EXPLICIT CALCULATION OF THE MASLOV-TYPE INDICES OCCURRING IN GUTZWILLER'S TRACE FORMULA; APPLICATION TO THE METAPLECTIC GROUP

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Abstract

The index appearing in Gutzwiller's trace formula for quantum systems with classical chaotic analogue is the Conley–Zehnder index for symplectic paths, familiar from the theory of periodic Hamiltonian orbits. That index is notoriously difficult to calculate for repetitions of prime periodic orbits. In this paper we use recent results of ours (the product formula and the relation with Morse's index of concavity) to give explicit and rigorous results. We apply our calculations to give, in addition, the Weyl representation of some metaplectic operators.

Introduction

Gutzwiller's formula and the Conley–Zehnder index

The physicist M.C. Gutzwiller [17] wrote down in the early 1970s a beautiful formula for calculating the semiclassical energy levels of a quantum system with chaotic classical analogue; this formula links the eigenvalues of the Hamiltonian operator with the periodic orbits of the corresponding classical system. Gutzwiller's approach was heuristic, and posed several difficult mathematical problems. Major contributions towards a rigorous proof and a better understanding of the mathematics underlying Gutzwiller's formula are nowadays to be found in the literature (see below for a non-exhaustive list of references). There is, however, a delicate point that has only been addressed by very few mathematicians (it has been addressed by many physicists, but unfortunately not in a very conclusive way). It is that of the precise definition of the "Maslov-type index" appearing in Gutzwiller's formula. Let me briefly recall what this formula looks like; my presentation will be formal and I do not claim any rigor at this point. Assuming that the Hamiltonian operator $\widehat{H}$ associated to the classical Hamiltonian H has a discrete sequence $E_0, E_1, ...E_j, ...$ of eigenvalues, one wants to evaluate the density of states $\rho(E) = \sum_{j=0}^{\infty} \delta(E - E_j)$. Gutzwiller's formula says that when $\hbar \to 0$ that density is approximated, up to the so-called "Weyl term" $\langle \rho(E) \rangle$, which counts the number of states in the phase-space region $H(z) \leq E$, by the sum

$$\frac{1}{\pi\hbar} \operatorname{Re} \sum_{\gamma} i^{-\sigma_\gamma} \frac{T_\gamma e^{\frac{i}{\hbar}\mathcal{A}_\gamma}}{\sqrt{|\det(P_\gamma - I)|}} \tag{1}$$

where γ runs over the set of all periodic orbits of H with energy E and period T_γ (including their repetitions). This set is assumed to be discrete, and to consist only of non-degenerate orbits so that the linearized Poincaré map P_γ for γ has no eigenvalue equal to one; $\mathcal{A}_\gamma$ is the classical action $\oint_\gamma pdx$. It is the number σ_γ that poses problems, and that will preoccupy us in this article. We mention that the nature of σ_γ is not even addressed in the otherwise excellent paper of Paul and Uribe [33], and is called "Maslov index" in Combescure *et al.* [4] (but this terminology is misleading because the consideration of simple examples shows that the index σ_γ does not have in general the additivity property $\sigma_{\gamma^{*(N)}} = N\sigma_\gamma$ enjoyed by *bona fide* Maslov indices of loops). It is indubitably the merit of Meinrenken [27, 28, 29] to have given the correct answer, and recognized that σ_γ is the *Conley–Zehnder* index $i_{\mathrm{CZ}}(\widetilde{P}_\gamma)$ of a certain symplectic path $\widetilde{P}_\gamma$ associated with the Poincaré map P_γ:

$$\sigma_\gamma = i_{\mathrm{CZ}}(\widetilde{P}_\gamma). \tag{2}$$

Discussion of some "known" results in the physical literature

There are many partial results in the literature about the calculation of the "Maslov type index" appearing in Gutzwiller's formula. For instance, Sugita [35] states correct results, but his approach seems to be *ad hoc* (he does not to give any clear and easily understandable prescription for obtaining his indices); in addition Sugita uses illicit mathematical methods such as divergent determinants of infinite matrices (some of his considerations however certainly deserve to be put on a rigorous mathematical basis). We note that Brack and Pletyukhov [2] have commented upon Sugita's results and given a tentative justification of them. We note that in Robbins [32] one can find valuable partial results; this also true of Creagh *et al.* [6] whose discussion of Lagrangian manifolds associated to some periodic orbits is very interesting, but their papers do not give a simple and universal prescription for the practical determination of the indices in all cases. We mention that Muratore-Ginanneschi [31] has also identified σ_γ as being the Conley–Zehnder index, which he relates to an older index constructed by Gelfand and Lidskii [8]; however, as is the case for Sugita, Muratore-Ginanneschi makes use of Feynman path integral techniques are not easy to justify mathematically.

Our review would not be complete if we didn't mention that deep and rich results for the Conley–Zehnder indices have been obtained by Long and his school (see for instance the papers [23, 24, 25] and the references therein); these results are of great theoretical interest but it seems however that they are not directly usable in our context.

Aims and scope of this work

This paper consists of two parts. In the first part (Sections 1–3) we review in some detail the theory of the Maslov and Conley–Zehnder indices, and of the extension of the latter which we have proposed in [14] (see [15] for a review). In addition we discuss two of the most useful consequences of our redefinition, the product formula

$$i_{\mathrm{CZ}}(\Sigma\Sigma') = i_{\mathrm{CZ}}(\Sigma) + i_{\mathrm{CZ}}(\Sigma') + \tfrac{1}{2}\operatorname{sign}(M_S + M_{S'})$$

(M_S the symplectic Cayley transform of the endpoint of Σ) and the formula

$$i_{\mathrm{CZ}}(\Sigma) = \mathrm{Mas}(\Sigma) - \operatorname{Inert} W_{xx}$$

relating the Conley–Zehnder index to the Maslov index and to Morse's index of concavity; these formulae will considerably simplify the practical calculations in most cases. We think this review is useful since the theory of the Conley–Zehnder index is perhaps not universally known outside a handful of experts working in the area of dynamical systems or symplectic topology. In the second part (Sections 4 and **??**) we set out to give explicit formulae for the calculation of the Conley–Zehnder index of periodic Hamiltonian orbits and of their repetitions in the most important cases; we use –as Sugita does in [35]– reduction to normal forms of the endpoints of the symplectic paths we consider. The interest of our calculations certainly goes well beyond their applications to Gutzwiller's formula, since they can be used in Hamiltonian mechanics and Morse theory (for which the Conley–Zehnder index was originally designed!) We finally apply our calculations to the metaplectic group $\mathrm{Mp}(n)$, and give some explicit formulae for the Weyl representation of some useful metaplectic operators.

Notation 1 *Symplectic spaces in general are denoted by (Z, ω); we reserve the notation σ for the standard symplectic form on $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$. By definition $\sigma(z, z') = p \cdot x' - p' \cdot x$ if $z = (x, p)$, $z' = (x', p')$. The standard symplectic matrix is $J = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{pmatrix}$ and we have $\sigma(z, z') = z' J z^T$.*

1 The Maslov index for Symplectic Paths

The study of symplectic paths imposes itself quite naturally when one studies the periodic orbits of Hamiltonian systems. For a detailed treatment and extensive references for the topics of this section and the following see [13] or our recent review [15].

1.1 Hamiltonian periodic orbits

Let H be a Hamiltonian function on $\mathbb{R}^{2n} \times \mathbb{R}$; we assume that $H \in C^{\infty}(\mathbb{R}^{2n} \times \mathbb{R})$. We denote by $(\phi^H_{t,t'})$ the time-dependent flow determined by Hamilton's equations $\dot{z} = J\partial_z H$: for fixed t' the function $t \longmapsto \phi^H_{t,t'}(z')$ is the solution curve passing through z' at time t'. We will set set $\phi^H_t = \phi^H_{t,0}$; when H is time-independent (ϕ^H_t) is the usual Hamiltonian flow. Let $z_0 \in \mathbb{R}^{2n}$ and $\tau > 0$ be such that $\phi^H_\tau(z_0) = z_0$ and set $\gamma^H_{z_0}(t) = \phi^H_t(z_0)$ for $t \in [0,\tau]$: $\gamma^H_{z_0}$ is the periodic orbit of H with origin z_0, we have $\gamma^H_{z_0}(0) = \gamma^H_{z_0}(\tau)$. The Jacobian matrix $S_t(z_0) = D\phi^H_t(z_0)$ is symplectic and satisfies the "variational equation"

$$\frac{d}{dt}S_t(z_0) = JD^2H(\phi^H_t(z_0),t)S_t(z_0) \quad , \quad S_0(z_0) = I \tag{3}$$

(D^2H the Hessian matrix of H). The matrix $S_\tau(z_0)$ is the *monodromy matrix* of the periodic orbit $\gamma^H_{z_0}$, and we have $S_{t+\tau}(z_0) = S_t(z_0)S_\tau(z_0)$ for all $t \in \mathbb{R}$. Observe that if the Hamiltonian is time-independent then $S_\tau(z_0)$ always has at least one (and hence at least two) eigenvalues equal to one: since $\phi^H_t \circ \phi^H_{t'} = \phi^H_{t+t'}$ we have

$$\tfrac{d}{dt'}\phi^H_t(\phi^H_{t'}(z_0))|_{t'=0} = S_t(z_0)X_H(z_0) = X_H(\phi^H_t(z_0));$$

Setting $t = \tau$ we get $S_\tau(z_0)X_H(z_0) = X_H(z_0)$ hence $X_H(z_0)$ is an eigenvector of $S_\tau(z_0)$ with eigenvalue one; the multiplicity of this eigenvalue is at least two since the eigenvalues of a symplectic matrix occur in quadruples $(\lambda, 1/\lambda, \bar{\lambda}, 1/\bar{\lambda})$.

Let $\gamma^H_{z_0}$ be a periodic orbit. When t goes from 0 to τ the Jacobian matrix $S_t(z_0) = D\gamma^H_{z_0}$ describes a path $\Sigma(z_0) \in \mathrm{Sp}(n)$ starting from the identity $I \in \mathrm{Sp}(n)$. It is customary to associate to that path two indices:

- The *Maslov index*, whose vocation is to count (algebraically) the number of times $\Sigma(z_0)$ intersects the "caustic" of $\mathrm{Sp}(n)$ consisting of all $S \in \mathrm{Sp}(n)$ such that $S\ell_P \cap \ell_P \neq 0$ where ℓ_P is the vertical Lagrangian plane $0 \times \mathcal{R}^n$;

- The *Conley–Zehnder* index, whose vocation is to give a count of the number of times the path $\Sigma(z_0)$ intersects the "Maslov cycle", which is the set of all $S \in \mathrm{Sp}(n)$ such that $\det(S - I) = 0$.

We review both notions below; we will see that the Conley–Zehnder index can in fact be interpreted as a Maslov index, not for the path $\Sigma(z_0)$ but rather for the path $I \oplus \Sigma(z_0)$ in a certain symplectic group associated to a $4n$-dimensional symplectic space (formula (15)).

1.2 The Maslov index on $\mathrm{Sp}(n)$

The Maslov index –or "Keller–Maslov" index as it should be more appropriately called– is one of the oldest intersection indices appearing in symplectic geometry. Its vocation is to give an algebraic count of the number of times a symplectic path intersects a certain variety of codimension one. See [11, 13, 21] for details and historical accounts.

1.2.1 The Maslov index for loops

We begin by constructing a continuous mapping $\mathrm{Sp}(n) \longrightarrow S^1$ that will be instrumental for the definition of both the Maslov and Conley–Zehnder indices. Let $U(n) = \mathrm{Sp}(n) \cap O(2n)$ the group of symplectic rotations. We have $U \in U(n)$ if and only if $U = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ and $A + iB \in U(n, \mathbb{C})$. We define a mapping $\rho : U(n) \longrightarrow S^1$ by the formula

$$\rho(U) = \det\nolimits_{\mathbb{C}}(U) = \det(A + iB). \tag{4}$$

We extend the mapping ρ into a mapping $\mathrm{Sp}(n) \longrightarrow S^1$ using polar decomposition: for $S \in \mathrm{Sp}(n)$ we write $S = UP$ where

$$U = S(S^T S)^{-1/2} \ , \ P = (S^T S)^{1/2}. \tag{5}$$

Both matrices U and P are symplectic: this is obvious for P since it is the square root of a positive definite symplectic matrix, hence $U = SP^{-1}$ is also symplectic. In addition $UJ = JU$ so that U is of the type $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ and we have $u = A + iB \in U(n, \mathbb{C})$.

Let now $\Gamma : [0, 1] \longrightarrow \mathrm{Sp}(n)$ be a loop in $\mathrm{Sp}(n)$: $\Gamma(0) = \Gamma(1)$. By definition, the Maslov index of Γ is the degree of the mapping $\rho \circ \Gamma : t \longmapsto \rho(\Gamma(t))$:

$$\mathrm{Mas}(\Gamma) = \deg(\rho \circ \Gamma). \tag{6}$$

It is thus explicitly calculated as follows: choose a continuous function $\theta : [0, 1] \longrightarrow \mathbb{R}$ such that $\rho(\Gamma(t)) = e^{i\theta(t)}$; then

$$\mathrm{Mas}(\Gamma) = \frac{\theta(1) - \theta(0)}{2\pi}. \tag{7}$$

Here is a basic example: the fundamental group $\pi_1(\mathrm{Sp}(n))$ is generated by the loop α defined by

$$\alpha(t) = e^{2\pi t J_1} \oplus I_{2n-2} \ \ , \ \ 0 \leq t \leq 1$$

where $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and I_{2n-2} is the identity on $\mathbb{R}^{2n-2}$ (this notation corresponds to a rearrangement of the coordinates of (x, p) as $(x_1, p_1, ..., x_n, p_n)$). Let r be an integer and denote by α^r the r-th repetition of α. We have

$$\mathrm{Mas}(\alpha^r) = r \tag{8}$$

hence $\mathrm{Mas} : \pi_1(\mathrm{Sp}(n)) \longrightarrow (\mathbb{Z}, +)$ is a group isomorphism.

1.2.2 The Maslov index for symplectic paths

The Maslov index defined above can be generalized to arbitrary paths starting from the identity. Let $\mathcal{C}(\mathrm{Sp}(n))$ be the set of all continuous paths $\Sigma : [0, 1] \longrightarrow$

$\mathrm{Sp}(n)$ such that $\Sigma(0) = I$. The Maslov index of $\Sigma \in \mathcal{C}(\mathrm{Sp}(n))$ is by definition the number

$$\mathrm{Mas}(\Sigma) = \frac{1}{2}(\mu(\Sigma\ell_P, \ell_P) + n) \tag{9}$$

where $\ell_P = 0 \times \mathbb{R}^n$ and μ is the Leray index on the Lagrangian Grassmannian $\mathrm{Lag}(n)$ of the standard symplectic space $(\mathbb{R}^{2n}, \sigma)$ (see de Gosson [10, 11, 13] where we extend Leray's [21] original definition, and apply our results to a general theory of Lagrangian and symplectic intersection indices). When the endpoint $\Sigma(1)$ is a free symplectic matrix, *i.e.* if $\Sigma(1) \cap \ell_P \cap \ell_P = 0$ (also see Subsection 2.2.4) then $\mathrm{Mas}(\Sigma) \in \mathbb{Z}$. We will need the three following results for our calculations in Section 3;

Example 2 *Assume $n = 1$; the formulae below all readily follow from (9):*

(i) ***Parabolic endpoint:*** *The path Σ is given by* $\Sigma(t) = \begin{pmatrix} 1 & \gamma t \\ 0 & 1 \end{pmatrix}$, $0 \leq t \leq 1$ *and $\gamma \neq 0$. We have*

$$\mathrm{Mas}(\Sigma) = 1 - \operatorname{sign}\gamma \tag{10}$$

where $\operatorname{sign}\gamma = 1$ *if* $\gamma > 0$ *and* $\operatorname{sign}\gamma = -1$ *if* $\gamma < 0$*;*

(ii) ***Elliptic endpoint****: The path Σ is given by* $\Sigma(t) = \begin{pmatrix} \cos\alpha t & \sin\alpha t \\ -\sin\alpha t & \cos\alpha t \end{pmatrix}$, $0 \leq t \leq 1$ *and $\alpha > 0$. In this case*

$$\mathrm{Mas}(\Sigma) = -\left[\frac{\alpha}{\pi}\right] \tag{11}$$

where $[\cdot]$ is the integer part function.

(iii) ***Hyperbolic endpoint****: The path Σ is given by* $\Sigma(t) = \begin{pmatrix} \cosh\beta t & \sinh\beta t \\ \sinh\beta t & \cosh\beta t \end{pmatrix}$, $0 \leq t \leq 1$ *and $\beta > 0$. Then*

$$\mathrm{Mas}(\Sigma) = 0. \tag{12}$$

2 The Conley–Zehnder index

The Conley–Zehnder is an index of symplectic paths generalizing the usual Morse index for closed geodesics on Riemannian manifolds. See Conley and Zehnder [5] and Hofer *et al.* [18]. In Subsection 2.2 we will extend this definition to paths whose endpoints are arbitrary –this extension is crucial for the practical calculations that will be done in this paper. (For those interested in recent developments in the theory of the Conley–Zehnder index, see [16].)

2.1 Definition of i_{CZ} and first properties

2.1.1 The index i_{CZ} on $\mathcal{C}^{\pm}(\mathrm{Sp}(n))$

Recall that $\mathcal{C}(\mathrm{Sp}(n))$ is the set of all continuous paths $[0, 1] \longrightarrow \mathrm{Sp}(n)$ starting from the identity. We denote by $\mathcal{C}^+(\mathrm{Sp}(n))$ (resp. $\mathcal{C}^-(\mathrm{Sp}(n))$) the subset of

$\mathcal{C}(\mathrm{Sp}(n))$ consisting of the paths with endpoint $\Sigma(1)$ either one of the sets $\mathrm{Sp}^{+}(n)$ or $\mathrm{Sp}^{-}(n)$ defined by

$$\begin{aligned}\mathrm{Sp}^{+}(n) &= \{S : \det(S-I) > 0\}\\ \mathrm{Sp}^{-}(n) &= \{S : \det(S-I) < 0\}.\end{aligned}$$

Here are two important properties of the sets $\mathrm{Sp}^{\pm}(n)$:

Sp1 $\mathrm{Sp}^{+}(n)$ and $\mathrm{Sp}^{-}(n)$ are arcwise connected;

Sp2 Every loop in $\mathrm{Sp}^{\pm}(n)$ is contractible to a point in $\mathrm{Sp}(n)$.

The complement of $\mathrm{Sp}^{+}(n) \cup \mathrm{Sp}^{-}(n)$ in $\mathrm{Sp}(n)$ is the set of all symplectic matrices having at least one eigenvalue equal to one; it is an algebraic variety with codimension 1. It is denoted by $\mathrm{Sp}^{0}(n)$ and sometimes called the "Maslov cycle" (the terminology is however also often used in the literature to denote other subsets of $\mathrm{Sp}(n)$, for instance the "caustic" consisting of symplectic matrices which are not free). We will write:

$$\mathcal{C}^{\pm}(\mathrm{Sp}(n)) = \mathcal{C}^{+}(\mathrm{Sp}(n)) \cup \mathcal{C}^{-}(\mathrm{Sp}(n)).$$

Let us introduce the following symplectic matrices S^{+} and S^{-}:

- $S^{+} = -I$; we have $S^{+} \in \mathrm{Sp}^{+}(n)$
- $S^{-} = \begin{pmatrix} L & 0 \\ 0 & L^{-1} \end{pmatrix}$ where $L = \mathrm{diag}(2, -1, ..., -1)$; we have $S^{-} \in \mathrm{Sp}^{-}(n)$.

Let ρ be the mapping $\mathrm{Sp}(n) \longrightarrow S^{1}$ defined in previous subsection. We obviously have $\rho(S^{+}) = (-1)^{n}$ and $\rho(S^{-}) = (-1)^{n-1}$.

We now have all we need to define the Conley–Zehnder index of a path $\Sigma \in \mathcal{C}^{\pm}(\mathrm{Sp}(n))$. We extend Σ into a path $\widetilde{\Sigma} : [0,2] \longrightarrow \mathrm{Sp}(n)$ by setting

$$\widetilde{\Sigma}(t) = \begin{cases} \Sigma(t) & \textit{if} \quad 0 \leq t \leq 1 \\ \Sigma'(t) & \textit{if} \quad 1 \leq t \leq 2 \end{cases}$$

where $\widetilde{\Sigma}(2) = S^{+}$ if $\Sigma(1) \in \mathrm{Sp}^{+}(n)$ and $\widetilde{\Sigma}(2) = S^{-}$ if $\Sigma(1) \in \mathrm{Sp}^{+}(n)$. When t varies from 0 to 2 the complex number $\rho(\widetilde{\Sigma}(t))$ varies from 1 to ± 1 hence $(\rho(\widetilde{\Sigma}(t)))$ describes a loop in S^{1}. By definition, the Conley–Zehnder index of Σ is the integer

$$i_{\mathrm{CZ}}(\Sigma) = \deg(\rho^{2} \circ \Gamma).$$

That this definition does not depend on the choice of the extension $\widetilde{\Sigma}$ follows from property (Sp2) of $\mathrm{Sp}^{\pm}(n)$.

Here is an elementary example; we will use it in Section 3 for the calculation of the index of a path with loxodromic endpoint:

Example 3 *Define a symplectic path by*

$$\Sigma_\alpha(t) = \begin{pmatrix} R_\alpha(t) & 0 \\ 0 & R_\alpha(t) \end{pmatrix} \text{ , } R_\alpha(t) = \begin{pmatrix} \cos\alpha t & \sin\alpha t \\ -\sin\alpha t & \cos\alpha t \end{pmatrix}$$

where $0 < \alpha < \pi$. *We have* $\det(\Sigma_\alpha(t) - 1) = (\det(R_\alpha(t) - I)^2 \geq 0$ *with equality only if* $t = 0$ *hence* $\Sigma_\alpha \in \mathcal{C}^+(\mathrm{Sp}(n))$. *We can define the extended path* $\widetilde{\Sigma}$ *by* $\widetilde{\Sigma}(t) = \Sigma_\alpha(t)$ *for* $0 \leq t \leq 1$ *and* $\widetilde{\Sigma}(t) = \Sigma_{\pi-\alpha}(t + 2\alpha - \pi)$ *for* $1 \leq t \leq 2$. *We obviously have* $U(t) = \widetilde{\Sigma}(t)$ *hence* $\det_{\mathbb{C}} U(t) = \det R_\alpha(t) = 1$. *It follows that*

$$i_{CZ}(\Sigma_\alpha) = 0. \tag{13}$$

2.1.2 Axiomatic description of i_{CZ}

The Conley–Zehnder index is the unique mapping $i_{\mathrm{CZ}} : \mathcal{C}^\pm(\mathrm{Sp}(n)) \longrightarrow \mathbb{Z}$ characterized by the three following properties (see [18]; also [14]):

CZ1 *Inverse path property*: We have $i_{\mathrm{CZ}}(\Sigma^{-1}) = -i_{\mathrm{CZ}}(\Sigma)$ (Σ^{-1} being defined by $\Sigma^{-1}(t) = (\Sigma(t))^{-1}$ for $t \in [0, 1]$);

CZ2 *Homotopy invariance:* $i_{\mathrm{CZ}}(\Sigma) = i_{\mathrm{CZ}}(\Sigma')$ if Σ and Σ' are homotopic in $\mathcal{C}^\pm(\mathrm{Sp}(n))$;

CZ3 *Action of* $\pi_1(\mathrm{Sp}(n))$: For every $\Gamma \in \pi_1(\mathrm{Sp}(n))$ we have $i_{\mathrm{CZ}}(\Gamma * \Sigma) = i_{\mathrm{CZ}}(\Sigma) + 2\,\mathrm{Mas}(\Gamma)$.

Remark 4 *Notice that (CZ2) implies the following: let* $\Sigma \in \mathcal{C}(\mathrm{Sp}(n))$ *and* Σ' *be a path joining the endpoint* $\Sigma(1) = S$ *to* $S' \in \mathrm{Sp}(n)$. *If* Σ' *lies in* $\mathcal{C}^+(\mathrm{Sp}(n))$ *or* $\mathcal{C}^-(\mathrm{Sp}(n))$ *then* $i_{CZ}(\Sigma * \Sigma') = i_{CZ}(\Sigma)$.

The Conley–Zehnder index has in addition the following properties:

CZ4 *Normalization*: Let J_1 be the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{Sp}(1)$. If $\Sigma_1 \in \mathcal{C}(Sp(2))$ is the path $t \longmapsto \begin{pmatrix} \cos\pi t & \sin\pi t \\ -\sin\pi & \cos\pi t \end{pmatrix}$ joining I to $-I$, then $i_{\mathrm{CZ}}(\Sigma_1) = 1$.

CZ5 *Dimensional additivity*: If $\Sigma_1 \in \mathcal{C}(\mathrm{Sp}(n_1))$ and $\Sigma_2 \in \mathcal{C}(\mathrm{Sp}(n_2), \mathbb{R})$ and $\Sigma = \Sigma_1 \oplus \Sigma_2 \in \mathcal{C}(\mathrm{Sp}(n_1 + n_2))$ then

$$i_{\mathrm{CZ}}(\Sigma) = i_{\mathrm{CZ}}(\Sigma_1) + i_{\mathrm{CZ}}(\Sigma_2). \tag{14}$$

Until, now we have assumed that the endpoint of the symplectic path Σ had no eigenvalue equal to one. Let us next show that we can relax this condition. The extension of the Conley–Zehnder index to all of $\mathcal{C}(\mathrm{Sp}(n))$ we obtain can take half-integer values (see the case of paths with parabolic endpoints in Section 3).

2.2 Extension of the Conley–Zehnder index

We review here the results in [14, 15].

2.2.1 i_{CZ} and the Leray index

Define the symplectic form $\sigma^{\ominus} = \sigma \oplus (-\sigma)$ on $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ where σ is the standard symplectic form $dp \wedge dx$ on $\mathbb{R}^{2n}$ (we write elements of $\mathbb{R}^{2n}$ as $z = (x, p)$). We denote by $\mathrm{Sp}^{\ominus}(2n)$ and $\mathrm{Lag}^{\ominus}(2n)$ the symplectic group and Lagrangian Grassmannian of $(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, \sigma^{\ominus})$. Let $\mu^{\ominus}$ be the Leray index on $\mathrm{Lag}^{\ominus}(2n)$. For $\Sigma \in \mathcal{C}(Sp(n))$ set $\Sigma^{\ominus} = I \oplus \Sigma$. We have $\Sigma^{\ominus} \in \mathcal{C}\,\mathrm{Sp}^{\ominus}(2n)$. In [14] we showed that:

CZ6 *Relation with the Leray index*: We have

$$i_{\mathrm{CZ}}(\Sigma) = \tfrac{1}{2}\mu^{\ominus}(\Sigma^{\ominus}\Delta, \Delta) \tag{15}$$

where $\Delta \in \mathrm{Lag}^{\ominus}(2n)$ is the diagonal $\Delta = \{(z, z) : z \in \mathbb{R}^{2n}\}$ of $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. (Compare formula (9) defining the Maslov index).

Formula (15) can be used to extend the original definition of i_{CZ} given in [5] and which is only valid for $\det(S - I) \neq 0$; we will take it as a *redefinition* of the Conley–Zehnder index. It follows from this redefinition that we can calculate the Maslov index of loops: it follows from the properties of the Leray index that for every $\Gamma \in \pi_1(\mathrm{Sp}(n))$

$$i_{\mathrm{CZ}}(\Gamma) = 2\,\mathrm{Mas}(\Gamma); \tag{16}$$

this formula is actually a particular case of property (CZ3) because it follows from the antisymmetry of the Leray index that if Σ_0 is the constant path (*i.e.* $\Sigma_0(t) = I$ for $0 \leq t \leq 1$) we have $i_{\mathrm{CZ}}(\Sigma_0) = 0$.

2.2.2 The product formula

In [14] we proved, using (15), the following important product formula:

CZ7 If Σ and Σ' are in $\mathcal{C}(\mathrm{Sp}(n))$ then the index of the product path $\Sigma\Sigma'$ defined by $\Sigma\Sigma'(t) = \Sigma(t)\Sigma'(t)$ is given by

$$i_{\mathrm{CZ}}(\Sigma\Sigma') = i_{\mathrm{CZ}}(\Sigma) + i_{\mathrm{CZ}}(\Sigma') + \tfrac{1}{2}\,\mathrm{sign}(M_S + M_{S'}) \tag{17}$$

where M_S is the "symplectic Cayley transform"

$$M_S = \tfrac{1}{2}J(S + I)(S - I)^{-1} \tag{18}$$

of the endpoint $S = \Sigma(1)$ of the symplectic path Σ; the matrix M_S is symmetric.

Formula (17) in particular implies that one can –in principle– calculate the index $i_{\mathrm{CZ}}(\Sigma^{*(N)})$ of a N times repeated orbit by induction; for instance

$$i_{\mathrm{CZ}}(\Sigma * \Sigma) = 2i_{\mathrm{CZ}}(\Sigma) + \tfrac{1}{2}\,\mathrm{sign}\, M_S.$$

It is however not directly obvious what a "nice formula" for $i_{\mathrm{CZ}}(\Sigma^{*N})$ could be.

2.2.3 Invariance under conjugation

The following simple result is the key to the reduction of the calculation of the Conley–Zehnder index to normal forms in Section 3; it says that two symplectic-conjugate paths have the same Conley–Zehnder index:

Proposition 5 *The Conley–Zehnder index is invariant under conjugation:*

$$i_{CZ}(S^{-1}\Sigma S) = i_{CZ}(\Sigma) \tag{19}$$

for every $S \in \mathrm{Sp}(n)$ *(the path* $S^{-1}\Sigma S$ *is defined by* $S^{-1}\Sigma S(t) = S^{-1}\Sigma(t)S$*).*

Proof. Clearly $S^{-1}\Sigma S \in \mathcal{C}(\mathrm{Sp}(n))$ if $\Sigma \in \mathcal{C}(\mathrm{Sp}(n))$. Since $\mathrm{Sp}(n)$ is connected there exists a path $t' \longmapsto S_{t'}$ $(0 \leq t' \leq 1)$ joining the identity $I \in \mathrm{Sp}(n)$ to $S = S_1$. The function $h(t,t') = S_{t'}^{-1}\Sigma(t)S_{t'}$ homotopes Σ on $S^{-1}\Sigma S$; since we have

$$\det(S_{t'}^{-1}\Sigma(t)S_{t'} - I) = \det(\Sigma(t) - I)$$

the function h is is a homotopy in $\mathcal{C}^{\pm}(\mathrm{Sp}(n))$ hence $i_{\mathrm{CZ}}(S^{-1}\Sigma S) = i_{\mathrm{CZ}}(\Sigma)$ in view of the property (CZ2) of he Conley–Zehnder index. ■

(It is easy to modify the proof above so that Proposition 5 still holds when S is replaced by an arbitrary symplectic path.)

It follows from Proposition 5 that the Conley–Zehnder index of a periodic Hamiltonian orbit is an intrinsic property of that orbit, and does not depend on the choice of origin of that orbit:

Corollary 6 *Let* $\gamma_{z_0}^H$ *be a* τ*-periodic orbit such that* $\gamma_{z_0}^H(0) = z_0$*. Let* $z_1 \in \gamma_{z_0}^H([0,\tau])$*. We have* $i_{CZ}(\gamma_{z_0}^H) = i_{CZ}(\gamma_{z_1}^H)$*.*

Proof. Let $z_1 = \phi_{t_0}^H(z_0)$, ϕ_t^H the flow determined by the Hamiltonian H. Writing $\phi_t^H = \phi_{t_0}^H \circ \phi_t^H \circ \phi_{-t_0}^H$ we have, using the chain rule,

$$\begin{aligned} D\phi_t^H(z) &= D\phi_{t_0}^H[\phi_t^H \circ \phi_{-t_0}^H(z)]D[\phi_t^H \circ \phi_{-t_0}^H(z)] \\ &= D\phi_{t_0}^H[\phi_t^H \phi_{-t_0}^H(z)]D\phi_t^H[\phi_{-t_0}^H(z_0)]D\phi_{-t_0}^H(z) \end{aligned}$$

Setting $z = z_1 = \phi_{t_0}^H(z_0)$ and $D\phi_t^H = S_t$ we get

$$S_t(z_1) = S_{t_0}(\phi_t^H(z_0))S_t(z_0)(S_{t_0}(z_0))^{-1} \tag{20}$$

hence, in particular $S_\tau(z_1)$ and $S_\tau(z_0)$ are conjugate:

$$S_\tau(z_1) = S_{t_0}(z_0)S_t(z_0)(S_{t_0}(z_0))^{-1}.$$

Let us denote the paths $t \longmapsto S_t(z_0)$ and $t \longmapsto S_t(z_1)$ $(0 \leq t \leq \tau)$ by Σ_0 and Σ_1, respectively, so that $i_{\mathrm{CZ}}(\gamma_{z_0}^H) = i_{\mathrm{CZ}}(\Sigma_0)$ and $i_{\mathrm{CZ}}(\gamma_{z_1}^H) = i_{\mathrm{CZ}}(\Sigma_1)$. Rewriting the equality (20) as

$$\begin{aligned} S_t(z_1) &= \Gamma(t)S_{t_0}(z_0)S_t(z_0)(S_{t_0}(z_0))^{-1} \\ \Gamma(t) &= S_{t_0}(\phi_t^H(z_0))(S_{t_0}(z_0))^{-1} \end{aligned}$$

we have $\Gamma(0) = \Gamma(\tau) = I$. We thus have

$$\Sigma_1 = \Gamma * S_{t_0}(z_0)\Sigma_0(S_{t_0}(z_0))^{-1}$$

hence

$$i_{\mathrm{CZ}}(\Sigma_1) = i_{\mathrm{CZ}}(S_{t_0}(z_0)\Sigma_0(S_{t_0}(z_0))^{-1}) + 2\,\mathrm{Mas}(\Gamma)$$

in view of property (CZ3) of the Conley–Zehnder index. By formula (19) in Proposition (5) above we have

$$i_{\mathrm{CZ}}(S_{t_0}(z_0)\Sigma_0(S_{t_0}(z_0))^{-1}) = i_{\mathrm{CZ}}(\Sigma_0)$$

hence

$$i_{\mathrm{CZ}}(\Sigma_1) = i_{\mathrm{CZ}}(\Sigma_0) + 2\,\mathrm{Mas}(\Gamma).$$

Let us show that the loop Γ is contractible to a point; then $\mathrm{Mas}(\Gamma) = 0$ and $i_{\mathrm{CZ}}(\Sigma_1) = i_{\mathrm{CZ}}(\Sigma_0)$. Define a function $h : [0, \tau] \times [0, t_0] \longrightarrow \mathrm{Sp}(n)$ by $h(t, t') = S_{t'}(\phi_t^H(z_0))(S_{t'}(z_0))^{-1}$. Fixing t' we have $h(0, t') = h(\tau, t') = I$; on the other hand $h(t, t_0) = S_{t_0}(\phi_t^H(z_0))(S_{t_0}(z_0))^{-1}$ and $h(t, 0) = I$, hence h homotopes Γ to the point $\{I\}$. ■

Corollary 6 thus allows us to speak about the Conley–Zehnder of a periodic Hamiltonian orbit; if γ^H is such an orbit we will write

$$i_{\mathrm{CZ}}(\gamma^H) = i_{\mathrm{CZ}}(\Sigma)$$

where Σ is a solution curve of the variational equation (3) without reference to the origin z_0.

2.2.4 Relation of i_{CZ} with Morse's index of concavity

Using (15) one obtains the following very useful result which links the Conley–Zehnder index to the Maslov index on $\mathrm{Sp}(n)$ when the endpoint S of Σ is a free symplectic matrix. Recall that $S \in \mathrm{Sp}(n)$ is said to be "free" if $S\ell_P \cap \ell_P = 0$ where $\ell_P = 0 \times \mathbb{R}^n$; identifying S with its matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in the canonical symplectic basis of $(\mathbb{R}^{2n}, \sigma)$ this condition is equivalent to $\det B \neq 0$. One calls the quadratic form

$$W(x, x') = \tfrac{1}{2} DB^{-1}x^2 - B^{-1}x \cdot x' + \tfrac{1}{2} B^{-1}Ax'^2 \tag{21}$$

the *generating function* of S (note that $(x, p) = S(x', p')$ is equivalent to $p = \partial_x W(x, x')$ and $p' = -\partial_{x'} W(x, x')$). We have proven in ([14]) the following important result which links the Conley–Zehnder index to the usual Maslov index:

Proposition 7 *Assume that the endpoint S of $\Sigma \in \mathcal{C}(\mathrm{Sp}(n))$ is free. Then*

$$i_{CZ}(\Sigma) = \mathrm{Mas}(\Sigma) - \tfrac{1}{2}(n + \operatorname{sign} W_{xx}) \tag{22}$$

where $\operatorname{sign} W_{xx}$ *is the signature of the Hessian matrix of the quadratic form* $x \longmapsto W(x,x)$*:*

$$W_{xx} = DB^{-1} - B^{-1} - (B^T)^{-1} + B^{-1}A. \tag{23}$$

When S *has no eigenvalue equal to one (i.e.* $\Sigma \in \mathcal{C}^+(\mathrm{Sp}(n))$ *or* $\Sigma \in \mathcal{C}^-(\mathrm{Sp}(n))$*) then*

$$i_{CZ}(\Sigma) = \mathrm{Mas}(\Sigma) - \operatorname{Inert} W_{xx}. \tag{24}$$

(Compare formulas (4.4) and (4.5) in Creagh *et al.* [6]).

Proposition 7 combined with the conjugation property in Proposition 5 provide us, as we will see in Subsection 3, with a powerful tool for calculating explicitly Conley–Zehnder indices. The idea is to find a symplectic matrix S such that the endpoint of $S\Sigma S^{-1}$ is free; it then suffices to apply formula (22) or (24) to the conjugate path $S\Sigma S^{-1}$.

The result above is more general than it seems at first sight; not only do free symplectic matrices form a dense subset of $\mathrm{Sp}(n)$, but we have the following result that shows that any symplectic matrix becomes free if multiplied by a suitable symplectic matrix:

Proposition 8 *For every* $S \in \mathrm{Sp}(n)$ *there exists* $R \in \mathrm{Sp}(n)$ *such that* RS *is free.*

Proof. Hofer and Zehnder show in [19] (Appendix A.1, p.270) that there exists $R' \in \mathrm{Sp}(n)$ such that

$$R'S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad \det A \neq 0.$$

Setting $R = JR'$ the symplectic matrix RS is free. ■

3 Conley–Zehnder index and Floquet theory

3.1 Symplectic Floquet theory

Consider a linear τ-periodic Hamiltonian system:

$$\dot{z}(t) = JH(t)z(t) \quad , \quad H(t+\tau) = H(t) \tag{25}$$

($H(t)$ is a real symmetric matrix depending continuously on $t \in \mathbb{R}$; the function $t \longmapsto z(t)$ is a solution of the Hamilton equations for the quadratic Hamilton function $H(z,t) = \frac{1}{2}H(t)z \cdot z$). Let S_t be the fundamental solution defined by

$$\tfrac{d}{dt}S_t = JH(t)S_t \quad , \quad S_0 = I; \tag{26}$$

we have $S_t \in \mathrm{Sp}(n)$ for all t. By definition S_τ is the monodromy matrix of the Hamiltonian system *(25)*. Let us now return to the question of whether the logarithm X of the monodromy matrix can be chosen real. A classical result is the following (Culver [7]): let Y be a real square matrix. Then there exists a real square matrix X such that $Y = e^X$ if and only if Y is nonsingular and each Jordan block of Y belonging to a negative eigenvalues occurs an even number of times. It follows that:

Lemma 9 *The square $S_{2\tau} = S_\tau^2$ of the monodromy matrix can be written as $S_{2\tau} = e^{2\tau X}$ where X is real.*

Proof. Since S_τ is real its eigenvalues must come in complex conjugate pairs. The only negative eigenvalues of $S_{2\tau}$ come from those which are purely imaginary. ∎

We observe that Montagnier *et al.* [30] have shown that one can relax the "period doubling" procedure in the Floquet decomposition; we will however not need their results here.

We have not exploited yet the fact that S_t is symplectic. Elaborating on an argument by Siegel and Moser [34] (p. 97–103), Wiesel and Pohlen [36] show that if X is a complex matrix such that $e^X \in \mathrm{Sp}(n)$ then there exist (complex) matrices F and D such that

$$X = FDF^{-1} \quad , \quad F^T JF = J \quad , \quad D^T J + JD = 0. \tag{27}$$

We can restate this result in the following way (we denote by $\mathrm{sp}(n)$ the Lie algebra of $\mathrm{Sp}(n)$):

Lemma 10 *Let X be a complex matrix such that $e^X \in \mathrm{Sp}(n)$. (i) We have $X^T J + JX = 0$. (ii) If X is real then $X \in \mathrm{sp}(n)$ and hence $e^{tX} \in \mathrm{Sp}(n)$ for all $t \in \mathbb{R}$.*

Proof. The statement (ii) immediately follows from (i) since the condition $X^T J + JX = 0$ is equivalent to $X \in \mathrm{sp}(n)$ when X is real. Differentiating the function $f(t) = e^{tX^T} J e^{tX} - J$ we have

$$\tfrac{d}{dt} f(t) = e^{tX^T}(X^T J + JX)e^{tX} = 0,$$

hence, using the relations (27),

$$X^T J + JX = (F^{-1})^T D^T F^T J + JFDF^{-1} = 0$$

hence (i) ∎

Combining Lemmas 9 and 10 we get the following symplectic Floquet decomposition result:

Proposition 11 *Let S_t be the solution of the variational equation (26). There exist symplectic matrices $\Pi_{t+\tau} = \Pi_t$ and $Q_t = e^{tX}$ ($X \in \mathrm{sp}(n)$) such that $S_t = \Pi_t Q_t$.*

Proof. Doubling the period τ if necessary we may assume that $S_\tau = e^{\tau X}$ with $X \in \mathrm{sp}(n)$. Writing $S_t = \Pi_t Q_t$ with $Q_t = e^{tX}$ we have $\Pi_t \in \mathrm{Sp}(n)$ and $\Pi_{t+\tau} = S_{t+\tau} e^{-(t+\tau)X} = S_t(S_\tau e^{-\tau X})e^{-tX} = \Pi_t$ since $S_\tau = e^{\tau X}$. ∎

3.2 Application to the calculation of $i_{\mathbf{CZ}}(\gamma^H)$

Let now γ^H be a a Hamiltonian periodic orbit. The discussion above implies that the Conley–Zehnder index of γ^H consists of two terms: a Maslov index, corresponding to the periodic part, and the Conley–Zehnder index of a "simple" path:

Corollary 12 *The Conley–Zehnder index $i_{CZ}(\gamma^H) = i_{CZ}(\Sigma)$ is given by*

$$i_{CZ}(\gamma^H) = i_{CZ}(\Sigma) + 2\operatorname{Mas}(\Gamma)$$

where $\Sigma \in \mathcal{C}(Sp(n))$ is the path defined by $\Sigma(t) = Q_{t\tau} = e^{t\tau X}$ and Γ the loop $\Gamma(t) = \Pi_{t\tau}$ $(0 \leq t \leq 1)$.

Proof. It immediately follows from Proposition (11) and property (CZ3) of the Conley–Zehnder index. ■

4 Some Explicit Calculations

We are going to calculate explicitly the Conley–Zehnder indices of a few symplectic paths of interest intervening in practical computations for the Gutzwiller formula after reduction to normal form. We are following here Sugita's classification [35]; the matrices we will write correspond to the choice of symplectic coordinates

$$(x,p) = (x_1, ..., x_n; p_1, ..., p_n).$$

The cases we will consider correspond to normal forms of symplectic mappings (respectively quadratic Hamiltonian functions); see Burgoyne and Cushman [3] or Laub and Meyer [20] for classical accounts extending and extending the classical symplectic diagonalization procedure of Williamson. More recently Long and Dong [24] have determined normal forms of symplectic matrices possessing eigenvalues on the unit circle. Also see Abraham–Marsden [1] for a discussion of normal forms from the point of view of equilibrium dynamics.

4.1 The case of prime orbits

Let $\gamma_{z_0}^H$ be a periodic Hamiltonian orbit with period τ; it is a prime orbit if $\gamma_{z_0}^H(t) \neq \gamma_{z_0}^H(0)$ if $0 < t < \tau$.

In what follows α and β denote real numbers; we assume $\beta > 0$.

4.1.1 Parabolic blocks

They correspond to eigenvalues $\lambda = 1$ and are of the type

$$S_{\mathrm{par}}(t) = \begin{pmatrix} 1 & \gamma t \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad S'_{\mathrm{par}}(t) = \begin{pmatrix} 1 & 0 \\ -\gamma t & 1 \end{pmatrix}$$

corresponding to, respectively, $H = \frac{1}{2}\gamma p^2$ and $H = \frac{1}{2}\gamma x^2$. Notice that since $S'_{\mathrm{par}}(t) = J^{-1}S_{\mathrm{par}}(t)J$ both paths are conjugate, hence they have the same

Conley–Zehnder index in view of Proposition 5. Since $S_{\mathrm{par}}(1)$ is free we can apply formula (22) in Proposition 7. The generating function of $S_{\mathrm{par}}(1)$ is $W(x,x') = \frac{1}{2}\gamma(x-x')^2$ hence $W_{xx} = 0$ so that

$$i_{\mathrm{CZ}}(\Sigma_{\mathrm{par}}) = \mathrm{Mas}(\Sigma_{\mathrm{par}}) - \frac{1}{2}.$$

Now, $\mathrm{Mas}(\Sigma_{\mathrm{par}}) = 1 - \operatorname{sign}\gamma$ (formula (10)) hence we have

$$i_{\mathrm{CZ}}(\Sigma_{\mathrm{par}}) = -\tfrac{1}{2}\operatorname{sign}\gamma. \tag{28}$$

The index $i_{\mathrm{CZ}}(\Sigma_{\mathrm{par}})$ is thus a *half-integer*.

4.1.2 Elliptic blocks

They correspond to eigenvalues $\lambda = e^{\pm i\alpha}$ with $0 < \alpha < 2\pi$ and $\alpha \neq \pi$:

$$S_{\mathrm{ell}} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix};$$

the associated Hamiltonian is $H = \frac{1}{2}\alpha(x^2+p^2)$ so that

$$\Sigma_{\mathrm{ell}}(t) = \begin{pmatrix} \cos\alpha t & \sin\alpha t \\ -\sin\alpha t & \cos\alpha t \end{pmatrix} \quad , \quad 0 \leq t \leq 1.$$

The symplectic matrix S_{ell} is free and its generating function is

$$W(x,x') = \frac{1}{2\sin\alpha}((x^2 + x'^2)\cos\alpha - 2xx')$$

and hence $W_{xx} = -\cot(\alpha/2)$. Since $\cot(\alpha/2) \neq 0$ we can calculate the Conley–Zehnder index of the path Σ_{ell} using formula (24), which reads here:

$$i_{\mathrm{CZ}}(\Sigma_{\mathrm{ell}}) = \mathrm{Mas}(\Sigma_{\mathrm{ell}}) - \mathrm{Inert}\left[-\cot\frac{\alpha}{2}\right]. \tag{29}$$

We have

$$\mathrm{Mas}(\Sigma_{\mathrm{ell}}) = -\left[\frac{\alpha}{\pi}\right] = \begin{cases} 0 \text{ if } 0 < \alpha < \pi \\ -1 \text{ if } \pi < \alpha < 2\pi \end{cases} \tag{30}$$

(formula (11)) and $\cot(\alpha/2) > 0$ if $0 < \alpha < \pi$ and $\cot(\alpha/2) < 0$ if $\pi < \alpha < 2\pi$; formula (29) thus yields

$$i_{\mathrm{CZ}}(\Sigma_{\mathrm{ell}}) = -1. \tag{31}$$

4.1.3 Hyperbolic blocks

They correspond to eigenvalues $\lambda = e^{\pm\beta}$; the Hamiltonian is here $H = -\beta px$. We have

$$S_{\mathrm{hyp}} = \begin{pmatrix} e^{\beta} & 0 \\ 0 & e^{-\beta} \end{pmatrix}$$

and the corresponding path is

$$\Sigma_{\mathrm{hyp}}(t) = \begin{pmatrix} e^{\beta t} & 0 \\ 0 & e^{-\beta t} \end{pmatrix} \quad , \quad 0 \leq t \leq 1.$$

Let $S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. The symplectic matrix

$$S\Sigma_{\mathrm{hyp}}(t)S^{-1} = \begin{pmatrix} \cosh \beta t & \sinh \beta t \\ \sinh \beta t & \cosh \beta t \end{pmatrix}$$

is free and the generating function of $S\Sigma_{\mathrm{hyp}}(1)S^{-1}$ is

$$W(x, x') = \frac{x^2}{2}(\coth \beta) - \frac{xx'}{\sinh \beta} + \frac{x'^2}{2}(\coth \beta);$$

it follows that

$$W_{xx} = \frac{\cosh \beta - 1}{\sinh \beta} > 0$$

hence, by formula (24), Proposition 5, and formula (12)

$$i_{\mathrm{CZ}}(\Sigma_{\mathrm{hyp}}) = \mathrm{Mas}(\Sigma_{\mathrm{hyp}}) = 0. \tag{32}$$

4.1.4 Inverse hyperbolic blocks

They correspond to eigenvalues $\lambda = -e^{\pm\beta}$:

$$S_{\mathrm{inv}} = \begin{pmatrix} -e^{\beta} & 0 \\ 0 & -e^{-\beta} \end{pmatrix};$$

since $\mathrm{Tr}(S_{\mathrm{inv}}) < -2$ there is no $X \in \mathrm{sp}(n)$ such that $S_{\mathrm{inv}} = e^X$. However, S_{inv} lies on the symplectic path determined by the time-dependent Hamiltonian defined by $H = \pi(p^2 + x^2)$ if $0 \leq t \leq 1/2$ and $H = 2\beta px$ if $1/2 \leq t \leq 1$. The corresponding path is given by $\Sigma_{\mathrm{inv}} = \Sigma_{\mathrm{inv}}^{(1)} * \Sigma_{\mathrm{inv}}^{(2)}$ with

$$\begin{cases} \Sigma_{\mathrm{inv}}^{(1)}(t) = \begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix} & \text{if } 0 \leq t \leq 1/2, \\ \Sigma_{\mathrm{inv}}^{(2)}(t) = \begin{pmatrix} -e^{\beta(2t-1)} & 0 \\ 0 & -e^{-\beta(2t-1)} \end{pmatrix} & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Since $\det(\Sigma_{\mathrm{inv}}^{(2)}(t) - I) > 0$ for all t the path $\Sigma_{\mathrm{inv}}^{(2)}$ stays forever in $\mathrm{Sp}^+(n)$ we thus have

$$i_{\mathrm{CZ}}(\Sigma_{\mathrm{inv}}^{(1)} * \Sigma_{\mathrm{inv}}^{(2)}) = i_{\mathrm{CZ}}(\Sigma_{\mathrm{inv}}^{(1)})$$

in view of Remark 4; using the normalization property (CZ4) of the Conley–Zehnder index we have $i_{\mathrm{CZ}}(\Sigma_{\mathrm{inv}}^{(1)}) = -1$ hence

$$i_{\mathrm{CZ}}(\Sigma_{\mathrm{inv}}) = -1. \tag{33}$$

4.1.5 Loxodromic blocks

They correspond to eigenvalues $\lambda = e^{\pm i\alpha \pm \beta}$ with $0 < \alpha < \pi$ and we have

$$S_{\text{lox}} = \begin{pmatrix} e^{-\beta}\cos\alpha & e^{-\beta}\sin\alpha & 0 & 0 \\ -e^{-\beta}\sin\alpha & e^{-\beta}\cos\alpha & 0 & 0 \\ 0 & 0 & e^{\beta}\cos\alpha & e^{\beta}\sin\alpha \\ 0 & 0 & -e^{\beta}\sin\alpha & e^{\beta}\cos\alpha \end{pmatrix};$$

so that

$$\Sigma_{\text{lox}}(t) = \begin{pmatrix} e^{-\beta t}\cos\alpha t & e^{-\beta t}\sin\alpha t & 0 & 0 \\ -e^{-\beta t}\sin\alpha t & e^{-\beta t}\cos\alpha t & 0 & 0 \\ 0 & 0 & e^{\beta t}\cos\alpha t & e^{\beta t}\sin\alpha t \\ 0 & 0 & -e^{\beta t}\sin\alpha t & e^{\beta t}\cos\alpha t \end{pmatrix}$$

the corresponding Hamiltonian is $H = \alpha(p_1x_2 - p_2x_1) - \beta(p_1x_1 + p_2x_2)$. We begin by noting that $\Sigma_{\text{lox}}(t) = \Sigma'(t)\Sigma_\alpha(t)$ with

$$\Sigma'(t) = \begin{pmatrix} e^{-\beta t} & 0 & 0 & 0 \\ 0 & e^{-\beta t} & 0 & 0 \\ 0 & 0 & e^{\beta t} & 0 \\ 0 & 0 & 0 & e^{\beta t} \end{pmatrix}$$

$$\Sigma_\alpha(t) = \begin{pmatrix} \cos\alpha t & \sin\alpha t & 0 & 0 \\ -\sin\alpha t & \cos\alpha t & 0 & 0 \\ 0 & 0 & \cos\alpha t & \sin\alpha t \\ 0 & 0 & -\sin\alpha t & \cos\alpha t \end{pmatrix}.$$

In view of the product formula (17) we have

$$i_{\text{CZ}}(\Sigma'\Sigma_\alpha) = i_{\text{CZ}}(\Sigma') + i_{\text{CZ}}(\Sigma_\alpha) + \tfrac{1}{2}\operatorname{sign}(M' + M_\alpha)$$

where M' (resp. M_α) is the symplectic Cayley transform (18) of $S' = \Sigma'(1)$ (resp. $S_\alpha = \Sigma_\alpha(1)$); now $i_{\text{CZ}}(\Sigma_\alpha) = 0$ (formula (13)); on the other hand $\Sigma' = \Sigma_{\text{hyp}} \oplus \Sigma_{\text{hyp}}$ hence, using successively the dimensional additivity property (CZ5) (formula (14)) and (32), we have $i_{\text{CZ}}(\Sigma') = 2i_{\text{CZ}}(\Sigma_{\text{hyp}}) = 0$. It follows that

$$i_{\text{CZ}}(\Sigma'\Sigma_\alpha) = \tfrac{1}{2}\operatorname{sign}(M' + M_\alpha).$$

A straightforward calculation now shows that

$$M' + M_\alpha = \frac{1}{2}\begin{pmatrix} 0 & 0 & \coth\frac{\beta}{2} & -\cot\frac{\alpha}{2} \\ 0 & 0 & \cot\frac{\alpha}{2} & \coth\frac{\beta}{2} \\ \coth\frac{\beta}{2} & \cot\frac{\alpha}{2} & 0 & 0 \\ -\cot\frac{\alpha}{2} & \coth\frac{\beta}{2} & 0 & 0 \end{pmatrix};$$

the eigenvalues of this matrix are the solutions of the characteristic equation

$$\left[\lambda^2 - \left(\cot^2\tfrac{\alpha}{2} + \coth^2\tfrac{\beta}{2}\right)\right]^2 = 0;$$

it follows that $\operatorname{sign}(M' + M_\alpha) = 0$ and hence

$$i_{\text{CZ}}(\Sigma_{\text{lox}}) = 0. \tag{34}$$

4.2 The Conley–Zehnder index of repeated orbits

For a path $\Sigma \in \mathcal{C}(\mathrm{Sp}(n))$ and N a positive integer we denote by Σ^{*N} the path $\Sigma * \cdots * \Sigma$. Clearly $\Sigma^{*N} \in \mathcal{C}(\mathrm{Sp}(n))$. (Observe however that the condition $\Sigma \in \mathcal{C}^{\pm}(\mathrm{Sp}(n))$ does not imply that $\mathcal{C}^{\pm}(\mathrm{Sp}(n))$).

4.2.1 The parabolic, hyperbolic, and loxodromic cases

Consider first the parabolic case. Let N be an integer; we have $\Sigma^{*N}_{\mathrm{par}}(t) = \Sigma_{\mathrm{par}}(Nt)$ hence the N-th repetition of a path with parabolic endpoint also has parabolic endpoint with γ replaced by $N\gamma$. Since γ and $N\gamma$ have same sign formula (34) yields:

$$i_{\mathrm{CZ}}(\Sigma^{*N}_{\mathrm{par}}) = -\tfrac{1}{2}\operatorname{sign}\gamma. \tag{35}$$

The same argument applies in the hyperbolic case: $\Sigma^{*N}_{\mathrm{hyp}}$ has parabolic endpoint with β replaced by $N\beta$; it follows that

$$i_{\mathrm{CZ}}(\Sigma^{*N}_{\mathrm{hyp}}) = 0. \tag{36}$$

Consider next the loxodromic case. It is easy to verify that the endpoint S^N_{lox} of $\Sigma^{*N}_{\mathrm{hyp}}$ is again loxodromic, but with α replaced by $N\alpha$ and β by $N\beta$. If $0 < N\alpha < \pi$ one is led back to the case $N = 1$; otherwise it suffices to view S^N_{lox} as a loxodromic endpoint with β replaced by $N\beta$ and α replaced by $\alpha' = \alpha - 2k_N\pi$ where k_N is an integer chosen so that $0 < \alpha' < \pi$. In either case we get the value

$$i_{\mathrm{CZ}}(\Sigma^{*N}_{\mathrm{lox}}) = 0. \tag{37}$$

4.2.2 The inverse hyperbolic case

Let us now study the inverse hyperbolic case. We will have to distinguish two cases.

- We first notice that if N is an even integer then the endpoint of Σ^N_{inv} is a symplectic path with hyperbolic endpoint; however one must also take into account the contribution of the N-th power of the path $\Sigma^{(1)}_{\mathrm{inv}}$ which is a loop. In fact, writing $N = 2k$ we have

$$\begin{cases} (\Sigma^{(1)}_{\mathrm{inv}}(t))^{2k} = \begin{pmatrix} \cos 4k\pi t & \sin 4k\pi t \\ -\sin 4k\pi t & \cos 4k\pi t \end{pmatrix} \ \textit{if}\ 0 \le t \le 1/2, \\ (\Sigma^{(2)}_{\mathrm{inv}}(t))^{2k} = \begin{pmatrix} -e^{k\beta(2t-1)} & 0 \\ 0 & -e^{-k\beta(2t-1)} \end{pmatrix} \ \textit{if}\ 1/2 \le t \le 1. \end{cases}$$

  $(\Sigma^{(1)}_{\mathrm{inv}})^{*2k} = \alpha^k$ where α is the generator of $\pi_1(\mathrm{Sp}(n))$ and $(\Sigma^{(2)}_{\mathrm{inv}})^{*2k} = \Sigma^{*2k}_{\mathrm{hyp}}$ so that $\Sigma^{*2k}_{\mathrm{inv}} = \alpha^k \Sigma^{*2k}_{\mathrm{hyp}}$. In view of property (CZ3) of the Conley–Zehnder index we have

$$i_{\mathrm{CZ}}(\Sigma^{*2k}_{\mathrm{inv}}) = i_{\mathrm{CZ}}(\Sigma^{*2k}_{\mathrm{hyp}}) + 2\,\mathrm{Mas}(\alpha^k)$$

  hence, using formulae (8) and (36):

$$i_{\mathrm{CZ}}(\Sigma^{*2k}_{\mathrm{inv}}) = 4k. \tag{38}$$

- Assume next that $N = 2k+1$. The same argument as above shows that we have $\Sigma_{\rm inv}^{*(2k+1)} = \alpha^k \Sigma'_{\rm inv}$ ($\Sigma'_{\rm inv}$ is $\Sigma_{\rm inv}$ with β replaced by $(2k+1)\beta$), hence, taking formula (33) into account:

$$i_{\rm CZ}(\Sigma_{\rm inv}^{*(2k+1)}) = -1 + 4k. \tag{39}$$

Remark 13 *Formulae (38), (39) show that $i_{CZ}(\Sigma_{inv}^{*N}) \equiv -1 \bmod 4$ in all cases. Therefore the number of repetitions of periodic orbits with inverse hyperbolic endpoint has no influence on the phase in Gutzwiller's formula.*

4.2.3 The elliptic case

Let $\Sigma_{\rm ell}$ be a symplectic path with elliptic endpoint. Suppose first that $N\alpha = 2k\pi$ for some integer k. Then $\Sigma_{\rm ell}^{*N}$ is a loop; in fact

$$\Sigma_{\rm ell}^{*N}(t) = \begin{pmatrix} \cos N\alpha t & \sin N\alpha t \\ -\sin N\alpha t & \cos N\alpha t \end{pmatrix} \ , \ 0 \leq t \leq 1$$

hence, taking formula (16) into account:

$$i_{\rm CZ}(\Sigma_{\rm ell}^{*N}) = 2\,{\rm Mas}(\Sigma_{\rm ell}^{*N}) = 4N_0$$

where N_0 is the smallest positive integer such that $N_0\alpha \equiv 0 \bmod 2\pi$. If $N\alpha \neq 2k\pi$ then the endpoint of $\Sigma_{\rm ell}^{*N}$ is again elliptic and we have (cf. formulae (29) and (30))

$$i_{\rm CZ}(\Sigma_{\rm ell}) = -\left[\frac{N\alpha}{\pi}\right] - {\rm Inert}\left[-\cot(\tfrac{1}{2}N\alpha)\right].$$

Let us consider the two following cases:

- $(2k-1)\pi < N\alpha < 2k\pi$. We then have $[N\alpha/\pi] = 2k-1$ and $-\cot(\frac{1}{2}N\alpha) > 0$ hence $i_{\rm CZ}(\Sigma_{\rm ell}) = -2k+1$;
- $2k\pi < N\alpha < (2k+1)\pi$. We then have $[N\alpha/\pi] = 2k$ and $-\cot(\frac{1}{2}N\alpha) < 0$ hence $i_{\rm CZ}(\Sigma_{\rm ell}) = -2k-1$.

In both cases this is the same thing as

$$i_{\rm CZ}(\Sigma_{\rm ell}) = -2\left[\frac{N\alpha}{2\pi}\right] - 1. \tag{40}$$

5 Application to the Metaplectic Group

The double covering of the symplectic group ${\rm Sp}(n)$ admits a faithful representation as unitary operators acting on the square integrable functions on $\mathbb{R}^n$ (see [11, 13, 21] and the references therein). The corresponding group is called the metaplectic group; we will denote it by ${\rm Mp}(n)$. The natural projection of ${\rm Mp}(n)$

onto $\mathrm{Sp}(n)$ is denoted ; it is chosen so that $\pi(F) = J$ where F is the $\hbar$-Fourier transform defined by

$$Ff(x) = \left(\tfrac{1}{2\pi i}\right)^n \int e^{-ix\cdot x'} f(x')d^n x'.$$

The formulae obtained in Section 3 will allow us to give explicit formulae for the Weyl symbols of metaplectic operators in "normal form" following the classification given above.

5.1 The Weyl representation of metaplectic operators

Let W be a quadratic form on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$W(x, x') = \tfrac{1}{2}Px^2 - Lx \cdot x' + \tfrac{1}{2}Qx^2 \tag{41}$$

with $P = P^T$, $Q = Q^T$, and $\det L \neq 0$. To W we associate the operator $\widehat{S}_{W,m} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ defined by

$$\widehat{S}_{W,m} f(x) = \left(\tfrac{1}{2\pi i}\right)^{n/2} \Delta(W) \int e^{-ix\cdot x'} f(x')d^n x'$$

where $\Delta(W) = i^m \sqrt{|\det L|}$, the integer m corresponds to a choice of the argument of $\det L$:

$$m\pi = \arg \det L \mod 2\pi.$$

The operators $\widehat{S}_{W,m}$ extend into unitary operators on $L^2(\mathbb{R}^n)$ and the inverse of $\widehat{S}_{W,m}$ is given by $(\widehat{S}_{W,m})^{-1} = \widehat{S}_{W^*,m^*}$ with $W^*(x, x') = -W(x', x)$ and $m^* = n - m$. The operators $\widehat{S}_{W,m}$ thus generate a group of unitary operators on $L^2(\mathbb{R}^n)$, the metaplectic group $\mathrm{Mp}(n)$. One defines a group homomorphism $\pi : \mathrm{Mp}(n) \longrightarrow \mathrm{Sp}(n)$ by specifying its value on the generators $\widehat{S}_{W,m}$: by definition $S_W = \pi(\widehat{S}_{W,m})$ is the free symplectic matrix characterized by

$$(x, p) = S_W(x', p') \Longleftrightarrow \begin{cases} p = \partial_x W(x, x') \\ p' = -\partial_{x'} W(x, x') \end{cases}.$$

One readily checks by an explicit calculation that

$$S_W = \begin{pmatrix} L^{-1}Q & L^{-1} \\ PL^{-1}Q - L^T & L^{-1}P \end{pmatrix};$$

equivalently, if $S_W = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a free symplectic matrix, we have $S_W = \pi(\pm\widehat{S}_{W,m})$ with

$$W(x, x') = \tfrac{1}{2}DB^{-1}x^2 - B^{-1}x \cdot x' + \tfrac{1}{2}B^{-1}Ax^2.$$

An essential result is the following:

Proposition 14 *Every* $\widehat{S} \in \mathrm{Mp}(n)$ *can be written as a product* $\widehat{S}_{W,m}\widehat{S}_{W',m'}$ *where* $S_W = \pi(\widehat{S}_{W,m})$ *and* $S_{W'} = \pi(\widehat{S}_{W',m'})$ *are such that* $\det(S_W - I) \neq 0$ *and* $\det(S_{W'} - I) \neq 0$*, that is* $S_W, S_{W'} \in \mathrm{Sp}^{\pm}(n)$*.*

Proof. The proof that every $\widehat{S}$ can be written $\widehat{S}_{W,m}\widehat{S}_{W',m'}$ can be found in [11]; it relies on the fact that the symplectic group acts transitively on pairs of transverse Lagrangian planes. That $\widehat{S}_{W,m}$ and $\widehat{S}_{W',m'}$ can be in addition be chosen such that $\det(S_W - I) \neq 0$ and $\det(S_{W'} - I) \neq 0$ was established in [12] (also see [13]). ■

In [12] (also see [13, 14, 15] for details and comments) we have shown that:

Theorem 15 *(i) Let* $S \in \mathrm{Sp}(n)$ *be such that* $\det(S - I) \neq 0$*. Then* $\widehat{S} \in \mathrm{Mp}(n)$ *has projection* $S = \pi(\widehat{S})$ *if and only if*

$$\widehat{S} = \left(\frac{1}{2\pi}\right)^n \frac{i^{\nu(\widehat{S})}}{\sqrt{|\det(S-I)|}} \int \exp\left[\frac{i}{2}M_S z^2\right] \widehat{T}(z) d^{2n}z \tag{42}$$

where $\widehat{T}(z)$ *is the Weyl–Heisenberg operator:*

$$\widehat{T}(z)f(x') = e^{i(px' - \frac{1}{2}px)} f(x' - x),$$

M_S *the symplectic Cayley transform (18) of* S*, and*

$$\nu(\widehat{S}) = i_{CZ}(\Sigma) \mod 4 \tag{43}$$

where $\Sigma \in \mathcal{C}^{\pm}(\mathrm{Sp}(n))$ *has endpoint* S*.*

In [9, 11] we have shown that if $\widehat{S} = \widehat{S}_{W,m}$ then $m = \mathrm{Mas}(\Sigma)$ where Σ is defined as above; hence, taking formula (24) in Proposition (7) into account:

Corollary 16 *Let* $\widehat{S}_{W,m}$ *be such that* $\det(S_W - I) \neq 0$*. Then*

$$\widehat{S}_{W,m} = \left(\frac{1}{2\pi}\right)^n \frac{i^{m - \mathrm{Inert}\, W_{xx}}}{\sqrt{|\det(S-I)|}} \int \exp\left[\frac{i}{2}M_S z^2\right] \widehat{T}(z) d^{2n}z \tag{44}$$

with $W_{xx} = P + Q - L - L^T$ *when* W *is given by (41).*

5.2 Explicit formulae

We will content ourselves her with the hyperbolic, inverse hyperbolic, and elliptic cases. The loxodromic case is calculated similarly.

5.2.1 The elliptic case

We have $S = S_{\mathrm{ell}} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$; a trivial calculation yields

$$\det(S_{\mathrm{ell}} - I) = 4\sin^2\frac{\alpha}{2}\ ,\ M_{\mathrm{ell}} = \frac{1}{2}\begin{pmatrix} \cot\frac{\alpha}{2} & 0 \\ 0 & \cot\frac{\alpha}{2} \end{pmatrix}$$

(M_{ell} the symplectic Cayley transform of S_{ell}) hence, in view of formulae (42), (43), and (31),

$$\widehat{S}_{\text{ell}} = \frac{1}{4\pi i |\sin \frac{\alpha}{2}|} \int \exp\left[\frac{i}{4}|z|^2 \cot \frac{\alpha}{2}\right] \widehat{T}(z) dz.$$

5.2.2 The hyperbolic and inverse-hyperbolic cases

Assume first that $S = S_{\text{hyp}} = \begin{pmatrix} e^{\beta} & 0 \\ 0 & e^{-\beta} \end{pmatrix}$ and

$$\det(S_{\text{hyp}} - I) = -4 \sinh^2 \frac{\beta}{2} \;,\; M_{\text{hyp}} = \frac{1}{2} \begin{pmatrix} 0 & \tanh \frac{\beta}{2} \\ \tanh \frac{\beta}{2} & 0 \end{pmatrix}$$

hence, taking formula (32) into account,

$$\widehat{S}_{\text{hyp}} = \frac{1}{4\pi \sinh \frac{\beta}{2}} \int \exp\left[\frac{i}{4} px \tanh \frac{\beta}{2}\right] \widehat{T}(z) dz.$$

Suppose next that $S = S_{\text{inv}} = \begin{pmatrix} -e^{\beta} & 0 \\ 0 & -e^{-\beta} \end{pmatrix}$. Then

$$\det(S_{\text{inv}} - I) = 4 \cosh^2 \frac{\beta}{2} \;,\; M_{\text{inv}} = -\frac{1}{2} \begin{pmatrix} 0 & \tanh \frac{\beta}{2} \\ \tanh \frac{\beta}{2} & 0 \end{pmatrix};$$

using (33) we get

$$\widehat{S}_{\text{inv}} = \frac{1}{4\pi i \cosh \frac{\beta}{2}} \int \exp\left[-\frac{i}{4} px \tanh \frac{\beta}{2}\right] \widehat{T}(z) dz.$$

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