# p-adic L-functions for modular forms over CM fields

by

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## ABSTRACT

We construct many variabled S-adic L-functions for weight 2 modular forms over CM fields, S being a finite set of primes away from the conductor of our form. This S-adic L-function is given by a measure on the Galois group of the maximal unramified-outside-S abelian extension of our CM ground field. We obtain this measure by playing the modular symbol game in an adelic language.

In chapter § 0 we recall the adelic definition of a modular form and fix notations. In chapter § 1 we define the harmonic form on the symmetric space associated with our modular form. In chapter § 2 we study the "periods"; these are first defined via an adelic integral, than after Lemma 1, we transform it to an archimeadian integral, and finally after Lemma 2, we show it is given by an itegral of our harmonic form against a cycle. Besides giving us a geometrical intuition, we can deduce from this interpretation that the module generated by these periods is finitely generated. In chapter § 3 we prove the crucial "Birch Lemma", expressing the critical value of the associated L-function as a linear combination of the above periods. In chapter § 4, we construct for each ideal $r$ a distribution $\mu^{(r)}$ on $\prod_{p \in S} O^*_p / \bar{O}^*_k$ with values in a certain module- the module of universal modular symbols that are Hecke eigen-symbols. In chapter § 5 we specialize this universal distribution with our modular form, and avaraging over all ideal classes, we use class field theory to get our distribution on the Galois group. We prove that the S-adic L-function interpolates the critical values of the classical zeta

function of the twists of our modular form by finite characters of conductor supported at S, and that it satisfies a similar functional equation

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§ 0. Notations (mainly those of [W]).

$k$ denotes a CM field, i.e. a totally imaginary qudratic extension of a totally real number field. We denote by $\infty_1 \ldots \infty_n$ the non-conjugate embeddings of $k$ into $\mathbb{C}$, $[k:\mathbb{Q}] = 2n$. We denote by $P$'s the primes of $k$, and we denote by $v$'s the places of $k$ whether finite or not.

$O_k$ = integers of $k$.

$k_v$ = completion of $k$ at $v$

$O_p$ = integers of $k_p$

$$k_{fin} = k \otimes_{\mathbb{Z}} \varprojlim_{N} \mathbb{Z}/N\mathbb{Z} = \text{finite adeles}$$

$$k_\infty = k \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{j=1}^{n} k_{\infty j} = \text{infinite adeles}$$

$$k_A = k_{fin} \times k_\infty = \text{the adeles}$$

$k_{\infty j}^+$ = real and positive elements of $k_{\infty j}^*$

$$k_\infty^+ = \prod_{j=1}^{n} k_{\infty j}^+$$

$k_{\infty j}^{sgn}$ = elements of absolute value 1 in $k_{\infty j}^*$

$$k_\infty^{sgn} = \prod_{j=1}^{n} k_{\infty j}^{sgn} \quad \text{so that } k_\infty^* = k_\infty^+ \cdot k_\infty^{sgn}.$$

Let $\omega$ denote a grossencharacter of k, i.e. a continuous homomorphism $\omega: k_A^* \to \mathbb{C}^*$ of the idele group $k_A^*$ into $\mathbb{C}^*$, which is trivial on $k^*$. Let $F$ denote its conductor. We denote by $\underline{\omega}$ the associated multiplicative function on ideals defined by $\underline{\omega}(P) = 0$ if $P|F$, $\underline{\omega}(P) = \omega(\pi_p)$ if $P\nmid F$ where $\pi_p$ is a uniformizer of $k_p$. We let $\omega_v$ denote the restriction of $\omega$ to $k_v^* \subseteq k_v^*$.

Let $|x|_A = \prod_v |x|_v$ be the normalized absolute value of $x \in k_A^*$, we can write $|\omega(x)| = |x|_A^\sigma$ with $\sigma = \sigma(\omega) \in \mathbb{R}$.

We fix a character $\psi: k_A \to \mathbb{C}^*$ of the adeles $k_A$, trivial on $k$ for definiteness let us take $\psi = \prod_v \psi_v$ with $\psi_{\infty j}(x) = e^{-2\pi i(x+\bar{x})}$ and with $\psi_p$ given by $k$:

$$\psi_p: k_p \xrightarrow{tr} \mathbb{Q}_p \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{R}/\mathbb{Z} \xrightarrow{\exp +2\pi i x} \mathbb{C}^*$$

We let $\underline{\partial}$ denote an idele representing the absolute different $\mathcal{D}$ of k, i.e. the associated ideal $(\underline{\partial})$ is $\mathcal{D}$ and for $v\nmid\mathcal{D}$, including $v = \infty_j$, $\underline{\partial}_v = 1$. So that $\underline{\partial}_p^{-1}\mathcal{O}_p$ is the orthogonal complement of $\mathcal{O}_p$ with respect to the pairing $x,y \to \psi_p(xy)$. Similarly we let $\underline{f}$ denote an idele representing $F$, the conductor of $\omega$; and we let $\underline{a}$ denote an idele representing $a$, the conductor of our modular form F.

We let $G$ denote the algebraic group $GL(2)/k$.
We denote by $G_k$, $G_v$, $G_{fin}$, $G_\infty$, $G_A$ the points of $G$ with values in $k$, $k_v$, $k_{fin}$, $k_\infty$, $k_A$ respectively. $G_{fin}$ and $G_\infty$ are viewed as subgroups of $G_A = G_{fin} \times G_\infty$ and for $g \in G_A$ we write $g_{fin}$, $g_\infty$ for its $G_{fin}$ and $G_\infty$ components. $Z_k$, $Z_v$, $Z_{fin}$, $Z_\infty$, $Z_A$ denote the centers of the above groups. We let $B = \{(x,y) \overset{def}{=} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in G\} \tilde{=} \mathbb{G}_m \ltimes \mathbb{G}_a$ and $B_k$, $B_v$, $B_{fin}$, $B_\infty$, $B_A$ its rational points, thus e.g. $B_A \tilde{=} k_A^* \ltimes k_A$ is the "adelic half plane". As a general rule, whenever we are given an element $g = \{g_v\}$ defined for some set of v's of some group, we add units for all the missing v's;
We define our level groups by:

$$K_{\infty j} = SU(2, k_{\infty j})$$

$$K_p = \{\begin{pmatrix} x & \partial_p^{-1} y \\ \underline{a}_p \partial_p z & w \end{pmatrix},\ x,y,z,w \in O_p,\ \det = xw - \underline{a}_p yz \in O_p^*\}$$

$$K_{fin} = \prod_p K_p;\quad K_\infty = \prod_{j=1}^{n} K_{\infty j};\quad K_A = K_{fin} \times K_\infty .$$

We define a $\mathbb{C}$-vector space $V$, the value space of our forms, by:

| | |
|---|---|
| $V_p = \mathbb{C} \cdot v_p$ | one dimensional, |
| $V_{\infty j} = \mathbb{C} \cdot v_{\infty j}^{1} \oplus \mathbb{C}\, v_{\infty j}^{0} \oplus \mathbb{C} \cdot v_{\infty j}^{-1}$ | three dimensional, |
| $V = \bigotimes_v V_v$ | $3^n$-dimensional . |

Thus $V$ has the basis $V^{e_1 \ldots e_n} = \bigotimes_p^n V_{\infty_j}^{e_i}$, $1 \geq e_j \geq -1$.

We define a <u>right</u> action $M$ of $K_A Z_A$ on $V$ as follows: for $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in K_{\infty j}$, $|a|^2 + |b|^2 = 1$, we let $M_{\infty j}\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ act on $V_{\infty j}$ via the symmetric square representation:

$$M_{\infty j}\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ -a\bar{b} & |a|^2 - |b|^2 & \bar{a}b \\ \bar{b}^2 & -2\overline{ab} & \bar{a}^2 \end{pmatrix};$$

for $k = (k_{\infty j}) \in K_\infty$ we set $M(k) = \bigotimes_{j=1}^{n} M_{\infty j}(k_{\infty j})$; we extend this action to all of $K_A Z_A$ by setting $M(kz) = M(k_\infty)$, $k \in K_A$, $z \in Z_A$.

We define a function $W\colon k_\infty^* \to V$ as follows:

$$W(x) = \bigotimes_{j=1}^{n} W_{\infty j}(x_{\infty j})$$

$$W_{\infty j}(x) = W_{\infty j}^{1}(x)\cdot V_{\infty j}^{1} + W_{\infty j}^{0}(x)\cdot V_{\infty j}^{0} + W_{\infty j}^{-1}(x)\ V_{\infty j}^{-1}$$

$$W_{\infty j}^{0}(x) = |x|^2 \cdot K_0(4\pi|x|)$$

$$W_{\infty j}^{\pm 1}(x) = \frac{1}{2}\cdot \left[\frac{1}{i}\cdot \operatorname{sgn}(x)\right]^{\pm 1}\cdot |x|^2 \cdot K_{\pm 1}(4\pi|x|).$$

Here $\operatorname{sgn}(x) = \frac{x}{|x|}$ is the projection of $k_{\infty j}^*$ onto $k_{\infty j}^{\mathrm{sgn}}$. $K_0$, $K_1$ are Hankel's functions [F].

Let $F$ denote our modular form; $F$ is a continous function from $G_{\mathbb{A}} = B_{\mathbb{A}} Z_{\mathbb{A}} K_{\mathbb{A}}$ into $V$, such that $F(gkz) = F(g)M(k)$ for $k \in K_{\mathbb{A}}$, $z \in Z_{\mathbb{A}}$, and

$$\underline{W}F(g) \overset{\text{def}}{=} F\left(g\begin{pmatrix} 0 & -\partial^{-1} \\ \underline{\partial a} & 0 \end{pmatrix}_{\text{fin}}\right) = \varepsilon_F \cdot F(g),\ \varepsilon_F = \pm 1.$$

Assume: that $F$ is an eigenform of all the Hecke operators $T_P$. For $P \nmid a$ we have $T_P F = \lambda_P \cdot F$, and the Hecke operator is defined by $T_P F(g) = \int_{K_P(\pi_P,0)K_P} F(gk)dk$, where $\pi_P$ is a uniformizer of $k_P$, $dk$ is the Haar measure normalized such that $\int_{K_P} dk = 1$. Since $K_P(\pi_P,0)K_P =$

$\pi_P(\pi_P^{-1},0)K_P \cup \bigsqcup_{u \bmod P} (\pi_P, u\underline{\partial}_P^{-1})K_P$ we get:

$$T_P F(g) = F(g(\pi_P^{-1},0)) + \sum_{u \bmod P} F(g(\pi_P, u\underline{\partial}_P^{-1}))$$

Assume further that $F$ is cuspidal at infinity, $\int_{k_{\mathbb{A}}/k} F(x,y)dy = 0$ for all $x \in k_{\mathbb{A}}^*$ so that $F$ has a Fourier expansion at infinity of the form: $F(x,y) = \sum_{\xi \in k^*} C((\xi x))W(\xi x_\infty)\cdot\psi(\xi y)$. (this restriction can be dropped, but it will simplify things considerably). Let us write $L_F(\omega) = \sum_{b} C(b)\cdot\underline{\omega}(b)$ for the associated L-function, here the sum is extended over all ideal $b$, but $C(b) = 0$ if $b$ is not integral, and $\underline{\omega}(b) = 0$ if $b$ is not prime to $F$. Note that $C(P) = \lambda_P \cdot \mathbb{N}P^{-1}$. Since $F$ is a Hecke eigenform, $L_F(\omega)$ has an Euler product, $L_F(\omega) = \prod_P \mathcal{L}_P(\mathbb{N}P^{-1}\underline{\omega}(P))^{-1}$ with $\mathcal{L}_P(T) = 1 - \lambda_P T + \mathbb{N}PT^2 = (1-\rho_P T)(1-\tilde{\rho}_P T)$ for $P \nmid a$.

Note that as in [W], everything is normalized so that the functional equation for finite $\omega$ has the form $L_F(\omega) = (-1)^n \cdot \varepsilon_F \cdot \tau(\omega)^2 \cdot L_F(\omega^{-1})$, (i.e. the critical value is at "S = 0"); here the Gaussain sums $\tau(\omega)$ are defined as follows: $\tau(\omega) = \prod_P \tau_P(\omega)$, for $P \nmid F$ $\tau_P(\omega) = \omega_P(\underline{\partial})$, and for $P|F$:

$$\tau_P(\omega) = |\underline{f}|_P^{1/2} \sum_{x \in (O_P/\underline{f}_P O_P)^*} \omega^{-1}(x\underline{\partial}_P^{-1}\underline{f}_P^{-1})\psi_P(x\underline{\partial}_P^{-1}\underline{f}_P^{-1})$$

$$= (1 - NP^{-1})\,|\underline{f}_P|^{-\frac{1}{2}}\omega_P(\underline{\partial}\underline{f}) \int_{O_P^*} \omega_P^{-1}(x)\psi_P(x\underline{\partial}_P^{-1}\underline{f}_P^{-1})\,d^*x,$$

(the multiplicative Haar measure $d^*x$ being normalized by $\int_{O_P^*} d^*x = 1$)

## § 1. <u>Associated harmonic form</u> (cf. [K] and [W])

Let $r_i$, $i = 1,\dots,h$, denote a set of finite ideles representing the class group $\underline{C\ell}_k$ of $k$.

Let $X = G_k \backslash G_A / K_A Z_\infty$, we have a natural map $\det\colon X \twoheadrightarrow k_A^* / k^* \cdot \prod_p \mathcal{O}_p^* \cdot k_\infty = \underline{C\ell}_k$. Decomposing $X$ into the fibers of this map we get:

$$X = G_k \backslash G_A / K_A Z_\infty = G_k \Big\backslash \bigcup_{i=1}^{h} G_k (r_i, 0) K_{fin} G_\infty \Big/ K_A Z_\infty = \bigcup_{i=1}^{h} X^{(r_i)}$$

with $X^{(r_i)} = \Gamma^{(r_i)} \backslash G_\infty / K_\infty Z_\infty$, $\Gamma^{(r_i)} = G_k \cap ((r_i, 0) K_{fin} (r_i^{-1}, 0) \cdot G_\infty)$.

We shall next associate with $F$ a harmonic form $\Omega_F$ on $X$.

Let $\quad H = G_\infty / Z_\infty K_\infty$. $\quad$ We use the projection

$B_\infty^+ = \{(x,y) \in B_\infty$ with $x \in k_\infty^+\} \xrightarrow{\sim} H$, as identification, and thus we have a group structure on $H = k_\infty^+ \ltimes k_\infty$, and we have coordinates $(x,y)$ on $H$. We have a Riemannian structure on $H_{\infty j}$ given by $ds^2 = \frac{1}{x^2}(dx^2 + dy\,\overline{dy})$ and $G_\infty / Z_\infty$ acts on $H$ as a group of isometries;
we denote this action by $\gamma \circ h$, $\gamma \in G_{\infty j}$, $h \in H_{\infty j}$.
For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\infty j}$, $h = (x,y) \in H_{\infty j}$, we define:

$$J(\gamma,h) = \begin{pmatrix} \operatorname{sgn}(\gamma) \cdot (\overline{cy+d}) & -\operatorname{sgn}(\gamma) \cdot \overline{cx} \\ cx & (cy+d) \end{pmatrix} \in K_\infty Z_\infty$$

where $\quad \operatorname{sgn}(\gamma) = \operatorname{sgn}(\det \gamma) \in k_\infty^{sgn}$.

An easy calculation gives the automorphy relation:
$J(\gamma_1\gamma_2,h) = J(\gamma_1,\gamma_2\circ h)J(\gamma_2,h)$.

On $H$ we define an n-form with values in $V^*$ = vector space dual to $V$, by:

$$\beta = \sum_{\substack{e_1\ldots e_n \\ 1\geq e_j\geq -1}} \bigwedge_{j=1}^{n} \beta_{\infty_j}^{e_j} \cdot v_{e_1\ldots e_n}$$, where $\{v_{e_1\ldots e_n}\}$ is the dual basis of $\{v^{e_1\ldots e_n}\}$, and

$$\beta_{\infty_j}^{e_j} = -\frac{dy_{\infty_j}}{x_{\infty_j}} \quad \text{if} \quad e_j = 1\,,$$

$$\beta_{\infty_j}^{e_j} = \frac{dx_{\infty_j}}{x_{\infty_j}} \quad \text{if} \quad e_j = 0,$$

$$\beta_{\infty} = \frac{dy_{\infty_j}}{x_{\infty_j}} \quad \text{if} \quad e_j = -1\,.$$

$\beta$ is defined in this manner to ensure that $\beta|_\gamma(h) = \beta(h)\cdot {}^tM(J(\gamma,h))$ for $\gamma \in G_\infty$, $h \in H$.

Fix $r \in k^*_{fin}$ a finite idele. With our modular form F we associate an n-form on $H$ given by $\Omega_F^{(r)}(h) = F(h\begin{pmatrix} r & 0 \\ 0 & 1\end{pmatrix}_{fin})\cdot\beta(h)$
Let $\Gamma^{(r)} = G_k \cap ((r,0)K_{fin}(r^{-1},0)G_\infty)$ it's a congruence subgroup of $G_k$ which we view as a discrete subgroup of $G_\infty$. An easy calculation gives $F(\gamma\circ h(r,0)) = F(h(r,0))M(J(\gamma,h))^{-1}$ $\gamma \in \Gamma^{(r)}$, and hence $\Omega_F^{(r)}$ is $\Gamma^{(r)}$ invariant, and can be viewed as a form on $X^{(r)} = \Gamma^{(r)}\backslash H$.

(Note that $X^{(r)}$ is not a manifold because the elliptic elements in $\Gamma^{(r)}$ give whole geodesics that are singular. But we can always find a normal subgroup of finite index $\Gamma_0^{(r)} \subseteq \Gamma^{(r)}$ that has no torsion, Then $X_0^{(r)} = \Gamma_0^{(r)} \backslash H$ is a manifold and $X^{(r)}$ is the quotient of $X_0^{(r)}$ by the finite group $\tilde{\Gamma}^{(r)} = \Gamma^{(r)}/\Gamma_0^{(r)}$. We can now view $\Omega_F^{(r)}$ as a form on $X_0^{(r)}$ invariant under $\tilde{\Gamma}^{(r)}$).

Moreover, the properties of Hankel's functions, $K_0 = \frac{1}{x}K_1 + K_1'$, $K_1 = -K_0'$ imply that $\Omega_F^{(r)}$ is harmonic, hence can be viewed as an element of $H^n(X^{(r)},\mathbb{C})$ (i.e. as an element of $H^n(X_0^{(r)},\mathbb{C})^{\overline{\Gamma}^{(r)}}$). We finally define $\Omega_F$ on $X$ by $\Omega_F|_{X^{(r_i)}} = \Omega_F^{(r_i)}$, $i = 1\ldots h$

We let $\overline{H} = H \cup \mathbb{P}^1(k)$.
For $h = (x,y)$ we set: $|h|_\infty = \prod_{j=1}^{n} x_{\infty j}^{-1}$, the "distance" of $h$ from $\infty$; $|h|_\eta = \prod_{j=1}^{n} x_{\infty j}^{-1}(|\eta-y|_{\infty j}^2 + x_{\infty j}^2)$, the "distance" of $h$ from $\eta \in k$. The topology on $\overline{H}$ is defined by taking for neighborhoods of $\eta \in \mathbb{P}^1(k)$ the sets $\{\eta\} \cup \{h \in H|\ |h|_\eta < r\}$, for all $r > 0$. It is easy to see that this topology is separated and that the action of $G_k$ on $\overline{H}$ is continuous. We let $\overline{X}^{(r)} = \Gamma^{(r)}\backslash \overline{H}$.

Remark:
Because of the estimates of Hankel's functions we have: $|F(h(r,0))| = O(|h|_\eta^\sigma)$ for all $\sigma \in \mathbb{R}$ if and only if $F(h(r,0))$ is cuspidal at $\eta$. By using the fact that $F$ is cuspidal at $\infty$, one gets that for $(f) = \mathcal{F}$ prime to $\mathfrak{a}$ (=conductor of $F$), $\alpha \in \mathcal{O}_{\mathcal{F}}^* = \prod_{P|\mathcal{F}} \mathcal{O}_P^*$, $r \in k_{fin}^*$ prime to $\mathcal{F}$:
$|F(r\underline{\partial f}x, -\alpha)| = O(|x|^\sigma)$ as $|x| \to 0$ or $\infty$, for all $\sigma \in \mathbb{R}$.

## § 2. The periods $L(r,\eta)$ (cf. [K])

We fix Haar measure $d^*x = \underset{v}{\otimes}\, d^*x_v$ on $k_A^*$ normalized by: $\int_{\mathcal{O}_p^*} dx_p = 1$ and $d^*x_{\infty j} = \frac{|d\theta \wedge dr|}{r}$ where $x_{\infty j} = re^{i\theta}$ in polar-coordinates. We let $F_0 : G_A \to \mathbb{C}$ denote the $V^{0\ldots 0} = \bigotimes_{j=1}^{n} V_{\infty j}^{0}$-component of $F : G_A \to V$. For $r \in k_A^*$, $\eta \in k_{fin}$, we define (if convergent, e.g. by the remark at the end of § 1):

$$L(r,\eta) = \frac{1}{(\mathcal{O}^*:E)} \int_{k_\infty \cdot \prod_p \mathcal{O}_p^*/E} F_0(r\underline{\partial}x, -\eta)\, d^*x$$

where $E$ is any subgroup of totally positive units, of finite index in $\mathcal{O}^*$, satisfying the congruence conditions:

$$(1-\varepsilon)\eta \in r_{fin} \prod_p \mathcal{O}_p \quad \text{for all} \quad \varepsilon \in E.$$

Lemma 1:

(i) $L(r,\eta)$ depends only on the ideal $(r)$.

(ii) $L(r,\eta)$ depends only on the image $\eta \in k_{fin}/r_{fin} \prod_p \mathcal{O}_p$.

(iii) $L(r,\eta) = L(r\xi, \eta\xi)$ for $\xi \in k^*$.

Pf. (i) follows since $F(r\underline{\partial}x,-\eta) = F((r\underline{\partial}x,-\eta)(u,0)) = F(r\underline{\partial}ux,-\eta)$ for $u \in \prod_p \mathcal{O}_p^*$, and $F_0(r\underline{\partial}x,-\eta) = F(r\underline{\partial}ux,-\eta)$ for $u \in k_\infty^{sgn}$.

(ii) follows since $F(r\underline{\partial}x,-\eta) = F((r\underline{\partial}x,-\eta)(1,-r^{-1}\underline{\partial}^{-1}x^{-1}\mu))$ $F(r\partial x,-\eta-\mu)$ for $\mu \in r \prod_p O_p$.

(iii) follows since $F(r\underline{\partial}x,-\eta) = F((\xi,0)(r\underline{\partial}x,-\eta)) =$ $F(\xi r\underline{\partial}x,-\xi\eta)$.

Thus if $\eta \in k$, which by (ii) we may assume without loss of generality, we have

$$L(r,\eta) = \frac{1}{(O^*:E)} \int_{k_\infty^+ \prod_p O_p/E} F_0((1,\eta)(r\underline{\partial}x,-\eta_{fin}))d^*x =$$

$$= \frac{1}{(O^*:E)} \int_{k_\infty^+/E} F_0(r\underline{\partial}x,\eta_\infty)d^*x \quad \text{an archimedian integral.}$$

We shall next describe some relative cycles in $\bar{X}^{(r\underline{\partial})}$ against which integrating $\Omega_F^{(r\underline{\partial})}$ we shall obtain $L(r,\eta)$, thus justifying the name "periods" for $L(r,\eta)$. Let

$$\bar{I} = \{(t_0,t_1,\dots t_{n-1}) \mid 0 \leq t_0 \leq \infty,\ 0 \leq t_1 \dots t_{n-1} \leq 1\}$$

$$I = \{(t_0,t_1\dots t_{n-1}) \in \bar{I} \mid 0 < t_0 < \infty\}$$

$$I_0 = \{(0,t_1\dots t_{n-1}) \in \bar{I}\},\quad I_\infty = \{(\infty,t_1\dots t_{n-1}) \in \bar{I}\}$$

so that $\bar{I} = I_0 \cup I \cup I_\infty$. Fix a basis $\varepsilon_1 \dots \varepsilon_{n-1}$ for $E$, and define $x\colon I \to k_\infty^+$ by $x(t)_{\infty j} = t_0 \prod_{k=1}^{n-1} (\varepsilon_k^{(\infty j)})^{t_k}$, so that $k_\infty^+ = \bigsqcup_{\varepsilon \in E} \varepsilon \cdot x(I)$. For $\eta \in k$ we define an n-simplex

$c(E,\eta) : \overline{I} \to \overline{H}$ by

$$c(E,\eta)[t] = \begin{cases} \eta & t \in I_0 \\ (x(t),\eta) & t \in I \\ \infty & t \in I_\infty \end{cases}$$

It is easily seen that $c(E,\eta)$ is continuous on $\overline{I}$, and smooth on $I$. We have $(1,(\varepsilon_k - 1)\eta)\circ c(E,\eta)[\{t \in \overline{I} \mid t_k = 0\}] = c(E,\eta)[\{t \in \overline{I} \mid t_k = 1\}]$. Thus if

$c^{(r\underline{\partial})}(E,\eta) : \overline{I} \xrightarrow{c(E,\eta)} \overline{H} \xrightarrow{\text{proj}} \Gamma^{(r\underline{\partial})}\backslash\overline{H} = \overline{X}^{(r\underline{\partial})}$, than $c^{(r\underline{\partial})}(E,\eta)$ is a cycle in $\overline{X}^{(r\underline{\partial})}$ relative to the boundary $\partial\overline{X}^{(r\underline{\partial})} = \Gamma^{(r\underline{\partial})}\backslash\mathbb{P}^1(k)$; $c^{(r\underline{\partial})}(E,\eta) \in H_n(\overline{X}^{(r\underline{\partial})}, \partial\overline{X}^{(r\underline{\partial})}; \mathbb{Z})$. Moreover, $c^{(r\underline{\partial})}(\eta) \overset{\text{def}}{=} \frac{1}{(O^*:E)} c^{(r\underline{\partial})}(E,\eta) \in H_n(\overline{X}^{(r\underline{\partial})}, \partial\overline{X}^{(r\underline{\partial})}; \mathbb{Q})$ is independent of $E$.

<u>Lemma 2</u>: $L(r,\eta) = \int_{c^{(r\underline{\partial})}(\eta)} \Omega_F^{(r\underline{\partial})}$.

<u>Pf</u>. We have:

$$\int_{c^{(r\underline{\partial})}(\eta)} \Omega_F^{(r\underline{\partial})} \qquad \text{integration in } \overline{X}^{r\underline{\partial}}$$

$$= \frac{1}{(O^*:E)} \int_{c(E,\eta)} \Omega_F^{(r\underline{\partial})} \qquad \text{integration in } \overline{H}$$

$$= \frac{1}{(O^*:E)} \int_I F((x(t),\eta)_\infty(r\underline{\partial},0))\cdot(c(E,\eta)^*\beta)(t) \qquad \text{integration in } I$$

Note that all the "y-components" are constant, $y_{\infty j} = \eta_{\infty j}$, thus

$$= \frac{1}{(O^*:E)} \int_I F_0(r\underline{\partial}x(t),\eta_\infty)\frac{dx(t)}{x(t)} = \frac{1}{(O^*:E)} \int_{k_\infty^+/E} F_0(r\underline{\partial}x, -\eta_{fin})d^*x$$

$= L(r,\eta)$.

**Corollary:** The $\mathbb{Z}$-module generated by all $L(r,\eta)$'s is finitely generated.

**Pf.** $H_n(\overline{X},\partial\overline{X};\mathbb{Z})$ is finitely generated and the denominators $\frac{1}{(\mathcal{O}^*:E)}$ are bounded.

§ 3 Twists and Mellin transforms ([M]'s and [K]'s generalization of the basic idea of [B], which really goes back as far as Dirichlet...).

Let $\omega$ be a finite character, so that $\omega_\infty$ is trivial. Let $F$ be its conductor. Write $\omega^s(x) = \omega(x)\cdot|x|_A^s$, so that on ideals $\underline{\omega}^s(b) = \underline{\omega}(b)\cdot \mathbb{N}b^{-s}$. Define the twist $F^\omega(x) = \sum_{\xi\in k^*} c((\xi x))\underline{\omega}((\xi x))\cdot W(\xi x_\infty)$. Fix finite ideles $r_1\dots r_h$ representing $\underline{C\ell}_k$ such that $r_i$ is prime to $F$.

Lemma Let $\Pi_2(\omega^s)=((2\pi)^{-2}\,\Gamma(s+1))^{2n}$. For Re s large we have:

$$(4\pi)^{2n}\Gamma_2(\omega^s)^{-1}\cdot L_E(\omega^s) = \sum_{i=1}^{h} \mathbb{N}(r_i)^{-s}\,\frac{1}{(\mathcal{O}^*:E)}\int_{k_\infty^+/E} F_0^\omega(r_i x_\infty)\cdot|x_\infty|^s d^*x_\infty$$

Pf. An easy calculation gives

$$\int_{k_A^*/k^*} F^\omega(x)\cdot|x|_A^s d^*x = \sum_b C(b)\underline{\omega}^s(b)\cdot\int_{k_\infty^*} W(x_\infty)\cdot|x_\infty|^s d^*x_\infty\cdot$$

$$= L_F(\omega^s)\cdot\frac{1}{(8\pi)^n}\,\Pi_2(\omega^2)\cdot V^{0\dots 0}$$

Thus only the $V^{0\dots 0}$-component $F_0$ gives a contribution and we get:

$$(8\pi)^{-n}\cdot\Gamma_2(\omega^s)\cdot L_F(\omega^s) = \int_{k_A^*/k^*} F_0^\omega(x)\cdot|x|_A^s d^*x =$$

$$= \sum_{i=1}^{h} \mathbb{N}(r_i)^{-s}\int_{k_\infty^*/\mathcal{O}^*} F_0^\omega(r_i x_\infty)\cdot|x_\infty|^s d^*x_\infty =$$

$$= (2\pi)^n\cdot\sum_{i=1}^{h} \mathbb{N}(r_i)^{-s}\,\frac{1}{(\mathcal{O}^*:E)}\cdot\int_{k_\infty^+/E} F_0^\omega(r_i x_\infty)\cdot|x_\infty|^s d^*x_\infty$$

Here $E$ is any subgroup of totally real units of finite index in $\mathcal{O}^*$, but we shall consider only $E$ satisfying the congruence conditions of § 2 in all that follows.

Lemma  Let $r \in k^*_{fin}$ be prime to $F$, i.e. $r_P = 1$ for all $P|F$.

For $x \in k^*_\infty$ we have:

$$F^\omega(rx) = \tau(\omega)\mathbb{N}F^{-\frac{1}{2}} \sum_{\alpha\in(\mathcal{O}_F/\hat{F})^*} \omega(\alpha r\underline{\partial}^{-1}\underline{f}^{-1})F(rx,-\alpha_F\underline{\partial}^{-1}\underline{f}^{-1}).$$

Pf. An application of Fourier inversion gives for $\xi \in k^*$:

$$\underline{\omega}((\xi)) = \tau(\omega)\mathbb{N}F^{-\frac{1}{2}} \sum_{\alpha\in(\mathcal{O}_F/\hat{F})^*} \omega(\alpha\underline{\partial}^{-1}\underline{f}^{-1})\psi(-\alpha_F\underline{\partial}^{-1}\underline{f}^{-1}\xi)$$

And so we get:

$$F^\omega(rx) = \sum_{\xi\in k^*} C((\xi r))\underline{\omega}((r))\underline{\omega}((\xi))W(\xi x) =$$

$$= \tau(\omega)\mathbb{N}F^{-\frac{1}{2}} \sum_{\alpha\in(\mathcal{O}_F/\hat{F})^*} \omega(\alpha r\underline{\partial}^{-1}\underline{f}^{-1}) \sum_{\xi\in k^*} C((\xi r))W(\xi x)\psi(-\alpha_F\underline{\partial}^{-1}\underline{f}^{-1}\xi) =$$

$$= \tau(\omega)\mathbb{N}F^{-\frac{1}{2}} \sum_{\alpha\in(\mathcal{O}_F/F)^*} \omega(\alpha r\underline{\partial}^{-1}\underline{f}^{-1})F(rx,-\alpha_F\underline{\partial}^{-1}\underline{f}^{-1}).$$

Birch Lemma [B]: $L_E(\omega) = \tau(\omega)\mathbb{N}F^{-\frac{1}{2}}(4\pi)^{2n} \sum_{i=1}^{h} \sum_{\alpha\in(\mathcal{O}_F/\hat{F})^*} \omega(\alpha r_i)L(r_i\underline{f},\alpha_F)$

Pf. Combining the last two lemmas we get for Re s large:

$$\tau(\omega)^{-1} N F^{\frac{1}{2}} (4\pi)^{-2n} \Gamma_2(\omega^s) \cdot L_F(\omega)^s =$$

$$\sum_{i=1}^{h} N(r_i)^{-s} \sum_{\alpha \in (O_F/\hat{F})^*} \omega(\alpha r_i \underline{\partial}^{-1} \underline{f}^{-1}) \frac{1}{(O^*:E)} \int_{k_\infty^+/E} F_0(r_i x_\infty, -\alpha_F \underline{\partial}^{-1} \underline{f}^{-1}) \cdot |x_\infty|^s \cdot d^* x_\infty$$

by the remark at the end of § 1 the right hand side converges for all $s$, at $s = 0$ we get:

$$\tau(\omega)^{-1} N F^{\frac{1}{2}} (4\pi)^{-2n} \cdot L_F(\omega) =$$

$$= \sum_{i=1}^{h} \sum_{\alpha \in (O_F/\hat{F})^*} \omega(\alpha r_i \underline{\partial}^{-1} \underline{f}^{-1}) \cdot \frac{1}{(O^*:E)} \int_{k_\infty^+/E} F_0(r_i x_\infty, -\alpha_F \underline{\partial}^{-1} \underline{f}^{-1}) d^* x_\infty$$

let $\xi \in k^*$ be such that $(\xi)_F = (\underline{\partial}^{-1} \underline{f}^{-1})_F$, multiplying $\alpha$ by $\xi_F \underline{\partial f} \in O_F^*$ we continue the equality

$$= \sum_{i=1}^{h} \sum_{\alpha \in (O_F/\hat{F})^*} \omega(\alpha r_i (\underline{\xi \partial f})_F \underline{\partial}^{-1} \underline{f}^{-1}) \frac{1}{(O^*:E)} \int_{k_\infty^+/E} F_0(r_i x_\infty, -\alpha_F \xi) d^* x_\infty$$

multiply the argument of $F_0$ by $(\xi^{-1}, 0)$ and use left $G_k$-invariance, then put $\xi^{-1} x$ for $x_\infty$

$$= \sum_{i=1}^{h} \sum_{\alpha \in (O_F/\hat{F})} \omega(\alpha r_i (\xi \underline{\partial})_F \underline{\partial}^{-1}) \frac{1}{(O^*:E)} \cdot \int_{k_\infty^+/E} F_0(r_i \xi_{fin}^{-1} x_\infty, -\alpha_F) d^* x_\infty$$

substitute $r_i (\xi \underline{\partial})_F^{-1} \underline{\partial} \xi_{fin}$ for $r_i$ we finally get

$$= \sum_{i=1}^{h} \sum_{\alpha \in (O_F/\hat{F})^*} \omega(\alpha r_i) \frac{1}{(O^*:E)} \int_{k_\infty^+/E} F_0(r_i \underline{\partial f} x_\infty, -\alpha_F) d^* x_\infty$$

## § 4 Universal modular symbol and associated distribution

([M]'s adelization of [M,S-D]).

Let S denote a finite set of primes. Let $L(S)$ denote by the $\mathbb{Z}[\rho_P^{-1}; P \in S]$-module generated by the symbols $\mathbb{L}(r,\eta)$, $r \in k^*_{fin}$, $\eta \in \prod_{P \in S} k_P$, subjected to the following relations:

Rel(i) : $\mathbb{L}(r,\eta)$ depends only on the ideal (r).

Rel(ii) : $\mathbb{L}(r,\eta)$ depends only on the image of $\eta$ in $\prod_{P \in S} k_P / r_P O_P$.

Rel(iii): $\mathbb{L}(r,\eta) = \mathbb{L}(r\xi, r\xi)$ for $\xi \in k^*$.

For $p \in S$ define the operator $R_p{}^{-1}$ acting on $L(S)$ by $R_p{-1}\mathbb{L}(r,\eta) = \mathbb{L}(rP^{-1},\eta)$. For r prime to S, define the operator $U_p$ by $U_p\mathbb{L}(r,\eta) = \sum_{u \bmod} \mathbb{L}(rp,\eta+u)$, and extend this operator to all of $L(S)$ vial Rel(iii) (here and in the following, $\sum_{u \bmod}$ means that we sum over $u \in O_p$ running through a complete set of representatives for the residue field $k(P)$). It is easy to see that these operators are well defined. We let $L^*(S) = L(S)/(\lambda_p - R_{p-1} - U_p)L(S)$. For the formal convinience we also define $R_p\mathbb{L}(r,\eta) = \mathbb{L}(rp,\eta)$ whenever $\eta_p \varepsilon {}^{k_P}/r_P O_P$ was given by the context as $\eta_p \in {}^{k_P}/Pr_P O_P$, and similarly we let $\ell_u\, \mathbb{L}(r,\eta) = \mathbb{L}(r,\eta+u)$ for $u \in k_p$; these are not operators because we can possibly have e.g. $\mathbb{L}(r,\eta) = 0$, $R_p\mathbb{L}(r,\eta) \neq 0$ , so whenever we have an expression involving $R_p$'s, $\ell_u$'s, and $\mathbb{L}(r,\eta)$'s we first apply the $R_p$'s and the $\ell_u$'s and only then look at the image of the resulting expression in $L^*(S)$. Thus by abuse of language we have the following Hecke relations:

$$(*) \qquad \wp_P + \tilde{\wp}_P = \lambda_P = R_P^{-1} + R_P \cdot \sum_{u \bmod P} \ell_u$$

$$(**) \qquad \wp_P \cdot \tilde{\wp}_P = \mathbb{N}P = \sum_{u \bmod P} \ell_u$$

when applied to $\mathbb{L}(r,\eta)$ with $r$ prime to $P$ ; (*) is just the relation $\lambda_P = R_{P-1} + U_P$ , and (**) follows from Rel(ii), $\mathbb{L}(r,\eta+u) = \mathbb{L}(r,\eta+u')$ for any $u,u' \in O_P$ .

Fixing $r \in k^*_{fin}$ prime to $S$ we shall define an $L^*(S)$-valued distribution $\mu^{(r)}$ on $O^*_S = \prod_{P \in S} O^*_P$ , by giving its value on "elementary sets". We write $S = S_0 \cup S_1$, $F = \prod_{P \in S_1} P^{e_P}$, $e_P > 0$, and let $\eta \in O^*_F = \prod_{P \in S_1} O^*_P$ extended to $\eta \in O_S$ by decreeing that $\eta_P = 0$ for $P \in S_0$; we let $\eta + (F)^* \overset{def}{=}$

$\prod_{P \in S} O^*_P \times \prod_{P \in S_1} (\eta + P^{e_P} O_P) \subseteq O^*_S$ . Every open set in $O^*_S$ is a finite union of such elementary open sets $\eta + (F)^*$'s.

<u>Definition</u>

$$\mu^{(r)}\left(\eta + (F)^*\right) = \Big[\prod_{P \in S_0} (1 - \wp_P^{-1} R_P)\Big]\Big[\prod_{P \in S} (1 - \wp_P^{-1} R_P^{-1}) \cdot \wp_P^{-\mathrm{ord}_P F} R_P^{\mathrm{ord}_P F}\Big] \mathbb{L}(r,\eta).$$

This depends only on the image of $\eta$ in $O^*_F/(1+(F))$ by Rel (ii).

Theorem $\mu^{(r)}$ is indeed a distribution:

$$\mu^{(r)}\left(\bigcup_{i=1}^{N} U_i\right) = \sum_{i=1}^{N} \mu^{(r)}(U_i) \text{ for disjoint open sets } U_i \subseteq O^*_S .$$

Pf. It's enough to check that

$$\text{(I)} \qquad \sum_{\substack{\eta' \bmod FP \\ \eta' \equiv \eta \bmod F}} \mu^{(r)}(\eta' + (FP)) = \mu^{(r)}(\eta + (F))$$

for $P \in S$, $F$ divisible by all $P \in S$, $\eta \in O_S^*$; and to check that

$$\text{(II)} \qquad \sum_{\substack{u_i \bmod P_i \\ u_i \neq 0}} \mu^{(r)}\Big(\eta + \sum_{i=1}^{e} u_i + (F \prod_{i=1}^{e} P_i)\Big) = \mu^{(r)}(\eta + (F)^*)$$

where $S_0 = \{P_1, \ldots, P_e\}$, $\eta \in O_F^*$, with the above convention $\eta_{P_i} = 0$.

We begin with (I), so that $S_0 = \emptyset$. Let $(-1)^d$ denote the Möbius function : $(-1)^d = 0$ if $P^2 | d$ some $P$, and $(-1)^d = (-1)^{\#\{P|d\}}$ if $d$ is square free.

Extend $R_P$ and $\rho_P$ by multiplicativity $R_d = \prod_{P|d} R_P^{\mathrm{ord}_P d}$, $\rho_d = \prod_{P|d} \rho_P^{\mathrm{ord}_P d}$. Then $\mu^{(r)}(\eta + (F)) =$

$$[\prod_{P \in S} (1 - \rho_P^{-1} R_P^{-1})] \cdot \rho_F^{-1} R_F L(r,\eta) = [\rho_F^{-1} \sum_{d|F} (-1)^d \rho_d^{-1} R_{Fd^{-1}}] L(r,\eta).$$

Choose $\xi \in k^*$ such that $(\xi)_S = F$, where we write $(\xi) = (\xi)_S (\xi)^S$ with $(\xi)^S$ prime to $S$. Write

$(\eta') = \eta + \xi u$ with $u \in O_P$ running through a complete set of representatives for the residue field $k(P)$. We have:

$$\sum_{\substack{\eta' \bmod FP \\ \eta' \equiv \eta \bmod F}} \mu^{(r)}(\eta'+(FP)) = \sum_{u \bmod P} \rho_{FP}^{-1} \sum_{d|FP} (-1)^d \rho_d^{-1} R_{FPd^{-1}} L(r,\eta+u\xi) =$$

writing $\sum_{d|FP}$ as $\sum_{\substack{d|F \\ P\nmid d}} + \sum_{\substack{d|FP \\ P|d}}$ and substituting $dP$ for $d$

in the second sum

$$= \rho_F^{-1} \sum_{\substack{d|F \\ P\nmid d}} (-1)^d \rho_d^{-1} \sum_{u \bmod P} \{\rho_P^{-1} L(rFPd^{-1},\eta+u\xi) - \rho_P^{-2} L(rFd^{-1},\eta+u\xi)\} =$$

by Rel(iii) we can divide $\xi$ and get

$$= \rho_F^{-1} \sum_{\substack{d|F \\ P\nmid d}} (-1)^d \rho_d^{-1} \sum_{u \bmod P} \{\rho_P^{-1} L(rPd^{-1}(\xi^{-1})^S,\eta\xi^{-1}+u) - \rho_P^{-2} L(rd^{-1}(\xi^{-1})^S,\eta\xi^{-1}+u)\}$$

using Hecke relations (*) and (**) for the first and second terms in { } respectively

$$= \rho_F^{-1} \sum_{\substack{d|F \\ P\nmid d}} (-1)^d \rho_d^{-1} \{\rho_P^{-1}(\rho_P+\tilde{\rho}_P) L(rd^{-1}(\xi^{-1})^S,\eta\xi^{-1}) - \rho_P^{-1} L(rd^{-1}(\xi^{-1})^S P^{-1},\eta\xi^{-1}) - \rho_P^{-2}(\rho_P \cdot \tilde{\rho}_P) L(rd^{-1}(\xi^{-1})^S,\eta\xi^{-1})\}$$

canceling terms inside { }, and using Rel(iii), to multiply by $\xi$, we get

$$= \rho_F^{-1} \sum_{\substack{d|F \\ P\nmid d}} (-1)^d \rho_d^{-1} \{L(rFd^{-1},\eta) - \rho_P L(rFd^{-1}P^{-1},\eta)\} =$$

$$= [\rho_F^{-1} \sum_{d|F} (-1)^d \rho_d^{-1} R_{Fd^{-1}}] L(r,\eta) = \mu^{(r)}(\eta+(F)) .$$

As to (II) we have with $S_0 = \{P_i\text{'s}\}$:

$$\sum_{\substack{u_i \bmod P_i \\ u_i \neq 0}} \mu^{(r)}\Big(\eta + \sum_{i=1}^{e} u_i + \big(F \prod_{i=1}^{e} P_i\big)\Big) =$$

$$= \Big[\rho_{F\Pi P_i}^{-1} \cdot \sum_{d \mid F\Pi P_i} (-1)^d \rho_d^{-1} R_{F(\Pi P_i)d^{-1}} \prod_{P_i} \sum_{\substack{u_i \bmod P_i \\ u_i \neq 0}} \ell_{u_i}\Big] L(r,\eta)$$

$$= \Big[\rho_{F\Pi P_i}^{-1} \cdot \sum_{d \mid F} (-1)^d \rho_d^{-1} R_{Fd^{-1}} \sum_{J \subseteq S_0} (-1)^{\#J} \rho^{-1}_{\substack{\Pi P_i \\ P_i \in J}} R_{\substack{\Pi P_i \\ P_i \in S_0 \setminus J}} \prod_{P_i} \sum_{\substack{u_i \bmod P_i \\ u_i \neq 0}} \ell_{u_i}\Big] L(r$$

$$= \Big[\rho_{F\Pi P_i}^{-1} \cdot \prod_{P \mid F} (1 - \rho_P^{-1} R_{P^{-1}}) \cdot R_F \prod_{P_i} \{(R_{P_i} - \rho_{P_i}^{-1}) \sum_{\substack{u_i \bmod P_i \\ u_i \neq 0}} \ell_{u_i}\}\Big] L(r,\eta) =$$

using the Hecke relations (*) and (**) we get

$$= \Big[\rho_{F\Pi P_i}^{-1} \prod_{P \mid F} (1 - \rho_P^{-1} R_P^{-1}) \cdot R_F \prod_{P_i} \{(-R_{P_i} + \rho_{P_i} + \tilde\rho_{P_i} - R_{P_i}^{-1})$$

$$- \rho_{P_i}^{-1} (\rho_{P_i} \tilde\rho_{P_i} - 1)\}\Big] L(r,\eta) =$$

$$= \prod_{P \mid F} \Big[\rho_P^{-\mathrm{ord}_P F} (1 - \rho_P^{-1} R_P^{-1}) R_P^{\mathrm{ord}_P F}\Big] \prod_{P_i} \Big[(1 - \rho_{P_i}^{-1} R_{P_i}^{-1})(1 - \rho_{P_i}^{-1} R_{P_i})\Big] L(r,\eta) =$$

$$= \Big[\prod_{P \in S} (1 - \rho_P^{-1} R_P^{-1}) \rho_P^{-\mathrm{ord}_P F} R_P^{\mathrm{ord}_P F} \cdot \prod_{P \in S_0} (1 - \rho_P^{-1} R_P)\Big] L(r,\eta) =$$

$$= \mu^{(r)}(\eta + (F)^*).$$

q.e.d.

Note that by Rel(i) and Rel(ii) we have for $\varepsilon \in O^*$, $L(r,\varepsilon\eta) = L(\varepsilon^{-1}r,\eta) = L(r,\eta)$. Hence $\mu^{(r)}(\varepsilon\eta+(F)^*) = \mu^{(r)}(\eta+(F)^*)$, $\varepsilon \in O^*$, and we can view $\mu^{(r)}$ as a distribution on $\prod_{P\in S} O_P^*/\overline{O}_k^*$, where $\overline{O}_k^*$ denote the closure of $O_k^*$ in $O_S^*$.

## § 5 Measure associated to a modular form

Let $F$ denote a modular form and let $L(r,\eta)$'s denote its periods. Fix a finite set of finite places $S$ away from $\mathfrak{a}$= conductors of $F$. By the remark at the end of § 1 the periods $L(r,\eta)$ converge for $\eta \in O_S$, $r \in k^*_{fin}$, and by Lemma 1 of § 2 these periods satisfy Rel(i), Rel(ii), Rel(iii) of § 4. Moreover, since $F$ is assumed to be a Hecke eigenform we have for $P \in S$, and $r$ prime to $P$,

$$\lambda_P \cdot L(r,\eta) = \frac{1}{(O^*:E)} \int_{k_\infty^+ \prod_P O_P/E} T_P F_0(r\underline{\partial}x,-\eta)\, d^*x =$$

$$= \frac{1}{(O^*:E)} \int_{k_\infty^+ \prod_P O_P^*/E} \{F_0(r\underline{\partial}P^{-1}x,-\eta) + \sum_{u \bmod P} F_0(r\underline{\partial}Px,-\eta-u)\}\, d^*x =$$

$$= L(rP^{-1},\eta) + \sum_{u \bmod P} L(rP,\eta+u) = [R_P^{-1} + R_P \cdot \sum_{u \bmod P} \ell_u] L(r,\eta).$$

and so $L(r,\eta)$ satisfy the extra Hecke relation (*) of § 4. Thus we have a well define map $\mathbb{L}(r,\eta) \mapsto L(r,\eta)$, $L^*(S) \to L_{S,F}$, where $L_{S,F}$ is the $\mathbb{Z}[\rho_P^{-1}; P \in S]$-module generated by the periods $L(r,\eta)$'s, $r \in k^*_{fin}$, $\eta \in O_S$. The construction of § 4 gives now for every $r \in k^*_{fin}$ an $L_{S,F}$-valued distribution on $O_S^*/\overline{O_k^*}$.

Let $k(1)$ denote the Hilbert class field of $k$, and let $k(S)$ denote the maximal abelian extension of $k$ unramified outside $S$. By means of the Artin symbol we have isomorphisms

$$\begin{array}{ccc}
O_S^*/\overline{O^*} \cong k^*\prod\limits_P O_P^* k_\infty^* \Big/ \overline{k^* \prod\limits_{P\notin S} O^* k_\infty^*} & \xrightarrow{\simeq} & \mathrm{Gal}(k(S)/k(1)) \\
\downarrow & & \downarrow \\
k_A^* \Big/ \overline{k^* \prod\limits_{P\notin S} O_P^* k_\infty^*} & \xrightarrow{\simeq} & \mathrm{Gal}(k(S)/k) \\
\downarrow & & \downarrow \\
\underline{Cl}_k \cong k_A^*/k^* \prod\limits_P O_P^* k_\infty^* & \xrightarrow{\simeq} & \mathrm{Gal}(k(1)/k)
\end{array}$$

We use these isomorphisms as identifications, and define a distribution on $G_S = \mathrm{Gal}(k(S)/k)$, by $\mu_F = \sum_{i=1}^{h} \delta_{r_i} * \mu_F^{(r_i)}$, where $r_1 \dots r_n \in k_{fin}^*$ represents $\underline{d}_k$ and are prime to $S$; that is for a locally constant function $g$ on $G_S$, we have

$$\int_{G_S} g \, d\mu = \sum_{i=1}^{h} \int_{O_S^*/\overline{O_k^*}} g(r_i\eta)\, d\mu^{(r_i)}(\eta),$$

The distribution $\mu_F$ is determined by its values on finite characters $\omega$. Let $\mathbb{Z}[\omega]$ denote the ring obtained by adjoining to $\mathbb{Z}$ the values of $\omega$, and let $L_{S,F}[\omega] = \mathbb{Z}[\omega] \otimes L_{S,F}$ .

<u>Theorem</u> For a finite character, $\omega : G_S \to \mathbb{Z}[\omega]$, $F$ = conductor of $\omega$, we have in $L_{S,F}[\omega]$:

$$\int_{G_S} \omega \, d\mu = \prod_{P\in S} (1-\rho_P^{-1}\underline{\omega}(P))(1-\rho_P^{-1}\underline{\omega}^{-1}(P)) \cdot \frac{NF^{\frac{1}{2}} \rho_F^{-1}}{\tau(\omega)(4\pi)^n} \cdot L_F(\omega)$$

<u>Pf.</u> $$\int_{G_F} \omega \, d\mu_F = \sum_{i=1}^{h} \omega(r_i) \int_{O_S^*/O^*} \omega(\eta) d\mu^{(r_i)}(\eta) =$$

$$= \sum_{i=1}^{h} \sum_{\eta\in(O_F/\hat{F})^*} \omega(r_i\eta)\mu^{(r_i)}(\eta+(F)^*) =$$

$$= \sum_{i=1}^{h} \sum_{\eta\in(O_F/\hat{F})} \omega(r_i\eta)\cdot\rho_F^{-1} \sum_{\substack{d|\Pi P\\ P\in S}} (-1)^d \rho_d^{-1} R_d^{-1} \sum_{\substack{d'|\Pi P\\ P\in S_0}} (-1)^{d'} \rho_{d'}^{-1} R_{d'}\cdot R_F\cdot L(r,\eta$$

without loss of generality we may assume $(d,F) = 1$, otherwise we get a "denominator" $Fd^{-1}$ and by Rel (ii) of § 4, $L(rd'Fd^{-1},\eta)$ depends only on the image $\eta_0 \in (O_F/F(F,d)^{-1})^*$ of $\eta$, but $\sum_{\substack{\eta\in(O_F/\hat{F})^*\\ \eta\equiv\eta_0 (\mathrm{mod}\ F(F,d)^{-1})}} \omega(\eta) = 0$; thus the above is equal to

$$= \rho_F^{-1} \sum_{\substack{d,d'|\Pi P\\ p\in S_0}} (-1)^d(-1)^{d'}\rho_d^{-1}\rho_{d'}^{-1} \sum_{i=1}^{h} \sum_{\eta\in(O_F/\hat{F})^*} \omega(r_i\eta)L(r_i d'd^{-1}F,\eta) =$$

by Birch lemma the last sum is independent of the choice of $r_i$'s and we may replace $r_i$ by $r_i d'd^{-1}$ obtaining

$$= \rho_F^{-1} \sum_{\substack{d|\Pi P\\ P\in S_0}} (-1)^d \rho_d^{-1}\underline{\omega}(d) \sum_{\substack{d'|\Pi P\\ P\in S_0}} (-1)^{d'}\rho_{d'}^{-1}\underline{\omega}(d')^{-1}.$$

$$\cdot \sum_{i=1}^{h} \sum_{\eta\in(O_F/\hat{F})^*} \omega(r_i\eta)L(r_iF,\eta) =$$

$$= \rho_F^{-1} \prod_{P\in S_0} (1-\rho_P^{-1}\underline{\omega}(P))(1-\rho_P^{-1}\underline{\omega}(P)^{-1}) \sum_{i=1}^{h} \sum_{\eta\in(O_F/\hat{F})^*} \omega(r_i\eta)L(r_iF,\eta) =$$

$$= \rho_F^{-1} \prod_{P \in S} (1-\rho_P^{-1}\underline{\omega}(P))(1-\rho_P^{-1}\underline{\omega}^{-1}(P))(\tau(\omega)NF^{-\frac{1}{2}}(4\pi)^{2n})^{-1}L_F(\omega)$$

by Birch Lemma. q.e.d.

Assume that the $\rho_P$'s, $P \in S$, can be chosen to be $P$-units, hence S-units. Let $\hat{L}_{S,F} = 0_S \otimes L_{S,F}$ denote the S-adic completion of $L_{S,F}$. By the result of § 2, $L_{S,F}$ is a finitely generated $\mathbb{Z}[\rho_P^{-1}; P \in S]$-module, hence by the above assumption $\hat{L}_{S,F}$ is a finitely generated $0_S$-module; and so if $0_S[g]$ is an $0_S$-algebra. finitely generated as an $0_S$-module, we can associate to very continuous function $g: G_S \to 0_S[g]$ the well defined integral of $g$ with respect to $\mu_F$, $\int_{G_F} g d\mu_F \in \hat{L}_{S,F}[g] = 0_S[g] \otimes_{0_S} \hat{L}_{S,F}$. In particular, for any continuous S-adic character, $\omega: G_S \to 0_S[\omega]$, we can define the S-adic L-functions, $L_{F,S}(\omega) = \int_{G_S} \omega d\mu_F \in \hat{L}_{S,F}[\omega]$.

**Remark:** If the $\rho_P$'s were not S-adic units the $\mu_F$ defined above would still be a distribution but would not be bounded. Nevertheless, it would have "moderate growth" and hence any analytic function (e.g. an S-adic character) could be integrated against it. But continuous functions could not be integrated and our S-adic L-function would have infinitely many zeros, cf. [V].

**Theorem:** We have the functional equation

$$L_{F,S}(\omega) = (-1)^n \cdot \varepsilon_F \cdot \omega(\underline{a}) \cdot L_{F,S}(\omega^{-1})$$

**Pf.** One way of proving this is by using the functional equation for $L_F(\omega)$. For finite characters $\omega$ we have by the previous theorem

$$L_{F,S}(\omega) = \frac{L_F(\omega)}{\tau(\omega)} \cdot (\text{inv.})$$

where (inv.) denotes a term invariant under $\omega \mapsto \omega^{-1}$. Using now the functional equation for $L_F(\omega)$, $\omega$ finite and $\tau(\omega)\cdot\tau(\omega^{-1}) = 1$, we obtain the functional equation for $L_{F,S}(\omega)$ for finite $\omega$'s. Since the measure $\mu$ is determined by its values on finite $\omega$'s we obtain the functional equation for all $\omega$'s.

A more direct proof is as follows. By using the functional equation.

$$F\left(g\begin{pmatrix}0 & -\partial^{-1}\\ \underline{\partial}\,\underline{a} & 0\end{pmatrix}_{fin}\right) = \varepsilon_F \cdot F(g)$$

one obtains for $\underline{f}$ such that $(\underline{f}) = F$ is prime to $a$, $r \in k^*_{fin}$ prime to $F$, and $\eta \in O^*_F$

$$F(\underline{a}\underline{\partial}^2 r^{-1} D^2 x^{-1}, -D_0\underline{a}\, r^{-1} D)\cdot\begin{pmatrix}0 & 0 & 1\\ 0 & -1 & 0\\ 1 & 0 & 0\end{pmatrix}$$

On the $V^{0\ldots0}$- component this reads:

$$F_0(r\underline{\partial}\underline{f}x, -\eta) = (-1)^n\cdot\varepsilon_F\cdot F_0(\underline{a}r^{-1}\underline{\partial}\underline{f}x^{-1}, \eta^{-1}).$$

Integrating this over $k^+_\infty \cdot \prod_p O^*_p/E$ with respect to $d^*x$

we get

$$L(rF,\eta) = (-1)^n \cdot \varepsilon_F \cdot L(\underline{a}r^{-1}F,-\eta^{-1}).$$

Hence we obtain a functional equation for our measures

$$\mu^{(r)}(\eta) = (-1)^n \cdot \varepsilon_F \cdot \mu^{(r^{-1}\underline{a})}(-\eta^{-1})$$

from which the functional equation for $L_{F,S}(\omega)$ follows immediately. q.e.d.

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