,a\bar{w}\rangle - i\langle aw,aq\rangle \\
Q^q_{zw} &= 2i\langle z,B(\bar{w},aq)\rangle + 2\langle z,a\langle a\bar{w},aq\rangle\rangle \\
Q^q_{ww} &= 2r(r(w,q),w) - r(r(w,w),q) - i\langle B(w,aq),a\bar{w}\rangle + i\langle ar(w,q),a\bar{w}\rangle - \\
&- ir(\langle aq,a\bar{w}\rangle,w) + ir(\langle aw,aq\rangle,w) - ir(\langle aw,a\bar{w}\rangle,q) + \frac{i}{2}\langle aq,ar(\bar{w},\bar{w})\rangle - \\
&- \frac{i}{2}\langle ar(w,w),aq\rangle + i\langle aw,ar(\bar{w},q)\rangle + \langle aw,a\langle a\bar{w},aq\rangle\rangle.
\end{aligned}
$$

Hence, all ingredients of the automorphism formula

$$
\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} (\Pi_p)_{(0,0)} & 2\Pi_q \\ \frac{1}{2}(\Psi_p)_{(0,0)} & \Psi_q \end{pmatrix}^{-1} \begin{pmatrix} P_e \\ \frac{1}{2}Q_e \end{pmatrix} \tag{29}
$$

are completely described.

7. The Heisenberg sphere in $\mathbb{C}^2$

In this section we want to demonstrate the obtained formula in the simple case of the sphere in $\mathbb{C}^2$. Let $Q = \{(z,w) \in \mathbb{C}^2 : \operatorname{Im} w = |z|^2\}$. Then any $\phi \in \operatorname{Aut}_{0,\mathrm{id}} Q$ can be described by Poincaré's formula (see [11])

$$
\begin{aligned}
f &= \frac{z+aw}{1-2i\bar{a}z-(r+i|a|^2)w} \\
g &= \frac{w}{1-2i\bar{a}z-(r+i|a|^2)w},
\end{aligned}
\tag{30}
$$

where $a \in \mathbb{C}$ and $r \in \mathbb{R}$.
We will now obtain ϕ by means of the procedure developped above.
We have

$$
\begin{aligned}
(\Pi_p)_{(0,0)} &= 1-4i\bar{a}z-rw-i|a|^2w-4\bar{a}^2z^2+2i\bar{a}rzw-2\bar{a}^2azw \\
\frac{1}{2}(\Psi_p)_{(0,0)} &= -i\bar{a}w-2\bar{a}^2zw-\bar{a}^2aw^2 \\
2(\Pi_q) &= -2a+6i|a|^2z-2rz+2arw+2ia^2\bar{a}w+4i\bar{a}rz^2+4a\bar{a}^2z^2+ \\
&\quad +2r^2zw+2|a|^4zw \\
(\Psi_q) &= 1-2i\bar{a}z-2rw+2a\bar{a}^2zw+2i\bar{a}rzw+r^2w^2+|a|^4w^2 \\
P_e &= z-aw-2i\bar{a}z^2-2rzw \\
\frac{1}{2}Q_e &= w-i\bar{a}zw-rw^2.
\end{aligned}
$$

Since

$$
\begin{pmatrix} (\Pi_p)_{(0,0)} & 2\Pi_q \\ \frac{1}{2}(\Psi_p)_{(0,0)} & \Psi_q \end{pmatrix} = \begin{pmatrix} 1-2i\bar{a}z & -2a-2rz+2i|a|^2z \\ -i\bar{a}w & 1-rw+i|a|^2w \end{pmatrix} N,
$$

with $N = 1-2i\bar{a}z-(r+i|a|^2)w$, and

$$
\begin{pmatrix} P_e \\ \frac{1}{2}Q_e \end{pmatrix} = \begin{pmatrix} 1-2i\bar{a}z & -2a-2rz+2i|a|^2z \\ -i\bar{a}w & 1-rw+i|a|^2w \end{pmatrix} \begin{pmatrix} z+aw \\ w \end{pmatrix},
$$

cancelling the corresponding matrices in the formula (29) we obtain the unique automorphism (30) with parameters (a,r).

8. Poincaré automorphisms

As it was shown in [6] the automorphisms from $\phi \in \mathrm{Aut}_{0,\mathrm{id}}\,Q$ with parameters (a,r,A,B) can be described by a much simpler "matrix fractional linear" formula which is similar to the Poincaré formula (30) if there exist a $\mathbb{C}^n$-valued bilinear form $\hat{A} : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n$ and a $\mathbb{C}^k$-valued hermitian form $\hat{r} : \mathbb{C}^k \otimes \bar{\mathbb{C}}^n \to \mathbb{C}^k$ such that

$$
\begin{aligned}
\langle \hat{A}(z,\zeta),\xi\rangle &= 2i\langle z, a\langle\xi,\zeta\rangle\rangle, \qquad (31)\\
\langle B(w,\zeta),\xi\rangle &= \hat{r}(w,\langle\xi,\zeta\rangle) \qquad (32)
\end{aligned}
$$

is satisfied for all $z, \zeta, \xi \in \mathbb{C}^n$, $w \in \mathbb{C}^k$. Then ϕ takes the form

$$
\begin{aligned}
(33) \qquad z^* &= (\mathrm{id} - \hat{A}(z,\cdot) - B(w,\cdot) - \frac{1}{2}\hat{A}(aw,\cdot))^{-1}(z + aw), \\
w^* &= (\mathrm{id} - 2i\langle z, a\bar{\cdot}\rangle - \hat{r}(w,\bar{\cdot}) - i\langle aw, a\bar{\cdot}\rangle)^{-1} w.
\end{aligned}
$$

This formula can be obtained from (29), as in the case of the Heisenberg sphere by cancelling appropriate matrices. We show in the Appendix that

$$
(34) \qquad \begin{pmatrix} (\Pi_p)_{(0,0)} & 2\Pi_q \\ \frac{1}{2}(\Psi_p)_{(0,0)} & \Psi_q \end{pmatrix} = \begin{pmatrix} \mathrm{id} - \hat{A}(\cdot, z) & -2a - 2B(\cdot, z) + \hat{A}(a\cdot, z) \\ -i\langle \cdot, a\bar{w}\rangle & \mathrm{id} + i\langle a\cdot, a\bar{w}\rangle - \hat{r}(\cdot, \bar{w}) \end{pmatrix} \times
$$
$$
\times \begin{pmatrix} \mathrm{id} - \hat{A}(z,\cdot) - B(w,\cdot) - \frac{1}{2}\hat{A}(aw,\cdot) & 0 \\ 0 & \mathrm{id} - 2i\langle z, a\bar{\cdot}\rangle - \hat{r}(w,\bar{\cdot}) - i\langle aw, a\bar{\cdot}\rangle \end{pmatrix}.
$$

Furthermore,

$$
\begin{pmatrix} P_e \\ \frac{1}{2}Q_e \end{pmatrix} = \begin{pmatrix} \mathrm{id} - \hat{A}(\cdot, z) & -2a - 2B(\cdot, z) + \hat{A}(a\cdot, z) \\ -i\langle \cdot, a\bar{w}\rangle & \mathrm{id} + i\langle a\cdot,, a\bar{w}\rangle - \hat{r}(\cdot, \bar{w}) \end{pmatrix} \begin{pmatrix} z + aw \\ w \end{pmatrix}.
$$

The theory of Poincaré automorphisms gives a complete description of the automorphisms of nondegenerate quadrics of codimension $k = 1, 2, n^2$ and of real associative quadrics (see [4] [5]). However, Palinchak [10] found a quadric in $\mathbb{C}^6$ of codimension 3 with a 9-dimensional $\mathrm{Aut}_{0,\mathrm{id}}$ group which does not contain Poincaré automorphisms.

9. Automorphisms of different types of rigid quadrics

We introduce the following terminology: A nondegenerate quadric Q will be called s-rigid if from $(Xz, sw) \in \mathfrak{g}_0$ follows that $s = t \cdot \mathrm{id}$ with $t \in \mathbb{R}$ (in particular, any hyperquadric is s-rigid); it will be called a-rigid, resp., r-rigid if $\mathcal{A} = \{0\}$, resp., $\mathcal{R} = \{0\}$.

Proposition 1. *If a nondegenerate quadric Q is a-rigid then it is also r-rigid.*

Proof. Consider P^p_w in (28). For $a = 0$ follows that $B(\cdot, p)$ is contained in $\mathcal{A}$ for all $p \in \mathbb{C}^n$. Hence, if there was some $B \neq 0$ then would exist some $p \in \mathbb{C}^n$ such that $B(\cdot, p) \neq 0$. □

Proposition 2. *If Q is a s-rigid nondegenerate quadric then $\mathrm{Aut}_{0,\mathrm{id}}\, Q$ consists of fractional linear mappings. If, moreover, $k > 1$ then Q is r-rigid.*

Proof. Consider Q^p_w in (28). Then $\mathfrak{s} \cong \mathbb{R}$ implies

$$
(35) \qquad \langle p, a\bar{w}\rangle = l(p)w,
$$

where l is a complex linear functional on $\mathbb{C}^n$. Setting in (35) $w = \langle z,\zeta\rangle$ one obtains a solution $\hat{A}$ of (31) corresponding to a: $\langle p, a\langle z,\zeta\rangle\rangle = l(p)\langle\zeta,z\rangle = \langle l(p)\zeta, z\rangle$, i.e., $\hat{A}(p,\zeta) = 2il(p)\zeta$. But then

$$\begin{aligned} z^* &= (1-2il(z)-il(aw))^{-1}(z+aw) \\ w^* &= (1-2il(z)-il(aw))^{-1}w \end{aligned}$$

is the uniquely determined automorphism corresponding to $(a,0)$.

Now we consider Q^q_w and set there $a = 0$. It follows

$$r(q,w) = \lambda(q)w, \tag{36}$$

where λ is a real linear functional on $\mathbb{R}^k$. Setting again $w = \langle z,\zeta\rangle$, one obtains $B(u,z) = \lambda(u)z$ and $\hat{r}(w,\omega) = \lambda(w)\bar{\omega}$.

Hence,

$$\begin{aligned} z^* &= (1-\lambda(w))^{-1}z \\ w^* &= (1-\lambda(w))^{-1}w \end{aligned}$$

is the automorphism corresponding to r.

From the symmetry of r follows $r(u,v) = \lambda(u)v = \lambda(v)u$, i.e., if $\mathfrak{s}\cong\mathbb{R}$ and $k>1$, then $\mathcal{R}=\{0\}$. □

10. Canonical parametrization of chains

Let $\Gamma_0 = \{z=0, \operatorname{Im} w = 0\}$ be the standard chain on Q. Then there exists a canonical family of parametrizations of Γ_0 which can be obtained from the standard parametrization $\{z=0, w=u : u\in\mathbb{R}^k\}$ by means of a "reparametrization" automorphism corresponding to parameters $(0,r)$.

From (29) and (23) we derive a simple equation for this reparametrization map:

Proposition 3. *The automorphism ϕ_r, corresponding to $(0,r)$ has the form*

$$\begin{aligned} z^* &= (\operatorname{id} - B(w,\cdot))^{-1}z \\ w^* &= (\operatorname{id} - 2r(w,\cdot) + 2r(r(w,\cdot),w) - r(r(w,w)\cdot))^{-1}(w - r(w,w)). \end{aligned}$$

Proof. At first we set in (23) $a = 0$. It follows

$$f(z,w) = \frac{\partial f}{\partial z}z$$

Setting in

$$(\Pi_p)_{(0,0)} = \frac{\partial f}{\partial z}$$

$a = 0$ one obtains immediately $f(z,w) = (\mathrm{id} - B(w,\cdot))^{-1} z$.
The expression for g can be derived by setting $a = 0$ in (29). □

The expression for the g-component in ϕ_r can be simplified if the following condition is satisfied:

Proposition 4. *Let ϕ_r be as in Proposition 3 and $\hat{r}(\cdot)$ be a linear map $\mathbb{C}^k \to \mathfrak{gl}(\mathbb{C},k)$ with*

$$\begin{aligned} \hat{r}(w)w &= r(w,w) \\ \hat{r}(w)^2 &= \hat{r}(r(w,w)), \end{aligned} \tag{37}$$

then

$$w^* = g(z,w) = (\mathrm{id} - \hat{r}(w))^{-1} w.$$

Proof. From Proposition 3 follows that g does not depend on z. The recursive formula for g therefore takes the simple form

$$(l-1) g_l(z,w) = \frac{\partial g_{l-1}}{w} r(w,w)$$

with $g_0 = 0$ and $g_1 = w$.
One easily verifies that $g_l := \hat{r}(w)^{l-1} w$ is the solution of the recursive equations. □

It follows

Proposition 5. *Let Q be a nondegenerate quadric and $r \in \mathcal{R}$ with the property $r(w,w) = \langle aw, a\bar{w}\rangle$ (resp. $r(w,w) = -\langle aw, a\bar{w}\rangle$). Then $\hat{r}(w) = \langle aw, a\bar{\cdot}\rangle$ (resp. $\hat{r}(w) = -\langle aw, a\bar{\cdot}\rangle$) satisfies (37).*

Proof. Set $r(w,w) = \langle aw, a\bar{w}\rangle$ and $\hat{r}(w) = \langle aw, a\bar{\cdot}\rangle$. Because of (3) then

$$\hat{r}(w)^2 = \langle aw, a\langle a\bar{\cdot}, aw\rangle\rangle = \frac{1}{2i}\langle A(aw,aw), a\bar{\cdot}\rangle.$$

On the other hand it follows from (27) that

$$\hat{r}(r(w,w) = \langle a\langle a\bar{\cdot}, aw\rangle a\bar{\cdot}\rangle = \frac{1}{2i}\langle A(aw,aw), a\bar{\cdot}\rangle.$$

□

Remark. The representation $r(w,w) = \langle aw, a\bar{w}\rangle$ is not unique. Moreover, there can exist tensors $\hat{r}$ satisfying (37) which cannot be obtained in the described manner.

For automorphisms corresponding to $(a,0)$ one can derive the following simple equation for the g-component:

Proposition 6. *Let $\Phi_a \in \mathrm{Aut}_{0,\mathrm{id}}\, Q$ with $r = 0$. Then*

$$w^* = (\mathrm{id} - 2i\langle z, a\bar{\cdot}\rangle - i\langle aw, a\bar{\cdot}\rangle)^{-1} w.$$

Proof. Set $d := 2i\langle z, a\bar{\cdot}\rangle + i\langle aw, a\bar{\cdot}\rangle$. We show by induction that $g_l = d^{l-1}w$. This implies the assertion.

For $l = 1$ we have $g_1 = w$. By inductive assumption then $g_{l-1} = d^{l-2}w$. Using the recursive formula (19) we come to

$$\begin{aligned}
(l-1)g_l &= \sum_{s=1}^{l-3} d^s(2i\langle A(z,z), a\bar{\cdot}\rangle + 2i\langle A(aw,z), a\bar{\cdot}\rangle + \\
&\quad +2\langle a\langle z, a\bar{w}\rangle, a\bar{\cdot}\rangle)d^{l-s-3}w + \\
&\quad + \sum_{s=1}^{l-3} d^s(-2\langle a\langle z, a\bar{w}\rangle, a\bar{\cdot}\rangle) - \langle a\langle aw, a\bar{w}\rangle, a\bar{\cdot}\rangle)d^{l-s-3}w + \\
&\quad +d^{l-2}(2i\langle z, a\bar{w}\rangle + i\langle aw, a\bar{w}\rangle) \\
&= \sum_{s=1}^{l-3} d^s(2i\langle A(z,z), a\bar{\cdot}\rangle + 2i\langle A(aw,z), a\bar{\cdot}\rangle - \\
&\quad -\langle a\langle aw, a\bar{w}\rangle, a\bar{\cdot}\rangle d^{l-s-3}w + \\
&\quad +d^{l-2}(2i\langle z, a\bar{w}\rangle + i\langle aw, a\bar{w}\rangle)
\end{aligned}$$

The assertion follows if we show that

$$2i\langle A(z,z), a\bar{\cdot}\rangle + 2i\langle A(aw,z), a\bar{\cdot}\rangle - \langle a\langle aw, a\bar{w}\rangle, a\bar{\cdot}\rangle) = d^2.$$

For d^2 we obtain

$$\begin{aligned}
d^2 &= -4\langle z, a\langle a\cdot, z\rangle\rangle - \langle aw, a\langle a\bar{\cdot}, aw\rangle\rangle - \\
&\quad -2\langle z, a\langle a\cdot, aw\rangle\rangle - 2\langle aw, \langle a\bar{\cdot}, z\rangle\rangle.
\end{aligned}$$

Because of (3) then

$$\begin{aligned}
-4\langle z, a\langle a\cdot, z\rangle\rangle &= 2i\langle A(z,z), a\bar{\cdot}\rangle \\
\text{and } -2\langle z, a\langle a\cdot, aw\rangle\rangle - 2\langle aw, \langle a\bar{\cdot}, z\rangle\rangle &= 2i\langle A(aw,z), a\bar{\cdot}\rangle.
\end{aligned}$$

Because of (27) and (3)

$$-\langle aw, a\langle a\bar{\cdot}, aw\rangle\rangle = \frac{1}{2}\langle A(aw, aw), a\bar{\cdot}\rangle = -\langle aw, \langle a\bar{\cdot}, aw\rangle\rangle.$$

□

Proposition 6 and Theorem 4 give a description of the chains including the canonical parameter:

Corollary 1. *The chains of the nondegenerate quadric Q have the following canonical parametrization*

$$\begin{aligned} f(u) &= a(\mathrm{id} - i\langle au, a\cdot\rangle)^{-1}u \\ g(u) &= (\mathrm{id} - i\langle au, a\cdot\rangle)^{-1}u \end{aligned}$$

with $u \in \mathbb{R}^k$.

Proof. The expression for g can be obtained by setting $z = 0$ and $w = u$ in the formula from Proposition 6. The expression for f follows then from Theorem 4. □

11. Appendix

For the proof of (34) we use the following facts:

Let $\mathfrak{A}$ be the set of all pairs $(D, d) \in \mathfrak{gl}(n, \mathbb{C}) \times \mathfrak{gl}(k, \mathbb{C})$ with the property $\langle Dz, z\rangle = d\langle z, z\rangle$. Then $\mathfrak{A}$ is an algebra with unit. It follows from (31) and (32) that

$$\begin{aligned} Da &= ad, \\ \hat{A}(Dz, \zeta) &= \hat{A}(z, D\zeta) = D\hat{A}(z, \zeta), \\ A(Dz, \zeta) &= A(z, D\zeta) = DA(z, \zeta), \\ B(dw, z) &= B(w, Dz) = DB(w, z), \\ \hat{r}(dw, \omega) &= d\hat{r}(w, \omega), \\ r(dw, \omega) &= dr(w, \omega). \end{aligned}$$

Moreover, (31) and (32) mean that

$$(\hat{A}(z, \cdot), 2i\langle z, a\bar{\cdot}\rangle) \in \mathfrak{A} \tag{38}$$

and

$$(B(w, \cdot), \hat{r}(w, \bar{\cdot})) \in \mathfrak{A}. \tag{39}$$

The equality of the left upper blocks is a consequence of the equalities

$$\begin{aligned}
&(40) & \mathrm{id} &= \mathrm{id},\\
&(41) & -2A(z,\cdot) &= -\hat{A}(z,\cdot)-\hat{A}(\cdot,z),\\
&(42) & -B(w,z) &= -B(w,z),\\
&(43) & -\frac{1}{2}\hat{A}(aw,\cdot) &= -A(aw,\cdot)+ia\langle\cdot,a\bar{w}\rangle,\\
&(44) & \hat{A}(\hat{A}(z,\cdot),z) &= 2A(A(z,\cdot),z)-A(A(z,z),\cdot),\\
&(45) & \frac{1}{2}\hat{A}(\hat{A}(aw,\cdot),z) &= -A(A(z,aw),\cdot)+A(A(z,\cdot),aw)+\\
& & & A(A(\cdot,aw),z)+2a\langle z,a\langle a\bar{w},\cdot\rangle\rangle\\
&(46) & \hat{A}(B(w,\cdot),z) &= A(B(w,\cdot),z)-A(B(w,z),\cdot)+iB(\langle\cdot,a\bar{w}\rangle,z)-\\
& & & -iB(\langle z,a\bar{w}\rangle,\cdot)+B(w,A(z,\cdot)),
\end{aligned}$$

Equations (40) and (42) are tautologies, (41) follows by symmetrization of (31). To prove (43) we show that $\frac{1}{2}\hat{A}(\cdot,aw)=ia\langle\cdot,a\bar{w}\rangle$ and apply (41). The latter equality follows from the fact that $(\hat{A}(p,\cdot),2i\langle p,a\bar{\cdot}\rangle)\in\mathfrak{A}$ and, that for any $(D,d)\in\mathfrak{A}$ and for any $a\in\mathcal{A}$ holds $Da=ad$. (44) can be obtained by the following sequence of equivalent transformations

$$\begin{aligned}
2A(A(z,\cdot),z)-A(A(z,z),\cdot) &= 2A(A(z,\cdot),z)-A(\cdot,\hat{A}(z,z))\\
&= 2A(A(z,\cdot),z)-A(\hat{A}(z,\cdot),z)\\
&= A(\hat{A}(\cdot,z),z)\\
&= \hat{A}(\cdot,A(z,z))\\
&= \hat{A}(\cdot,\hat{A}(z,z))\\
&= \hat{A}(z,\hat{A}(\cdot,z))\\
&= \hat{A}(\hat{A}(z,\cdot),z).
\end{aligned}$$

In (45) we use the identity

$$2a\langle z,a\langle a\bar{w},\cdot\rangle\rangle=-i\hat{A}(z,a\langle\cdot,a\bar{w}\rangle)=-\frac{1}{2}\hat{A}(\hat{A}(z,\cdot),aw).$$

The right hand side of (45) takes then the form

$$
\begin{aligned}
& -\frac{1}{2}A(\hat{A}(z,aw),\cdot) - \frac{1}{2}A(\hat{A}(aw,z),\cdot) + +\frac{1}{2}A(\hat{A}(z,\cdot),aw) + \frac{1}{2}A(\hat{A}(\cdot,z),aw) + \\
& +\frac{1}{2}A(\hat{A}(\cdot,aw),z) + \frac{1}{2}A(\hat{A}(aw,\cdot),z) - -\frac{1}{2}\hat{A}(\hat{A}(z,\cdot),aw) \\
= & -\frac{1}{2}\hat{A}(z,A(aw,\cdot)) - \frac{1}{2}\hat{A}(aw,A(z,\cdot)) + \frac{1}{2}\hat{A}(z,A(\cdot,aw)) + \frac{1}{2}\hat{A}(\cdot,A(z,aw)) + \\
& +\frac{1}{2}\hat{A}(\cdot,A(aw,z)) + \frac{1}{2}\hat{A}(aw,A(\cdot,z)) - \frac{1}{2}\hat{A}(\hat{A}(z,\cdot),aw) \\
= & \hat{A}(\cdot,A(z,aw)) - \frac{1}{2}\hat{A}(\hat{A}(z,\cdot),aw) \\
= & \hat{A}(\cdot,A(z,aw)) - \frac{1}{2}\hat{A}(\cdot,\hat{A}(z,aw)) \\
= & \frac{1}{2}\hat{A}(\cdot,\hat{A}(aw,z)) \\
= & \frac{1}{2}\hat{A}(\hat{A}(aw,\cdot),z).
\end{aligned}
$$

It remains to prove (46). Using the symmetry of A and (39) we see that $A(B(w,\cdot),z) - A(B(w,z),\cdot) = 0$. Therefore, the right hand side of (46) takes the form

$$
\begin{aligned}
& iB(\langle\cdot,a\bar{w}\rangle,z) - iB(\langle z,a\bar{w}\rangle,\cdot) + B(w,A(z,\cdot)) \\
= & \frac{1}{2}B(2i\langle\cdot,a\bar{w}\rangle,z) - \frac{1}{2}B(2i\langle z,a\bar{w}\rangle),\cdot) + B(w,A(z,\cdot)) \\
= & \frac{1}{2}B(w,\hat{A}(\cdot,z)) - \frac{1}{2}B(w,\hat{A}(z,\cdot)) + B(w,A(z,\cdot)) \\
= & B(w,\hat{A}(\cdot,z)) \\
= & \hat{A}(B(w,\cdot),z).
\end{aligned}
$$

The equality of the right upper blocks is a consequence of the equalities

$$
\begin{aligned}
&(47) & -2a &= -2a \\
&(48) & -2B(\cdot,z) &= -2B(\cdot,z) \\
&(49) & 4ia\langle z,a\bar{\cdot}\rangle + \hat{A}(a\cdot,z) &= 2A(a\cdot,z)+2ia\langle z,a\bar{\cdot}\rangle \\
&(50) & 2a\hat{r}(w,\bar{\cdot}) &= 2B(w,a\cdot) \\
&(51) & 2ia\langle aw,a\bar{\cdot}\rangle &= 2ia\langle aw,a\bar{\cdot}\rangle \\
&(52) & 4iB(\langle z,a\bar{\cdot}\rangle,z) &= 4iB(\langle z,a\bar{\cdot}\rangle,z) \\
&(53) & -2i\hat{A}(a\langle z,a\bar{\cdot}\rangle,z) &= -2A(A(z,a\cdot),z)+A(A(z,z),a\cdot) \\
&(54) & 2B(\hat{r}(w,\bar{\cdot}),z) &= B(w,B(\cdot,z))-B(\cdot,B(w,z))+ \\
&& &\quad +2B(r(\cdot,w),z) \\
&(55) & -\hat{A}(a\hat{r}(w,\bar{\cdot}),z)+ & \\
&& +2iB(\langle aw,a\bar{\cdot}\rangle,z) &= -B(w,A(a\cdot,z))-iB(\langle a\cdot,a\bar{w}\rangle,z)- \\
&& &\quad -iB(\cdot,a\langle z,\bar{w}\rangle)+iB(w,a\langle z,\bar{\cdot}\rangle)- \\
&& &\quad -B(\cdot,A(aw,z))+2iB(\langle z,a\bar{\cdot}\rangle,aw)+ \\
&& &\quad +iB(\langle aw,a\bar{\cdot}\rangle,z)-4ia\langle z,B(\bar{w},a\bar{\cdot})\rangle+ \\
&& &\quad +2ia\langle z,ar(\bar{w},\bar{\cdot})\rangle+A(B(w,z),a\cdot)- \\
&& &\quad -A(B(w,a\cdot),z)+A(B(\cdot,z),aw)+ \\
&& &\quad +A(B(\cdot,aw),z)-ia\langle B(w,z),a\bar{\cdot}\rangle+ \\
&& &\quad +ia\langle B(\cdot,z),a\bar{w}\rangle \\
&(56) & -i\hat{A}(a\langle aw,a\bar{\cdot}\rangle,z) &= -ia\langle z,A(a\bar{w},a\bar{\cdot}\rangle-A(A(aw,,a\cdot),z)+ \\
&& &\quad +A(A(aw,z),a\cdot)-A(A(a\cdot,z),aw)- \\
&& &\quad -ia\langle A(z,aw),a\bar{\cdot}\rangle+ia\langle A(z,a\cdot),a\bar{w}\rangle- \\
&& &\quad -a\langle aw,a\langle a\bar{\cdot},z\rangle\rangle+a\langle a\cdot,a\langle a\bar{w},z\rangle\rangle.
\end{aligned}
$$

The equalities (47), (48), (51), (52) are tautologies. Using (38) and (39), one easily proves (49) and (50).

The equality (53) is a consequence of

$$
\begin{aligned}
-2A(A(z,a\cdot),z)+A(A(z,z),a\cdot) &= -A(\hat{A}(z,a\cdot),z)-A(\hat{A}(a\cdot,z),z)+A(A(z,z),a\cdot) \\
&= -A(a\cdot,A(z,z))-A(\hat{A}(a\cdot,z),z)+ \\
&\quad +A(A(z,z),a\cdot) \\
&= -\hat{A}(a\cdot,A(z,z)) \\
&= -\hat{A}(\hat{A}(z,a\cdot),z) \\
&= -2i\hat{A}(a\langle z,a\bar{\cdot},z).
\end{aligned}
$$

To prove (54) we notice that $B(w,B(\cdot,z))=B(\cdot,B(w,z))$. Then we obtain

$$
\begin{aligned}
2B(r(\cdot,w),z) &= B(\hat{r}(w,\bar{\cdot}),z)+B(\hat{r}(\cdot,\bar{w}),z) \\
&= B(\hat{r}(w,\bar{\cdot}),z)+B(w,B(\cdot,z)) \\
&= B(\hat{r}(w,\bar{\cdot}),z)+B(\cdot,B(w,z)) \\
&= 2B(\hat{r}(w,\bar{\cdot}),z).
\end{aligned}
$$

The left hand side of (55) equals $\hat{A}(B(w,a\cdot),z)-\hat{A}(aw,B(\cdot,z))$. The right hand side can be transformed in the following way

$$
\begin{aligned}
&-\frac{1}{2}B(w,\hat{A}(a\cdot,z))-\frac{1}{2}B(w,\hat{A}(z,a\cdot))-\frac{1}{2}\hat{A}(a\cdot,B(w,z))-\frac{1}{2}\hat{A}(z,B(\cdot,aw))+ \\
&\frac{1}{2}\hat{A}(z,B(w,a\cdot))-\frac{1}{2}B(\cdot,\hat{A}(aw,z))-\frac{1}{2}B(\cdot,\hat{A}(z,aw))+\hat{A}(z,B(\cdot,aw))+ \\
&+\frac{1}{2}\hat{A}(aw,B(\cdot,z))-2\hat{A}(z,B(\cdot,aw))+\frac{1}{2}\hat{A}(z,a\hat{r}(\cdot,\bar{w}))+\frac{1}{2}\hat{A}(z,a\hat{r}(w,\bar{\cdot}))+ \\
&+\frac{1}{2}\hat{A}(B(w,z),a\cdot)+\frac{1}{2}\hat{A}(a\cdot,B(w,z))-\frac{1}{2}\hat{A}(B(w,a\cdot),z)-\frac{1}{2}\hat{A}(z,B(w,a\cdot))+ \\
&+\frac{1}{2}\hat{A}(B(\cdot,z),aw)+\frac{1}{2}\hat{A}(aw,B(\cdot,z))-\frac{1}{2}B(w,\hat{A}(z,a\cdot))+\frac{1}{2}\hat{A}(B(\cdot,aw),z)+ \\
&+\frac{1}{2}\hat{A}(z,B(\cdot,aw))+\frac{1}{2}B(\cdot,\hat{A}(z,aw))
\end{aligned}
$$

This equals to

$$-\frac{1}{2}\hat{A}(B(w,a\cdot),z)-\frac{1}{2}\hat{A}(B(w,z),a\cdot)-\frac{1}{2}\hat{A}(B(w,a\cdot),z)-\frac{1}{2}\hat{A}(B(\cdot,z),aw)+$$
$$+\frac{1}{2}\hat{A}(B(w,z),a\cdot)-\frac{1}{2}\hat{A}(B(\cdot,aw),z)-\frac{1}{2}\hat{A}(B(\cdot,z),aw)+\hat{A}(B(\cdot,z),aw)+$$
$$+\frac{1}{2}\hat{A}(B(\cdot,aw),z)-2\hat{A}(B(\cdot,z),aw)+\hat{A}(B(\cdot,z),aw)+\frac{1}{2}\hat{A}(B(w,z),a\cdot)+$$
$$+\frac{1}{2}\hat{A}(B(w,a\cdot),z)-\frac{1}{2}\hat{A}(B(w,a\cdot),z)-\frac{1}{2}\hat{A}(B(w,z),a\cdot)+\frac{1}{2}\hat{A}(B(\cdot,z),aw)+$$
$$+\frac{1}{2}\hat{A}(B(\cdot,aw),z)-\frac{1}{2}\hat{A}(B(w,z),a\cdot)+\frac{1}{2}\hat{A}(B(\cdot,aw),z)+\frac{1}{2}\hat{A}(B(\cdot,z),aw)+$$
$$+\frac{1}{2}\hat{A}(B(\cdot,z),aw)$$

Cancelling appropriate terms and using the identity

$$\hat{A}(B(w,z),a\cdot)=\hat{A}(B(\cdot,z),aw)$$

we obtain an expression which coincides with the left hand side of (55)
It remains to prove (56). We transform the right hand side:

$$-\frac{i}{2}a\langle z,\hat{A}(a\bar{w},\bar{a\cdot})\rangle-\frac{i}{2}a\langle z,\hat{A}(\bar{a\cdot},a\bar{w})\rangle-\frac{1}{2}A(\hat{A}(aw,a\cdot),z)-\frac{1}{2}A(\hat{A}(a\cdot,aw),z)+$$
$$+\frac{1}{2}A(\hat{A}(aw,z),a\cdot)+\frac{1}{2}A(\hat{A}(z,aw),a\cdot)-\frac{1}{2}A(\hat{A}(a\cdot,z),aw)-\frac{1}{2}A(\hat{A}(z,a\cdot),aw)-$$
$$-\frac{i}{2}a\langle\hat{A}(z,aw),\bar{a\cdot}\rangle-\frac{i}{2}a\langle\hat{A}(aw,z),\bar{a\cdot}\rangle+\frac{i}{2}a\langle\hat{A}(z,a\cdot),a\bar{w}\rangle+\frac{i}{2}a\langle\hat{A}(a\cdot,z),a\bar{w}\rangle+$$
$$+\frac{i}{2}a\langle\hat{A}(aw,z),\bar{a\cdot}\rangle-\frac{i}{2}a\langle\hat{A}(a\cdot,z),a\bar{w}\rangle.$$

The terms $\frac{i}{2}a\langle\hat{A}(aw,z),\bar{a\cdot}\rangle$ as well as $\frac{i}{2}a\langle\hat{A}(a\cdot,z),a\bar{w}\rangle$ with positive and negative sign cancel out.
Using the identities

$$\frac{1}{2}A(\hat{A}(aw,a\cdot),z)=\frac{1}{2}\hat{A}(aw,A(a\cdot,z))=\frac{1}{2}\hat{A}(aw,A(z,a\cdot))=\frac{1}{2}A(\hat{A}(aw,z),a\cdot)$$

and

$$\frac{1}{2}A(\hat{A}(z,aw),a\cdot)=\frac{1}{2}\hat{A}(z,A(aw,a\cdot))=\frac{1}{2}\hat{A}(z,A(a\cdot,aw))=\frac{1}{2}A(\hat{A}(z,a\cdot),aw)$$

two more pairs cancel out.
Thus, we obtain

$$-\frac{i}{2}a\langle z,\hat{A}(a\bar{w},a\bar{\cdot})\rangle-\frac{i}{2}a\langle z,\hat{A}(a\bar{\cdot},a\bar{w})\rangle-\frac{1}{2}\hat{A}(a\cdot,A(aw,z))-\frac{1}{2}\hat{A}(a\cdot,A(z,aw))-$$
$$-\frac{i}{2}a\langle\hat{A}(z,aw),a\bar{\cdot}\rangle+\frac{i}{2}a\langle\hat{A}(z,a\cdot),a\bar{w}\rangle.$$

This can be transformed to

$$\begin{aligned}
&-a\langle z,a\langle a\bar{w},a\cdot\rangle\rangle+a\langle z,a\langle a\bar{\cdot},aw\rangle\rangle-a\langle z,a\langle a\bar{\cdot},aw\rangle\rangle-a\langle z,a\langle a\bar{w},a\cdot\rangle\rangle-\\
&-\frac{1}{2}\hat{A}(a\cdot,\hat{A}(aw,z))-\frac{1}{2}\hat{A}(a\cdot,\hat{A}(z,aw))\\
=\;&-2a\langle z,a\langle a\bar{w},a\cdot\rangle\rangle-\frac{1}{2}\hat{A}(a\cdot,\hat{A}(z,aw))-\frac{1}{2}\hat{A}(a\cdot,\hat{A}(aw,z))\\
=\;&\frac{1}{2}\hat{A}(a\cdot,\hat{A}(z,aw))-\frac{1}{2}\hat{A}(a\cdot,\hat{A}(z,aw))-\frac{1}{2}\hat{A}(a\cdot,\hat{A}(aw,z))\\
=\;&\frac{1}{2}\hat{A}(a\cdot,\hat{A}(aw,z))
\end{aligned}$$

The latter expression equals to the term at the left hand side of (56).
The equality of the left lower blocks is a consequence of the equalities

$$-i\langle\cdot,a\bar{w}\rangle=-i\langle\cdot,a\bar{w}\rangle \tag{57}$$
$$i\langle\hat{A}(z,\cdot),a\bar{w}\rangle=-2\langle z,a\langle a\bar{w},\cdot\rangle\rangle \tag{58}$$
$$i\langle B(w,\cdot),a\bar{w}\rangle=\frac{i}{2}\langle B(w,\cdot),a\bar{w}\rangle-\frac{i}{2}\langle\cdot,ar(\bar{w},\bar{w})\rangle+ +ir(\langle\cdot,a\bar{w}\rangle,w) \tag{59}$$
$$\frac{i}{2}\langle\hat{A}(aw,\cdot),a\bar{w}\rangle=-\langle aw,a\langle a\bar{w},\cdot\rangle\rangle. \tag{60}$$

(57) is a tautology, (58) a direct consequence of (38) and (60) holds because of (31). In order to prove (59) we have to show

$$\langle B(w,\cdot),a\bar{w}\rangle=-\langle\cdot,ar(\bar{w},\bar{w})\rangle+2r(\langle\cdot,a\bar{w}\rangle,w).$$

We transform the right hand side as follows

$$\begin{aligned}
-\langle\cdot,ar(\bar{w},\bar{w})\rangle+2r(\langle\cdot,a\bar{w}\rangle,w)&=-\langle\cdot,ar(\bar{w},\bar{w})\rangle+2\langle\cdot,\overline{ar(w,w)}\rangle\\
&=\langle\cdot,a\overline{\hat{r}(w,\bar{w})}\rangle\\
&=\hat{r}(w,\overline{\langle\cdot,a\bar{\cdot}\rangle})\\
&=\langle B(w,\cdot),a\bar{w}\rangle.
\end{aligned}$$

The equality of the right lower blocks is a consequence of the equalities

$$\begin{aligned}
&(61) & \mathrm{id} &= \mathrm{id}\\
&(62) & -2i\langle z, a\bar{\cdot}\rangle &= -2i\langle z, a\bar{\cdot}\rangle\\
&(63) & -\hat r(w,\bar{\cdot}) - \hat r(\cdot,\bar w) &= -2r(\cdot, w)\\
&(64) & -i\langle aw, a\bar{\cdot}\rangle + i\langle a\bar{\cdot}, a\bar w\rangle &= -i\langle aw, a\bar{\cdot}\rangle + i\langle a\bar{\cdot}, a\bar w\rangle\\
&(65) & 2i\hat r(\langle z, a\bar{\cdot}\rangle, \bar w) &= 2i\langle z, B(\bar w, a\bar{\cdot})\rangle\\
&(66) & 2\langle a\langle z, a\bar{\cdot}\rangle, a\bar w\rangle &= 2\langle z, a\langle a\bar w, a\cdot\rangle\rangle\\
&(67) & \hat r(\hat r(w,\bar{\cdot}),\bar w) &= 2r(r(w,\cdot),w) - r(r(w,w),\cdot)\\
&(68) & -i\langle a\hat r(w,\bar{\cdot}), a\bar w\rangle + i\hat r(\langle aw, a\bar{\cdot}\rangle, \bar w) &= -i\langle B(w, a\cdot), a\bar w\rangle + i\langle ar(w,\cdot), a\bar w\rangle -\\
& & &\quad -ir(\langle a\cdot, a\bar w\rangle, w) + ir(\langle aw, a\bar{\cdot}\rangle, w) -\\
& & &\quad -ir(\langle aw, a\bar w\rangle, \cdot) + \frac{i}{2}\langle a\cdot, ar(\bar w, \bar w)\rangle -\\
& & &\quad -\frac{i}{2}\langle ar(w,w), a\bar{\cdot}\rangle + i\langle aw, ar(\bar w, \bar{\cdot})\rangle\\
&(69) & \langle a\langle aw, a\bar{\cdot}\rangle, a\bar w\rangle &= \langle aw, a\langle a\bar w, a\bar{\cdot}\rangle\rangle.
\end{aligned}$$

The equalities (61), (62), (64) are tautological, (63) follows from (4) and (32). (65) is a consequence of (39), (66) follows from (38). (31) and (38) imply (69) (67) can be obtained by the following transformations of the right hand side:

$$\begin{aligned}
2r(r(w,\cdot),w) - r(r(w,w),\cdot) &= r(\hat r(w,\bar{\cdot}), w) + r(\hat r(\cdot,\bar w), w) - r(\hat r(w,\bar w),\cdot)\\
&= r(\hat r(\cdot,\bar w), w)\\
&= \hat r(\cdot, \overline{r(w,w)})\\
&= \hat r(\cdot, \hat r(\bar w, w))\\
&= \hat r(\hat r(w,\bar{\cdot}), \bar w).
\end{aligned}$$

It remains to prove (68). Since

$$-i\langle a\hat r(w,\bar{\cdot}), a\bar w\rangle = -i\langle B(w, a\cdot), a\bar w\rangle,$$

these terms cancel out immediately. In the remainding terms on the right hand side we express r by $\hat r$.

$$\frac{i}{2}\langle a\hat{r}(w,\bar{\cdot}), a\bar{w}\rangle + \frac{i}{2}\langle a\hat{r}(\cdot,\bar{w}), a\bar{w}\rangle - \frac{i}{2}\hat{r}(\langle a\cdot, a\bar{w}\rangle, \bar{w}) - \frac{i}{2}\hat{r}(w, \langle a\bar{w}, a\cdot\rangle) +$$
$$+\frac{i}{2}\hat{r}(\langle aw, a\bar{\cdot}\rangle, \bar{w}) + \frac{i}{2}\hat{r}(w, \langle a\bar{\cdot}, aw\rangle - \frac{i}{2}\hat{r}(\langle aw, a\bar{w}\rangle, \bar{\cdot}) - \frac{i}{2}\hat{r}(\cdot, \langle a\bar{w}, aw\rangle) +$$
$$+\frac{i}{2}\langle a\cdot, a\hat{r}(\bar{w}, w)\rangle - \frac{i}{2}\langle ar(w,\bar{w}), a\bar{\cdot}\rangle + \frac{i}{2}\langle aw, a\hat{r}(\bar{w}, \cdot)\rangle + \frac{i}{2}\langle aw, a\hat{r}(\bar{\cdot}, w)\rangle$$

Using the identities

$$\begin{aligned}
\frac{i}{2}\langle a\hat{r}(w,\bar{\cdot}), a\bar{w}\rangle &= \frac{i}{2}\hat{r}(w, \langle a\bar{w}, a\cdot\rangle \\
\frac{i}{2}\langle a\hat{r}(\cdot,\bar{w}), a\bar{w}\rangle &= \frac{i}{2}\hat{r}(\cdot, \langle a\bar{w}, aw\rangle \\
-\frac{i}{2}\hat{r}(\langle a\cdot, a\bar{w}\rangle, \bar{w}) &= +\frac{i}{2}\langle a\cdot, a\hat{r}(\bar{w}, w)\rangle \\
\frac{i}{2}\hat{r}(w, \langle a\bar{\cdot}, aw\rangle &= \frac{i}{2}\langle ar(w,\bar{w}), a\bar{\cdot}\rangle \\
\frac{i}{2}\hat{r}(\langle aw, a\bar{w}\rangle, \bar{\cdot}) &= \frac{i}{2}\langle aw, a\hat{r}(\bar{\cdot}, w)\rangle \\
\frac{i}{2}\langle aw, a\hat{r}(\bar{w}, \cdot)\rangle &= \frac{i}{2}\hat{r}(\langle aw, a\bar{\cdot}\rangle, \bar{w})
\end{aligned}$$

and cancelling out the corresponding terms in the right hand side of (68) we obtain

$$\hat{r}(\langle aw, a\bar{\cdot}\rangle, \bar{w}),$$

which coincides with the remaining term on the left hand side.

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