Topology of Complete Intersections

Fuquan FANG

Nankai Institute of Mathematics
Nankai University
Tianjin 300071

P.R.CHINA

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

GERMANY

MPI 96-27

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§1. Introduction

The main aim of this paper is to give several homeomorphism and homotopy classification theorems for complete intersections without singularities. Partial results here have been announced in [6]. Recall that a complete intersection is the transversal intersection of some complex hypersurfaces defined by the homogeneous polynomials in a complex projective space. Below we will use $X_n(\mathbf{d})$ to denote the complete intersection with multi degree $\mathbf{d} = (d_1, ..., d_r)$ and complex dimension n. We call the product $d_1 \cdots d_r := d$ the total degree. It is a classical observation of R.Thom that the differential topology of $X_n(\mathbf{d})$ is determined by the multi degree and dimension n. Lefschetz hyperplane section Theorem asserts that the inclusion

$$X_n(\mathbf{d}) \to CP^{n+r}$$

is an n-equivalence.

In lower dimensions, the topology of complete intersections are well understood by the general theory of differential topology. For instance, $X_1(\mathbf{d})$ is a complex curve of genus $g = 1 - \frac{d}{2}(r + 2 - \sum_{i=1}^{r} d_i)$.
$X_2(\mathbf{d})$ is a simply connected complex surface. The homotopy and homeomorphism type is determined by its intersection form[8]. An interesting example of Ebeling [5] asserts that there are two complete intersections with the same homeomorphism type but not diffeomorphic.
$X_3(\mathbf{d})$ is a simply connected 3-dimensional complex manifold with all homology groups torsion free. A complete classification of such kinds of manifolds was done by C.T.C.Wall[20] and Jupp[10].
$n = 4$ is the first nontrivial dimension in which we can not refer to any classical classification theory. In [7], S.Klaus and I proved that two 4-dimensional complete intersections are homeomorphic if and only if their total degrees, Euler numbers and all Pontryagin numbers agree. Even in this special dimension, the homotopy classification for complete intersections is still open.

On the other hand, some general classification for complete intersections were carried out under certain restriction about the total degree d. For example, under the assumption

*Supported in partial by Sonderforschungsbereich 343 at Universität Bielefeld and the Max-Planck-Institut fuer Mathematik at Bonn. The author would like to thank both for their hospitality.

that for all prime p with $p(p-1) \leq n+1$, the total degree d is divisible by $p^{[(2n+1)/(2p-1)]+1}$, Traving[19](c.f: [12]) proved that two complete intersections with the same total degree d are diffeomorphic if and only if their Euler numbers and all Pontryagin classes agree. For the homotopy classification, Libgober and Wood[14] proved that, if the dimension n is odd and the total degree d has no prime factors less than $\frac{n+3}{2}$, then two n-dimensional complete intersections with total degree d are homotopy equivalent if and only if their Euler numbers agree. They made a further conjecture [16] for the case of n is even. In this situation, the topology becomes much more complicated. Some more details can be looked up in [15][16].

In this paper, the following homotopy classification theorem will be proved. The proof confirms also Libgober-Wood's conjecture.

Theorem 1.1. *Let n be even, $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ be two complete intersections with the same total degree d. Suppose that d has no small prime factors less than $\frac{n+3}{2}$. Then $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are homotopy equivalent if and only if they have the same Euler charateristic and signatures.*

Once the homotopy type of two complete intersections are the same, Sullivan's charateristic variety theory can be applied to handle the problem of when they are homeomorphic or diffeomorphic. For a complete intersection $X_n(\mathbf{d})$, the Pontryagin classes p_i must be an integral multiple of x^{2i}, where x is a 2-dimensional generator of $X_n(\mathbf{d})$. Thus we can compare Pontryagin classes of two different complete intersections with the same total degree d just by means of these integers.

Theorem 1.2. *Let $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ be two homotopy equivalent complete intersections. If d is odd and $n \neq 2^i - 2$, then they are homeomorphic to each other if and only if their Pontryagin classes agree.*

The proof of the above theorem can not be extended to the case of d even. We shall explain this with more details in §3. Combining this with the homotopy classification Theorem above and [14] for n odd, the homeomorphism classification in the case of $n \neq 2^i - 2$ and d has no prime factors less than $\frac{n+3}{2}$. With a little bit more argument we have the following corollary.

Corollary 1.3. *If $n \geq 3$, $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are two complete intersections with the same total degree d. Suppose that d has no small prime factors less than $\frac{n+3}{2}$. Then $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are homeomorphic if and only if their Pontryagin classes and Euler numbers agree.*

When I had finished this paper, M.Kreck pointed out to me that under the same assumption of the above corollary, S.Stolz has a unpublished version to assert the two complete intersections are even diffeomorphic.

Another very natural question is to ask, if $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are diffeomorphic/or homeomorphic/or homotopy equivalent, is $X_n(\mathbf{d}, a)$ and $X_n(\mathbf{d}', a)$ diffeomorphic for a natural number a?

To phrase our relevant results on this question, we make a convention, namely, we say two $2n$-dimensional manifolds M and N are *S-diffeomorphic(homeomorphic, homotopy equivalent)* if there are integers r and s so that $M\#rS^n \times S^n$ and $N\#sS^n \times S^n$ are diffeomorphic(homeomorphic, homotopy equivalent). The following theorem answer the above question partially.

Theorem 1.4. *Let $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ be two S-diffeomorphic(homeomorphic, homotopy equivalent) complete intersections. If $a_1, \cdots, a_k$ are positive integers satisfying*

$$max\{a_1, \cdots, a_k\} \leq min\{\mathbf{d}, \mathbf{d}'\}.$$

Then $X_n(\mathbf{d}, a_1, \cdots, a_k)$ and $X_n(\mathbf{d}', a_1, \cdots, a_k)$ are S-diffeomorphic(homeomorphic, homotopy equivalent).

Without loss of generality we can always assume that the multi degree $\mathbf{d}$ does not contain 1. By the above theorem, if $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are S-equivalent, then so are $X_n(\mathbf{d}, 2, \cdots, 2)$ and $X_n(\mathbf{d}', 2, \cdots, 2)$.

The project of the current paper was begun during my visit to Univerty of Mainz. The discussions with M.Kreck proved to be valuable. I would like to thank him for his warm hospitality.

§2. Homotopy type

For n odd, every smooth complete intersection can be splitted as the connected sum $K\#rS^n \times S^n\#N$ in the topological category, where K satisfies $H_n(K) = 0$ and N is $(n-1)$-connected with $H_n(N) \cong Z \oplus Z$. K is called the topological core of the complete intersection. We let $K_n(\mathbf{d})$ denote the corresponding topological cores of $X_n(\mathbf{d})$. When $n = 1, 3$ or 7, the piece N will be $S^n \times S^n$. For other n, this holds if and only if either there is homological trivial embedded n-sphere in $X_n(\mathbf{d})$ with nontrivial normal bundle or the Kervaire invariant of a well-defined quadratic function on $H^n(X_n(\mathbf{d}), Z_2)$ vanishes. The Kervaire invariant of complete intersection was well investigated by several authors during 70's. The main results are:

Proposition 2.1.(J.Wood [21]) *There is no homological trivial n-sphere in $X_n(\mathbf{d})$ with nontrivial normal bundle if and only if*

(**a**) : *The binomial coefficient $\begin{pmatrix} m+l \\ m+1 \end{pmatrix}$ is even, where $n = 2m+1, \neq 1, 3, 7$ and l is the number of even entries in* $\mathbf{d}$.

If (a) holds, then there is a well defined quadratic function

$$q : H^n(X_n(\mathbf{d}), Z_2) \to Z_2$$

by $q(x)$ the normal bundle of an embedded n-sphere representing the dual of x. The Kervaire invariant is just the Arf invariant of q. We denote it by $K(X_n)$.

Theorem 2.2.(Browder[3], Morita, Wood[21]) *If d is odd,*

$$K_n(X_n(\mathbf{d})) = \begin{array}{l} 0 \text{ } if\, d = \pm 1(mod 8) \\ 1 \text{ } if\, d = \pm 3(mod 8) \end{array}$$

If d is even, $K(X_n(\mathbf{d}) = 1$ if and only if $n = 1(mod8)$, $l = 2$ and d is not divisible by 8.

Note in the case of $K(X_n(\mathbf{d}) = 1$, the piece N is the Kervaire manifold.

For n even, the situation is quite different, the rank of $H_n(K_n(\mathbf{d}))$ can never be zero since not all element of $H_n(X_n(\mathbf{d}))$ are spherical. By [15], one can get a topological splitting and topological core $K_n(\mathbf{d})$ which has rank $H_n(K_n(\mathbf{d})) \leq 5$. The precise value of this minimal rank depends on the type of the intersection form as well as the total degree d. It is readily to see that, at least up to homotopy this topological cores is unique.

For n odd, the cohomology ring $H^*(K_n(\mathbf{d})) \cong Z[x,y]/\{x^{\frac{n+1}{2}} = dy, y^2 = 0\}$. If d has no prime factors less than $\frac{n+3}{2}$, it is proved in [14] that $K_n(\mathbf{d})$ has the homotopy type of the $2n$-skeleton of E, the homotopy fibre of $x^{\frac{n+1}{2}} : CP^\infty \to K(Z_d, n+1)$. Thus it depends only on d, the total degree. The similar method does not work in the case of n even. But Libgober and Wood make a conjecture that the same conclusion holds when n is even. This is more or less equivalent to Theorem 1.1. We will give a proof of this fact base on surgery theory of F. Quinn and Freedman.

Let us recall some notations and main results in the form useful for our purpose in the surgery theory of Quinn.

Let M be a manifold of dimension $2n$ and N a codimension 2 submanifold. Let $C = M - intU$ where U denote a tubular neighborhood of N. We say N is taut if the pair $(C, \partial C)$ is $(n-1)$-connected. It is proved in [11] that every codimension 2 homology class can be represented by an embedded taut submanifold. This was generalized by F.Quinn which we will introduce now adapted for our purpose.

Let $f : M \to X$ be a map transversal to a CW subcomplex $Y \subset X$ where Y has a 2 dimensional normal bundle. Let $E(f,Y)$ and $E(f, X-Y)$ denote the fibre spaces over M

$$\begin{array}{ccc} E(f,Y) & \longrightarrow & Y \\ \downarrow & & \downarrow inclu \\ M & \xrightarrow{f} & X \end{array}$$

and

$$\begin{array}{ccc} E(f,X-Y) & \longrightarrow & X-Y \\ \downarrow & & \downarrow inclu \\ M & \xrightarrow{f} & X \end{array}$$

f is called almost canonical with respect to Y if the natural maps $f^{-1}(Y) \to E(f,Y)$ and $f^{-1}(X-Y) \to E(f, X-Y)$ are $(n-1)$ and n equivalences respectively. When f is a homotopy equivalence, the maps $f : f^{-1}(Y) \to Y$ and $f : f^{-1}(X-Y) \to X-Y$ are $n-1$ and n equivalences. The following theorem of F.Quinn is very useful for us.

Theorem (F.Quinn[17]). *Let $Y \subset X$ have a dimension 2 normal bundle neighborhood. For every map $f : M \to X$ is homotopic holding the boundary fixed to an almost canonical one with respect to Y.*

Proof of Theorem 1.1. Obivious we need only to show the sufficiency. Let $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ be two complete intersections with the same total degree d and same signature, where n is even and d has no prime factors less than $\frac{n+3}{2}$. First we shall show that there is integers s and t so that $X_n(\mathbf{d})\#sS^n \times S^n$ and $X_n(\mathbf{d}')\#tS^n \times S^n$ are homotopy equivalent.

By [14], the cores $K_{n+1}(\mathbf{d})$ and $K_{n+1}(\mathbf{d}')$ are homotopy equivalent by the hypothesis on d. In this case, d is of course odd and so the Kervaire invariant of $X_{n+1}(\mathbf{d})$ and $X_{n+1}(\mathbf{d}')$are the same. Without loss of generality, assume that $rankH_{n+1}(X_{n+1}(\mathbf{d})) \geq rankH_{n+1}(X_{n+1}(\mathbf{d}'))$. Then there is an integer r so that $X_{n+1}(\mathbf{d}) \simeq X_{n+1}(\mathbf{d}')\#rS^{n+1} \times S^{n+1}$. Let $f : X_{n+1}(\mathbf{d}) \to X_{n+1}(\mathbf{d}')\#rS^{n+1} \times S^{n+1}$ denote such a homotopy equivalence.

Notice both $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are taut submanifolds of $X_{n+1}(\mathbf{d})$ and $X_{n+1}(\mathbf{d}')\#rS^{n+1}\times S^{n+1}$ representing the $2n$-dimensional homology generators respectively. By Quinn's theorem above, we can assume that f is almost canonical with respect to the submanifold $X_n(\mathbf{d}')$. Since f is a homotopy equivalence, it follows that $f^{-1}(X_n(\mathbf{d}'))$ is also a taut submanifold of $X_{n+1}(\mathbf{d})$ representing the codimension 2 homological generator. The uniqueness theorem of Freedman[9] says that $X_n(\mathbf{d})$ and $f^{-1}(X_n(\mathbf{d}'))$ are stably diffeomorphic. Note the tautness is invariant under sum some trivial $S^n \times S^n$. Thus we may assume that $f^{-1}(X_n(\mathbf{d}')) \cong X_n(\mathbf{d})\#r'S^n \times S^n$.

This gives us a map $f : X_n(\mathbf{d})\#r'S^n \times S^n \to X_n(\mathbf{d}')$ which is a n-equivalence. Moreover, f is a degree one map since the two complete intersection have the same total degree. It follows that the sublattice $Kerf_* \subset H_n(X_n(\mathbf{d})\#r'S^n \times S^n)$ is unimodular. Notice that $Kerf_*$ consists of spherical elements. This can be seen by looking at the commutative square of the Hurewicz homomorphisms and so it is of even type. The signature of this sublattice is exactly the difference of the target and source manifolds. Thus it is zero. This shows that the unimodular lattice $Kerf_*$ is isomorphic to the sum of some copy of the hyperbolic plane H, say mH. As in [16], there is a decomposition $M\#mS^n \times S^n = X_n(\mathbf{d})\#r'S^n \times S^n \to X_n(d')$. Moreover, as f is null homotopy when restricted to $mS^n \times S^n - disc$, we get a map $f' : M \to X_n(\mathbf{d}')$ which is in fact a homotopy equivalence. This proves that $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are S-homotopy equivalent.

In the case of $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ have the same Euler numbers, the number $m = r'$. That is, by summing the same copy of $S^n \times S^n$, the two complete intersections are homotopy equivalent.

By the same argument as in [16] Proposition 3.3, the cores of these two completes intersections are homotopy equivalent. Thus $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are also homotopy equivalent. This completes the proof. ♣

Remark: The proof above shows in fact also that the conjecture in [16] p126 holds true.

As pointed out in [16], the condition about d in Theorem 1.1 is sharp. By using

K-theory one can get some more strong restrictions to the multi degrees of two homotopy equivalent complete intersections. To illustrate our method, we give a proof of the following proposition.

Proposition 2.3. *Let $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ be two complete intersections with homotopy equivalent cores. Let l and l' denote the numbers of even entires in $\mathbf{d}$ and $\mathbf{d}'$ respectively. Then $l - l'$ is divisible by $2^{f(n)-1}$. Here $f(n)$ is as the following table*

1	2	3	4	5	6	7	8	$\cdots$	$m+8$
1	2	2	3	3	3	3	4	$\cdots$	$f(m)+4$

Proof: Note that the stable normal bundle of $X_n(\mathbf{d})$ is $H^{d_1}\oplus\cdots H^{d_r}-(n+r+1)H$, here H is the Hopf line bundle over $X_n(\mathbf{d})$. If $K_n(\mathbf{d})$ and $K_n(\mathbf{d}')$ are homotopy equivalent, by the decompositions there are two $(n-1)$-connected almost parallelizable manifolds, saying M and M' such that $X_n(\mathbf{d})\#M$ and $X_n(\mathbf{d}')\#M'$ are homotopy equivalent. We warn that M and M' are not necessarily smoothable. By Atiyah [2], the stable normal spherical fibrations of $X_n(\mathbf{d})\#M$ and $X_n(\mathbf{d}')\#M'$ are equivalent up to fibre homotopy. The restriction on $(2n-2)$-skeleton of these normal spherical fibrations are exactly the restriction of the stable normal bundles of $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ since M and M' are almost parallelizable. In particular, the restrictions of the normal bundles of $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ to $CP^{[\frac{n}{2}]}$, the subcomplex of the complete intersections are proper fibrewise homotopy equivalent. In other words, viewed as the element of the J-group $J(CP^{[\frac{n}{2}]})$, $H^{d_1}\oplus\cdots H^{d_r}-(n+r+1)H$ and $H^{d'_1}\oplus\cdots H^{d'_{r'}}-(n+r'+1)H$ are the same.

Consider the canonical S^1-fibration $\pi : RP^{2[\frac{n}{2}]+1}\to CP^{[\frac{n}{2}]}$. The complex line bundle $\pi^*(H^{d_i})$ has trivial first Chern class if and only if d_i is even as $H^2(RP^{2[\frac{n}{2}]+1})\cong Z_2$. Moreover, when d_i is odd, $\pi^*(H^{d_i})\cong\pi^*(H)$. Thus the difference $\pi^*(H^{d_1}\oplus\cdots H^{d_r}-(n+r+1)H)-\{H^{d'_1}\oplus\cdots H^{d'_{r'}}-(n+r'+1)H\}=\pi^*(l'-l)H\in J(RP^{2[\frac{n}{2}]+1})$ is zero. Note $\pi^*(H)=2\eta\in KO(RP^{2[\frac{n}{2}]+1})$, where η is the Hopf real line bundle. Thus $2(l'-l)$ must be a multiple of the order of the J-group $J(RP^{2[\frac{n}{2}]+1})$ which is $2^{f(2[\frac{n}{2}]+1)}$[1]. This completes the proof. ♣

§3. Sullivan's characteristic variety

This section is devoted to a proof of Theorem 1.2 by using Sullivan's charateristic variety theory[18]. It had received considerable attentions that when two homotopy equivalent manifolds are homeomorphic or diffeomorphic. Sullivan's charateristic variety theory is a very powerful approach to this problem. For reader's convenience, we recall several main results in this theory adapted for our purpose which will be used below.

Let M be an oriented PL m-manifold whose oriented boundary is the disjoint union of n-copies of closed oriented $(m-1)$-manifolds L(with the induced orientations). We call the polyhedron V obtained from M by identifying these copies of L to one another a *Z_n-manifold*. We denote by $L\subset V$ by δV, the Bockstein of V.

A finite disjoint union of Z_n-manifolds for various n's and of various dimensions is called a *variety*. If X is a polyhedron, a *singular variety* in X is a piecewise linear map, $f : V\to X$, of a variety V to X.

The Z_n manifold provides a nice model for Z_n-homology classes since every Z_n-manifold V carries a well-defined fundamental class in $H_m(V; Z_n)$. Clearly closed manifold is a Z_n-manifold for each n with Bockstein the empty.

For a homotopy equivalence $f : L \to M$, where L, M are closed PL manifold. Let $V \to M$ be an embedded connected singular Z_n-manifold of dimension v. Assuming that M, V and δV are all simply connected and $dimM \geq 3$. If $v = 2s$ is even, then f can be deformed to a map f' so that:
(i): f' is transversal regular to $(V, \delta V)$ with $U = f'^{-1}(V)$ and $\delta U = f'^{-1}(\delta V)$.
(ii): $f'^{-1} : \delta U \to \delta V$ is a homotopy equivalence.
(iii): $f' : U \to V$ is s-connected.

Let $K_s = ker f'_* \subset H_s(U, Z)$. This is a unimodular form. Moreover, when s is even, it is of even type and so its signature is divisible by 8. When s is odd, one has an Arf invariant in Z_2.

By Sullivan, the *splitting obstruction* $\theta_f(V)$ of $f : L \to M$ along V is defined as the the Arf invariant of K_s if s is odd, $\frac{signK_s}{8}(modn)$ if $s \neq 2$ even and $\frac{signK_s}{8}(mod2n)$ if $s = 2$.

In general the splitting invariants $\theta_f(V)$ of a nonconnected singular variety V is defined as the collection of the corresponding invariant along these connected components.

The characteristic variety Theorem(Sullivan[18]) *Let $f : L \to M$ be a homotopy equivalence between two simply connected PL n-manifolds L and M. Assuming $n \geq 6$. Then there is a (charateristic) singular variety in M, $V \to M$, so that f is homotopic to a PL homeomorphism if and only if the splitting invariants of f along V is identically zero.*

To apply this theorem, it is important to get a charateristic variety for the given manifold. There is no a general way to define it. For the complex projective space CP^n, as noted in [18], the charateristic variety is the union $CP^2 \cup CP^3 \cup \cdots \cup CP^{n-1} \subset CP^n$. We will show below that, for a complete intersection $X_n(\mathbf{d})$ where $n = 2m+1$ and d is odd, the collection of hypersections $X_3(\mathbf{d}) \cup X_5(\mathbf{d}) \cup \cdots \cup X_{2[\frac{m}{2}]+1}(\mathbf{d}) \cup X_{2[\frac{m}{2}]+2}(\mathbf{d}) \cup \cdots \cup X_{n-1}(\mathbf{d}) \subset X_n(\mathbf{d})$ and $CP^2 \cup CP^4 \cup \cdots \cup CP^{2[\frac{m}{2}]} \subset X_n(\mathbf{d})$ representing the generators of $4i(1 \leq i \leq [\frac{m}{2}])$ dimensional homology groups is a characteristic variety. Thus by Sullivan's theorem above, two odd dimensional homotopy equivalent complete intersections with odd degree are PL homeomorphic if and only if these splitting invariants along these singular manifolds are the same.

Lemma 3.1. *If n and d are both odd. $n \geq 5$. Then*

$$V = \cup_{i=1}^{[\frac{m}{2}]} X_{2i+1}(\mathbf{d}) \cup \cup_{i=2[\frac{m}{2}]+2}^{n-1} X_i(\mathbf{d}) \cup \cup_{i=1}^{[\frac{m}{2}]} CP^{2i} \subset X_n(\mathbf{d})$$

is a charateristic variety.

Proof: Note that $X_n(\mathbf{d})$ has no odd dimensional homology and all homology groups are no torsion free. The K-homology group $KO_{-1}(X_n(\mathbf{d})) \otimes Z_{(odd)} = 0$ since $\Sigma X_n(\mathbf{d})$ has a cell decomposition with only odd dimensional cells. Moreover, $Sq^2 : H^2(X_n(\mathbf{d}), Z_2) \to$

$H^4(X_n(\mathbf{d}), Z_2)$ is an isomorphism. By the proof of Sullivan's charateristic variety Theorem(refer to [18]page 33, 34), we need only to show the variety satisfies:
(i). A basis of $\oplus_{i\geq 1, \neq \frac{n-1}{2}} H_{4i+2}(X_n(\mathbf{d}), Z_2)$ can be represented by the fundamental classes of this variety.
(ii). The image of the oriented bordism classes of the variety under the natural maps S_* and I_* below in the groups $\Omega^{so}_{4*}(X_n(\mathbf{d})) \otimes_{\Omega^{so}_*} Z_{(odd)}$ and $\oplus_{i\geq 1} H_{4i}(X_n(\mathbf{d}))$ are basis, where

$$I_* : \Omega^{so}_{4*}(X_n(\mathbf{d})) \to \Omega^{so}_{4*}(X_n(\mathbf{d}) \otimes_{\Omega^{so}_*} Z_{(odd)}/torsion$$

is the natural projection and

$$S_* : \Omega^{so}_{4*}(X_n(\mathbf{d})) \stackrel{fundamental\ class}{\longrightarrow} \oplus_{i\geq 1} H_{4i}(X_n(\mathbf{d}))/torsion.$$

(i) is clearly satisfied by our variety since d is odd. To verify (ii), note that $\Omega^{so}_{4*}(X_n(\mathbf{d})) \otimes_{\Omega^{so}_*} Z_{(odd)} \cong H_{4*}(X_n(\mathbf{d}), \Omega^{so}_*) \otimes_{\Omega^{so}_*} Z_{(odd)}$ is torsion free. As all $4i$ dimensional homology generators are represented by our variety, this completes the proof. ♣

The characteristic variety for n even is more complicated since we have to count the middle dimensional homology and represent them by singular manifolds.

Let $x \in H^2(X_n(\mathbf{d})$ be a generator where n is even. We use h denote $x^{\frac{n}{2}} \cap [X_n(\mathbf{d})]$. By [15], the image of Hurewicz homomorphism $\pi_n(X_n(\mathbf{d})) \to H_n(X_n(\mathbf{d})) := H$ is the orthogonal complement $h^\perp$. Let $\beta \in H_n(X_n(\mathbf{d}))$ satisfy $u \cdot h = 1$. Then $H = h^\perp + Z\beta$. Notice this is not an orthogonal decomposition. By [15], every element in $h^\perp$ can be represented by an embedded n-sphere with stably trivial normal bundle if $n \geq 2$. β can be represented by an embedded $CP^{\frac{n}{2}}$ with normal bundle $(\frac{n}{2}+r)H - \sum_1^r H^{d_i}$. Choose a basis of $h^\perp$ and represent them by embedded n spheres $\alpha_1, \cdots, \alpha_k$. Similar to Lemma 2.2 it is readily to check the following lemma. We omit the details.

Lemma 3.2. *Let $n = 2m \geq 2$ and d be odd. Then*

$$V = \cup_{i=1}^{[\frac{m}{2}]-1} X_{2i+1}(\mathbf{d}) \cup \cup_{i=2[\frac{m}{2}]}^{n-1} X_i(\mathbf{d}) \cup \cup_{i=1}^{[\frac{m}{2}]} CP^{2i} \cup \beta(CP^m) \cup \cup_{i=1}^k \alpha_i(S^{2m}) \subset X_n(\mathbf{d})$$

is a charateristic variety.

In general, we can also write down a charateristic variety for a complete intersection with d even. But it is difficult to compute the Arf type splitting invariant. Now we are ready to show the theorem 1.2.

Proof of Theorem 1.2. We need only to show the sufficiency. Let $f : X_n(\mathbf{d}) \to X_n(\mathbf{d}')$ be a homotopy equivalence. By the charateristic variety Theorem we need to show the spliiting invariant $\theta_f(V) = 0$, where V denote the variety defined above.

Let's consider first the case of n odd. Notice that the splitting invariant along $4i$-dimension subvariety, saying $X_{2i}(\mathbf{d}')$(or CP^{2i}), is the difference $Signf^{-1}(X_{2i}(\mathbf{d}')) - SignX_{2i}(\mathbf{d}')$. Now $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ have the same Pontryagin classes. Applying Hirzebruch signature Theorem, it is easy to show that all splitting invariants along $4i(1 \leq i \leq \frac{n}{2})$ dimensional subvarieties vanish.

The only difficulty is to show the Arf type splitting invariants are all zero along V. Fortunately the main difficulty have been overcome by Browder and Wood. When d is odd, by [3] or [21], the Kervaire invariants of $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are well defined and its value depends only on the total degree $d(mod8)$(independent of the dimension). Let us now show the splitting invariant along $X_{n-2}(\mathbf{d}')$ vanishes.

By Quinn's theorem, we can assume that f is almost canonical with respect to $X_{n-1}(\mathbf{d}')$. Thus $f^{-1}(X_{n-1}(\mathbf{d}'))$ is a taut submanifold of $X_n(\mathbf{d})$ representing the dual of the generator $x \in H^2(X_n(\mathbf{d}))$. Freedman's theorem[9] applies to claim that $f^{-1}(X_{n-1}(\mathbf{d}'))$ and $X_{n-1}(\mathbf{d})$ are stably diffeomorphic. Consider the restricted map $g : f^{-1}(X_{n-1}(\mathbf{d}')) \to X_{n-1}(\mathbf{d}')$, applying Quinn's and Freedman's Theorems again we can deform g to get a taut submanifold $g^{-1}(X_{n-2}(\mathbf{d}'))$ which is stably diffeomorphic to $X_{n-2}(\mathbf{d})$. When $n-2 \neq 1, 3$ or 7, the Kervaire invariant is a stably diffeomorphic invariant by the geometric definition. Therefore, with the exception of $n = 1, 3$ or 7, the Kervaire invariant of $g^{-1}(X_{n-2}(\mathbf{d}'))$ is the same as that of $X_{n-2}(\mathbf{d})$ and so as $X_{n-2}(\mathbf{d}')$. By the naturality of the splitting obstruction, the splitting invariants of g and f along $X_{n-2}(\mathbf{d}')$ are the same. Notice the splitting invariant of g along $X_{n-2}(\mathbf{d}')$ is the difference of the Kervaire invariants of $g^{-1}(X_{n-2}(\mathbf{d}'))$ and $X_{n-2}(\mathbf{d}')$ which is identically zero. This proves the splitting invariant along $X_{n-2}(\mathbf{d}')$ vanishes. Continuing this process we can show that the splitting invariants along $X_i(\mathbf{d}')$(i odd) is zero if $i \geq 8$.

When $i = 7$, we have to take care of the framing. Notice that the Kervaire invariant is a framed bordism invariant. If $f : X_8(\mathbf{d}) \to X_8(\mathbf{d}')$ is a degree 1 map and 8-equivalence. The transversal preimage $f^{-1}(X_7(\mathbf{d}'))$ is normal bordant to $X_7(\mathbf{d})$ since they both represent the 14-th dimensional homology generator. Thus the splitting obstruction along $X_7(\mathbf{d}')$ is zero too. The case of $i = 3$ is identically. One can also refer to [7] for this detail. This completes the proof in the case of n odd.

For $n \neq 2$ even, everything applies identically except we have to count the splitting invariants along the subvarieties $\alpha_i(S^n)$ and $CP^{\frac{n}{2}}$ if $n = 0(mod4)$. If $n = 0(mod4)$, these splitting invariant along α_i is the signature of its transversal preimage $f^{-1}(\alpha_i)$ which is zero since its all Pontryagin classes are zero. The splitting invariant along $CP^{\frac{n}{2}}$ is $Sign f^{-1}(\beta) - 1$. By using Hirzebruch signature Theorem one can check directly this is zero. For $n = 2(mod4)$ but $n \neq 2^i - 2$, the splitting invariant along α_i is the Kervaire invariant of $f^{-1}(\alpha_i)$, which is a smooth framed manifold of dimension $n \neq 2^i - 2$ and thus its Kervaire invariant vanishes [4].

Now Sullivan's Theorem applies to conclude our Theorem. ♣

Proof of Corollary 1.3. By [14], Theorem 1.1 and 1.2, we need only to consider the case of n even and to show the sufficiency. Note that $X_{n+1}(\mathbf{d})$ and $X_{n+1}(\mathbf{d}')$ are S-homotopy equivalent. Thus there is an integer such that, $X_{n+1}(\mathbf{d})\#rS^{n+1} \times S^{n+1}$ and $X_{n+1}(\mathbf{d}')$ are homotopy equivalent. It is easy to check that all of the Pontryagin classes of these two manifolds are the same too. Applying Theorem 1.2(with a slightly extension but identically proof) we have that $X_{n+1}(\mathbf{d})\#rS^{n+1} \times S^{n+1}$ and $X_{n+1}(\mathbf{d}')$ are homeomorphic. Notice that $X_n(\mathbf{d}) \subset X_{n+1}(\mathbf{d})\#rS^{n+1} \times S^{n+1}$ and $X_n(\mathbf{d}') \subset X_{n+1}(\mathbf{d}')$

are taut submanifolds. The Freedman's Theorem[9] applies to conclude that $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are stably homeomorphic. With the exception of $\mathbf{d}$ or $\mathbf{d}' = (1), (2), (2,2)$ or (3), the complete intersection $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ can split out a factor $S^n \times S^n$(c.f: [15]). Now applying the cancellation Theorem[12] the proof follows. ♣

§4. Proof of Theorem 1.4.

This section is devoted to prove the Theorem 1.4. The main idea is to use the branched covering $X_n(\mathbf{d}, a) \to X_n(\mathbf{d})$ with branch set $X_{n-1}(\mathbf{d}, a)$ constructed in [21].

Proof of Theorem 1.4. Without loss of generality we consider only the case of $n \geq 4$. By induction we can assume $k = 1$. By [21], $X_n(\mathbf{d}', a)$ is an a-fold branched cover over $X_n(\mathbf{d}')$ with branch set $X_{n-1}(\mathbf{d}, a)$. Let $p' : X_n(\mathbf{d}', a) \to X_n(\mathbf{d}')$ be this cover. If $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are S-equivalent(homotopy, homeomphism or diffeomorphism). Assuming $rankH_n(X_n(\mathbf{d})) \geq rankH_n(X_n\mathbf{d}'))$. Thus there is a degree one map $f : X_n(\mathbf{d}) \to X_n(\mathbf{d}')$ which is an n-equivalence. By [17], we can assume that f is almost canonical with respect to $X_{n-1}(\mathbf{d}', a)$. Let $Y_{n-1} = f^{-1}(X_{n-1}(\mathbf{d}', a))$. Pulling back this covering to $X_n(\mathbf{d})$ we get a covering $\pi : Y_n \to X_n(\mathbf{d})$ with branch set Y_{n-1} and a map $g : Y_n \to X_n(\mathbf{d}', a)$ so that the following diagram commutes

$$\begin{array}{ccc} Y_{n-1} & \xrightarrow{f} & X_{n-1}(\mathbf{d}', a) \\ \downarrow\cap & & \downarrow\cap \\ Y_n & \xrightarrow{g} & X_n(\mathbf{d}', a) \\ \downarrow\pi & & \downarrow p' \\ X_n(\mathbf{d}) & \xrightarrow{f} & X_n(\mathbf{d}') \end{array}$$

Note that Y_n is a connected manifold, g is a degree one map. Notice that $g : Y_n - Y_{n-1} \to X_n(\mathbf{d}', a) - X_{n-1}(\mathbf{d}', a)$ is n-connected and the latter space is $(n-1)$-connected. Moreover, $f : Y_{n-1} \to X_{n-1}(\mathbf{d}', a)$ is a $(n-1)$-equivalence. It is easy to check that g is a $(n-1)$-equivalence. By Alexander duality, $H_q(Y_n, Y_{n-1}) \cong H^{2n-q}(Y_n - Y_{n-1}) = 0$ if $q \neq 0, n$ or $2n$. Note that the Euler class of the normal circle bundle of Y_{n-1} in Y_n is a generator of 2-dimensional cohomology group. By applying Gysin exact sequence it is easy to show that $H^{n-1}(Y_n) \cong Z$ if $n-1$ is even and 0 if $n-1$ is odd. Thus by the diagram above it is readily to check that g is an n-equivalence. When n is even, one can check that the signature of Y_n is the same as that of $X_n(\mathbf{d}', a)$. Thus Y_n and $X_n(\mathbf{d}', a)$ are S-equivalent(c.f the proof of Theorem 1.1).

Now we want to show that Y_n and $X_n(\mathbf{d}, a)$ are stably diffeomorphic. The theorem will follow from this directly. First notice that Y_{n-1} and $X_{n-1}(\mathbf{d}, a)$ are stably diffeomorphic[9] since both of them represent the dual of $ax \in H^2(X_n(\mathbf{d}))$, where x is a generator. Thus we have a homotopy

$$h : X_n(\mathbf{d}) \times I \to CP^N$$

(N large) such that $h_0^{-1}(X_{N-1}(a)) = X_{n-1}(\mathbf{d}, a)$ and $h_1^{-1}(X_{N-1}(a)) = Y_{n-1}$, where $X_{N-1}(a)$ is a hypersurface of degree a. (h_0 and h_1 indicate the restriction of h at the two boundary.) By Quinn[17], we can deform h relatively to the boundary to get an almost canonical map with respect to $X_{N-1}(a)$. Set $W = h^{-1}(X_{N-1}(a))$. W is a manifold with boundary $X_{n-1}(\mathbf{d}, a)$ and Y_{n-1} and the map $h : W \to X_{N-1}(a) \times I$ is a $(n-1)$-equivalence. Thus W is a $(n-2)$-connected cobordism, i.e., $H_q(W, Y_{n-1}) = H_q(W, X_{n-1}(\mathbf{d}, a)) = 0$ if $q \leq n-2$. In particular, the complement of W in $X_n(\mathbf{d}) \times I$ is cyclic of order a. Now we consider the branched cover M over $X_n(\mathbf{d}) \times I$ with branch set W. The boundary of M is the union of $X_n(\mathbf{d}, a)$ and Y_n with opposite orientations. It is easy to show that $H_q(M, Y_n) = H_q(M, X_n(\mathbf{d}, a)) = 0$ for $q \leq n-1$. Moreover, note that each embedded n-sphere in W has trivial normal bundle by the covering. Applying the handle subtraction technique[13] it follows that there are two integers s and t so that $Y_n \# sS^n \times S^n$ and $X_n(\mathbf{d}, a) \# tS^n \times S^n$ are h-cobordant. Thus Y_n and $X_n(\mathbf{d}, a)$ are stably diffeomorphic. This completes the proof. ♣

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Nankai Institute of Mathematics, Nankai University, Tianjin 300071, P.R.China

Current address: Max-Planck-Institut für Mathematik, Gottfried-Claren Strasse 26, D-53225 Bonn, Germany
e-mail: fang@mpim-bonn.mpg.de