Isotropic automorphisms of non-quadratic elliptic CR-manifolds in $\mathbb{C}^4$ *†

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Abstract

We use an analogue (formerly constructed by the authors) of the Chern-Moser normal form for 2-codimensional real-analytic non-quadratic CR-manifolds in $\mathbb{C}^4$ with elliptic Levi form to describe local isotropic holomorphic automorphisms. We prove that apart from very special explicit family of exceptional manifolds any isotropic automorphism is uniquely determined by the restriction of its Jacobian at the origin to the complex tangent space.

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1 Introduction

Let M be an embedded real-analytic CR-manifold in $\mathbb{C}^N$. We are interested in a description of the isotropic automorphisms of M with respect to some point $p \in M$, i.e., the germs of biholomorphic mappings at p that preserve p and M. According to a result by Belošapka these germs are uniquely determined by their 2-jets if M is Levi non-degenerate at p [B2]. In the case when M is a Levi-nondegenerate hypersurface, not equivalent to a quadric, Belošapka and Loboda [B1, L1] proved that any such automorphism is uniquely determined by the pseudo-unitary part of its first order derivatives at p (see [E2] for a simple proof). According to a result by Ežov [E1], for a given isotropic automorphism Φ there exist coordinates such that the equation of M has Chern-Moser normal form and Φ is linear. Kružilin and Loboda [KL] proved that, if M, in addition, is strictly pseudoconvex, then there exist normal coordinates where all isotropic automorphisms are linear, simultaneously.

The main tool in the study of these problems was the Chern-Moser normal form. Analogues of the normal form for CR-manifolds of higher codimension were constructed in the following situations:

1. M is rigid at p, i.e., the osculating quadric to M at p has only the trivial automorphisms (see [ES3]). If the CR-codimension k and the CR-dimension n of M satisfy $3 \leq k \leq n^2 - 3$ rigidity is a property of general position. It follows easily that any isotropic automorphism is automatically linear if the equation of M has normal form (see [ES2]).

2. $M \subset \mathbb{C}^4$ has codimension 2. This case is exceptional because there exist only three types of (non-degenerate) Levi forms up to linear coordinate change. These three types are called hyperbolic, elliptic and parabolic. This is similar to the situation of hypersurfaces where one has finitely many Levi forms that are characterised by their signature. In difference to the above described rigid CR-manifolds, and again in resemblance with hypersurfaces, the osculating quadrics have large automorphism groups.

   M is of hyperbolic type at p if, in suitable coordinates z_1, z_2, w_1, w_2, the equation of M takes the form

$$v_1 = |z_1|^2 + O(3)$$
$$v_2 = |z_2|^2 + O(3),$$

where $w_j = u_j + iv_j$ and $p = 0$. The normal form was constructed by Loboda [L2]. Schmalz proved in [S1] that there exist normal coordinates where the isotropic automorphisms are simultaneously linear if M is not semi-quadratic, i.e., of the form

$$v_1 = |z_1|^2$$
$$v_2 = |z_2|^2 + N^2(z_2, \bar{z}_2, u_2).$$

In the latter case any isotropic automorphism splits into an automorphism of the sphere $v_1 = |z_1|^2$ and an automorphism of the strictly pseudoconvex hypersurface $v_2 = |z_2|^2 + N^2(z_2, \bar{z}_2, u_2)$ in $\mathbb{C}^2$.

The hyperbolicity is a property that is stable with respect to small perturbations. In particular, the set of hyperbolic points at M is open.

3. The only other stable situation in the said sense is the elliptic situation. A point $p \in M \subset \mathbb{C}^4$ is of elliptic type if, in suitable coordinates z_1, z_2, w_1, w_2, the equation of M takes the form

$$v_1 = \operatorname{Re} z_1 \bar{z}_2 + O(3)$$
$$v_2 = \operatorname{Im} z_1 \bar{z}_2 + O(3).$$

The automorphisms of elliptic CR-manifolds will be studied in this paper. We will make use of the normal form constructed by the authors in [ES1]. Thus, M will be given by one complex equation

$$V = z_1 \bar{z}_2 + N(z, \bar{z}, \eta, \bar{\eta}),$$

where $z_1, z_2, \omega_1, \omega_2$ are coordinates in $\mathbb{C}^4$ and

$$V = \frac{1}{2i}(\omega_1 - \bar{\omega}_2), \quad \eta = \frac{1}{2}(\omega_1 + \bar{\omega}_2),$$

The expansion of the complex valued function N into polynomials N_{kl} of degree k with respect to z and l with respect to $\bar{z}$ and with coefficients that are analytic functions of η equals

$$N = \sum_{k>1} (N_{k1} + N_{k1}) + N_{22} + N_{32} + N_{23} + N_{33} + \sum_{k,l} N_{kl},$$

where the polynomials N_{kl} meet the conditions

$$\begin{aligned}
\frac{\partial N_{k1}}{\partial \bar{z}_2} &= 0 & \frac{\partial N_{1k}}{\partial z_1} &= 0, \quad \text{für } k \geq 2 \\
\frac{\partial^3 N_{21}}{\partial z_1 \partial \bar{z}_1 \partial z_2} &= 0 & \frac{\partial^3 N_{12}}{\partial \bar{z}_2 \partial \bar{z}_1 \partial z_2} &= 0 \\
\frac{\partial^4 N_{22}}{\partial z_1 \partial z_2 \partial \bar{z}_1 \partial \bar{z}_2} &= 0 & \left.\frac{\partial^4 N_{22}}{(\partial z_1)^2 (\partial \bar{z}_2)^2}\right|_{\bar{\eta}=0} &= 0 \\
\left.\frac{\partial^5 N_{32}}{(\partial z_1)^3 (\partial \bar{z}_2)^2}\right|_{\bar{\eta}=0} &= 0 & \left.\frac{\partial^5 N_{23}}{(\partial z_1)^2 (\partial \bar{z}_2)^3}\right|_{\bar{\eta}=0} &= 0 \\
\left.\frac{\partial^6 N_{33}}{(\partial z_1)^3 (\partial \bar{z}_2)^3}\right|_{\bar{\eta}=0} &= 0. & &
\end{aligned} \tag{1}$$

By $\mathcal{N}$ we will denote the space of all polynomials that satisfy these conditions. The space $\mathcal{R}$ spanned by all monomials that do not satisfy these conditions is then a direct complement of $\mathcal{N}$. Thus, one of the properties of this decomposition $\mathcal{N} \oplus \mathcal{R}$ is that any monomial either belongs to $\mathcal{N}$ or to $\mathcal{R}$. Another property, that we will frequently employ, is that the $\mathcal{N}$-space is stable under multiplication by any function in $(z, \bar{z}, \eta, \bar{\eta})$. It is worth mentioning that the latter is not true for the $\mathcal{R}$-space.
In this paper we attempt to find out when any isotropic automorphism $\phi = (f, g)$: $\quad z^* = f(z, w), \quad w^* = g(z, w)$ of an elliptic CR-manifold M is uniquely determined by the first order derivatives of f with respect to z at 0. In other words, if we denote by $\operatorname{Aut}_{0,\mathrm{id}} M$ the subgroup of germs of isotropic automorphisms $\phi = (f, g)$ satisfying

$$\left.\frac{\partial f}{\partial z}\right|_{\substack{z=0\\ w=0}} = \mathrm{id},$$

then the latter statement reduces to: *For which* M $\operatorname{Aut}_{0,\mathrm{id}} M = \{\mathrm{id}\}$?

Though the technical details are very different from the hyperbolic case, considered in [S1], the rough scheme of our approach is similar, namely

1. One shows that any automorphism $\phi \in \mathrm{Aut}_{0,\mathrm{id}} M$ can be embedded into some one-parametric subgroup ϕ^t. We have

$$\operatorname{Im} g^t(z,\omega) - f_1^t(z,\omega)\overline{f_2^t(z,\omega)} - N(f^t, \bar f^t, \operatorname{Re} g^t)\Big|_{\operatorname{Im}\omega = z_1\bar z_2 + N} \equiv 0, \quad (2)$$

   where $\operatorname{Im}\omega = \frac{1}{2i}(\omega_1 - \bar\omega_2)$, $\operatorname{Im} g^t = \frac{1}{2i}(g_1^t - \bar g_2^t)$, $\operatorname{Re} g^t = \frac{1}{2}(g_1^t + \bar g_2^t)$.
   It follows that the derivative of the left hand side with respect to t for $t = 0$ must vanish. Denote this function by $\mathcal{P}(z, \bar z, \eta, \bar\eta)$. It can be expressed by

$$\mathcal{P}(z, \bar z, \eta) = 2\operatorname{Re}\chi(z_1\bar z_2 - V + N(z, \bar z, \eta, \bar\eta))|_{\operatorname{Im}\omega = z_1\bar z_2 + N}\,, \qquad (3)$$

   where χ is a vector field that is called infinitesimal automorphism. It is now sufficient to prove that there are no non-vanishing vector fields such that

$$2\operatorname{Re}\chi(V - z_1\bar z_2 - N(z, \bar z, \eta, \bar\eta))|_{\operatorname{Im}\omega = z_1\bar z_2 + N} \equiv 0.$$

   Since this first step is a literal repetition of the arguments from the hyperbolic case [S1] the details will be omitted.

2. At the second step we will study the action of vector fields on N. Similar to the hyperbolic case, we will introduce the notions of weight and defect. If the vector field χ does not vanish its lower weighted components coincide with those from infinitesimal automorphisms of the quadric. To study the action of χ on N we introduce the so-called *weight-defect diagrams* (see Section 3 below). This construction is designed to show that if N_γ is the non-vanishing term of lowest weight of N and $\chi \not\equiv 0$ then $\mathcal{P}_{\gamma_0+1}$ does not vanish.

2 Preliminaries

The elliptic quadric Q is given by the equation $V = z_1\bar{z}_2$. The group $\mathrm{Aut}_{0,\mathrm{id}}\, Q$ consists of the mappings

$$z_j^* = \frac{z_j + a_j\omega_j}{1-\delta_j}$$
$$\omega_j^* = \frac{\omega_j}{1-\delta_j},$$

where $\delta_j = 2iz_j\hat{a}_j - (r_j + ia_j\hat{a}_j)\omega_j$; $a_j \in \mathbb{C}$, $r_1 = \bar{r}_2 = r \in \mathbb{C}$, $j = 1, 2$. By $\hat{a} = \widehat{(a_1, a_2)}$ we denote the vector $(\bar{a}_2, \bar{a}_1)$ ([ES1]).
The corresponding infinitesimal automorphisms are

$$\sum_{j=1,2} (a_j\omega_j + 2i\hat{a}_j z_j^2 + r_j z_j\omega_j)\frac{\partial}{\partial z_j} + (2i\hat{a}_j z_j\omega_j + r_j\omega_j^2)\frac{\partial}{\partial \omega_j}.$$

Definition 1 *By the following convention we assign a weight γ to each monomial in a formal power series or vector field:*

$$\gamma(z_j) = 1, \qquad \gamma(\omega_j) = \gamma(\eta) = \gamma(\bar{\eta}) = 2$$
$$\gamma\left(\frac{\partial}{\partial z_j}\right) = -1, \qquad \gamma\left(\frac{\partial}{\partial \omega_j}\right) = \gamma\left(\frac{\partial}{\partial \eta}\right) = \gamma\left(\frac{\partial}{\partial \bar{\eta}}\right) = -2,$$

and the weight of a product is the sum of the weights of its factors.

Let M be an elliptic CR-manifold given in normal form. Then any automorphism $\phi \in \mathrm{Aut}_{0,\mathrm{id}}\, M$ can be considered as a renormalisation, i.e., a change to (the same) normal coordinates. Therefore, ϕ has the form

$$z_j^* = \frac{z_j + a_j\omega_j + \tilde{f}_j}{1-\delta_j}$$
$$\omega_j^* = \frac{\omega_j + \tilde{g}_j}{1-\delta_j}.$$

Let N_{γ_0+2} be the non-vanishing component of N with minimal weight $\gamma_0+2 > 2$ then $\tilde{f}_j$ contains only components of weight $\geq \gamma_0 + 2$ and $\tilde{g}_j$ contains only components of weight $\geq \gamma_0 + 3$.

Analogously, an infinitesimal automorphism χ equals to an infinitesimal automorphism of the quadric up to components of weight $\leq \gamma_0$.
The Lie algebra of infinitesimal automorphisms, corresponding to $\mathrm{Aut}_{0,\mathrm{id}}\, M$ will be denoted by $\mathfrak{aut}_{\,0,\mathrm{id}}$.

Definition 2 *For any monomial Π in $z_1, z_2, \bar z_1, \bar z_2, \eta, \bar\eta$ the 1-defect $d_1(\Pi)$ is defined as the difference of the degrees with respect to z_1 and with respect to $\bar z_2$.*
By $\bar 1$-defect $d_{\bar 1}(\Pi)$ we will denote the difference of the degrees with respect to $\bar z_1$ and with respect to z_2.
The total defect $d(\Pi)$ is the sum of d_1 and $d_{\bar 1}$.

We split the component

$$2\,\mathrm{Re}\,\chi_1 = 2\,\mathrm{Re} \sum_{j=1,2} (a_j \omega_j + 2i\hat a_j z_j^2)\frac{\partial}{\partial z_j} + (2i\hat a_j z_j \omega_j)\frac{\partial}{\partial \omega_j}$$

of weight 1 in $2\,\mathrm{Re}\,\chi$ into four parts

$$2\,\mathrm{Re}\,\chi_1 = \chi_1^1 + \chi_1^{\bar 1} + \chi_1^2 + \chi_1^{\bar 2},$$

with

$$\begin{aligned}
\chi_1^1 &= i\bar a_2 \left(2z_1^2\frac{\partial}{\partial z_1} + 2(\eta + iV)z_1\frac{\partial}{\partial \omega_1} - i(\eta - iV)\frac{\partial}{\partial \bar z_2}\right) \qquad (4)\\
\chi_1^{\bar 1} &= \overline{\chi_1^1} = -ia_2\left(2\bar z_1^2\frac{\partial}{\partial \bar z_1} - 2(\bar\eta - i\bar V)\bar z_1\frac{\partial}{\partial \bar\omega_1} + i(\bar\eta + i\bar V)\frac{\partial}{\partial z_2}\right)\\
\chi_1^2 &= i\bar a_1\left(2z_2^2\frac{\partial}{\partial z_2} + 2(\bar\eta + i\bar V)z_2\frac{\partial}{\partial \omega_2} - i(\bar\eta - i\bar V)\frac{\partial}{\partial \bar z_1}\right)\\
\chi_1^{\bar 2} &= \overline{\chi_1^2} = -ia_1\left(2\bar z_2^2\frac{\partial}{\partial \bar z_2} + 2(\eta - iV)\bar z_2\frac{\partial}{\partial \bar\omega_2} + i(\eta + iV)\frac{\partial}{\partial z_1}\right).
\end{aligned}$$

Each of these fields increases by 1 the defect indicated by the upper index when acting on a monomial.
Set $\nu = z_1 \bar z_2$. Then any monomial $\Pi(z_1, z_2, \bar z_1, \bar z_2, \eta, \bar\eta)$ can be uniquely written (up to a constant multiplier) as

$$\Pi = z_1^{d_1}\bar z_1^{\,d_{\bar 1}}\nu^\alpha\eta^\beta\bar\nu^{\bar\alpha}\bar\eta^{\bar\beta}.$$

Notice that the defects $d_1, d_{\bar{1}}$ might be negative. We consider ν as a new independent variable, i.e. we performed a non-linear coordinate change. Denote by $\mathcal{F}_{(d_1,d_{\bar{1}})}$ the linear space of polynomials $z_1^{d_1}\bar{z}_1{}^{d_{\bar{1}}}p(\eta,\bar{\eta},\nu,\bar{\nu})$. We introduce an isomorphism

$$\begin{aligned} \pi_{(d_1,d_{\bar{1}})}: \qquad\qquad \mathcal{F}_{(d_1,d_{\bar{1}})} &\to \mathcal{F}_{(0,0)} \\ z_1^{d_1}\bar{z}_1{}^{d_{\bar{1}}}p(\eta,\bar{\eta},\nu,\bar{\nu}) &\mapsto p(\eta,\bar{\eta},\nu,\bar{\nu}). \end{aligned}$$

Thus $\pi_{(d_1,d_{\bar{1}})}$ acts by multiplying with $z_1^{-d_1}\bar{z}_1^{-d_{\bar{1}}}$. We will write π instead of $\pi_{(d_1,d_{\bar{1}})}$ if it is clear or not important what are the defects.
The defects measure the "disbalance" of a polynomial. Thus the linear mapping π can also be considered as a projection of the space of all polynomials onto the space of balanced polynomials. A monomial Π can be recovered from its balanced part if one memorises the defects of Π.
For polynomials from $\mathcal{F}_{(0,0)}$ it is convenient to pass to coordinates

$$\begin{aligned} s &= \eta + i\nu, \\ t &= \eta - i\nu. \end{aligned}$$

Then

$$\Pi = \left(\frac{s-t}{2i}\right)^{\alpha}\left(\frac{s+t}{2}\right)^{\beta}\left(\frac{\bar{s}-\bar{t}}{-2i}\right)^{\bar{\alpha}}\left(\frac{\bar{s}+\bar{t}}{2}\right)^{\bar{\beta}}.$$

We will need the push-forwards of certain vector fields (acting as linear mappings from $\mathcal{F}_{(d_1,d_{\bar{1}})}$ to $\mathcal{F}_{(d'_1,d'_{\bar{1}})}$) with respect to π.

Lemma 1

$$\begin{aligned} \pi_*\left(\frac{\partial}{\partial z_1}\right) &= \left(d_1 \,\mathrm{id} + \frac{s-t}{2}\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\right) \\ \pi_*\left(\frac{\partial}{\partial \bar{z}_2}\right) &= i\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right) \\ \pi_*\left(\frac{\partial}{\partial \omega_1}\right) &= \frac{1}{2}\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t} - i\frac{\partial}{\partial V}\right) \\ \pi_*\left(\frac{\partial}{\partial \bar{\omega}_2}\right) &= \frac{1}{2}\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t} + i\frac{\partial}{\partial V}\right). \end{aligned}$$

Proof. We only verify the first identity, the other verifications are similar. Let Π be a polynomial of defect $(d_1, d_{\bar{1}})$ and $\Psi(s,t,\bar{s},\bar{t}) = \pi(\Pi)$. Then

$$\pi_*\left(\frac{\partial}{\partial z_1}\right)\Psi = \pi_{(d_1-1,d_{\bar{1}})} \circ \left(\frac{\partial}{\partial z_1}\right) \circ \pi^{-1}_{(d_1,d_{\bar{1}})}\Psi =$$

$$=\pi_{(d_1-1,d_{\bar{1}})}\left(\frac{\partial}{\partial z_1} z_1^{d_1}\bar{z}_1^{d_{\bar{1}}}\Psi(s,t,\bar{s},\bar{t})\Big|_{\substack{s=\eta+iz_1\bar{z}_2\\ t=\eta-iz_1\bar{z}_2}}\right) =$$

$$=\pi_{(d_1-1,d_{\bar{1}})}\left(d_1 z_1^{d_1-1}\bar{z}_1^{d_{\bar{1}}}\Psi(s,t,\bar{s},\bar{t}) + z_1^{d_1}\bar{z}_1^{d_{\bar{1}}} i\bar{z}_2\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\Psi(s,t,\bar{s},\bar{t})|_{\substack{s=\eta+iz_1\bar{z}_2\\ t=\eta-iz_1\bar{z}_2}}\right)$$

$$= \left(d_1 \,\mathrm{id} + \frac{s-t}{2}\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\right)\Psi.$$

$\square$

In particular we obtain

$$\pi_*(\chi_1^1) = i\bar{a}_2\left(2d_1\,\mathrm{id} + 2s\frac{\partial}{\partial s} + [2iV - (s-t)]\frac{\partial}{\partial t}\right) \tag{5}$$

and

$$\pi_*(\chi_1^1)|_{V=\nu} = 2i\bar{a}_2\left(d_1\,\mathrm{id} + s\frac{\partial}{\partial s}\right). \tag{6}$$

The introduced projection π is adapted to the study of action of the vector fields that increase the 1- and $\bar{1}$-defects χ_1^1 and $\chi_1^{\bar{1}}$ (this corresponds to the parameter a_2 in χ_1, cp. (4)). For the study of the action of the vector fields χ_1^2 and $\chi_1^{\bar{2}}$ we would use similar mappings

$$\begin{aligned} \pi'_{(d_1,d_{\bar{1}})} : \qquad & \mathcal{F}_{(d_1,d_{\bar{1}})} \to \mathcal{F}_{(0,0)} \\ & z_2^{-d_{\bar{1}}}\bar{z}_2^{-d_1} p(\eta,\bar{\eta},\nu,\bar{\nu}) \mapsto p(\eta,\bar{\eta},\nu,\bar{\nu}). \end{aligned}$$

Analogously as above, one obtains

$$\pi'_*(\chi_1^{\bar{2}})|_{V=\nu} = 2\left(t\frac{\partial}{\partial t} - d_1\,\mathrm{id}\right).$$

3 Weight-defect diagrams

Let M be a non-quadratic elliptic manifold with first non-quadratic term N_{γ_0+2}. Then the difference of any infinitesimal automorphism χ and the corresponding infinitesimal automorphism of the quadric begins, at earliest, with the component of weight γ_0+1. Consider the infinitesimal automorphism

$$\chi = \chi_1 + \chi_2 + \chi_{\gamma_0+1} + \cdots$$

Suppose, the parameter a in χ_1 was different from 0. Then, by a linear coordinate change one can achieve that $a_2 = i$. Let Δ be the maximal and δ be the minimal total defect that occur in the lowest non-quadratic term N_{γ_0+2}. Furthermore, let δ_1 and $\delta_{\bar 1}$ be the maximal 1-defect resp. $\bar 1$-defect in $N_{\gamma_0+2,\Delta}$ and $-\delta_1'$ resp. $-\delta_{\bar 1}'$ the minimal 1-defect resp. $\bar 1$-defect in $N_{\gamma_0+2,\delta}$. We will analyse when the hypothesis $a \neq 0$ leads to a contradiction to identity (3) in its component of weight γ_0+3

$$\mathcal{P}_{\gamma_0+3} \equiv 0.$$

It is convenient to write this in the form of a diagram:

$$\begin{tikzcd}
\tilde{\mathcal{P}}_{\gamma_0+3,\delta_1+1,\Delta-\delta_1} & & \tilde{\mathcal{P}}_{\gamma_0+3,\delta_1,\Delta-\delta_1+1} & & & & \tilde{\mathcal{P}}_{\gamma_0+3,\Delta-\delta_{\bar 1}+1,\delta_{\bar 1}} & & \tilde{\mathcal{P}}_{\gamma_0+3,\Delta-\delta_{\bar 1},\delta_{\bar 1}+1} \\
& N_{\gamma_0+2,\delta_1,\Delta-\delta_1} \arrow[ul, "\chi_1^1"'] \arrow[ur, "\chi_1^{\bar 1}"'] \arrow[dl, "\chi_1^2"] \arrow[dr, "\chi_1^{\bar 2}"'] & & N_{\gamma_0+2,\delta_1-1,\Delta-\delta_1+1} \arrow[ul, "\chi_1^1"'] \arrow[dl, "\chi_1^2"] & \cdots & N_{\gamma_0+2,\Delta-\delta_{\bar 1}+1,\delta_{\bar 1}-1} \arrow[ur, "\chi_1^{\bar 1}"'] \arrow[dr, "\chi_1^{\bar 2}"'] & & N_{\gamma_0+2,\Delta-\delta_{\bar 1},\delta_{\bar 1}} \arrow[ul, "\chi_1^1"'] \arrow[ur, "\chi_1^{\bar 1}"'] \arrow[dl, "\chi_1^2"] \arrow[dr, "\chi_1^{\bar 2}"'] & \\
\tilde{\mathcal{P}}_{\gamma_0+3,\delta_1,\Delta-\delta_1-1} & & \tilde{\mathcal{P}}_{\gamma_0+3,\delta_1,\Delta-\delta_1} & & & & \tilde{\mathcal{P}}_{\gamma_0+3,\Delta-\delta_{\bar 1},\delta_{\bar 1}-1} & & \tilde{\mathcal{P}}_{\gamma_0+3,\Delta-\delta_{\bar 1}-1,\delta_{\bar 1}} \\
& {} \arrow[ul, "\chi_1^1"'] \arrow[ur, "\chi_1^{\bar 1}"'] & & {} \arrow[ul, "\chi_1^1"'] & & {} \arrow[ur, "\chi_1^{\bar 1}"'] & & {} \arrow[ul, "\chi_1^1"'] \arrow[ur, "\chi_1^{\bar 1}"'] &
\end{tikzcd}$$

. . . .

$$\begin{tikzcd}
& N_{\gamma_0+2,\delta+\delta_{\bar 1}',-\delta_{\bar 1}'} \arrow[dl, "\chi_1^2"] \arrow[dr, "\chi_1^{\bar 2}"'] & & N_{\gamma_0+2,\delta+\delta_{\bar 1}'-1,-\delta_{\bar 1}'+1} \arrow[dl, "\chi_1^2"] & \cdots & N_{\gamma_0+2,-\delta_1'+1,\delta+\delta_1'-1} \arrow[dr, "\chi_1^{\bar 2}"'] & & N_{\gamma_0+2,-\delta_1',\delta+\delta_1'} \arrow[dl, "\chi_1^2"] \arrow[dr, "\chi_1^{\bar 2}"'] & \\
\tilde{\mathcal{P}}_{\gamma_0+3,\delta+\delta_{\bar 1}',-\delta_{\bar 1}'-1} & & \tilde{\mathcal{P}}_{\gamma_0+3,\delta+\delta_{\bar 1}'-1,-\delta_{\bar 1}'} & & & & \tilde{\mathcal{P}}_{\gamma_0+3,-\delta_1',\delta+\delta_1'-1} & & \tilde{\mathcal{P}}_{\gamma_0+3,-\delta_1'-1,\delta+\delta_1'}
\end{tikzcd}$$

Here,

$$\tilde{\mathcal{P}}_{\gamma_0+3,\alpha+1,d-\alpha} = N_{\gamma_0+2,\alpha,d-\alpha}\frac{\partial}{\partial V}\chi_1^1(\nu - V)\bigg|_{V=\nu} + \chi_{\gamma_0+1,\alpha+1,d-\alpha}(\nu - V)\bigg|_{V=\nu}.$$

By $\chi_{\gamma_0+1,\alpha+1,d-\alpha}$ we mean the component of $2\,\mathrm{Re}\,\chi_{\gamma_0+1}$ that produces by applying to $\nu - V$ a term of defect $(\alpha+1, d-\alpha)$. Since $\nu - V$ has defect $(0,0)$, the part $\chi_{\gamma_0+1,\alpha+1,d-\alpha}$ is itself the vector field of the weight γ_0+1 and defect $(\alpha+1, d-\alpha)$.
This diagram illustrates that $\mathcal{P}_{\gamma_0+3,\alpha+1,d-\alpha}$ is a sum of $\tilde{\mathcal{P}}_{\gamma_0+3,\alpha+1,d-\alpha}$ and of the result of applying the vector fields indicated at the arrows to the term at the base point of the arrow pointing to $\tilde{\mathcal{P}}_{\gamma_0+3,\alpha+1,d-\alpha}$.
The dots mean that this diagram should be continued for all total defects $\delta \le d \le \Delta$. For $\delta < d < \Delta$ there are (at most) four arrows pointing to $\tilde{\mathcal{P}}_{\gamma_0+3,\alpha+1,d-\alpha}$ corresponding to the results of applying χ_1^1, χ_1^2, $\chi_1^{\bar{1}}$, $\chi_1^{\bar{2}}$. Of course, the length and the place of the lines in the diagram depends on d since maximal and minimal 1- and $\bar{1}$-defects are different for different total defects d. Also we should mention that the introduced diagram consists either of terms with odd total defect or of terms with even total defect.
Later (see Lemma 4) we will show that the assumption $a_2 \neq 0$, that produces non-vanishing $\nwarrow$ and $\nearrow$ arrows in the diagram ultimates $a_1 = 0$, that makes the $\searrow$ and $\swarrow$ arrows vanish. Thus, the weight-defect diagram always splits into a number of single-line diagrams with only $\nwarrow$ and $\nearrow$ (or, $\searrow$ and $\swarrow$) arrows remaining.
We will work with a version of the weight-defect diagram that is adapted to the coordinates s, t. Denote $\pi N_{\gamma_0+2,\alpha,d-\alpha}$ by $\mathfrak{N}_{\gamma_0+2,\alpha,d-\alpha}$ and $\pi_*\chi_{\gamma_0+1,\alpha+1,d-\alpha}$ by $X_{\gamma_0+1,\alpha+1,d-\alpha}$. Since

$$\begin{aligned}\pi N_{\gamma_0+2,\alpha,d-\alpha}\frac{\partial}{\partial V}\chi_1^1(\nu - V)\bigg|_{V=\nu} &= \pi N_{\gamma_0+2,\alpha,d-\alpha}\frac{\partial}{\partial V}2z_1(1-V)\bigg|_{V=\nu}\\ &= -\,2\mathfrak{N}_{\gamma_0+2,\alpha,d-\alpha}\end{aligned}$$

we will include this term into the result of the operator at the arrow applied to $\mathfrak{N}_{\gamma_0+2,\alpha,d-\alpha}$. Then at the tips of the arrows there remain only the terms $\mathfrak{X}_{\gamma_0+3,\alpha+1,d-\alpha} = X_{\gamma_0+1,\alpha+1,d-\alpha}(\nu - V)|_{\nu=V}$.

$$\begin{tikzcd}[column sep=tiny]
\mathfrak{X}_{\gamma_0+3,\delta_1+1,\Delta-\delta_1} & & \mathfrak{X}_{\gamma_0+3,\delta_1+1,\Delta-\delta_1+1} & & & & \mathfrak{X}_{\gamma_0+3,\Delta-\delta_{\bar 1}+1,\delta_{\bar 1}} & & \mathfrak{X}_{\gamma_0+3,\Delta-\delta_{\bar 1},\delta_{\bar 1}} \\
& \mathfrak{N}_{\gamma_0+2,\delta_1,\Delta-\delta_1} \arrow[ul, "{2s\frac{\partial}{\partial s}+ 2(\delta_1-1)\mathrm{id}}"] \arrow[ur, "{2\bar s\frac{\partial}{\partial \bar s}+ 2(\Delta-\delta_1)\mathrm{id}}"'] & & \mathfrak{N}_{\gamma_0+2,\delta_1-1,\Delta-\delta_1+1} \arrow[ul, "{2s\frac{\partial}{\partial s}+ 2(\delta_1-2)\mathrm{id}}"'] & \cdots & \mathfrak{N}_{\gamma_0+2,\Delta-\delta_{\bar 1}+1,\delta_{\bar 1}-1} \arrow[ur, "{2\bar s\frac{\partial}{\partial \bar s}+ 2(\delta_{\bar 1}-1)\mathrm{id}}"] & & \mathfrak{N}_{\gamma_0+2,\Delta-\delta_{\bar 1},\delta_{\bar 1}} \arrow[ul, "{2s\frac{\partial}{\partial s}+ 2(\Delta_1-\delta_{\bar 1}-1)\mathrm{id}}"] \arrow[ur, "{2\bar s\frac{\partial}{\partial \bar s} +2\delta_{\bar 1}\mathrm{id}}"'] &
\end{tikzcd}$$

We interpret the upper left corner of the diagram as

$$2\left(s\frac{\partial}{\partial s} + (\delta_1 - 1)\,\mathrm{id}\right)\mathfrak{N}_{\gamma_0+2,\delta_1,\Delta-\delta_1} + X_{\gamma_0+1,\delta_1+1,\Delta-\delta_1}(\nu - V)|_{V=\nu} = 0 \quad (7)$$

and the upper right corner as

$$2\left(\bar{s}\frac{\partial}{\partial \bar{s}} + \delta_{\bar 1}\,\mathrm{id}\right)\mathfrak{N}_{\gamma_0+2,\Delta-\delta_{\bar 1},\delta_{\bar 1}} + X_{\gamma_0+1,\Delta-\delta_{\bar 1},\delta_{\bar 1}+1}(\nu - V)|_{V=\nu} = 0. \quad (8)$$

For certain defects the operator in front of $\mathfrak{N}_{\gamma_0+2,\Delta-\delta_{\bar 1},\delta_{\bar 1}}$ is non-degenerate. This implies immediately that, for $\mathfrak{N}_{\gamma_0+2,\Delta-\delta_{\bar 1},\delta_{\bar 1}} \neq 0$ The second summand in the equations above must be different from zero. It is now a crucial observation that

$$2\,\mathrm{Re}\,\chi_{\gamma_0-1}|_{V=\nu}(\nu - V)$$

always contains a component that belongs to the space $\mathcal{R}$, namely for

$$\chi = \sum_{j=1,2} f_j\frac{\partial}{\partial z_j} + g_j\frac{\partial}{\partial \omega_j}$$

$2\,\mathrm{Re}\,\chi|_{V=\nu}(\nu - V)$ contains, for example,

$$f_1(z,\eta)\bar{z}_2, \quad \overline{f_2(z,\eta)}z_1, \quad \frac{1}{2i}g_1(z,\eta), \quad \frac{i}{2}\overline{g_2(z,\eta)}.$$

Moreover, the field χ can be completely recovered from the $\mathcal{R}$-part of

$$2\,\mathrm{Re}\,\chi|_{V=\nu}(\nu - V).$$

Our strategy is to analyse when the first summand in the equations can contain non-normal terms. This will either lead to a contradiction with the assumption $a_2 \neq 0$, or else, will determine the terms N_{γ_0+2} and χ_{γ_0+1}.

Lemma 2 *If M has an infinitesimal automorphism χ with $a_2 \neq 0$ then for the maximal 1-defect δ_1 and the maximal $\bar{1}$-defect $\delta_{\bar{1}}$ that occur in $N_{\gamma_0+2,\Delta}$ one has*

$$-2 \leq \delta_1 \leq 1 \tag{9}$$
$$-1 \leq \delta_{\bar{1}} \leq 0 \tag{10}$$

Proof. Suppose $\delta_1 \geq 2$. Then the operator

$$s\frac{\partial}{\partial s} + (\delta_1 - 1)\,\mathrm{id}$$

is invertible. It follows from (7) and from $N_{\gamma_0+2,\delta_1,\Delta-\delta_1} \neq 0$ that $X_{\gamma_0+1} \neq 0$ and

$$\chi_1^1(N_{\gamma_0+2,\delta_1,\Delta-\delta_1}) \notin \mathcal{N}.$$

According to (4), χ_1^1 consists of 3 summands. The first two of them preserve $\mathcal{N}$, and only the third one

$$-i(\eta - iV)\frac{\partial}{\partial \bar{z}_2}$$

can produce non-normal terms from normal ones. This happens exactly when

$$\frac{\partial}{\partial \bar{z}_2} N_{\gamma_0+2,\delta_1,\Delta-\delta_1} \notin \mathcal{N}.$$

Since $\delta_1 \geq 2 > 0$, $N_{\gamma_0+2,\delta_1,\Delta-\delta_1}$ is divisible by $z_1^{\delta_1}$. The part $N_{\mathcal{R}}$ in $N_{\gamma_0+2,\delta_1,\Delta-\delta_1}$ that contributes to $\mathcal{R}$ must be divisible by $\bar{z}_2^2$, hence by ν^2 and not by $\bar{z}_1$. Otherwise, the terms $N_{\mathcal{R}}$ and $\frac{\partial}{\partial \bar{z}_2} N_{\mathcal{R}}$ would be simultaneously normal or non-normal. On the other hand it cannot be divisible by a higher power of ν because then $\frac{\partial}{\partial \bar{z}_2} N_{\mathcal{R}}$ would be normal. Thus, the only possibility (cp. Table 2 in Appendix) is

$$N_{\mathcal{R}} = z_1^{\delta_1} z_2^{\delta_1-\Delta} \nu^2 P(\eta, \bar{\eta}),$$

where $P(\eta, \bar{\eta})$ is a homogeneous polynomial of degree $(\gamma_0 - 2\delta_1 + \Delta + 2)/2$. Consequently,

$$-i\eta\frac{\partial}{\partial \bar{z}_2} N_{\mathcal{R}} = -iz_1^{\delta_1+1}\nu\eta P(\eta, \bar{\eta}).$$

Since this cancels with the non-normal part of $2\,\mathrm{Re}\,\chi_{\gamma_0+1}(\nu - V)$, we obtain

$$2\,\mathrm{Re}\,\chi_{\gamma_0-1}(\nu - V)|_{V=\nu} = iz_1^{\delta_1+1} z_2^{\delta_1-\Delta}\nu(\eta + i\nu)P(\eta + i\bar{\nu}, \bar{\eta} + i\bar{\nu}),$$

and

$$\chi_{\gamma_0+1} = i z_1^{\delta_1+2} z_2^{\delta_1-\Delta}\omega_1 P(\omega_1,\omega_2)\frac{\partial}{\partial z_1}.$$

This means that $X_1|_{V=\nu}(\nu - V)$ contains a monomial

$$\#(\bar{s}-\bar{t})^{\delta_1-\Delta}(s-t)s^{n-1}\bar{t}^{k+\Delta-\delta_1}$$

of a certain bidegree in (s,t) and $(\bar{s},\bar{t})$ variables which we set equal to (n,k). According to (7), in $\mathfrak{N}_{\gamma_0+2,\delta_1,\Delta-\delta_1}$ is then a summand

$$\#\left(s\frac{\partial}{\partial s}+(\delta_1-1)\,\mathrm{id}\right)^{-1}\left[(\bar{s}-\bar{t})^{\delta_1-\Delta}(s-t)s^{n+1}\bar{t}^{k}\right] = \\ \#\left(\frac{s^n}{n+\delta_1-1}-\frac{ts^{n-1}}{n+\delta_1-2}\right)(\bar{s}-\bar{t})^{\delta_1-\Delta}\bar{t}^{k+\Delta-\delta_1},$$

i.e., $N_{\gamma_0+2,\delta_1,\Delta-\delta_1}$ contains a monomial

$$\frac{c}{2(n+\delta_1-1)(n+\delta_1-2)}z_1^{\delta_1}z_2^{\delta_1-\Delta}\eta^n\bar{\eta}^{k+\Delta-\delta_1},$$

that is harmonic and does not belong to $\mathcal{N}$. The contradiction proves the statement of the lemma for $\delta_1 \geq 2$.
Suppose now that $\delta_{\bar{1}} \geq 1$. Instead of equation (7) from the left upper corner of the weight-defect diagram we use now equation (8) from the right upper corner. Similar reasoning as above will lead to a contradiction to $\mathcal{P}_{\gamma_0+3,\Delta-\delta_{\bar{1}},\delta_{\bar{1}}+1} \neq 0$.
For $\delta_{\bar{1}} \geq 1$ the operator

$$\bar{s}\frac{\partial}{\partial\bar{s}}+\delta_{\bar{1}}\,\mathrm{id}$$

is invertible and consequently

$$2\left(\bar{s}\frac{\partial}{\partial\bar{s}}+\delta_{\bar{1}}\,\mathrm{id}\right)\mathfrak{N}_{\gamma_0+2,\Delta-\delta_{\bar{1}}} \neq 0.$$

The latter expression must cancel with $2\,\mathrm{Re}\,\bar{X}_{\gamma_0+1}(\nu-V)|_{V=\nu}$ that has non-normal components. Therefore, $\chi_1^1 N_{\gamma_0+2,\Delta-\delta_{\bar{1}},\delta_{\bar{1}}}$ also contains non-normal components that can arise only from

$$i\bar{\eta}\frac{\partial}{\partial z_2}N_{\gamma_0+2,\Delta-\delta_{\bar{1}},\delta_{\bar{1}}}.$$

According to Table 3 in the Appendix, there are the following possibilities:
i. if $\beta = \delta_{\bar{1}} - \Delta \geq 0$ then it is a component

$$Q = \bar{z}_1^{\delta_{\bar{1}}} \bar{z}_2^{\beta} \bar{\nu}(P_1(\eta, \bar{\eta}) + \nu P_2(\eta, \bar{\eta}))$$

in $N_{\gamma_0+2,\Delta-\delta_{\bar{1}},\delta_{\bar{1}}}$.
ii. if $\beta = -1$ then it is a component

$$Q = z_1 \bar{z}_1^{\delta_{\bar{1}}} \bar{\nu} P_2(\eta, \bar{\eta}).$$

Consider the first option. Then

$$i\bar{\eta}\frac{\partial}{\partial z_2} Q = i\bar{z}_1^{\delta_{\bar{1}}+1} \bar{z}_2^{\beta} \bar{\eta}(P_1(\eta, \bar{\eta}) + \nu P_2(\eta, \bar{\eta}))$$

and

$$\chi_{\gamma_0+1,\Delta-\delta_{\bar{1}},\delta_{\bar{1}}} = 2\bar{z}_1^{\delta_{\bar{1}}+1} \bar{z}_2^{\beta} \bar{\omega}_1 P_1(\bar{\omega}_2, \bar{\omega}_1) \frac{\partial}{\partial \bar{\omega}_2} - i\bar{z}_1^{\delta_{\bar{1}}+1} \bar{z}_2^{\beta+1} \bar{\omega}_1 P_2(\bar{\omega}_2, \bar{\omega}_1) \frac{\partial}{\partial \bar{z}_2}$$

i.e.,

$$X_{\gamma_0+1,\Delta-\delta_{\bar{1}},\delta_{\bar{1}}}(\nu - V)|_{V=\nu} = 2\left(\frac{s-t}{2i}\right)^{\beta} \bar{s} P_1(t, \bar{s}) - i\left(\frac{s-t}{2i}\right)^{\beta+1} \bar{s} P_2(t, \bar{s}).$$

Then $\mathfrak{N}_{\gamma_0+2,\Delta-\delta_{\bar{1}},\delta_{\bar{1}}})$ contains a monomial

$$\frac{1}{2}\left(\bar{s}\frac{\partial}{\partial \bar{s}} + \delta_{\bar{1}}\,\mathrm{id}\right)^{-1} c_1 (s-t)^{\beta} \bar{s}^k t^{n-\beta} = \frac{c_1}{2(k+\delta_{\bar{1}})}(s-t)^{\beta} \bar{s}^k t^{n-\beta}$$

or,

$$\frac{1}{2}\left(\bar{s}\frac{\partial}{\partial \bar{s}} + \delta_{\bar{1}}\,\mathrm{id}\right)^{-1} c_2 (s-t)^{\beta+1} \bar{s}^k t^{n-\beta-1} = \frac{c_2}{2(k+\delta_{\bar{1}})}(s-t)^{\beta+1} \bar{s}^k t^{n-1}.$$

It follows that there is a non-normal component

$$\frac{c_1}{2(k+\delta_{\bar{1}})} \bar{z}_1^{\delta_{\bar{1}}} \bar{z}_2^{\beta} \eta^n \bar{\eta}^k, \quad \text{resp.,} \quad \frac{c_2}{2(k+\delta_{\bar{1}})} \bar{z}_1^{\delta_{\bar{1}}} \bar{z}_2^{\beta+1} \eta^{n-1} \bar{\eta}^k$$

in $N_{\gamma_0+2,\Delta-\delta_{\bar{1}},\delta_{\bar{1}}}$. Analogously, in case ii. we would obtain a non-normal component

$$\frac{c_2}{2(k+\delta_{\bar{1}})} \bar{z}_1^{\delta_{\bar{1}}} \nu \eta^n \bar{\eta}^k$$

in $N_{\gamma_0+2,1,\delta_{\bar{1}}}$.
The contradiction proves the assertion of the lemma for $\delta_{\bar{1}} \geq -1$.
Suppose now $\delta_1 \leq -3$. Then $\mathfrak{N}_{\gamma_0+2,\delta_1,\Delta-\delta_1}$ is divisible by $\nu^{-\delta_1}$. So χ_1^1 does not produce non-normal terms and therefore

$$\chi_{\gamma_0+1,\delta_1+1,\Delta-\delta_1} = 0.$$

It follows that $\mathfrak{N}_{\gamma_0+2,\delta_1,\Delta-\delta_1}$ belongs to the kernel of $s\frac{\partial}{\partial s} + (\delta_1 - 1)\,\mathrm{id}$ and, therefore, has the form

$$\mathfrak{N}_{\gamma_0+2,\delta_1,\Delta-\delta_1} = s^{-\delta_1+1}\Pi(t,\bar{s},\bar{t}).$$

But this polynomial is never divisible by $s - t = 2i\nu$, because it does not vanish for $s = t$. The contradiction proves the lemma for this case. Notice that this argument remains true for $\delta_1 \leq -1$ if $\mathfrak{N}_{\gamma_0+2,\delta_1,\Delta-\delta_1}$ does not contain monomials that produce non-normal components.
Analogously we deal with situation $\delta_{\bar{1}} \leq -2$. Then $\mathfrak{N}_{\gamma_0+2,\Delta-\delta_{\bar{1}},\delta_{\bar{1}}}$ is divisible by $\bar{\nu}^{-\delta_{\bar{1}}}$. Consequently, $\chi_1^{\bar{1}}$ does not produce non-normal which implies

$$\chi_{\gamma_0+1,\Delta-\delta_{\bar{1}},\delta_{\bar{1}}+1} = 0.$$

Then, $\mathfrak{N}_{\gamma_0+2,\Delta-\delta_{\bar{1}},\delta_{\bar{1}}}$ belongs to the kernel of $\bar{s}\frac{\partial}{\partial \bar{s}} + \delta_{\bar{1}}$, i.e.,

$$\mathfrak{N}_{\gamma_0+2,\Delta-\delta_{\bar{1}},\delta_{\bar{1}}} = \bar{s}^{-\delta_{\bar{1}}}\Pi(s,t,\bar{t}),$$

that is, though, never divisible by $\bar{s} - \bar{t} = -2i\nu$. This completes the proof of the lemma. □

We obtain an analogous result concerning the minimal 1 and $\bar{1}$ defects that occur in the polynomial of minimal total defect by a simple dualisation: The (anti-holomorphic) coordinate change $(z_1, z_2, \omega_1, \omega_2) \mapsto (\bar{z}_2, \bar{z}_1, \bar{\omega}_2, \bar{\omega}_1)$ transforms the normalised equation $V = \nu + N$ into a normalised equation with respect to the new coordinates (and the new complex structure). Denote by $\delta_{\bar{2}}$ the maximal 1-defect and by δ_2 the the maximal $\bar{1}$-defect that occur in the component of maximal total defect in the lowest non-quadratic term. Then $-\delta_{\bar{2}}$ is the minimal 1-defect and $-\delta_2$ the minimal $\bar{1}$-defect in the component of minimal total defect in the lowest non-quadratic term of the initial manifold. Reformulating Lemma 2 for δ_2 and $\delta_{\bar{2}}$ we obtain

Lemma 3 *If M has an infinitesimal automorphism with $a_1 \neq 0$ then*

$$-2 \leq \delta_{\bar{2}} \leq 1 \tag{11}$$
$$-1 \leq \delta_2 \leq 0 \tag{12}$$

Remark. In the latter proofs we have only used the fact that there is only one arrow, $\nwarrow$, at the left corner of the upper line (upper left corner) of the weight-defect diagram pointing at $\mathcal{P}_{\gamma_0+3,\delta_1+1,\Delta-\delta_1}$, and, also, that there is only one arrow, $\nearrow$, at the right corner of the upper line (upper right corner) of the weight-defect diagram pointing at $\mathcal{P}_{\gamma_0+3,\Delta-\delta_{\bar{1}},\delta_{\bar{1}+1}}$. The same situation happens, of course, at the most upper line that provides the most left entry $N_{\gamma_0+2,d}$ (left upper corner) and, analogously, for the right upper corner and, respectively in Lemma 3) , for the right lower and left lower corners of the diagram. Thus, inequalities (9) - (12) applied to those corners are also valid.

4 Results

Theorem 1 *For any real-analytic, non-quadratic, elliptic CR-manifold M the group* $\mathrm{Aut}_{0,\mathrm{id}}\, M \cong \mathrm{id}$ *or* $\mathbb{C}$.

In dependence on whether $\mathrm{Aut}_{0,\mathrm{id}}\, M \cong \mathrm{id}$ or $\mathbb{C}$ we will distinguish two types of manifolds.

Definition 3 *M is called non-exceptional if* $\mathrm{Aut}_{0,\mathrm{id}}\, M \cong \mathrm{id}$. *If* $\mathrm{Aut}_{0,\mathrm{id}}\, M \cong \mathbb{C}$ *the manifold M is called exceptional.*

In particular, if inequalities (9) or (10) and (11) or (12) are not satisfied then M is non-exceptional. Thus, the property to be non-exceptional is a property of general position. In fact we do not know whether exceptional manifolds exist. For the time being we can only spot the candidates for exceptional manifolds and determine their first non-quadratic terms, which, as we shall see, determine the entire defining equation of the manifold.
It turns out that the first non-quadratic term $N_{\gamma_0+2} = \sum_{d=\delta}^{\Delta} N_{\gamma_0+2,d}$ of the defining equation in normal form of an exceptional M consists of solutions of the equation

$$\chi_{\gamma_0+1,d+1}(\nu - V)|_{V=\nu} + \chi_1 N_{\gamma_0+2,d}|_{V=\nu} + \chi_1(\nu - V)|_{V=N_{\gamma_0+2,d}} = 0 \qquad (13)$$

that can be interpreted as a single line of the weight-defect diagram corresponding to the total defect d with $\nwarrow$ and $\nearrow$ arrows only.
We consider (13) as a system of equations for

$$\chi_{\gamma_0+1,d+1} = \sum_{j=0}^{\delta_1+\delta_{\bar{1}}-d+1} \chi_{\gamma_0+1,\delta_1-j+1,d-\delta_1+j}$$

and

$$N_{\gamma_0+2,d}=\sum_{j=0}^{\delta_1+\delta_{\bar{1}}-d} N_{\gamma_0+2,\delta_1-j,d-\delta_1+j}.$$

It is more convenient to consider system (13) in coordinates $(s,t,\bar{s},\bar{t})$ in $\mathcal{F}_{(0,0)}$. Let $\mathfrak{N}^{n,k}_{\gamma_0+2,\delta_1-j,d-\delta_1+j}$ be the component of degree n in s,t and degree k in $\bar{s},\bar{t}$ of $\mathfrak{N}_{\gamma_0+2,\delta_1-j,d-\delta_1+j}$. Analogously, let $X^{n-1,k}_{\gamma_0+1,\delta_1-j+1,d-\delta_1+j}$ be the component of bidegree $(n-1,k)$ of $X_{\gamma_0+1,\delta_1-j+1,d-\delta_1+j}$ (the degrees of $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ are set equal to -1).

Theorem 2 *The expressions below form a basis of solutions of (13)*

1. *For $d=-2$, and bidegree $(n,0)$, $n\geq 5$*

$$\begin{aligned}X^{n-1,0}_{\gamma_0+1,-1,0}=&-(n-3)(n-4)\Bigg[(\eta+iV)^{n-1}\left(d_1\,\mathrm{id}-\frac{s-t}{2}\left(\frac{\partial}{\partial s}-\frac{\partial}{\partial t}\right)\right)+\\&+\frac{1}{2}(n-1)(\eta-iV)^{n-2}(s-t)^2\left(\frac{\partial}{\partial s}-\frac{\partial}{\partial t}\right)-\\&-\frac{1}{2}(\eta-iV)^{n-1}(s-t)\left(\frac{\partial}{\partial s}+\frac{\partial}{\partial t}+i\frac{\partial}{\partial V}\right)\Bigg]\end{aligned}$$

here and below in this theorem a vector field X is considered as a differential operator applied to the polynomials obtained by homomorphism $\pi_{(d-1,d-2)}$ from a polynomial of defect $(d_1,d-2)$).

$$\begin{aligned}\mathfrak{N}^{n,0}_{\gamma_0+2,-2,0}=&-\frac{i}{12}(n-4)(n-3)(n-2)t^n+\frac{i}{8}(n-4)(n-3)(2n-3)st^{n-1}-\\&-\frac{i}{4}(n-4)(n-3)(n-1)s^2t^{n-2}+\\&+\frac{i}{24}(n-2)(n-1)(2n-9)s^3t^{n-3}+\\&+\frac{i}{4}(n-3)s^{n-1}t-\frac{i}{4}(n-4)s^n\end{aligned}\tag{14}$$

2. For $d=-1$, and bidegree $(n,0)$, $n \geq 4$

$$\begin{aligned}X^{n-1,0}_{\gamma_0+1,0,0} =& 4(n-3)\Big[n(\eta+iV)^{n-1}\left(d_1 \operatorname{id} + \frac{s-t}{2}\left(\frac{\partial}{\partial s}-\frac{\partial}{\partial t}\right)\right)+\\ &+ n(\eta-iV)^{n-1}\frac{s-t}{2}\left(\frac{\partial}{\partial s}-\frac{\partial}{\partial t}\right)+\\ &+(\eta+iV)^n\left(\frac{\partial}{\partial s}+\frac{\partial}{\partial t}-i\frac{\partial}{\partial V}\right)+\\ &+(\eta-iV)^n\left(\frac{\partial}{\partial s}+\frac{\partial}{\partial t}+i\frac{\partial}{\partial V}\right)\Big]\end{aligned}$$

$$\begin{aligned}\mathfrak{N}^{n,0}_{\gamma_0+2,-1,0} =& \frac{1}{2}(n-3)(n-2)t^n-(n-3)nst^{n-1}+\\ &+\frac{1}{2}(n-1)ns^2t^{n-2}-ns^{n-1}t+(n-3)s^n\end{aligned} \tag{15}$$

3. For $d=-m$ and bidegree (n,k), $n \geq m+3$, $k > m \geq 0$

$$\begin{aligned}X^{n-1,k}_{\gamma_0+2,-m+1} =& X^{n-1,k}_{\gamma_0+2,1,-m} =\\ =& \frac{(n-1)!}{(n-m-3)!}(\eta+iV)^{n-1}(\bar\eta+i\bar V)^{k-m}\times\\ &\times(\bar t-\bar s)^m\left(-2\delta_1 \operatorname{id} + (s-t)\left(\frac{\partial}{\partial s}-\frac{\partial}{\partial t}\right)\right)\end{aligned}$$

$$\begin{aligned}&\mathfrak{N}^{n,k}_{\gamma_0+2,0,-m} =\\ &= \frac{(n-3)!}{2(n-m-3)!}\left[-(n-2)s^n+(n-1)s^{n-1}t-st^{n-1}\right](\bar t-\bar s)^m\bar t^{k-m}\\ &\qquad = (s-t)^2P^{(0)}_{n-2}(s,t)(\bar t-\bar s)^m\bar t^{k-m}\end{aligned}$$

$$\mathfrak{N}^{n,k}_{\gamma_0+2,-j,-m+j} = (s-t)^{j+2}P^{(j)}_{n-j-2}(s,t)(\bar t-\bar s)^{m-j}\bar t^{k-m+j}, \tag{16}$$

where

$$\frac{1}{2}\left(s\frac{\partial}{\partial s}-(j+1)\operatorname{id}\right)(s-t)^{j+2}P^{(j)}_{n-j-2}(s,t)+(s-t)^{j+1}P^{(j-1)}_{n-j-1}(s,t)=0.$$

4. For $d=-m+1$ and bidegree (n,k), $n \geq m+2$, $k > m \geq 0$

$$X^{n-1,k}_{\gamma_0+2,-m+2} = X^{n-1,k}_{\gamma_0+2,2,-m} =$$
$$= \frac{n!}{(n-m-2)!}(\eta + iV)^{n-1}(\bar{t}-\bar{s})^m \left(-2\delta_1 \operatorname{id} + (s-t)\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\right)$$

$$\mathfrak{N}^{n,k}_{\gamma_0+2,1,-m} =$$
$$= \frac{(n-2)!}{2(n-m-2)!}\left[-(n-1)s^n + ns^{n-1}t - t^n\right](\bar{t}-\bar{s})^m \bar{t}^{k-m}$$
$$= (s-t)^2 P^{(0)}_{n-2}(s,t)(\bar{t}-\bar{s})^m \bar{t}^{k-m}$$

$$\mathfrak{N}^{n,k}_{\gamma_0+2,-j+1,-m+j} = (s-t)^{j+2} P^{(j)}_{n-j-2}(s,t)(\bar{t}-\bar{s})^{m-j}\bar{t}^{k-m+j}, \tag{17}$$

where

$$\frac{1}{2}\left(s\frac{\partial}{\partial s} - j \operatorname{id}\right)(s-t)^{j+2}P^{(j)}_{n-j-2}(s,t) + (s-t)^{j+1}P^{(j-1)}_{n-j-1}(s,t) = 0.$$

Theorem 3 *For exceptional M each polynomial $\mathfrak{N}_{\gamma_0+2,d_1,d_2}$ is a linear combination of the elements of the basis listed in Theorem 2.*

In fact, only polynomials (16) and (17) can occur in a common linear combination. Because of of the defect-bidegree partition, if (14) or (15) occurs in a linear combination then the whole combination is a multiple of such particular term.
One can extend a weight-defect diagram for all weights $\gamma \geq \gamma_0$. The system of equations analogous to (13) for arbitrary $\gamma \geq \gamma_0$ takes the form

$$\chi_{\gamma+1,d+1}(\nu - V)|_{V=\nu} + \chi_1 N_{\gamma+2,d}|_{V=\nu} + \chi_1(\nu - V)|_{V=N_{\gamma+2,d}} = P(N_{<\gamma+2}, \chi_{<\gamma+1}). \tag{18}$$

We consider (18) as a system of non-homogeneous equations with the right hand side P. Having a basis of solutions of the homogeneous system determined in Theorem 2, we conclude with the following

Theorem 4 *For exceptional M each polynomial $\mathfrak{N}_{\gamma+2,d_1,d_2}$ is a sum of a linear combination of the elements of the basis listed in Theorem 2 (with γ_0 substituted by γ) and the unique solution $\tilde{N}_{\gamma+2,d}$ of (18) with the conditions that $\tilde{N}_{\gamma+2,d}$ does not contain monomials of the form $\#\bar{z}_2^2\nu^2\eta^{n-4}$, $\#\bar{z}_2\nu^3\eta^{n-4}$, $\#\nu^2\eta^{n-2}z_2^m\bar{\eta}^{k-m}$, $\#z_1\nu^2\eta^{n-2}z_2^m\bar{\eta}^{k-m}$.*

The uniqueness of the particular solution in the above theorem means that there could be none (and, most likely, to happen so), in case there is a solution, we generally determine how to choose the particular one.

5 The case $\delta_1 = -2$

In this section we consider the case when $N_{\gamma_0+2,\Delta}$ has maximal 1-defect $\delta_1 = -2$. It follows from Lemma 2 that we may restrict ourselves to the case $d_{\bar{1}} = \Delta + 2 \leq \delta_{\bar{1}} \leq 0$. Then $N_{\gamma_0+2,-2,\Delta+2}$ consists of terms

$$\bar{z}_2^2 z_2^{-\Delta-2} P(\nu,\bar{\nu},\eta,\bar{\eta}).$$

According to Table 2 in the Appendix, the only monomials Ψ in this expression such that $\frac{\partial}{\partial \bar{z}_2}\Psi$ belongs to $\mathcal{R}$ are

$$\Psi_1 = \bar{z}_2^2 z_2^m \eta^{n-2}\bar{\eta}^{k-m}, \quad m \geq 1 \tag{19}$$

$$\Psi_2 = \bar{z}_2^2 \bar{\nu}\eta^{n-2}\bar{\eta}^{k-1} \tag{20}$$

$$\Psi_3 = \bar{z}_2^2 \nu^2\eta^{n-4}. \tag{21}$$

If $N_{\gamma_0+2,-2,\Delta+2}$ does not contain monomials of the type (19), (20) or (21) then $N_{\gamma_0+2,-2,\Delta+2}$ belongs to the kernel of $\left(s\frac{\partial}{\partial s} - 3\,\mathrm{id}\right)$, and we obtain a contradiction by the same argument as in Lemma 2.
Suppose now that $N_{\gamma_0+2,-2,\Delta+2}$ contains a monomial

$$\Psi_1 = \#\bar{z}_2^2 z_2^m \eta^{n-2}\bar{\eta}^{k-m}.$$

Then $\Delta = -m-2$ and therefore there are no monomials of type Ψ_2, Ψ_3 in $N_{\gamma_0+2,-2,-m}$.
Let us remark that we set n for the degree with respect to the holomorphic variables s, t and k for the degree with respect to the antiholomorphic variables $\bar{s}, \bar{t}$ in $\pi(\Psi_1)$. These degrees are preserved by the vector fields that

we apply below. Further on we will always refer to the pair (n,k) as to the bidegree of a polynomial. The homogeneous part of bidegree (n,k) of a polynomial Ψ will be denoted by $\Psi^{n,k}$.
We consider the part of bidegree (n,k) of the upper line in the weight-defect diagram starting from the left corner. It consists of the terms $\mathfrak{N}^{n,k}_{\gamma_0+2,-2,-m},\dots,$ $\mathfrak{N}^{n,k}_{\gamma_0+2,-2-m,0}$. According to Tables 2 and 3 in the Appendix, for $1\le j\le m-1$, both

$$\frac{\partial}{\partial\bar z_2}N^{n,k}_{\gamma_0+2,-2-j,-m+j},\quad\text{ and }\quad\frac{\partial}{\partial z_2}N^{n,k}_{\gamma_0+2,-3-j,-m+j+1}$$

always belong to $\mathcal{N}$. Therefore, $\mathfrak{X}^{n,k}_{\gamma_0+3,-2,-m+1},\dots,\mathfrak{X}^{n,k}_{\gamma_0+3,-m,-1}$ vanish and the field $\mathfrak{X}^{n,k}_{\gamma_0+3,-1,-m}$, is determined by the $\mathcal{R}$-contribution of $\mathfrak{N}^{n,k}_{\gamma_0+2,-2,-m}$ and $\mathfrak{X}^{n,k}_{\gamma_0+3,-1-m,0}$ and $\mathfrak{X}^{n,k}_{\gamma_0+3,-2-m,1}$ are determined by the $\mathcal{R}$-contributions of $\mathfrak{N}^{n,k}_{\gamma_0+2,-2-m,0}$. Thus, we will be able to calculate all contributions to $\mathcal{P}^{\backslash,\|}_{\gamma_\prime+\ni,-\infty-\Updownarrow,\prime}$ in two different ways, starting from the left, and, also, from the right corner. This will lead to a contradiction.
The $\mathcal{R}$-term produced by $i\eta\frac{\partial}{\partial\bar z_2}$ equals

$$\#\bar z_2 z_2^m\eta^{n-1}\bar\eta^{k-m}.$$

Then

$$\chi^{n-1,k}_{\gamma_0+1,-1,-m}=\#z_2^m\omega_1^{n-1}\omega_2^{k-m}\frac{\partial}{\partial z_1}$$
$$X_{\gamma_0+1,-1,-m}(\nu-V)|_{V=\nu}=\#(s-t)s^{n-1}(\bar s-\bar t)^m\bar t^{k-m}.$$

¿From the equation

$$2\left(s\frac{\partial}{\partial s}-3\,\mathrm{id}\right)\mathfrak{N}^{n,k}_{\gamma_0+2,-2,-m}=-X^{n-1,k}_{\gamma_0+1,-1,-m}(\nu-V)|_{V=\nu}$$

we obtain that $n\neq 4$. Only is this case there exist a solution that is uniquely determined by the condition that $\mathfrak{N}^{n,k}_{\gamma_0+2,-2,-m}$ must be divisible by $s-t$ (because of the negative 1-defect). It follows

$$\mathfrak{N}^{n,k}_{\gamma_0+2,-2,-m}=\#[(n-4)s^n-(n-3)s^{n-1}t+s^3t^{n-3}](\bar t-\bar s)^m\bar t^{k-m}.$$

Notice that this term is divisible by $(s-t)^2$ but not by $(s-t)^3$ (corresponding to the 1-defect -2). For convenience in calculations we set the coefficient

$$\#=\frac{(n-5)!}{(n-m-5)!}$$

Since the terms $\mathfrak{X}^{n,k}_{\gamma_0+3,-2,-m+1}, \ldots, \mathfrak{X}^{n,k}_{\gamma_0+3,-m,-1}$ vanish, we obtain

$$\begin{aligned}\mathfrak{N}^{n,k}_{\gamma_0+2,-2-j,-m+j} =& \\ -\left(s\frac{\partial}{\partial s} - (3+j)\,\mathrm{id}\right)^{-1} & \left(\bar{s}\frac{\partial}{\partial \bar{s}} - (m-j)\,\mathrm{id}\right)\mathfrak{N}^{n,k}_{\gamma_0+2,-3-j,-m+j+1} + \\ & + \lambda s^{j+3} t^{n-j-3} P(\bar{s},\bar{t}),\end{aligned}$$

where the inverse operator is defined for polynomials that do not contain monomials of degree $3+j$ in s. The number λ will be uniquely determined by the condition

$$\mathfrak{N}_{\gamma_0+2,-2-j,-m+j}|_{s=t} = 0$$

(divisibility by ν). It follows by induction that, for $1 \leq j \leq m-1$,

$$\begin{aligned}\mathfrak{N}^{n,k}_{\gamma_0+2,-2-j,-m+j} = \frac{m!(n-5-j)!}{2(m-j)!(n-m-5)!}\big[(n-4-j)s^n - (n-3)s^{n-1}t + \\ + \lambda^{(j)}_{j+3} s^{j+3} t^{n-j-3} + \cdots + \lambda^{(j)}_{3} s^3 t^{n-3}\big](\bar{t}-\bar{s})^{m-j}\bar{t}^{k-m+j}\end{aligned}$$

and, moreover, that $\mathfrak{N}^{n,k}_{\gamma_0+2,-2-j,-m+j}$ is divisible exactly by $(s-t)^{j+2}$. For $j = m-1$ we write the polynomial in brackets in the form

$$(s-t)^{m+1}\left(c_0\eta^{n-m-1} + c_1(s-t)\eta^{n-m-2} + (s-t)^2 P_{n-m-2}(s,t)\right),$$

with $c_0 \neq 0$. Then the $\mathcal{R}$-component of

$$i\bar{\eta}\frac{\partial}{\partial z_2} N^{n,k}_{\gamma_0+2,-1-m,-1}$$

equals

$$ic_0\,\bar{z}_2^{m+1}\eta^{n-m-1}\bar{\eta}^k + ic_1\,\bar{z}_2^{m+1}\nu\eta^{n-m-2}\bar{\eta}^k.$$

This implies

$$\chi^{n-1,k}_{\gamma_0+1,-m-1,0} = 2c_0\,\bar{z}_2^{m+1}\bar{\omega}_2^{n-m-1}\bar{\omega}_1^k\frac{\partial}{\partial \bar{\omega}_2} - ic_1\,\bar{z}_2^{m+2}\bar{\omega}_2^{n-m-2}\bar{\omega}_1^k\frac{\partial}{\partial \bar{z}_2}$$

and, therefore,

$$X^{n-1,k}_{\gamma_0+1,-m-1,0}(\nu - V)|_{V=\nu} = c_0'(s-t)^{m+1}t^{n-m-1}\bar{s}^k + c_1'(s-t)^{m+2}t^{n-m-2}\bar{s}^k.$$

On the other hand, one can start to determine $N^{n,k}_{\gamma_0+2,\Delta}$ from the right upper corner in the weight-defect diagram. We observe that the only terms in $N^{n,k}_{\gamma_0+2,-m-2,0}$ that contribute into $\mathcal{R}$ are

$$\begin{aligned} \bar z_2^{m+2}\bar\nu\Pi(\eta,\bar\eta) &\xrightarrow{\bar\eta\frac{\partial}{\partial z_2}} \bar z_2^{m+2}\bar z_1\bar\eta\Pi(\eta,\bar\eta)\\ \bar z_2^{m+2}\nu\bar\nu\Pi(\eta,\bar\eta) &\xrightarrow{\bar\eta\frac{\partial}{\partial z_2}} \bar z_2^{m+2}\nu\bar z_1\bar\eta\Pi(\eta,\bar\eta). \end{aligned}$$

This leads to

$$X^{n-1,k}_{\gamma_0+1,-m-2,1}(\nu - V)|_{V=\nu} = b_0'(s-t)^{m+2}t^{n-m-2}\bar s^k + b_1'(s-t)^{m+3}t^{n-m-3}\bar s^k$$

and

$$\mathfrak{N}^{n,k}_{\gamma_0+2,-m-2,0} = b_0(s-t)^{m+2}t^{n-m-2}\bar s^k + b_1(s-t)^{m+3}t^{n-m-3}\bar s^k + \#(s,t)\bar t^k.$$

Finally, we consider the second equation from the right upper corner in the diagram

$$\begin{aligned} 2\left(\bar s\frac{\partial}{\partial\bar s} - \mathrm{id}\right)\mathfrak{N}^{n,k}_{\gamma_0+2,-m-1,-1} + 2\left(s\frac{\partial}{\partial s} - (m+3)\,\mathrm{id}\right)\mathfrak{N}_{\gamma_0+2,-m-2,0} + \\ + \mathfrak{X}^{n,k}_{\gamma_0+3,-m-1,0} = 0. \end{aligned}$$

It suffices to take into account the part of degree k in $\bar s$.

$$\begin{aligned} 2b_0(s-t)^{m+1}[-s+(m+3)t]t^{n-m-2}\bar s^k + 2b_1(m+3)(s-t)^{m+2}t^{n-m-2}\bar s^k + \\ + c_0(s-t)^{m+1}t^{n-m-1}\bar s^k + c_1(s-t)^{m+2}t^{n-m-2}\bar s^k = 0. \end{aligned}$$

Since $c_0 \neq 0$, the term at the left hand side cannot vanish. The contradiction completes the prove in the case $N_{\gamma_0,-2,-m}$ contained a polynomial Ψ_1.

We consider now the case when $N_{\gamma_0+2,-2,\Delta+2}$ contains monomials of type

$$\Psi_2 = \bar z_2 z_2 \bar z_1 \eta^{n-2}\bar\eta^{k-1}.$$

We have $\Delta = -2$ and, therefore, $N_{\gamma_0+2,\Delta} = N_{\gamma_0+2,-2,0}$. The part of $\mathfrak{N}_{\gamma_0+2,-2,0}$ with bidegree (n,k) ($n \geq 2$, $k \geq 1$) does not depend on occurrences of Ψ_3 because the bidegree of $\pi\Psi_3$ is $(n,0)$ with $n \geq 2$. We will determine $\mathfrak{N}^{n,k}_{\gamma_0+2,-2,0}$ in two ways and compare the results.

The $\mathcal{R}$-component of $\chi_1^1 N^{n,k}_{\gamma_0+2,}$ equals

$$c\bar{z}_2 z_2 \bar{z}_1 \eta^{n-1}\bar{\eta}^{k-1}.$$

The above expression can be compensated only by the following components of $\chi^{n-1,k}_{\gamma_0+1,-1,0}(\nu - V)|_{V=\nu}$:

$$c^{(1)}\omega_1^{n-1}\omega_2^k \frac{\partial}{\partial z_1}(\nu - V)|_{V=\nu}$$
$$c^{(2)}\bar{z}_2\bar{\omega}_1^k\bar{\omega}_2^{n-1}\frac{\partial}{\partial\bar{\omega}_2}(\nu - V)|_{V=\nu}$$

These terms also contain the non-normal expressions

$$c^{(1)}\bar{z}_2\eta^{n-1}\bar{\eta}^k, \quad \text{resp.} \quad \frac{-c^{(2)}}{2i}\bar{z}_2\eta^{n-1}\bar{\eta}^k$$

that must cancel each other. Therefore, $c^{(2)} = 2ic^{(1)} = 2kc$ and

$$\chi^{n-1,k}_{\gamma_0+1,-1,0} = c\left(\omega_1^{n-1}\omega_2^k\frac{\partial}{\partial z_1} + 2i\bar{z}_2\bar{\omega}_1^k\bar{\omega}_2^{n-1}\frac{\partial}{\partial\bar{\omega}_2}\right).$$

One computes

$$\pi\chi^{n-1,k}_{\gamma_0+1,-1,0}(\nu - V)|_{V=\nu} = c(s-t)(s^{n-1}\bar{t}^k + t^{n-1}\bar{s}^k).$$

It follows

$$2\left(s\frac{\partial}{\partial s} - 3\,\mathrm{id}\right)\mathfrak{N}^{n,k}_{\gamma_0+2,-2,0} - c(s-t)(s^{n-1}\bar{t}^k + t^{n-1}\bar{s}^k). \tag{22}$$

This equation determines $\mathfrak{N}^{n,k}_{\gamma_0+2,-2,0}$ up to summands of degree 3 in s. It has no solution for $n = 3, 4$, thus, in this case we obtain a contradiction. Since $N^{n,k}_{\gamma_0+2,-2,0}$ must be divisible by $\bar{\nu}$ (thus, vanish for $\bar{s} = \bar{t}$), one has to add to the solution uniquely determined terms of degree 3 in t. This leads to the explicit expression

$$\mathfrak{N}^{n,k}_{\gamma_0+2,-2,0} = c'\Big[-((n-4)s^n + (n-3)ts^{n-1} - s^3t^{n-3})\bar{t}^k + \\ + \frac{(n-3)(n-4)}{6}(-s^3t^{n-3} + 3st^{n-1} - 2t^n)\bar{s}^k\Big]. \tag{23}$$

We construct now a contradiction by an analogous study of the $\mathcal{R}$-component of $\chi_1^{\bar{1}} N^{n,k}_{\gamma_0+2,-2,0}$ that has the form

$$i\bar{\eta}\frac{\partial}{\partial z_2} N^{n,k}_{\gamma_0+2,-2,0} = \#\bar{z}_2^2\eta^{n-2}\bar{z}_1\bar{\eta}^k.$$

This determines

$$\chi^{n-1,k}_{\gamma_0+1,-2,1} = -\#\bar{z}_2^3\bar{\omega}_2^{n-2}\bar{z}_1\bar{\omega}_1^{k-1}\frac{\partial}{\partial z_2}.$$

and,

$$X^{n-1,k}_{\gamma_0+1,-2,1}(\nu - V)|_{V=\nu} = -\#(s-t)^3 t^{n-3}\bar{s}^k.$$

For $\mathfrak{N}^{n,k}_{\gamma_0+2,-2,0}$ we have the equation

$$2\bar{s}\frac{\partial}{\partial \bar{s}}\mathfrak{N}^{n,k}_{\gamma_0+2,-2,0} = \#(s-t)^3 t^{n-1}\bar{s}^k.$$

The solution is unique up to some term that does not contain $\bar{s}$ and that is determined by the condition that $\mathfrak{N}^{n,k}_{\gamma_0+2,-2,0}$ is divisible by $(s-t)^2$. The explicit expression is

$$\mathfrak{N}^{n,k}_{\gamma_0+2,-2,0} = \#(s-t)^2 t^{n-2}(\bar{t}^k - \bar{s}^k).$$

This term does not contain summands of degree 3 with respect to s whereas the above obtained expression (23) does. The desired contradiction proves that in this situation also $a_2 = 0$.

It remains to study the case when $N_{\gamma_0+2,-2,\Delta+2}$ contains terms of type (21) and does not contain terms of type (19) or (20). Then $N_{\gamma_0+2,-2,\Delta+2}$ has $\bar{1}$-defect 0. Since $\chi_{\gamma_0+1,-1,0}$ and therefore $N_{\gamma_0+2,-2,0}$ are determined by the $\mathcal{R}$ contribution of

$$\eta\frac{\partial}{\partial \bar{z}_2}\Psi_3$$

the bidegree of $\mathfrak{N}_{\gamma_0+2,-2,0}$ equals $(n,0)$ (the bidegree of $\pi\Psi_3$). It follows that $N_{\gamma_0+2,-2,0}$ does not depend on $z_2, \bar{z}_1, \bar{\eta}$.
Thus $N_{\gamma_0+2,-2,\Delta+2} = N_{\gamma_0+2,-2}(z_1, \bar{z}_2, \eta)$. The $\mathcal{R}$-component of $\chi_1^1(N^{n,0}_{\gamma_0+2,-2})|_{V=\nu}$ equals

$$\varkappa\bar{z}_2\nu^2\eta^{n-3}. \tag{24}$$

By means of a linear coordinate change

$$z' = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & \bar{c} \end{pmatrix} z, \qquad \omega' = \omega$$

one can control the coefficient $\varkappa$ in (24).
We obtain

$$\chi^{n-1,0}_{\gamma_0+1,-1,\Delta+2} = \mu \left[\omega_1^{n-1} \frac{\partial}{\partial z_1} + 2i(n-1)\bar{z}_2^2 \bar{\omega}_2^{n-2} \frac{\partial}{\partial \bar{z}_2} - 2i\bar{z}_2 \bar{\omega}_2^{n-1} \frac{\partial}{\partial \bar{\omega}_2} \right].$$

Here, we used that there are no non-normal terms of the type $\bar{z}_2\eta^{n-1}$, or $\bar{z}_2\nu\eta^{n-2}$. The coefficient $\varkappa$ evaluates to

$$\varkappa = 2(n-1)(n-2)\mu.$$

The expression $X^{n-1,0}_{\gamma_0+1,-1,\Delta+2}(\nu - V)|_{V=\nu}$ equals

$$\frac{\mu}{2i}(s-t)(s^{n-1} - (n-1)(s-t)t^{n-2} - t^{n-1}).$$

It follows

$$\mathfrak{N}^{n,0}_{\gamma_0+2,-2} = \left(s\frac{\partial}{\partial s} - 3\,\mathrm{id} \right)^{-1} \frac{\mu i}{4}(s-t)[s^{n-1} - (n-1)(s-t)t^{n-2} - t^{n-1}].$$

Without loss of generality we set $\mu = -(n-3)(n-4)$.

$$\mathfrak{N}^{n,0}_{\gamma_0+2,-2} = -\frac{i}{12}(n-4)(n-3)(n-2)t^n + \frac{i}{8}(n-4)(n-3)(2n-3)st^{n-1} - \\ - \frac{i}{4}(n-4)(n-3)(n-1)s^2t^{n-2} + \frac{i}{4}(n-3)s^{n-1}t - \frac{i}{4}(n-4)s^n + \lambda s^3 t^{n-3},$$

where λ is determined by the fact that $\mathfrak{N}^{n,0}_{\gamma_0+2,-2}|_{s=t} = 0$ (divisibility by ν). Thus,

$$\lambda = \frac{i}{24}(n-2)(n-1)(2n-9).$$

It follows that $\mathfrak{N}^{n,0}_{\gamma_0+2,-2} = 0$ for $n = 4$ and $\mathfrak{N}^{n,0}_{\gamma_0+2,-2} \neq 0$ for $n \geq 5$. This construction leads to the solution (14) in Theorem 2.

6 The case $\delta_1 = -1$

In this section we study the case $\delta_1 = -1$. The only monomials Ψ in $N_{\gamma_0+2,-1,\delta+1}$ for that $\eta\frac{\partial}{\partial \bar z_2}\Psi$ does not belong to the space $\mathcal{N}$ are

$$\Psi_1 = \bar z_2 \nu z_2^m \eta^{n-2}\bar\eta^{k-m}, \quad m \geq 1,\, n \geq 2,\, k \geq m \tag{25}$$

$$\Psi_2 = \bar z_2 \nu \eta^{n-2}\bar\nu\bar\eta^{k-1}, \qquad n \geq 2,\, k \geq 1 \tag{26}$$

$$\Psi_3 = \bar z_2 \nu^3 \eta^{n-4}, \qquad n \geq 4. \tag{27}$$

If $N_{\gamma_0+2,-1,\delta+1}$ does not contain such monomials then again, by the argument from Lemma 2, we obtain a contradiction to $\mathcal{P} \equiv 0$.
Suppose now that $N_{\gamma_0+2,-1,\delta+1}$ contains a monomial of type

$$\Psi_1 = \#\bar z_2 \nu z_2^m \eta^{n-2}\bar\eta^{k-m}.$$

Then $\Delta = -m-1 \leq -2$. It follows that there are no monomials of type Ψ_2, Ψ_3 in $N_{\gamma_0+2,-1,-m}$. As in the previous section, we have to consider components of bidegree (n,k) of the upper line in the weight-defect diagram. It consists of the terms $\mathfrak{N}^{n,k}_{\gamma_0+2,-1,-m}, \ldots, \mathfrak{N}^{n,k}_{\gamma_0+2,-1-m,0}$. The polynomial $\mathfrak{N}^{n,k}_{\gamma_0+2,-1-m,0}$ can be determined as in the previous section with m substituted by $m-1$:

$$\mathfrak{N}^{n,k}_{\gamma_0+2,-m-1,0} = b_0(s-t)^{m+1}t^{n-m-1}\bar s^k + b_1(s-t)^{m+2}t^{n-m-2}\bar s^k + P(s,t)\bar t^k.$$

The contribution into $\mathcal{R}$ of

$$\eta\frac{\partial}{\partial \bar z_2}N^{n,k}_{\gamma_0+2,-m-1,0}$$

equals zero. Therefore,

$$2\left(s\frac{\partial}{\partial s} - (m+2)\,\mathrm{id}\right)\mathfrak{N}^{n,k}_{\gamma_0+2,-m-1,0} = 2b_0(s-t)^m[-s+(m+2)t]t^{n-m-1}\bar s^k + \\ + 2b_1(m+2)(s-t)^{m+1}t^{n-m-1}\bar s^k + \#(s,t)\bar t^k$$

is the complete contribution of $N^{n,k}_{\gamma_0+2,-m-1,0}$ into $\mathcal{P}^{n,k}_{\gamma_0+3,-m,0}$. It must cancel with $\mathfrak{X}^{n,k}_{\gamma_0+3,-m,0}$ and

$$2\left(\bar s\frac{\partial}{\partial \bar s} - \mathrm{id}\right)\mathfrak{N}^{n,k}_{\gamma_0+2,-m,-1}.$$

To determine $\mathfrak{N}^{n,k}_{\gamma_0+2,-m,-1}$ we start from the left corner of the diagram. As in the previous section we find

$$\begin{aligned}\mathfrak{N}^{n,k}_{\gamma_0+2,-1,-m} =&\#(s^n - s^2 t^{n-2})(\bar t - \bar s)^m \bar t^{k-m}\\ \mathfrak{N}^{n,k}_{\gamma_0+2,-2,-m+1} =&\#(s^n - (n-2)s^3 t^{n-3} + (n-3)s^2 t^{n-2})(\bar t - \bar s)^{m-1}\bar t^{k-m+1} +\\ &+ \#'((n-4)s^n - (n-3)s^{n-1}t + s^3 t^{n-3})(\bar t - \bar s)^{m-1}\bar t^{k-m+1}.\end{aligned}$$

We observe that both summands in $\mathfrak{N}^{n,k}_{\gamma_0+2,-2,-m+1}$ are divisible by $(s-t)^2$ and $\mathfrak{N}^{n,k}_{\gamma_0+2,-2,-m+1}$ is divisible by $(s-t)^3$ for

$$\#(n-2) + \#'(n-4) = 0. \tag{28}$$

By induction we obtain

$$\mathfrak{N}^{n,k}_{\gamma_0+2,-m,-1} = (s-t)^m[c_0\eta^{n-m} + c_1(s-t)\eta^{n-m-1} + (s-t)^2\#(s,t)](\bar t - \bar s)\bar t^{k-1},$$

where $c_0 = 0$ exactly under the condition (28). Thus, the contribution of $\mathfrak{N}^{n,k}_{\gamma_0+2,-m,-1}$ into $\mathcal{P}^{n,k}_{\gamma_0+3,-m,0}$ equals to

$$2\left(\bar s\frac{\partial}{\partial \bar s} - \mathrm{id}\right)\mathfrak{N}^{n,k}_{\gamma_0+2,-m,-1} = (s-t)^m[c_0\eta^{n-m} + c_1(s-t)\eta^{n-m-1} + (s-t)^2\#'(s,t)]\bar t^k.$$

As in the previous section

$$X^{n-1,k}_{\gamma_0+1,-m,0}(\nu - V)|_{V=\nu} = c_0'(s-t)^m t^{n-m}\bar s^k + c_1'(s-t)^{m+1}t^{n-m-1}\bar s^k.$$

Adding up all listed contributions to $\mathcal{P}_{\gamma_0+3,-m,0}$ we obtain the identity:

$$\begin{aligned}c_0'(s-t)^m t^{n-m}\bar s^k + c_1'(s-t)^{m+1}t^{n-m-1}\bar s^k + 2b_0(s-t)^m[-s+(m+2)t]t^{n-m-1}\bar s^k +\\ + 2b_1(m+2)(s-t)^{m+1}t^{n-m-1}\bar s^k + P(s,t)\bar t^k +\\ + (s-t)^m[c_0\eta^{n-m} + c_1(s-t)\eta^{n-m-1} + (s-t)^2\#(s,t)]\bar t^k \equiv 0.\end{aligned} \tag{29}$$

Comparing the coefficients at $(s-t)^m t^{n-m}\bar s^k$ we conclude that $c_0' = 0$, $b_0 = 0$. This implies that

$$P(s,t) = -2b_1(s-t)^{m+1}t^{n-m-1} + (s-t)^{m+2}\#(s,t).$$

Identity (29) takes the form

$$\begin{aligned}c_1'(s-t)^{m+1}t^{n-m-1}\bar s^k + 2b_1(m+2)(s-t)^{m+1}t^{n-m-1}\bar s^k - 2b_1(s-t)^{m+1}t^{n-m-1}\bar t^k +\\ + (s-t)^{m+2}\#(s,t)\bar t^k + (s-t)^{m+1}[c_1\eta^{n-m-1} + (s-t)^2\#(s,t)]\bar t^k \equiv 0.\end{aligned}$$

We see that the coefficient at $(s-t)^{m+1}\bar{t}^k$ does not vanish unless $c_1 = c_1' = b_1 = 0$ because η^{n-m-1} cannot cancel with $\bar{t}^k$. Contradiction.
Let's now consider the the situation when $N_{\gamma_0+2,\Delta}$ contains a monomial

$$\Psi_2 = \bar{z}_2 \nu \eta^{n-2} \bar{\nu} \bar{\eta}^{k-1}.$$

Then $\Delta = -1$ and, therefore, $N_{\gamma_0+2,\Delta} = N_{\gamma_0,-1,0}$.
The $\mathcal{R}$-component of $\chi_1^1 N^{n,k}_{\gamma_0+2,-1}$ equals $\#\bar{z}_1\bar{z}_2\nu\eta^{n-2}\bar{\eta}^k$, and, hence,

$$\chi^{n-1,k}_{\gamma_0+1,-1,1} = \left(\#\bar{z}_1\bar{z}_2^2\bar{\omega}_1^k\bar{\omega}_2^{n-2}\frac{\partial}{\partial\bar{z}_2}\right).$$

It follows that

$$X^{n-1,k}_{\gamma_0+1,0,0}(\nu - V)|_{V=\nu} = \#(s-t)^2 t^{n-2}\bar{s}^k.$$

Therefore, the equation

$$2\left(\bar{s}\frac{\partial}{\partial\bar{s}}\right)\mathfrak{N}^{n,k}_{\gamma_0+2,-1}) + X^{n-1,k}_{\gamma_0+1,0,0}(\nu - V)|_{V=\nu} = 0$$

implies

$$\mathfrak{N}^{n,k}_{\gamma_0+2,-1} = \#(s-t)^2 t^{n-2}(\bar{s}^k - \bar{t}^k).$$

(we have added $-\#(s-t)^2t^{n-2}\bar{t}^k$ from the kernel, in order to satisfy the normal form condition $N^{n,k}_{\gamma_0+2,-1}|_{\bar{s}=\bar{t}} = 0$).
On the other hand $\chi_1^{\bar{1}}\Psi_2$ contains the $\mathcal{R}$-component $c\nu\bar{\nu}\eta^{n-1}\bar{\eta}^{k-1}$ which implies

$$\chi^{n-1,k}_{\gamma_0+1,0,0} = c' z_1\omega_1^{n-1}\omega_2^k\frac{\partial}{\partial z_1} - c'\bar{z}_2\bar{\omega_1}^k\bar{\omega_2}^{n-1}\frac{\partial}{\partial\bar{z}_2}.$$

The contradiction proves that $\mathcal{P}_{\gamma_0+3} \neq 0$ if $N_{\gamma_0+2,-1}$ contains a monomial of the form (26).
Suppose now that $N_{\gamma_0+2,-1}$ contains a monomial Ψ_3. Since Ψ_3 does not depend on the variables $z_2, \bar{z}_1, \bar{\eta}$ one can repeat the argument from the previous section to show that the entire term $N_{\gamma_0+2,-1}$ does not depend on them. The $\mathcal{R}$-component of $\chi_1^1 N_{\gamma_0+2,-1}$ equals to $c\nu^3\eta^{n-3}$. It follows that

$$\chi^{n-1,0}_{\gamma_0+1} = \mu\left(n\omega_1^{n-1}z_1\frac{\partial}{\partial z_1} + n\bar{\omega_2}^{n-1}\bar{z}_2\frac{\partial}{\partial\bar{z}_2} + 2\omega_1^n\frac{\partial}{\partial\omega_1} + 2\bar{\omega}_2^n\frac{\partial}{\partial\bar{\omega}_2}\right),$$

with $\mu = \frac{3c}{n(n-1)(n-2)}$. By an appropriate coordinate change, as in the previous section, we can achieve that $X^{n-1,0}_{\gamma_0+1}(\nu - V)|_{V=\nu}$ takes the form

$$2(n-3)\left[(n-2)s^n - ns^{n-1}t + nst^{n-1} - (n-2)t^n\right].$$

Then

$$\begin{aligned}\mathfrak{N}^{n,0}_{\gamma_0,-1} =& \left(s\frac{\partial}{\partial s} - 2\,\mathrm{id}\right)^{-1}(n-3)\left[(n-2)s^n - ns^{n-1}t + nst^{n-1} - (n-2)t^n\right] \\ =&(n-3)s^n - ns^{n-1}t - (n-3)nst^{n-1} + \frac{(n-3)(n-2)}{2}t^n + \lambda s^2 t^{n-2},\end{aligned}$$

where $\lambda = \frac{n(n-1)}{2}$ is determined by the condition that $\mathfrak{N}_{\gamma_0+2,-1}$ is divisible by $(s-t)$.
For $n = 3$ the polynomial $\mathfrak{N}^{n,0}_{\gamma_0+2,-1} = 0$ whereas, for $n \geq 4$, $\mathfrak{N}^{n,0}_{\gamma_0,-1} \neq 0$. This represents the solution (15) of Theorem (2).

7 The case $\delta_1 = 0$

Suppose $\delta_1 = 0$. According to Table 2, $\chi^1_1 N_{\gamma_0,0,\Delta}$ has a non-trivial $\mathcal{R}$-component if and only if $N_{\gamma_0,0,\Delta}$ contains a monomial

$$\Psi = \# z_2^m \nu^2 \eta^{n-2}\bar{\eta}^{k-m}, \quad m = -\Delta \geq 0,\ n \geq 2,\ k \geq m.$$

Any solution $(\mathfrak{N}^{n,k}_{\gamma_0+2,\Delta}, X^{n-1,k}_{\gamma_0+1,\Delta+1})$ of the upper line in the weight-defect diagram of bidegree (n,k)

$$X^{n-1,k}_{\gamma_0+1,\Delta+1}(\nu - V)|_{V=\nu} + (X^1_1 + X^{\bar{1}}_{\bar{1}} - 2\,\mathrm{id})\mathfrak{N}^{n,k}_{\gamma_0+1,\Delta} = 0 \tag{30}$$

is the sum of a particular solution and of a solution of the system (30) that satisfies the extra condition

$$\left(s\frac{\partial}{\partial s} - \mathrm{id}\right)\mathfrak{N}^{n,k}_{\gamma_0+1,0,\Delta} = 0 \tag{31}$$

which is equivalent to the absence of Ψ in $N^{n,k}_{\gamma_0+1,0,\Delta}$. We start with the construction of a particular solution. The field $X_{\gamma_0+1,1,-m}$ is uniquely determined by the $\mathcal{R}$-contribution

$$\Psi \xrightarrow{\bar{\eta}\frac{\partial}{\partial z_2}} \# z_1 z_2^m \nu \eta^{n-1}\bar{\eta}^{k-m}$$

and equals

$$X_{\gamma_0+1,1,-m} = \# z_1^2 z_2^m \omega_1^{n-1} \omega_2^{k-m} \frac{\partial}{\partial z_1},$$

thus,

$$X^{n-1,k}_{\gamma_0+1,1,-m}|_{V=\nu} = \#(s-t)s^{n-1}(\bar{s}-\bar{t})^m \bar{t}^{k-m}.$$

Now we determine $\mathfrak{N}^{n,k}_{\gamma_0+2,0,-m}$ up to a summand from the kernel of $s\frac{\partial}{\partial s} - \mathrm{id}$ by resolving the equation

$$2\left(s\frac{\partial}{\partial s} - \mathrm{id}\right)\mathfrak{N}^{n,k}_{\gamma_0+2,0,-m} + c(s^n - ts^{n-1})(\bar{s}-\bar{t})^m \bar{t}^{k-m} = 0.$$

For $c = \frac{(n-1)!}{(n-m-3)!}$ we obtain the particular solution

$$\mathfrak{N}^{n,k}_{\gamma_0+2,0,-m} = \frac{(n-3)!}{2(n-m-3)!}[-(n-2)s^n + (n-1)s^{n-1}t - st^{n-1}](\bar{s}-\bar{t})^m \bar{t}^{k-m}.$$

Then

$$\begin{aligned} 2\left(\bar{s}\frac{\partial}{\partial \bar{s}} - m\,\mathrm{id}\right)\mathfrak{N}^{n,k}_{\gamma_0,0,-m} &= \\ = \frac{m(n-3)!}{(n-m-3)!}&[-(n-2)s^n + (n-1)s^{n-1}t - st^{n-1}](\bar{s}-\bar{t})^{m-1}\bar{t}^{k-m+1}. \end{aligned}$$

Since $\chi_1^{\bar{1}} N^{n,k}_{\gamma_0+2,0,-m}$ belongs to $\mathcal{N}$ there is no contribution of the field to the terms of defect $(-1,-m+1)$ and we have

$$\mathfrak{N}^{n,k}_{\gamma_0+2,-1,-m+1} = -\left(s\frac{\partial}{\partial s} - \mathrm{id}\right)^{-1}\left(\bar{s}\frac{\partial}{\partial \bar{s}} - m\,\mathrm{id}\right)\mathfrak{N}_{\gamma_0+2,0,-m}.$$

¿From this we obtain

$$\begin{aligned} \mathfrak{N}^{n,k}_{\gamma_0,-1,-m+1} = \frac{m(n-4)!}{2(n-m-3)!}&[(n-3)s^n - (n-1)s^{n-1}t + (n-1)s^2t^{n-2} - \\ &(n-3)st^{n-1}](\bar{s}-\bar{t})^{m-1}\bar{t}^{k-m+1}. \end{aligned}$$

Consecutively, this procedure leads to the expressions for all terms

$$\begin{aligned} \mathfrak{N}^{n,k}_{\gamma_0+2,-j,-m+j} &= \left(\mu_n^{(j)} s^n + \mu_{n-1}^{(j)} s^{n-1}t + \sum_{\alpha=1}^{j+1} \mu_\alpha^{(j)} s^\alpha t^{n-\alpha}\right)(\bar{s}-\bar{t})^{m-j}\bar{t}^{k-m+j} \\ &= (s-t)^{j+2}P_{n-j-2}(s,t)(\bar{s}-\bar{t})^{m-j}\bar{t}^{k-m+j}, \qquad (32) \end{aligned}$$

in particular

$$\mathfrak{N}^{n,k}_{\gamma_0+2,-m,0} = (s-t)^{m+2} P_{n-m-2}(s,t)\bar{t}^k.$$

Since $\bar{s}\frac{\partial}{\partial \bar{s}}\mathfrak{N}^{n,k}_{\gamma_0+2,-m,0} = 0$, hence the sequence terminates at this term. Hence, we have obtained a particular solution of (30), that leads to (16) of Theorem 2.

We consider now the situation when (31) is satisfied, i.e., $\mathfrak{N}^{n,k}_{\gamma_0+2,0,-m}$ is linear in s. Then $n-1$ is the degree in t. Since st^{n-1} contains η^{n-1} the term $N^{n,k}_{\gamma_0+2,0,-m}$ must be divisible by $z_2^m\bar{\nu}$ if $m>0$ in order to have normal form. For $m=0$ it must be divisible by $\bar{\nu}^2$. Thus, we obtain

$$\mathfrak{N}^{n,k}_{\gamma_0+2,0,-m} = st^{n-1}(\bar{s}-\bar{t})^{m+1} Q_{k-m-1}(\bar{s},\bar{t}), \quad m>0 \tag{33}$$

$$\mathfrak{N}^{n,k}_{\gamma_0+2,0,0} = st^{n-1}(\bar{s}-\bar{t})^{2} Q_{k-2}(\bar{s},\bar{t}), \qquad m=0. \tag{34}$$

Consider, at first, the case $m=0$. Since $\delta_{\bar{1}} \leq 0$, the term $N^{n,k}_{\gamma_0+2,0}$ consists of just one component $N^{n,k}_{\gamma_0+2,0,0}$. According to (34), $\bar{\eta}\frac{\partial}{\partial z_2}N^{n,k}_{\gamma_0+2,0}$ is divisible by $\bar{z}_1^2 z_2$ and therefore $\chi_1^{\bar{1}}N^{n,k}_{\gamma_0+2,0}$ is normal. This implies that $\chi^{n-1,k}_{\gamma_0+1,0,1}=0$. But, since

$$\bar{s}\frac{\partial}{\partial \bar{s}}(\bar{s}-\bar{t})^2 Q_{k-2}(\bar{s},\bar{t}) = (\bar{s}-\bar{t})Q_{k-1}(\bar{s},\bar{t}) \neq 0$$

we have $\mathcal{P}_{\gamma_0+3} \not\equiv 0$.

Now, let $m=1$. Because of $\delta_{\bar{1}} \leq 0$

$$N^{n,k}_{\gamma_0+2,-1} = N_{\gamma_0+2,0,-1} + N_{\gamma_0+2,-1,0}.$$

According to the conditions of the normal form we obtain that the expression $\bar{\eta}\frac{\partial}{\partial z_2}N^{n,k}_{\gamma_0+2,-1,0}$ is also normal. In order to achieve $\mathcal{P}_{\gamma_0+3} \equiv 0$ the term $\chi_1^{\bar{1}}N_{\gamma_0,-1,0}$ must vanish. Therefore, $N^{n,k}_{\gamma_0+2,-1,0}$ is divisible by $\bar{z}_2\nu^2$. This implies

$$\mathfrak{N}^{n,k}_{\gamma_0+2,-1,0} = (s-t)^3 P_{n-3}(s,t)\bar{t}^k. \tag{35}$$

Then $\eta\frac{\partial}{\partial \bar{z}_2}N^{n,k}_{\gamma_0+2,-1,0}$ and $\chi_1^1 N^{n,k}_{\gamma_0+2,-1,0}$ are also normal. Therefore

$$2\left(s\frac{\partial}{\partial s} - 2\,\mathrm{id}\right)(s-t)^3 P_{n-3}(s,t)\bar{t}^k = (s-t)^2 P_{n-2}(s,t)\bar{t}^k. \tag{36}$$

On the other hand,

$$2\left(\bar{s}\frac{\partial}{\partial \bar{s}} - \mathrm{id}\right) st^{n-1}(\bar{s}-\bar{t})^2 Q_{k-2}(\bar{s},\bar{t}) = st^{n-1}(\bar{s}-\bar{t})Q_{k-1}(\bar{s},\bar{t}) \tag{37}$$

is not divisible by $(s-t)^2$ nor by $\bar{t}^k$. This means that the expressions (36) and (37) cannot cancel each other. Therefore, $\chi^{n-1,k}_{\gamma_0+1,0,0}$ must be different from zero which implies that $N_{\gamma_0,0,-1}$ contains

$$\Psi = \#\eta^n z_2 \bar{\nu} \bar{\eta}^{k-2},$$

so that

$$-i\bar{\eta}\frac{\partial}{\partial z_2}\Psi = c\eta^n \bar{\nu} \bar{\eta}^{k-1}$$

does not belong to $\mathcal{N}$. This defines χ_{γ_0+1} uniquely, up to a constant factor

$$\chi^{n-1,k}_{\gamma_0+1} = c'\left[n\omega_1^{n-1}\omega_2^k z_1\frac{\partial}{\partial z_1} + n\bar{\omega}_2^{n-1}\bar{\omega}_1^k \bar{z}_2\frac{\partial}{\partial \bar{z}_2} + 2\omega_1^n\omega_2^k\frac{\partial}{\partial \omega_1} + 2\bar{\omega}_2^n\bar{\omega}_1^k\frac{\partial}{\partial \bar{\omega}_2}\right]$$

with $c = 2kc'$.
Then

$$\begin{aligned}
X^{n-1,k}_{\gamma_0+1,0}(\nu - V)|_{V=\nu} &= \\
&= \frac{-ic'}{2}\left[(n-1)s^n\bar{t}^k - ns^{n-1}t\bar{t}^k + nst^{n-1}\bar{s}^k - (n-1)t^n\bar{s}^k\right] \\
&= \frac{-ic'}{2}\left[s^{n-1}\bar{t}^k\left((n-1)s - nt\right) + t^{n-1}\bar{s}^k\left(ns - (n-1)t\right)\right].
\end{aligned}$$

We see that

$$\begin{aligned}
\mathcal{P}^{n,k}_{\gamma_0+3,0,0} = (s-t)^2 P_{n-2}(s,t)\bar{t}^k + st^{n-1}(\bar{s}-\bar{t})Q_{k-1}(\bar{s},\bar{t}) + \\
+ \frac{-ic'}{2}\left[s^{n-1}\bar{t}^k\left((n-1)s - nt\right) + t^{n-1}\bar{s}^k\left(ns - (n-1)t\right)\right] \not\equiv 0.
\end{aligned}$$

Contradiction.
Let $m \geq 2$. Then $N^{n,k}_{\gamma_0+2,0,-m}$ is divisible by $z_2^m\bar{\nu}$, so $\bar{\eta}\frac{\partial}{\partial z_2}N^{n,k}_{\gamma_0+2,0,-m}$ is divisible by $z_2^{m-1}\bar{\nu} \in \mathcal{N}$. It follows that $\chi_1^{\bar{1}} N^{n,k}_{\gamma_0+2,0,-m} \in \mathcal{N}$. According to Table 2, the $\nwarrow$ arrows of the upper line of the weight-defect diagram may contribute into $\mathcal{R}$ only if they are applied to $N^{n,k}_{\gamma_0+2,-1,-m+1}$ and $N^{n,k}_{\gamma_0+2,-2,-m+2}$. The corresponding solutions $\mathfrak{N}^{n,k}_{\gamma_0+2,-j,-m+j}$ until $j = m-1$ can be extracted from the already considered cases $\delta_1 = -1$ (with m substituted by $m-1$) and $\delta_1 = -2$ (with m substituted by $m-2$). In particular,

$$\begin{aligned}
\mathfrak{N}^{n,k}_{\gamma_0+2,-m+1,-1} = (s-t)^{m-1}\big(c_0\eta^{n-m+1} + c_1(s-t)\eta^{n-m} + \\
+ (s-t)^2 P_{n-m-1}(s,t)\big)(\bar{t}-\bar{s})\bar{t}^{k-1} + \tilde{\mathfrak{N}}^{n,k}_{\gamma_0+2,-m+1,-1},
\end{aligned}$$

where $\tilde{\mathfrak{N}}^{n,k}_{\gamma_0+2,-m+1,-1}$ is a solution of the upper line of the diagram when the $\nwarrow$ arrows do not contribute into $\mathcal{R}$. Starting with

$$\tilde{\mathfrak{N}}^{n,k}_{\gamma_0+2,0,-m} = \mathfrak{N}^{n,k}_{\gamma_0+2,0,-m} = st^{n-1}(\bar{s}-\bar{t})^{m+1}Q_{k-m-1}(\bar{s},\bar{t})$$

and inductively solving the equation

$$\left(s\frac{\partial}{\partial s}-(j+2)\,\mathrm{id}\right)\tilde{\mathfrak{N}}^{n,k}_{\gamma_0+2,-j-1,-m+j+1}+\left(\bar{s}\frac{\partial}{\partial \bar{s}}-(m-j)\,\mathrm{id}\right)\tilde{\mathfrak{N}}^{n,k}_{\gamma_0+2,-j,-m+j}=0,$$

we find that

$$\tilde{\mathfrak{N}}^{n,k}_{\gamma_0+2,-j-1,-m+j+1} = (s-t)^{j+1}st^{n-j-2}(\bar{s}-\bar{t})^{m-j}Q_{k-m+j}(\bar{s},\bar{t})$$

and therefore

$$\tilde{\mathfrak{N}}^{n,k}_{\gamma_0+2,-m+1,-1} = (s-t)^{m-1}st^{n-m}(\bar{s}-\bar{t})^{2}Q_{k-2}(\bar{s},\bar{t})$$

Hence,

$$2\left(\bar{s}\frac{\partial}{\partial \bar{s}}-\mathrm{id}\right)\mathfrak{N}^{n,k}_{\gamma_0+2,-m+1,-1} = (s-t)^{m-1}\big(c_0\eta^{n-m+1}+c_1(s-t)\eta^{n-m}+ \\ +(s-t)^2P_{n-m-1}(s,t)\big)\bar{t}^k+(s-t)^{m-1}st^{n-m}(\bar{s}-\bar{t})Q_{k-1}(\bar{s},\bar{t}), \quad (38)$$

with $Q_{k-1}(\bar{s},\bar{t})\neq 0$. As we have previously computed, the contribution of $\mathfrak{N}^{n,k}_{\gamma_0+2,-m,0}$ and $X^{n-1,k}_{\gamma_0+1,-m+1,0}$ into $\mathcal{P}^{n,k}_{\gamma_0+3,-m+1,0}$ equals to

$$c_0'(s-t)^{m-1}t^{n-m+1}\bar{s}^k+c_1'(s-t)^m t^{n-m}\bar{s}^k+ \\ +2b_0(s-t)^{m-1}[-s+(m+1)t]t^{n-m}\bar{s}^k+2b_1(m+1)(s-t)^m t^{n-m}\bar{s}^k+P(s,t)\bar{t}^k.$$

We see that this expression cannot cancel with (38) because nothing can cancel with $(\bar{s}-\bar{t})Q_{k-1}(\bar{s},\bar{t})$ in the last summand of (38). Contradiction.

8 The case $\delta_1 = 1$

As in the previous section we distinguish two cases in dependence whether the $\nwarrow$-arrow in the left upper corner of the diagram contributes into $\mathcal{R}$ or not, i.e.,

$$\eta\frac{\partial}{\partial \bar{z}_2}N^{n,k}_{\gamma_0+2,1,-m}\in\mathcal{N} \quad (39)$$

holds or not.
First, suppose (39) is not valid. According to Table 2, this implies that $N^{n,k}_{\gamma_0+2,1,-m}$ must contain a monomial of the form

$$\Psi = \# z_1 \nu^2 \eta^{n-2} z_2^m \bar{\eta}^{k-m},$$

where $m = -\Delta + 1 \geq 0$. The $\mathcal{R}$-component of

$$\chi^1_1 N^{n,k}_{\gamma_0+2,1,-m}$$

is then

$$\# z_1^2 z_2^m \nu \eta^{n-1} \bar{\eta}^{k-m}.$$

It follows

$$\chi^{n-1,k}_{\gamma_0+1,2,-1} = \# z_1^3 z_2^m \omega_1^{n-1} \omega_2^{k-m} \frac{\partial}{\partial z_1},$$

and furthermore,

$$X^{n-1,k}_{\gamma_0+1,2,-1}|_{V=\nu}) = \#(s-t)s^{n-1}(\bar{s}-\bar{t})^m \bar{t}^{k-m}.$$

Resolving the equation

$$2s\frac{\partial}{\partial s}\mathfrak{N}^{n,k}_{\gamma_0+2,0,-m} + \#(s^n - ts^{n-1})(\bar{s}-\bar{t})^m \bar{t}^{k-m} = 0$$

with $\# = n(n-1)$ we find

$$\mathfrak{N}^{n,k}_{\gamma_0+2,1,-m} = \frac{1}{2}[ns^{n-1}t - (n-1)s^n - t^n](\bar{s}-\bar{t})^m \bar{t}^{k-m}.$$

¿From the equations

$$\left(s\frac{\partial}{\partial s} - j + 1\,\mathrm{id}\right)\mathfrak{N}^{n,k}_{\gamma_0+2,-j,-m+j+1}) + \left(\bar{s}\frac{\partial}{\partial \bar{s}} - (m-j)\,\mathrm{id}\right)\mathfrak{N}^{n,k}_{\gamma_0+2,1-j,-m+j}) = 0$$

one determines by induction the expressions

$$\mathfrak{N}^{n,k}_{\gamma_0+2,-j,-m+j+1} = \left(\lambda^{(j+1)}_n s^n + \lambda^{(j+1)}_{n-1} s^{n-1}t + \sum_{\alpha=0}^{j+1}\lambda^{(j+1)}_\alpha s^\alpha t^{n-\alpha}\right)(\bar{s}-\bar{t})^{m-j-1}\bar{t}^{k-m+j+1}$$

that must be divisible exactly by $(s-t)^{j+2}$. These expressions are uniquely defined up to a multiple of the solution (32) (with m substituted by $m-1$). Eventually, we obtain for $j=m$

$$\mathfrak{N}^{n,k}_{\gamma_0+2,1-m,0} = (s-t)^{m+1} P_{n-m-1}(s,t)\bar{t}^k \tag{40}$$

This belongs to the kernel of $\bar{s}\frac{\partial}{\partial \bar{s}}$ and, hence, we have found a particular solution of the upper line of the weight-defect diagram that corresponds to non-trivial $\chi^{n-1,k}_{\gamma_0+1,2,-m}$ and leads to (17) of the Theorem 2.
Suppose now that (39) is satisfied, i.e.

$$s\frac{\partial}{\partial s}\mathfrak{N}^{n,k}_{\gamma_0+2,1,-m} = 0.$$

It follows from (39) and the condition of the normal form that

$$\mathfrak{N}^{n,k}_{\gamma_0+2,1,-m} =\# t^n(\bar{s}-\bar{t})^{m+1} Q_{k-m-1}(\bar{s},\bar{t}) \quad \text{for} \quad m \geq 1$$

and

$$\mathfrak{N}^{n,k}_{\gamma_0+2,1,0} =\# t^n(\bar{s}-\bar{t})^{2} Q_{k-2}(\bar{s},\bar{t}) \qquad \text{for} \quad m = 0.$$

Consider the case $m=0$. Then $N^{n,k}_{\gamma_0+2,1,0}$ takes the form indicated above and $\chi^{\bar{1}}_1 N^{n,k}_{\gamma_0+2,1,0}$ is divisible by $z_2\bar{\nu}$ and, therefore belongs to $\mathcal{N}$. This implies that $\chi^{n-1,k}_{\gamma_0+1,1,1}$ vanishes. But

$$\bar{s}\frac{\partial}{\partial \bar{s}}\mathfrak{N}^{n,k}_{\gamma_0+2,1,0} = \# t^n(\bar{s}-\bar{t})Q_{k-1}(\bar{s},\bar{t}) \neq 0$$

implies then $\mathcal{P}_{\gamma_0+3} \neq 0$.
Now, let $m=1$. Then, because of $\delta_{\bar{1}} \leq 0$,

$$N^{n,k}_{\gamma_0+2,0} = N^{n,k}_{\gamma_0+2,1,-1} + N^{n,k}_{\gamma_0+2,0,0},$$

where

$$\mathfrak{N}^{n,k}_{\gamma_0+2,1,-1} = t^n(\bar{s}-\bar{t})^2 Q_{k-2}(\bar{s},\bar{t}).$$

Therefore

$$\left(\bar{s}\frac{\partial}{\partial \bar{s}} - 1\,\mathrm{id}\right)\mathfrak{N}^{n,k}_{\gamma_0+2,1,-1} = t^n(\bar{s}-\bar{t})Q_{k-1}(\bar{s},\bar{t}). \tag{41}$$

For $N^{n,k}_{\gamma_0+2,0,0}$, Table 3 indicates that $\bar{\eta}\frac{\partial}{\partial z_2}N^{n,k}_{\gamma_0+2,0,0} \in \mathcal{N}$. Therefore,

$$\bar{s}\frac{\partial}{\partial \bar{s}}\mathfrak{N}^{n,k}_{\gamma_0+2,0,0} = 0.$$

Thus, $\mathfrak{N}^{n,k}_{\gamma_0+2,0,0}$ must be divisible by ν^2. By subtracting a solution of type (32) with $m = 0$ we can achieve that it is divisible even by ν^3. Therefore,

$$\begin{aligned} \mathfrak{N}^{n,k}_{\gamma_0+2,0,0} =&(s-t)^3 P_{n-3}(s,t)\bar{t}^k \\ \left(s\frac{\partial}{\partial s} - 1\,\mathrm{id}\right)\mathfrak{N}^{n,k}_{\gamma_0+2,0,0} =&(s-t)^2 P_{n-2}(s,t)\bar{t}^k. \end{aligned} \tag{42}$$

Since the expressions (41) and (41) cannot cancel each other this implies that $\chi^{n-1,k}_{\gamma_0+1,1,0} \neq 0$. As $N^{n,k}_{\gamma_0+2,0,0}$ is divisible by ν^3 the expression $\eta\frac{\partial}{\partial \bar{z}_2}N^{n,k}_{\gamma_0+2,0,0}$ belongs to $\mathcal{N}$. Thus, the $\mathcal{R}$-component can arise only from $\bar{\eta}\frac{\partial}{\partial z_2}N^{n,k}_{\gamma_0+2,1,-1}$. According to Table 2 this can only happen if $N^{n,k}_{\gamma_0+2,1,-1}$ contains a monomial

$$\Psi = \# z_1 z_2 \nu^2 \eta^{n-2}.$$

But then $k = 0$ that contradicts to the assumption that $\mathfrak{N}^{n,k}_{\gamma_0+2,1,-1}$ is divisible by $\bar{s}\bar{t}^2$, i.e., $k \geq 2$.
We study now the case $m \geq 2$. Then $N^{n,k}_{\gamma_0+2,1,-m}$ is divisible by $z_1 z_2^m \bar{\nu}$, so $\bar{\eta}\frac{\partial}{\partial z_2}N^{n,k}_{\gamma_0+2,0,-m}$ is divisible by $z_1 z_2^{m-1}\bar{\nu} \in \mathcal{N}$. It follows that $\chi_1^{\bar{1}} N^{n,k}_{\gamma_0+2,1,-m} \in \mathcal{N}$. After subtracting, if needed, a multiple of the solution (32) we may assume that the $\nwarrow$ arrows of the upper line of the weight-defect diagram may contribute into $\mathcal{R}$ only if they are applied to $N^{n,k}_{\gamma_0+2,-1,-m+2}$ and $N^{n,k}_{\gamma_0+2,-2,-m+3}$. The corresponding solutions $\mathfrak{N}^{n,k}_{\gamma_0+2,-j+1,-m+j}$ until $j = m-1$ can be extracted from the already considered cases $\delta_1 = -1$ (with m substituted by $m-2$) and $\delta_1 = -2$ (with m substituted by $m-3$). In particular,

$$\begin{aligned} \mathfrak{N}^{n,k}_{\gamma_0+2,-m+2,-1} = (s-t)^{m-2}\big(&c_0\eta^{n-m+2} + c_1(s-t)\eta^{n-m+1} + \\ &+ (s-t)^2 P_{n-m}(s,t)\big)(\bar{t}-\bar{s})\bar{t}^{k-1} + \tilde{\mathfrak{N}}^{n,k}_{\gamma_0+2,-m+2,-1}, \end{aligned}$$

where $\tilde{\mathfrak{N}}^{n,k}_{\gamma_0+2,-m+2,-1}$ is a solution of the upper line of the diagram when the $\nwarrow$ arrows do not contribute into $\mathcal{R}$. Starting with

$$\tilde{\mathfrak{N}}^{n,k}_{\gamma_0+2,1,-m} = \mathfrak{N}^{n,k}_{\gamma_0+2,1,-m} = t^n(\bar{s}-\bar{t})^{m+1}Q_{k-m-1}(\bar{s},\bar{t})$$

and inductively solving the equation

$$\left(s\frac{\partial}{\partial s}-(j+1)\,\mathrm{id}\right)\tilde{\mathfrak{N}}^{n,k}_{\gamma_0+2,-j,-m+j+1}+\left(\bar{s}\frac{\partial}{\partial\bar{s}}-(m-j)\,\mathrm{id}\right)\tilde{\mathfrak{N}}^{n,k}_{\gamma_0+2,-j+1,-m+j}=0,$$

we find that

$$\tilde{\mathfrak{N}}^{n,k}_{\gamma_0+2,-j,-m+j+1}=(s-t)^{j+1}t^{n-j-1}(\bar{s}-\bar{t})^{m-j}Q_{k-m+j}(\bar{s},\bar{t})$$

and therefore

$$\tilde{\mathfrak{N}}^{n,k}_{\gamma_0+2,-m+2,-1}=(s-t)^{m-2}t^{n-m+2}(\bar{s}-\bar{t})^{2}Q_{k-2}(\bar{s},\bar{t})$$

Hence,

$$\begin{aligned}2\left(\bar{s}\frac{\partial}{\partial\bar{s}}-\mathrm{id}\right)\mathfrak{N}^{n,k}_{\gamma_0+2,-m+2,-1}&=(s-t)^{m-2}\big(c_0\eta^{n-m+2}+c_1(s-t)\eta^{n-m+1}+\\&+(s-t)^2P_{n-m}(s,t)\big)\bar{t}^k+(s-t)^{m-2}t^{n-m-2}(\bar{s}-\bar{t})Q_{k-1}(\bar{s},\bar{t}),\quad(43)\end{aligned}$$

with $Q_{k-1}(\bar{s},\bar{t})\neq 0$. As we have previously computed, the contribution of $\mathfrak{N}^{n,k}_{\gamma_0+2,-m+1,0}$ and $X^{n-1,k}_{\gamma_0+1,-m+2,0}$ into $\mathcal{P}^{n,k}_{\gamma_0+3,-m+2,0}$ equals to

$$\begin{aligned}&c_0'(s-t)^{m-2}t^{n-m+2}\bar{s}^k+c_1'(s-t)^{m-1}t^{n-m+1}\bar{s}^k+\\&+2b_0(s-t)^{m-2}[-s+mt]t^{n-m+1}\bar{s}^k+2b_1m(s-t)^{m-1}t^{n-m+1}\bar{s}^k+P(s,t)\bar{t}^k.\end{aligned}$$

We see that this expression cannot cancel with (43) because nothing can cancel with $(\bar{s}-\bar{t})Q_{k-1}(\bar{s},\bar{t})$ in the last summand of (43). Contradiction.

9 Description of $\mathrm{Aut}_{0,\mathrm{id}}$

In this section we prove that the dimension of $\mathrm{Aut}_{0,\mathrm{id}}\,M$ equals to zero or two and in case it is equal to two then $\mathrm{Aut}_{0,\mathrm{id}}\,M\cong\mathbb{C}$.

Lemma 4 *If for some $\chi\in\mathfrak{aut}_{\,0,\mathrm{id}}M$, $a_2\neq 0$ then $a_1=0$.*

Proof. If in one of the lines of the weight-defect diagram d_1 attains values ≤ -2 then $\delta_{\bar{2}}\geq 2$ and this would contradict to inequality (11) and would imply $a_1=0$. Therefore, since $\delta_{\bar{2}}\leq 1$, the solution $N_{\gamma_0+2,\Delta}$ of the upper line of the weight-defect diagram does not contain a multiple of (14).

Analogously, using inequality (12) we conclude that $N_{\gamma_0+2,\Delta}$ does not contain neither a multiple of (32) for $m \geq 1$, nor a multiple of (40) for $m \geq 1$. Thus, $\mathfrak{N}^{n,0}_{\gamma_0+2,\Delta}$ can be equal

$$\begin{aligned}\mathfrak{N}^{n,0}_{\gamma_0+2,\Delta} =&\mathfrak{N}^{n,0}_{\gamma_0+2,-1,0} = (n-3)s^n - ns^{n-1}t - (n-3)nst^{n-1} + \\ &+ \frac{(n-3)(n-2)}{2}t^n + \frac{n(n-1)}{2}s^2t^{n-2}, \\ \mathfrak{N}^{n,k}_{\gamma_0+2,\Delta} =&\mathfrak{N}^{n,k}_{\gamma_0+2,0,0} = \frac{1}{2}[(n-1)s^{n-1}t - (n-2)s^n - st^{n-1}]\bar{t}^k \\ \mathfrak{N}^{n,k}_{\gamma_0+2,\Delta} =&\mathfrak{N}^{n,k}_{\gamma_0+2,1,0} = \frac{1}{2}[ns^{n-1}t - (n-1)s^n - t^n]\bar{t}^k\end{aligned}$$

Consider the first option. We see that $\eta\frac{\partial}{\partial z_1}N^{n,0}_{\gamma_0+2,-1,0}$ belongs to $\mathcal{N}$, and also $\chi_1^{\bar{2}}N^{n,0}_{\gamma_0+2,-1,0} \neq 0$. From the weight-defect diagram it follows that in order to cancel $\chi_1^{\bar{2}}N^{n,0}_{\gamma_0+2,-1,0}$ the term $N^{n,0}_{\gamma_0+2,-3,0} \neq 0$ (The term $N^{n,0}_{\gamma_0+2,-2,1}$ does not contribute into $\mathcal{P}^{n,0}_{\gamma_0+3,-2,0}$ because of the bidegree $(n,0)$. Thus, $\delta_{\bar{2}} \geq 3$ that contradicts to (11). For the second and the third options $0 \neq \chi_1^2 N^{n,k}_{\gamma_0+2,\epsilon,0} \in \mathcal{N}$ ($\epsilon = 0, 1$). According to the weight-defect diagram, this implies that at least one of the polynomials $N^{n,k}_{\gamma_0+2,\epsilon,-2}$ or $N^{n,k}_{\gamma_0+2,\epsilon+1,-3}$ does not vanish. This means that $\delta_2 \leq -2$. Contradiction with (12). $\square$

Corollary 1 *The weight-defect diagram always separates into one-line diagrams.*

Lemma 5 *For any $\chi \in \mathfrak{aut}_{0,\mathrm{id}}$, the parameter $r = 0$.*

Proof. If $a = (a_1, a_2) = 0$ then $\Phi = \exp\chi$ is by Theorem 3 in [ES1] is a birational transformation

$$\Phi: \qquad \begin{aligned} z_1' &= \frac{z_1}{1-r\omega_1} \qquad & \omega_1' &= \frac{\omega_1}{1-r\omega_1} \\ z_2' &= \frac{z_2}{1-r\omega_2} \qquad & \omega_2' &= \frac{\omega_2}{1-r\omega_2}. \end{aligned}$$

Considering N_{γ_0+2} and N_{γ_0+4} we immediately conclude that, for $r \neq 0$, the transformation Φ changes the equation of M, so $\Phi \notin \mathrm{Aut}\, M$. For, say, $a_2 \neq 0$, according to the previous lemma, we have $a_1 = 0$. We now consider the next level of the weight-defect diagram, that corresponds to the weight $\gamma_0 + 4$. We

observe that the lowest weighted component in χ that depends on r is χ_2 with
$$2\,\mathrm{Re}\,\chi_2 = 2\,\mathrm{Re}\sum_{j=1,2} r_j z_j\omega_j\frac{\partial}{\partial z_j} + r_j\omega_j^2\frac{\partial}{\partial\omega_j},$$
where $r_1 = r = \bar r_2$. The operator $2\,\mathrm{Re}\,\chi_2$ preserves the space $\mathcal{N}$ and is, obviously, nondegenerate for $r \neq 0$. We distinguish two components of $2\,\mathrm{Re}\,\chi_2$, namely:
$$\chi_2^1 = r\omega_1 z_1\frac{\partial}{\partial z_1} + r\bar\omega_2\bar z_2\frac{\partial}{\partial\bar z_2} + r\omega_1^2\frac{\partial}{\partial\omega_1}$$
and
$$\chi_2^{\bar 1} = \bar r\bar\omega_1\bar z_1\frac{\partial}{\partial\bar z_1} + \bar r\omega_2 z_2\frac{\partial}{\partial z_2} + \bar r\omega_2^2\frac{\partial}{\partial\omega_2},$$
that, respectively, increase by 1 the degrees in (s,t) and $(\bar s,\bar t)$ variables. Starting with admissible $\mathfrak{N}^{n,k}_{\gamma_0+2,\Delta}$, provided by Theorem 2, we are going to show that it will be impossible to determine $\mathfrak{N}^{n+1,k}_{\gamma_0+3,\Delta-1}$ via the identity $\mathcal{P}^{n+1,k}_{\gamma_0+4,\Delta-1}\equiv 0$.
Without loss of generality we may assume that $\mathfrak{N}^{n+1,k}_{\gamma_0+3,\delta_1+j,\Delta-\delta_1-j-1} = 0$ for $j \geq 0$ because, otherwise we could have subtracted an appropriate solution of the homogeneous equation analogous to the upper line of the weight-defect diagram that we considered above. We see that the operator $\pi_*(\chi_2^1)$ increases the degree in s by one. Therefore, applied to $\mathfrak{N}^{n,k}_{\gamma_0+2,\delta_1,\Delta-\delta_1}$ it produces $\# s^4t^{n-3}$ for $\mathfrak{N}^{n,k}_{\gamma_0+2,\delta_1,\Delta-\delta_1}$ of the type (14), $\# s^3t^{n-2}$ for the type (15), $\# s^2t^{n-1}$ for (16) and $\# st^n$ for (17) in $\mathcal{P}^{n+1,k}_{\gamma_0+4,\delta_1,\Delta-\delta_1}$. Each of these terms belongs to the kernel of $\left(s\frac{\partial}{\partial s} - (\delta_1-1)\,\mathrm{id}\right)$, the operator representing the $\nwarrow$-arrow of the weight-defect diagram, that is applied to $\mathfrak{N}^{n+1,k}_{\gamma_0+3,\delta_1-1,\Delta-\delta_1}$ and points at $\mathcal{P}^{n+1,k}_{\gamma_0+4,\delta_1,\Delta-\delta_1}$. Thus we cannot solve the equation $\mathcal{P}^{n+1,k}_{\gamma_0+4,\delta_1,\Delta-\delta_1}\equiv 0$ because
$$\left(s\frac{\partial}{\partial s} - (\delta_1-1)\,\mathrm{id}\right)\mathfrak{N}^{n+1,k}_{\gamma_0+3,\delta_1-1,\Delta-\delta_1}$$
does not contain terms from the kernel of $\left(s\frac{\partial}{\partial s} - (\delta_1-1)\,\mathrm{id}\right)$ but on the other hand
$$\pi_*(\chi_2^1)\mathfrak{N}^{n,k}_{\gamma_0+2,\delta_1,\Delta-\delta_1}$$
does. Contradiction. Therefore, $r = 0$. $\square$

This lemma demonstrates that an isotropic automorphism of a non-quadratic elliptic manifold can be uniquely reconstructed by its Jacobian matrix at the origin.

Corollary 2 $\mathrm{Aut}_{0,\mathrm{id}}\, M \cong \{\mathrm{id}\}$ *or* $\mathbb{C}$.

Proof of Theorem 1 If $\mathfrak{aut}_{\,0,\mathrm{id}} \ni \chi \neq 0$ then one of the parameters a_1 or a_2 in χ does not vanish. Then the other one vanishes, and so does r. Then $\mathrm{Aut}_{0,\mathrm{id}} \cong \mathbb{C}$.
We conclude with a few questions that remain open. Can any isotropic automorphisms of a non-exceptional manifold be linearised in normal coordinates? Can the whole group $\mathrm{Aut}_0\, M$ be equivariantly linearised? Which "sprouts" from the list in Theorem 2 give rise to exceptional manifolds? Are there exceptional manifolds that are torsion-free in the sense of [SS] (In [EIS] also the notion semi-flat was used)?

10 Appendix

Below we display the list of all monomials belonging to $\mathcal{R}$. By Π we denote an arbitrary polynomial in the indicated variables. The index is the degree of the polynomial.

Bidegree in $(z, \bar{z})$	Monomial in $\mathcal{R}$
$(1,1)$	all but ν
$(2,1)$	$z_1 z_2 \bar{z}_1 \Pi(\eta,\bar{\eta})$, $\Pi_2(z)\bar{z}_2\Pi(\eta,\bar{\eta})$
$(1,2)$	$z_2 \bar{z}_1 \bar{z}_2 \Pi(\eta,\bar{\eta})$, $z_1 \Pi_2(\bar{z})\Pi(\eta,\bar{\eta})$
$(2,2)$	$\nu^2\Pi(\eta)$, $\nu\bar{\nu}\Pi(\eta,\bar{\eta})$
$(3,2)$	$z_1\nu^2\Pi(\eta)$
$(2,3)$	$\bar{z}_2\nu^2\Pi(\eta)$
$(3,3)$	$\nu^3\Pi(\eta)$
$(l,1)$ $l \geq 3$	$\Pi_l(z)\bar{z}_2\Pi(\eta,\bar{\eta})$
$(1,l)$ $l \geq 3$	$z_1\Pi_l(\bar{z})\Pi(\eta,\bar{\eta})$
$(l,0)$	$\Pi_l(z)\Pi(\eta,\bar{\eta})$
$(0,l)$	$\Pi_l(\bar{z})\Pi(\eta,\bar{\eta})$

Table 1.

By direct inspection one obtains a list of normal monomials that produce non-normal monomials after applying $\frac{\partial}{\partial \bar{z}_2}$ resp. $\frac{\partial}{\partial z_2}$. Just consider when the pre-image of a monomial from $\mathcal{R}$ does not belong to $\mathcal{R}$. We summarise the results in the following tables.

Monomial Ψ	$\frac{\partial}{\partial \bar{z}_2}\Psi$	Defects
$\Pi_k(z)\bar{z}_2^2\Pi(\eta,\bar{\eta})$ $\Pi_k \neq z_1$ $(\Pi_k,\Pi) \neq (z_1^2,\Pi(\eta))$ $(\Pi_k,\Pi) \neq (z_1^3,\Pi(\eta))$	$\Pi_k(z)\bar{z}_2\Pi(\eta,\bar{\eta})$	$d_1 \geq -2,\ d_{\bar{1}} \leq 0$ $d_1 = -2 + \deg_{z_1}\Pi_k$
$\bar{\nu}\bar{z}_2^2\Pi(\eta,\bar{\eta})$	$\bar{\nu}\bar{z}_2\Pi(\eta,\bar{\eta})$	$d_1 = -2,\ d_{\bar{1}} = 0$
$\nu\bar{\nu}\bar{z}_2\Pi(\eta,\bar{\eta})$	$\nu\bar{\nu}\Pi(\eta,\bar{\eta})$	$d_1 = -1,\ d_{\bar{1}} = 0$
$\bar{z}_2^2\nu^2\Pi(\eta)$	$\bar{z}_2\nu^2\Pi(\eta)$	$d_1 = -2,\ d_{\bar{1}} = 0$
$\bar{z}_2\nu^3\Pi(\eta)$	$\nu^3\Pi(\eta)$	$d_1 = -1,\ d_{\bar{1}} = 0$

Table 2.

Monomial Ψ	$\frac{\partial}{\partial z_2}\Psi$	Defects
$z_2\nu\Pi(\eta,\bar{\eta})$	$\nu\Pi(\eta,\bar{\eta})$	$d_1 = 0,\ d_{\bar{1}} = -1$
$z_2\bar{\nu}\Pi(\eta,\bar{\eta})$	$\bar{\nu}\Pi(\eta,\bar{\eta})$	$d_1 = 0,\ d_{\bar{1}} = -1$
$z_1z_2\bar{\nu}\Pi(\eta,\bar{\eta})$	$z_1\bar{\nu}\Pi(\eta,\bar{\eta})$	$d_1 = 1,\ d_{\bar{1}} = -1$
$z_2\bar{z}_2\bar{\nu}\Pi(\eta,\bar{\eta})$	$\bar{z}_2\bar{\nu}\Pi(\eta,\bar{\eta})$	$d_1 = -1,\ d_{\bar{1}} = -1$
$z_1z_2\Pi_k(\bar{z})\Pi(\eta,\bar{\eta})$ $\Pi_k \neq \bar{z}_1\bar{z}_2,\ k \geq 2$	$z_1\Pi_k(\bar{z})\Pi(\eta,\bar{\eta})$	$d_1 \leq 1,\ d_{\bar{1}} \geq -1$
$z_2\nu^2\Pi(\eta)$	$\nu^2\Pi(\eta)$	$d_1 = 0,\ d_{\bar{1}} = -1$
$z_2\nu\bar{\nu}\Pi(\eta,\bar{\eta})$	$\nu\bar{\nu}\Pi(\eta,\bar{\eta})$	$d_1 = 0,\ d_{\bar{1}} = -1$
$z_1z_2\nu^2\Pi(\eta)$	$z_1\nu^2\Pi(\eta)$	$d_1 = 1,\ d_{\bar{1}} = -1$
$z_2\bar{z}_2\nu^2\Pi(\eta)$	$\bar{z}_2\nu^2\Pi(\eta)$	$d_1 = -1,\ d_{\bar{1}} = -1$
$z_2\nu^3\Pi(\eta)$	$\nu^3\Pi(\eta)$	$d_1 = 0,\ d_{\bar{1}} = -1$
$z_2\Pi_k(\bar{z})\Pi(\eta,\bar{\eta})$ $\Pi_k \neq \bar{z}_1\bar{z}_2,\ k \geq 2$	$\Pi_k(\bar{z})\Pi(\eta,\bar{\eta})$	$d_1 \leq 0,\ d_{\bar{1}} \geq -1$

Table 3.

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