Adiabatic Limits and Spectral Geometry of Foliations

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Abstract

We study spectral asymptotics for the Laplace operator on differential forms on a Riemannian foliated manifold equipped with a bundle-like metric in the case when the metric is blown up in directions normal to the leaves of the foliation. The asymptotical formula for the eigenvalue distribution function is obtained. The relationships with the spectral theory of leafwise Laplacian and with the noncommutative spectral geometry of foliations are discussed.

Introduction

Let $(M,\mathcal{F})$ be a closed foliated manifold, $\dim M = n$, $\dim \mathcal{F} = p$, $p+q=n$, equipped with a Riemannian metric g_M. We assume that the foliation $\mathcal{F}$ is Riemannian, and the metric g_M is bundle-like. Let $F = T\mathcal{F}$ be an integrable distribution of p-planes in TM, and $H = F^{\perp}$ be the orthogonal complement to F. So we have a decomposition of TM into a direct sum:

$$TM = F \bigoplus H. \tag{1}$$

The decomposition (1) induces the decomposition of the metric

$$g_M = g_F + g_H. \tag{2}$$

Define a one-parameter family g_h of metrics on M by the formula

$$g_h = g_F + h^{-2} g_H, 0 < h \leq 1. \tag{3}$$

For any $h > 0$, we have the Laplace operator on differential forms defined by the metric g_h:

$$\Delta_h = d^*_{g_h} d + d d^*_{g_h}, \tag{4}$$

where d is the de Rham differential:

$$d : C^\infty(M, \Lambda^k T^* M) \to C^\infty(M, \Lambda^{k+1} T^* M), \tag{5}$$

$d^*_{g_h}$ is the adjoint with respect to the metric on $C^\infty(M, \Lambda T^* M)$ induced by g_h. The operator Δ_h is a self-adjoint, elliptic differential operator with the positive definite, scalar principal symbol in the Hilbert space $L^2(M, \Lambda T^* M, g_h)$. By the standard perturbation theory, there are (countably many) analytic functions $\lambda_i(h)$ such that, for any $h > 0$

$$\text{spec } \Delta_h = \{\lambda_i(h) : i = 0, 1, \ldots\}. \tag{6}$$

The main result of the paper is an asymptotical formula for the eigenvalue distribution function $N_h(\lambda)$ of the operator Δ_h:

$$N_h(\lambda) = \sharp\{\lambda_i(h) : \lambda_i(h) \leq \lambda\}. \tag{7}$$

Theorem 0.1 *If $(M, \mathcal{F})$ be a Riemannian foliation, equipped with a bundle-like Riemannian metric g_M. Then the asymptotical formula for $N_h(\lambda)$ has the following form:*

$$N_h(\lambda) = h^{-q} \frac{(4\pi)^{-q/2}}{\Gamma((q/2)+1)} \int_{-\infty}^{\lambda} (\lambda - \tau)^{q/2} d_\tau N_{\mathcal{F}}(\tau) + o(h^{-q}), h \to 0, \tag{8}$$

where $N_{\mathcal{F}}(\lambda)$ is the spectrum distribution function of the tangential Laplace operator

$$\Delta_F : C^\infty(M, \Lambda T^* M) \to C^\infty(M, \Lambda T^* M). \tag{9}$$

We refer the reader to Section 5 for a detailed formulation of this Theorem. We stated also the asymptotical formula for the trace of an operator $f(\Delta_h)$ for any function $f \in C_c(\mathbf{R})$ (see Theorems 3.1 and 5.1 below).

The study of asymptotical behaviour of geometric objects (like as harmonic forms, eta-invariants etc.) associated with a family of Riemannian metrics on fibrations as the metrics become singular was stimulated by Witten's work on adiabatic limits [28]. For further developments see, for instance, [22, 9, 11, 12] and references there.

In the spectral theory of differential operators, problems in question are related with the Born-Oppenheimer approximation which consist in that the Schrodinger operator for polyatomic molecule is considered in the semiclassical limit where the mass ratio of electronic to nuclear mass tends to zero (see, for instance, [16] and references there). In particular, the result on semiclassical asymptotics for spectrum distribution function in a fibration case is, essentially, due to [3].

The investigation of semiclassical spectral asymptotics for foliations was started by the author in [17, 18, 20]. There we considered the problem in the operator setting, that is, we studied spectral asymptotics for the self-adjoint hypoelliptic operator A_h of the form

$$A_h = A + h^m B, \tag{10}$$

where A is a tangentially elliptic operator of order $\mu > 0$ with the positive tangential principal symbol, and B be a differential operator of order m on M with the positive, holonomy invariant transversal principal symbol and obtianed an asymptotical formula for spectrum distribution function of this operator when h tends to zero.

In this work, we adapted our results on semiclassical spectral asymptotics to the geometric setting of adiabatic limits on foliations.

The main observation related with the asymptotical formula (8) is that its right-hand side depends only on leafwise spectral data of the tangential Laplace operator Δ_F. So, in a case when the foliation $\mathcal{F}$ is nonamenable, there might to be a $\lambda > 0$ such that

$$\lim_{h\to 0} h^q N_h(\lambda) = 0. \tag{11}$$

The formula (11) allows, in particular, to introduce spectral characteristics $r_k(\lambda)$ related with adiabatic limits which are nontrivial in the nonamenable case. We hope that some invariants of the function $r_k(\lambda)$ introduced above near $\lambda = 0$ might to be independent of the choice of metric on M (otherwise speaking, to be coarse invariants), and, moreover, be topological or homotopic invariants of foliated manifolds (just as in the case of Novikov-Shubin invariants [13]). We discuss these questions and their relationships with the spectral theory of leafwise Laplacian and with noncommutative spectral geometry of foliations in Section 7.

The organization of the paper is as follows.

In Section 1, we recall some facts on pseudodifferential operators on foliated manifolds.

In Section 2, we summarize some necessary properties of the Laplace operator on a foliated manifold.

In the Sections 3 and 4, we formulate and prove the asymptotical formula for $\mathrm{tr} f(\Delta_h)$ when h tends to zero for any function $f \in C_c(\mathbf{R})$.

In Section 5, we rewrite the asymptotical formula of Section 3 in terms of spectral characteristics of the operator Δ_F. In particular, this provides a proof of the main

Theorem 0.1 on an asymptotic behaviour of the eigenvalue distribution function.

Finally, in Section 6 we discuss some facts and examples related with the asymptotical behaviour of individual eigenvalues of the operator Δ_h when h tends to zero, and, as mentioned above, Section 7 is devoted to a discussion of various aspects of the main asymptotical formula (8).

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1 Pseudodifferential operators on foliations

Here we recall some facts on pseudodifferential operators on foliated manifolds. The main references here are [19, 20].

Let $(M, \mathcal{F})$ be a compact foliated manifold, F be a distribution of tangent planes to $\mathcal{F}$. The embedding $F \subset TM$ induces an embedding of differential operators $Diff^{\mu}(\mathcal{F}) \subset Diff^{\mu}(M)$, and differential operators on M obtained in such a way is called tangential differential operators.

More generally, let E be an Hermitian vector bundle on M. We say that a linear differential operator A of order μ acting on $C^{\infty}(M, E)$ is a tangential operator, if, in any foliated chart $\kappa \ : \ I^p \times I^q \to M$ ($I = (0,1)$ is the open interval) and any trivialization of the bundle E over it, A is of the form

$$A = \sum_{|\alpha| \leq \mu} a_{\alpha}(x, y) D_x^{\alpha}, (x, y) \in I^p \times I^q, \tag{12}$$

with a_{α}, being matrix valued functions on $I^p \times I^q$.

Let $Diff^{\mu}(\mathcal{F}, E)$ denote the set of all tangential differential operators of order μ acting in $C^{\infty}(M, E)$.

Now we introduce the classes $Diff^{m,\mu}(M, \mathcal{F}, E)$ by taking compositions of tangential differential operators of order μ and differential operators of order m on M. That is, we say that $A \in Diff^{m,\mu}(M, \mathcal{F}, E)$ if A is of the form

$$A = \sum_{\alpha} B_{\alpha} C_{\alpha}, \tag{13}$$

where $B_{\alpha} \in Diff^{m}(M, E)$, $C_{\alpha} \in Diff^{\mu}(\mathcal{F}, E)$.

From symbolic calculus, it can be easily seen that:

(1) if $A_1 \in Diff^{m_1,\mu_1}(M, \mathcal{F}, E)$, $A_2 \in Diff^{m_2,\mu_2}(M, \mathcal{F}, E)$, then $A_1 \circ A_2 \in Diff^{m_1+m_2,\mu_1+\mu_2}(M, \mathcal{F}, E)$;

(2) if $A \in Diff^{m,\mu}(M, \mathcal{F}, E)$, then the adjoint $A^* \in Diff^{m,\mu}(M, \mathcal{F}, E)$.

Classes $Diff^{m,\mu}(M, \mathcal{F}, E)$ can be extended to bigraded classes of pseudodifferential operators $\Psi^{m,\mu}(M, \mathcal{F}, E)$, which contain, for instance, parametrices for elliptic operators from the classes $Diff^{m,\mu}(M, \mathcal{F}, E)$. We don't give its definition here, referring to [19](see also [20]) for details and will be restricted by an introduction of classes of differential operators.

Now we recall the definition of a scale of Sobolev type spaces $H^{s,k}(M, \mathcal{F}, E), s \in \mathbf{R}, k \in \mathbf{R}$, corresponding to classes of differential operators introduced above.

The space $H^{s,k}(\mathbf{R}^n, \mathbf{R}^p, \mathbf{C}^r)$ consists of all $\mathbf{C}^r$-valued tempered distributions $u \in S'(\mathbf{R}^n, \mathbf{C}^r)$ such that $\tilde{u} \in L^2_{\text{loc}}(\mathbf{R}^n, \mathbf{C}^r)$ ($\tilde{u}$ the Fourier transform) and

$$\|u\|^2_{s,k} = \int\int |\tilde{u}(\xi, \eta)|^2 (1 + |\xi|^2 + |\eta|^2)^s (1 + |\xi|^2)^k d\xi d\eta < \infty. \tag{14}$$

The identity (14) serves as a definition of a norm $\| \|_{s,k}$ in the space $H^{s,k}(\mathbf{R}^n, \mathbf{R}^p, \mathbf{C}^r)$.

The space $H^{s,k}(M, \mathcal{F}, E)$ consists of all $u \in \mathcal{D}'(M, E)$ such that, for any foliated coordinate chart $\kappa : I^p \times I^q \to U = \kappa(I^p \times I^q) \subset M$, any trivialization of the bundle E over it, and for any $\phi \in C^\infty_c(U)$, the function $\kappa^*(\phi u)$ belongs to the space $H^{s,k}(\mathbf{R}^n, \mathbf{R}^p, \mathbf{C}^r)$ (r = rank E). Fix some finite covering $\{U_i : i = 1, \dots, d\}$ of M by foliated coordinate patches with the foliated coordinate charts $\kappa_i : I^p \times I^q \to U_i = \kappa_i(I^p \times I^q)$ and trivializations of the bundle E over them, and a partition of unity $\{\phi_i \in C^\infty(M) : i = 1, \dots, d\}$ subordinate to this covering. A scalar product in $H^{s,k}(M, \mathcal{F}, E)$ is defined by the formula

$$(u, v)_{s,k} = \sum_{i=1}^{d} (\kappa^*(\phi_i u), \kappa^*(\phi_i v))_{s,k}, u, v \in H^{s,k}(M, \mathcal{F}, E). \tag{15}$$

We have the following result on the action of differential operators of class $Diff^{m,\mu}(M, \mathcal{F}, E)$ in the spaces $H^{s,k}(M, \mathcal{F}, E)$ (see [19, 20] for a proof in the scalar case).

Proposition 1.1 *An operator $A \in Diff^{m,\mu}(M, \mathcal{F}, E)$ defines a linear bounded operator from $H^{s,k}(M, \mathcal{F}, E)$ to $H^{s-m,k-\mu}(M, \mathcal{F}, E)$ for any $s \in \mathbf{R}$, $k \in \mathbf{R}$.*

Finally, the scale of Sobolev type spaces introduced above allows us to formulate a Garding inequality for tangentially elliptic operators (for the proof, see [19]).

Proposition 1.2 *If A is tangentially elliptic operator of order μ with the positive tangential principal symbol, then, for any $s \in \mathbf{R}, k \in \mathbf{R}$, there exist constants $C_1 > 0$ and C_2 such that*

$$Re\ (Au, u)_{s,k} \geq C_1 \|u\|^2_{s,k+\mu/2} - C_2 \|u\|^2_{s,-\infty}, u \in C^\infty(M, E). \tag{16}$$

2 Geometric operators on Riemannian foliations

Here we summarize some necessary properties of the Laplace operator on a foliated manifold.

As above, $(M, \mathcal{F})$ denotes a closed foliated Riemannian manifold, $\dim M = n$, $\dim \mathcal{F} = p$, $p + q = n$, equipped with a Riemannian metric g_M, $F = T\mathcal{F}$ be an integrable distribution of p-planes in TM. Recall that we choose the orthogonal complement H to F, so

$$F \bigoplus H = TM. \tag{17}$$

The decomposition (17) induces a bigrading on ΛT^*M by the formula

$$\Lambda^k T^*M = \bigoplus_{i=0}^{k} \Lambda^{i,k-i} T^*M, \tag{18}$$

where

$$\Lambda^{i,j} T^*M = \Lambda^i F^* \bigotimes \Lambda^j H^*. \tag{19}$$

Now we transfer the family Δ_h to a fixed Hilbert space $L^2(M, \Lambda T^*M, g)$. For this goal we introduce the isometry

$$\Theta_h : L^2(M, \Lambda T^*M, g_h) \to L^2(M, \Lambda T^*M, g), \tag{20}$$

where, for $u \in L^2(M, \Lambda^{i,j} T^*M, g_h)$, we have

$$\Theta_h u = h^j u. \tag{21}$$

The operator Δ_h in the Hilbert space $L^2(M, \Lambda T^*M, g_h)$ corresponds under the isometry Θ_h to the operator

$$L_h = \Theta_h \Delta_h \Theta_h^{-1} \tag{22}$$

in the Hilbert space $L^2(M, \Lambda T^*M) = L^2(M, \Lambda T^*M, g)$.

De Rham differential d inherits the decomposition (17) in the form

$$d = d_F + d_H + \theta. \tag{23}$$

Here the tangential de Rham differential d_F and the transversal de Rham differential d_H are first order differential operators, and θ is zeroth order. Moreover, the operator d_F doesn't depend on a choice of the orthogonal complement H (see, for instance, [25]).

Then we have the following assertion on the form of the operator L_h.

Lemma 2.1 ([11]) *We have*

$$L_h = d_h\delta_h + \delta_h d_h, \tag{24}$$

where

$$d_h = d_F + hd_H + h^2\theta, \tag{25}$$

and

$$\delta_h = \delta_F + h\delta_H + h^2\theta^*, \tag{26}$$

is the adjoint, where δ_F*,* δ_H *and* θ^* *are the adjoints to* d_F*,* d_H *and* θ *respectively. Here we consider the adjoints taken in the Hilbert space* $L^2(M, \Lambda T^*M)$*.*

By Lemma 2.1, the operator L_h is of the following form:

$$L_h = \Delta_F + h^2\Delta_H + h^4\Delta_{-1,2} + hK_1 + h^2K_2 + h^3K_3, \tag{27}$$

where

- The operator

$$\Delta_F = d_F\delta_F + \delta_F d_F \in Diff^{0,2}(M, \mathcal{F}, \Lambda T^*M) \tag{28}$$

  is the tangential Laplacian in the space $C^\infty(M, \Lambda T^*M)$.

- The operator

$$\Delta_H = d_H\delta_H + \delta_H d_H \in Diff^{2,0}(M, \mathcal{F}, \Lambda T^*M) \tag{29}$$

  is the transversal Laplacian in the space $C^\infty(M, \Lambda T^*M)$.

- $\Delta_{-1,2} = \theta\theta^* + \theta^*\theta \in Diff^{0,0}(M, \mathcal{F}, \Lambda T^*M))$.
- $K_1 = d_F\delta_H + \delta_H d_F + \delta_F d_H + d_H\delta_F \in Diff^{1,0}(M, \mathcal{F}, \Lambda T^*M))$.
- $K_2 = d_F\theta^* + \theta^* d_F + \delta_F\theta + \theta\delta_F \in Diff^{0,0}(M, \mathcal{F}, \Lambda T^*M))$.
- $K_3 = d_H\theta^* + \theta^* d_H + \delta_H\theta + \theta\delta_H \in Diff^{1,0}(M, \mathcal{F}, \Lambda T^*M))$.

From now on, we will assume that $(M, \mathcal{F})$ is a Riemannian foliation with a bundle-like metric g_M, that is, it satisfies one of the following equivalent conditions (see [25]):

1. $(M, \mathcal{F})$ locally has the structure of Riemannian submersion;

2. for any $X \in F$ we have
$$\nabla^{\mathcal{F}}_X g_H = 0, \tag{30}$$
where $\nabla^{\mathcal{F}}$ is a Bott connection on H;

3. the distribution H is totally geodesic.

The following Lemma states the main specific property of geometrical operators on Riemannian foliated manifold.

Lemma 2.2 *If $(M, \mathcal{F})$ is a Riemannian foliation with a bundle-like metric g_M, then the operators*
$$d_F\delta_H + \delta_H d_F \text{ and } \delta_F d_H + d_H\delta_F \tag{31}$$
belong to the class $Diff^{0,1}(M, \mathcal{F}, \Lambda T^*M))$. In particular, we have
$$K_1 \in Diff^{0,1}(M, \mathcal{F}, \Lambda T^*M)). \tag{32}$$

For any $h > 0$, the operator L_h is a formally self-adjoint, elliptic operator in $L^2(M, \Lambda T^*M)$ with the positive principal symbol. The following Proposition is a refinement of the classical Garding inequality for the operator L_h in $H^{s,k}(M, \mathcal{F}, \Lambda T^*M)$

Proposition 2.3 *Under current hypotheses, there exists constants $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ such that for any $h > 0$ small enough we have the following inequality:*
$$(L_h u, u) \geq (1 - C_1h^2)(\Delta_F u, u) + C_2h^2\|u\|^2_{1,0} - C_3\|u\|^2, u \in C^\infty(M, \Lambda T^*M)). \tag{33}$$

Proof. By (27), we have
$$\begin{aligned}(L_h u.u) &= (\Delta_F u, u) + h^2(\Delta_H u, u) + h^4(\Delta_{-1,2}u, u)\\ &+ h(K_1u, u) + h^2(K_2u, u) + h^3(K_3u, u),\\ &\quad u \in C^\infty(M, \Lambda T^*M).\end{aligned} \tag{34}$$

It clear that $(\Delta_{-1,2}u, u) \geq 0$. By Proposition 1.1, we have
$$(K_2u, u) \geq -C_4\|u\|^2, (K_3u, u) \geq -C_5\|u\|^2_{1,0}. \tag{35}$$

So we obtain
$$\begin{aligned}(L_h u.u) &\geq (\Delta_F u, u) + h^2(\Delta_H u, u) + h(K_1u, u)\\ &- C_4h^2\|u\|^2 - C_5h^3\|u\|^2_{1,0}.\end{aligned} \tag{36}$$

The operator $\Delta_F + \Delta_H$ is an second order elliptic operator with the positive principal symbol, so, by the standard Garding inequality, we have

$$((\Delta_F + \Delta_H)u, u) \geq C_6\|u\|^2_{1,0} - C_7\|u\|^2, \tag{37}$$

that implies the estimate

$$\begin{aligned}(L_h u.u) &\geq (1-h^2)(\Delta_F u, u) + C_7 h^2\|u\|^2_{1,0} + h(K_1 u, u) \\ &- C_8\|u\|^2.\end{aligned} \tag{38}$$

Finally, we make use of the inequality

$$|(K_1 u, u)| \leq C_7\|u\|_{0,1}\|u\| \leq C_8(h\|u\|^2_{0,1} + h^{-1}\|u\|^2) \tag{39}$$

and the tangential Garding estimate (see Proposition 1.2)

$$\|u\|^2_{0,1} \leq C_9((\Delta_F u, u) + \|u\|^2), \tag{40}$$

that completes immediately the proof.

Remark. In some cases, it is sufficient to use more crude estimate

$$(L_h u, u) \geq C_1\|u\|^2_{0,1} + C_2 h^2\|u\|^2_{1,0} - C_3\|u\|^2, u \in C^\infty(M, \Lambda T^*M)), \tag{41}$$

which follows from (33), if we apply the standard Sobolev norm estimate

$$(\Delta_F u, u) \leq C_{10}\|u\|^2_{0,1}. \tag{42}$$

Let $H_h(t) = \exp(-tL_h), t \geq 0$, be the parabolic semigroup, generated by the operator L_h:

$$H_h(t) : C^\infty(M, \Lambda T^*M) \to C^\infty(M, \Lambda T^*M). \tag{43}$$

For any $t > 0$, the operator $H_h(t)$ is an operator with a smooth kernel. Proposition 2.3 implies the following norm estimates for operators of this semigroup in the spaces $H^{s,k}(M, \mathcal{F}, \Lambda T^*M)$ (see also [20]).

Proposition 2.4 *We have the following estimates:*

$$\|H_h(t)u\|_{r,k} \leq C_{rsk} t^{(s-k-r)/2} h^{s-r}\|u\|_s, u \in C^\infty(M, \Lambda T^*M), \tag{44}$$

if $r > s, h \in (0,1], 0 < t \leq 1$, *and the estimate*

$$\|H_h(t)u\|_{s,k} \leq C_{sk} t^{-k/2}\|u\|_s, u \in C^\infty(M, \Lambda T^*M). \tag{45}$$

if $r = s, h \in [0,1], 0 < t \leq 1$, *where the constants don't depend on* t *and* h.

3 Asymptotical formula for the functions of the Laplace operator

Form now on, we will assume that $(M, \mathcal{F})$ is a Riemannian foliation, equipped with a bundle-like Riemannian metric g_M. In this Section, we state the asymptotical formula for $\mathrm{tr} f(\Delta_h)$ when h tends to zero for any function $f \in C_c(\mathbf{R})$.

We will denote by $G_{\mathcal{F}}$ the holonomy groupoid of $(M, \mathcal{F})$. Recall that $G_{\mathcal{F}}$ is equipped with the source and the target maps $s, r : G_{\mathcal{F}} \to M$. We will make use of the standard notation: $G_{\mathcal{F}}^{(0)} = M$ is the set of objects, $G_{\mathcal{F}}^x = \{\gamma \in G_{\mathcal{F}} : r(\gamma) = x\}, x \in M$. Recall that $G_{\mathcal{F}}^x$ is the covering of the leaf through the point x, associated with the holonomy group of the leaf. We will identify a point $x \in M$ with the identity element in G_x^x. Finally, we will denote by λ_L the Riemannain volume form on each leaf L of $\mathcal{F}$ and by λ^x its lift to a measure on the holonomy covering $G_{\mathcal{F}}^x, x \in M$.

For any vector bundle E on M, we denote by $C_c^\infty(G_{\mathcal{F}}, E)$ the space of all smooth, compactly supported sections of the vector bundle $(s,r)^*(E^* \otimes E)$ over $G_{\mathcal{F}}$. In other words, for any $k \in C_c^\infty(G_{\mathcal{F}}, E)$, its value at a point $\gamma \in G_{\mathcal{F}}$ is a linear map $k(\gamma) : E_{s(\gamma)} \to E_{r(\gamma)}$. We will use a correspondence between tangential kernels $k \in C_c^\infty(G_{\mathcal{F}}, E)$ and tangential operators $K : C^\infty(M, E) \to C^\infty(M, E)$ via the formula

$$Ku(x) = \int_{G_{\mathcal{F}}^x} k(\gamma) u(s(\gamma)) d\lambda^x(\gamma), u \in C^\infty(M, E). \tag{46}$$

Now we introduce a notion of a principal h-symbol of the operator Δ_h. It is well-known (see, for instance, [23, 25]) that the conormal bundle H^* to the foliation $\mathcal{F}$ has a partial (Bott) connection, which is flat along the leaves of the foliation. So we can lift the foliation $\mathcal{F}$ to the foliation $\mathcal{F}_H$ in the conormal bundle H^*. The leaf $\tilde{L}_\nu$ of the foliation $\mathcal{F}_H$ through a point $\nu \in H^*$ is diffeomorphic to the holonomy covering $G_{\mathcal{F}}^x$ of the leaf $L_x, x = \pi(\nu)$ of the foliation $\mathcal{F}$ through the point x (here $\pi : H^* \to M$ is the bundle map) and has a trivial holonomy.

Denote by

$$\Delta_{\mathcal{F}_H} : C^\infty(H^*, \pi^* \Lambda T^* M) \to C^\infty(H^*, \pi^* \Lambda T^* M) \tag{47}$$

the lift of the leafwise Laplacian Δ_F to tangentially elliptic operator on H^* with respect to $\mathcal{F}_H$.

Remark. If we fix $x \in M$, the restriction of the foliation $\mathcal{F}_H$ on H_x^* is a linear model of the foliation $\mathcal{F}$ in some neighborhood of the leaf L_x through a point x, so the restriction Δ_x of $\Delta_{\mathcal{F}_H}$ on H^*,

$$\Delta_x : C^\infty(H_x^*, \pi^* \Lambda T^* L \bigotimes \Lambda H_x^*) \to C^\infty(H_x^*, \pi^* \Lambda T^* L \bigotimes \Lambda H_x^*), \tag{48}$$

is the model operator for the tangential Laplacian Δ_F at the "point" $x \in M/\mathcal{F}$.

Definition. The principal h-symbol of the operator Δ_h is a tangentially elliptic operator

$$\sigma_h(\Delta_h) : C^\infty(H^*, \pi^*\Lambda T^*M) \to C^\infty(H^*, \pi^*\Lambda T^*M) \tag{49}$$

on H^* with respect to the foliation $\mathcal{F}_H$, given by the formula

$$\sigma_h(\Delta_h) = \Delta_{\mathcal{F}_H} + g_H, \tag{50}$$

where g_H is the scalar multiplication operator by the function $g_H(\nu), \nu \in H^*$.

The holonomy groupoid $G_{\mathcal{F}_H}$ of the lifted foliation $\mathcal{F}_H$ consists of all triples $(\gamma, \nu, \eta) \in G_{\mathcal{F}} \times H^* \times H^*$ such that $s(\gamma) = \pi(\nu)$, $r(\gamma) = \pi(\eta)$ and $(dh^*_\gamma)^{-1}(\nu) = \eta$, where dh^*_γ is codifferential of the holonomy map, with the source map $s : G_{\mathcal{F}_H} \to H^*, s(\gamma, \nu, \eta) = \nu$ and the target map $r : G_{\mathcal{F}_H} \to H^*, r(\gamma, \nu, \eta) = \eta$.The projection $\pi : H^* \to M$ induces the map $\pi_G : G_{\mathcal{F}_H} \to G_{\mathcal{F}}$ by

$$\pi_G(\gamma, \nu, \eta) = \gamma, (\gamma, \nu, \eta) \in G_{\mathcal{F}_H}. \tag{51}$$

Denote by $\mathrm{tr}_{\mathcal{F}_H}$ the trace on the von Neumann algebra $W^*(G_{\mathcal{F}_H}, \pi^*\Lambda T^*M)$ of all tangential operators on H^* with respect to the foliation $\mathcal{F}_H$, given by a holonomy invariant measure $dx\, d\nu$ on H^* [6]. For any tangentially elliptic operator K on $(H^*, \mathcal{F}_H)$, given by the tangential kernel $k \in C^\infty_c(G_{\mathcal{F}_H}, \pi^*\Lambda T^*M)$, $k = k(\gamma, \nu, \eta)$ we have

$$\mathrm{tr}_{\mathcal{F}_H}(K) = \int_{H^*} \mathrm{Tr}_{\pi^*\Lambda T^*M}\ k(x, \nu, \nu) dx d\nu. \tag{52}$$

Theorem 3.1 *For any function $f \in C_c(\mathbf{R})$, we have the asymptotical formula*

$$tr\ f(\Delta_h) = (2\pi)^{-q} h^{-q} tr_{\mathcal{F}_H}\ f(\sigma_h(\Delta_h)) + O(h^{1-q}), h \to 0. \tag{53}$$

We will prove this Theorem in the next Section, and now we conclude the Section with some remarks.

Remarks. (1) In a case of the Schrodinger operator on a compact manifold M with an operator-valued potential $V \in \mathcal{L}(H)$ with a Hilbert space H such that $V(x)^* = V(x)$ (a fibration case)

$$H_h = -h^2\Delta + V(x), x \in M, \tag{54}$$

the corresponding asymptotical formula has the following form:

$$\mathrm{tr}\ f(\Delta_h) = (2\pi)^{-n} h^{-n} \int\ Tr\ f(h(x, \xi)) dx d\xi + o(h^{-n}), h \to 0+, \tag{55}$$

where $h(x,\xi)$ is the operator-valued principal h-symbol

$$h(x,\xi) = |\xi|^2 + V(x), (x,\xi) \in T^*M. \tag{56}$$

So the formula (53) has the same form as (55) with the difference that the usual integration over the base and the fibrewise trace are replaced by the integration in a sense of the noncommutative integration theory [6].

(2) We don't make an essential use of a operator-valued symbolic calculus. Indeed, it is a difficult problem to develop such a calculus in a general case. The introduction of the principal h-symbol of the operator Δ_h allow us to simplify the final asymptotical formula and also some algebraic calculations (see below for a passage from an asymptotical formula for tr $\exp(-t\Delta_h)$ to an asymptotical formula for tr $f(\Delta_h)$ with an arbitrary function $f \in C_c^\infty(\mathbf{R})$.

4 Proof of Theorem 3.1

In this Section, we prove Theorem 3.1, concerning an asymptotical behaviour of tr $f(\Delta_h)$ when h tends to zero.

First of all, let us note that, without loss of generality, we may consider an asymptotical behaviour of tr $f(L_h)$. The proof of Theorem 3.1 relies on a comparison of the operator L_h with some operator $\bar{L}_h$ of the almost product structure as in [20] with a subsequent use of results of [20] on semiclassical spectral asymptotics for elliptic operators on foliated manifolds.

So let the operator $\bar{L}_h \in Diff^{2,0}(M,\mathcal{F},\Lambda T^*M))$ be given by the formula

$$\bar{L}_h = \Delta_F + h^2\Delta_H. \tag{57}$$

The operators L_h and $\bar{L}_h$ are generators parabolic semigroups of linear bounded operators in the space $L^2(M,\Lambda T^*M)$ denoted by

$$\begin{aligned} H_h(t) &= e^{-tL_h}, t \geq 0, & (58)\\ \bar{H}_h(t) &= e^{-t\bar{L}_h}, t \geq 0, & (59)\end{aligned}$$

respectively. It is clear that, indeed, these operators are smoothing operators when $t > 0$.

The operator $\bar{L}_h$ satisfies the conditions of [20], that is, it is of the form

$$\bar{L}_h = A + h^2B, \tag{60}$$

where $A = \Delta_F$ is a second order tangentially elliptic operator with the scalar, positive tangential principal symbol, and $B = \Delta_H$ be a second order differential operator on M with the scalar, positive, holonomy invariant transversal principal symbol. Indeed, it is easy to see that the transversal principal symbol of operator Δ_H, which is the restriction of its principal symbol from T^*M to the conormal bundle H^*, is given by the formula

$$\sigma(\nu) = g_{H^*}(\nu)I, \nu \in H^*, \tag{61}$$

and its holonomy invariance is equivalent to the assumption on the foliation $\mathcal{F}$ to be Riemannian (see (30)).

Remark. The only necessary property which we need from holonomy invariance condititon is the fact that the commutator $[A, B]$, which, by general symbolic calculus, belongs to the class $Diff^{2,1}(M, \mathcal{F}, \Lambda T^*M)$, is an operator of the class $Diff^{1,2}(M, \mathcal{F}, \Lambda T^*M)$, and this fact can be checked by a straightforward calculation and looks very similar to the second assertion of Lemma 2.2.

By [20], the operators of the parabolic semigroup $\bar{H}_h(t)$ satisfy the same estimate as in Proposition 2.4.

$$\|\bar{H}_h(t)u\|_{r,k} \leq C_{r,s,k}t^{(s-k-r)/2}h^{s-r}\|u\|_s, u \in C^\infty(M, \Lambda T^*M), \tag{62}$$

if $r > s, h \in (0,1], 0 < t \leq 1$, and the estimate

$$\|\bar{H}_h(t)u\|_{s,k} \leq C_{sk}t^{-k/2}\|u\|_s, u \in C^\infty(M, \Lambda T^*M). \tag{63}$$

if $r = s, h \in [0,1], 0 < t \leq 1$, where the constants don't depend on t and h.

Now we want to compare the semigroups $\bar{H}_h(t)$ and $H_h(t)$. First, we state the norm estimates for the difference $H_h(t) - \bar{H}_h(t)$.

Proposition 4.1 *We have the estimate*

$$\|(H_h(t) - \bar{H}_h(t))u\|_{r,k} \leq C_{r,s,k}t^{(s-k-r)/2}h^{s-r-1}\|u\|_s, u \in C^\infty(M, \Lambda T^*M), \tag{64}$$

if $r > s, h \in (0,1], 0 < t \leq 1$, *and the estimate*

$$\|(H_h(t) - \bar{H}_h(t))u\|_{s,k} \leq C_{sk}t^{-k/2}\|u\|_s, u \in C^\infty(M, \Lambda T^*M). \tag{65}$$

if $r = s, h \in [0,1], 0 < t \leq 1$, *where the constants don't depend on* t *and* h.

Proof. For a proof, we make use of the Duhamel formula

$$(H_h(t) - \bar{H}_h(t))u = \int_0^t H_h(\tau)(\bar{L}_h - L_h)\bar{H}_h(t-\tau)ud\tau. \tag{66}$$

We know the norm estimates for operators $H_h(t)$ and $\bar{H}_h(t)$ (see Propositions 2.4 and (62)) and the explicit formula for the difference $\bar{L}_h - L_h$:

$$L_h - \bar{L}_h = h^4\Delta_{-1,2} + hK_1 + h^2K_2 + h^3K_3. \tag{67}$$

from where Proposition is proved in a usual way.

Now we pass from the Sobolev estimates for the operator $H_h(t) - \bar{H}_h(t)$ to pointwise and trace estimates.

Proposition 4.2 *Under current hypotheses, we have the estimates*

$$|tr(H_h(t) - \bar{H}_h(t))| \leq Ch^{1-q}. \tag{68}$$

Proof. For the proof, we make use the following proposition (see [20] for a scalar case):

Proposition 4.3 *Let $(M, \mathcal{F})$ be a compact foliated manifold, E be an Hermitian vector bundle on M. For any $s > p/2$ and $k > q/2$, there is a continuous embedding*

$$H^{s,k}(M, \mathcal{F}, E) \subset C(M, E). \tag{69}$$

Moreover, for any $s > p/2$ and $k > q/2$, there is a constant $C_{s,k} > 0$ such that, for each $\lambda \geq 1$,

$$\sup_{x \in M} |u(x)| \leq C_{s,l}\lambda^{q/2}(\lambda^{-s}\|u\|_{s,k} + \|u\|_{0,k+s}), u \in H^{s,k}(M, \mathcal{F}, E). \tag{70}$$

Denote by $H_h(t,x,y)$ ($\bar{H}_h(t,x,y)$) the integral kernels of operators $H_h(t)$ ($\bar{H}_h(t)$) respectively. Then, by Propositions 4.1 and 4.3, we obtain:

$$|H_h(t,x,x) - \bar{H}_h(t,x,x)| \leq Ch^{1-q}, x \in M. \tag{71}$$

that, due to the well-known formula for the trace of an integral operator K in the Hilbert space $L^2(M, \Lambda T^*M)$ with a smooth kernel $k(x,y)$:

$$\text{tr } K = \int_M \text{Tr } k(x,x)dx, \tag{72}$$

immediately completes the proof.

Denote by $h_{\mathcal{F}}(t,\gamma) \in C^\infty(G_{\mathcal{F}}, \Lambda T^*M)$ the tangential kernel of the smoothing tangential operator $\exp(-t\Delta_F)$.

Proposition 4.4 *For any $t > 0$, we have the asymptotical formula*

$$tr\ e^{-tL_h} = (2\pi)^{-q}h^{-q}\int_M(\int_{H_x^*} e^{-tg_H(\nu)}d\nu)\ Tr_{\Lambda T^*M}\ h_{\mathcal{F}}(t,x)dx + O(h^{1-q}), h\to 0. \quad (73)$$

Proof. By Propositions 2.4 and 4.3, we have the estimate

$$\mathrm{tr}\ e^{-tL_h} \leq Ch^{-q}, h\to 0. \quad (74)$$

Moreover, by Proposition 4.2, asymptotics of traces of the operators $H_h(t)$ and $\bar{H}_h(t)$ when h tends to zero have the same leading terms (of order h^{-q}), and we can apply the asymptotical formula of [20] to complete the proof.

Remarks. (1) Since

$$\int_{H_x^*} e^{-tg_H(\nu)}d\nu = \pi^{q/2}t^{-q/2}, \quad (75)$$

the formula (73) can be rewritten in a simpler form:

$$\mathrm{tr}\ e^{-tL_h} = (4\pi t)^{-q/2}h^{-q}\int_M \mathrm{Tr}_{\Lambda T^*M}\ h_{\mathcal{F}}(t,x)dx + O(h^{1-q}), h\to 0. \quad (76)$$

From (76), we can also obtain an asymptotical formula for the spectrum distribution function, but it is more convenient for us to use the formula in the form (73).

(2) For any $x \in M$, the restriction $h_{\mathcal{F}}(t,\gamma) \in C^\infty(G_{\mathcal{F}}^x, \Lambda T^*M)$ of $h_{\mathcal{F}}$ on $G_{\mathcal{F}}^x$ is the kernel of the operator $\exp(-t\Delta_x)$, where Δ_x the restriction of Δ_F on $G_{\mathcal{F}}^x$ (see also Section 5). This fact doesn't extend to more general functions $f(\Delta_F)$ (see [19]), and this is closely related with so-called spectrum coincidence theorems and with appearance of nonstandard asymptotical formula (11).

Proof of Theorem 3.1. The tangential kernel $h_{\mathcal{F}_H}(t) \in C^\infty(G_{\mathcal{F}_H}, \pi^*\Lambda T^*M)$ of the operator $\exp(-t\Delta_{\mathcal{F}_H})$ is related with the tangential kernel $h_{\mathcal{F}}(t) \in C^\infty(G_{\mathcal{F}}, \Lambda T^*M)$ of operator $\exp(-t\Delta_F)$ by the formula

$$h_{\mathcal{F}_H}(t,\gamma,\nu,\eta) = \pi_G^* h_{\mathcal{F}}(t,\gamma). \quad (77)$$

The essential difference of the case of Riemannian foliation from the general one consists in the fact that the operators $\Delta_{\mathcal{F}_H}$ and g_H considered as operators on H^* commutes. In particular, we have

$$e^{-t\sigma_h(\Delta_h)} = e^{-tg_H(\nu)}e^{-t\Delta_{\mathcal{F}_H}}, t > 0. \quad (78)$$

So the formula (73) can be rewritten in terms of the notation of this Section as follows:

$$\operatorname{tr} e^{-tL_h} = h^{-q}\operatorname{tr}_{\mathcal{F}_H} e^{-t\sigma_h(\Delta_h)} + O(h^{1-q}), h \to 0. \tag{79}$$

From where, using standard approximation arguments, the theorem follows immediately.

Remark. The passage from the operator L_h to the operator $\bar{L}_h$ resembles the passage from the Riemannian connection on M to the almost product connection as in [1, 25].

5 Formulation in terms of leafwise spectral characteristics

Here we will write the asymptotical formula (53) in terms of spectral characteristics of the operator Δ_F. In particular, we obtain a proof of the main theorem on an asymptotic behaviour of the eigenvalue distribution function.

Recall that Δ_F denotes the tangential Laplacian in the space $C^\infty(M, \Lambda T^*M)$. Let us restrict the operator Δ_F to the leaves of the foliation $\mathcal{F}$ and lift the restrictions to holonomy coverings of leaves. We obtain the family

$$\Delta_x : C_c^\infty(G_{\mathcal{F}}^x, r^*\Lambda T^*M) \to C_c^\infty(G_{\mathcal{F}}^x, r^*\Lambda T^*M) \tag{80}$$

of Laplacians on holonomy coverings of leaves. By the hypotheses of Riemannian foliation, the operator Δ_x is formally self-adjoint in $L^2(G_{\mathcal{F}}^x, r^*\Lambda T^*M)$, that, in turn, implies its essential self-adjointness in this Hilbert space (with initial domain $C_c^\infty(G_{\mathcal{F}}^x, r^*\Lambda T^*M)$) for any $x \in M$. For each $\lambda \in \mathbf{R}$, the kernel $e(\gamma, \lambda), \gamma \in G_{\mathcal{F}}$ of the spectral projections of the operators Δ_x, corresponding to the semiaxis $(-\infty, \lambda]$ define an element of the von Neumann algebra $W^*(G_{\mathcal{F}}, \Lambda T^*M)$. The section $e(\gamma, \lambda)$ is a leafwise smooth section of the bundle $(s^*\Lambda T^*M)^* \otimes r^*\Lambda T^*M$ over $G_{\mathcal{F}}$.

We introduce the spectrum distribution function $N_{\mathcal{F}}(\lambda)$ of the operator Δ_F by the formula

$$N_{\mathcal{F}}(\lambda) = \int_M \operatorname{Tr}_{\Lambda T^*M}\, e(x, \lambda)dx, \lambda \in \mathbf{R}. \tag{81}$$

By [19], for any $\lambda \in \mathbf{R}$, the function $\operatorname{Tr}_{\Lambda T^*M}\, e(x, \lambda)$ is a bounded measurable function on M, therefore, the spectrum distribution function $N_{\mathcal{F}}(\lambda)$ is well-defined and takes finite values.

Theorem 5.1 *For any function $f \in C^\infty(\mathbf{R})$, we have the following asymptotic formula:*

$$tr\ f(L_h) = h^{-q}\frac{(4\pi)^{-q/2}}{\Gamma(q/2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma^{q/2-1}f(\tau+\sigma)\ d\sigma\ dN_{\mathcal{F}}(\tau) + O(h^{1-q}), h \to 0. \quad (82)$$

Proof. Let $E_{g_H}(\tau)$ and $E_\Delta(\sigma)$ denote the spectral projections of the operators g_H and $\Delta_{\mathcal{F}_H}$ in $L^2(H^*, \pi^*\Lambda T^*M)$. Then, since these operators commute, we have

$$\begin{aligned} f(\sigma_h(\Delta_h)) &= f(\Delta_F + g_H) \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\tau+\sigma)\ dE_{g_H}(\tau)\ dE_\Delta(\sigma) \end{aligned}$$

is a tangential operator on H^* with respect to the foliation $\mathcal{F}_H$, which tangential kernel has the form

$$k_{f(\sigma_h(\Delta_h))}(\gamma,\nu,\eta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\tau+\sigma)\ dE_{g_H}(\tau)(\nu)\ dE_\Delta(\gamma,\sigma). \quad (83)$$

So we obtain

$$\begin{aligned} tr_{\mathcal{F}_H} f(\sigma_h(\Delta_h)) &= \int_M\int_{H_x^*} \mathrm{Tr}_{\Lambda T^*M}\ k_{f(\sigma_h(\Delta_h))}(x,\nu)dxd\nu \\ &= \int_M\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\tau+\sigma)\ (\int_{H_x^*} dE_{g_H}(\tau)(\nu)\ d\nu) \\ &\quad d_\sigma(\mathrm{Tr}_{\Lambda T^*M}\ E_\Delta(x,\sigma))\ d\tau\ dx, \end{aligned} \quad (84)$$

from where, taking into account that

$$E_{g_H}(\tau)(\nu) = \chi_{\{g_H(\nu)\le\tau\}} I_{\pi^*\Lambda T^*M} \quad (85)$$

and

$$\int_{H_x^*} E_{g_H}(\tau)(\nu)\ d\nu = volume\{\nu \in H^* : g_H(\nu) \le \tau\} = \omega_q \tau^{q/2}, \quad (86)$$

where

$$\omega_q = \frac{\pi^{q/2}}{\Gamma((q/2)+1)} \quad (87)$$

is the volume of the unit ball in $\mathbf{R}^q$, we immediately obtain the desired formula.

In a particular case when f is a characteristic function of the semiaxis $(-\infty, \lambda)$, Theoremla 5.1 gives the asymptotic formula for the spectrum distribution function $N_h(\lambda)$.

Theorem 5.2 *Under current hypothesis, we have*

$$N_h(\lambda) = h^{-q}\frac{(4\pi)^{-q/2}}{\Gamma((q/2)+1)}\int_{-\infty}^{\lambda}(\lambda-\tau)^{q/2}\, dN_{\mathcal{F}}(\tau) + o(h^{-q}), h \to 0 \tag{88}$$

for any $\lambda \in \mathbf{R}$.

Theorem 0.1 is, exactly, Theorem 5.2 formulated in terms of the operator Δ_h.

6 Limits of eigenvalues

Here we discuss the asymptotical behaviour of individual eigenvalues of the operator Δ_h when h tends to zero.

As usual, we will, equivalently, consider the operator L_h instead Δ_h. Moreover, we will consider eigenvalues of this operator on differential k-forms. Therefore, we will write L_h^k for the restriction of the operator L_h on $C^\infty(M, \Lambda^k T^*M)$ $k = 1, \dots, n$, omitting k where it is not essential.

For any $h > 0$, L_h is an analytic family of type (B) of self-adjoint operators in sence of [15]. Therefore, for $h > 0$, the eigenvalues of L_h depends analytically on h. Thus there are (countably many) analytic functions $\lambda_i(h)$ such that

$$\text{spec } L_h = \{\lambda_i(h) : i = 1, 2, \dots\}, h > 0. \tag{89}$$

Moreover, by [15], the functions $\lambda_i(h)$ satisfy the following equality

$$\lambda_i'(h) = ((dL_h/dh)v_h, v_h), \tag{90}$$

where v_h is a normalized eigenvector associated with the eigenvalue $\lambda_i(h)$.

Proposition 6.1 *Under current hypotheses, for ant i, there exists a limit*

$$\lim_{h\to 0+} \lambda_i(h) = \lambda_{\lim,i}. \tag{91}$$

Moreover, if v_h is a normalized eigenform associated with the eigenvalue $\lambda_i(h)$, then we have the estimates

$$\|v_h\|_{0,1} < C_1, \ h\|v_h\|_{1,0} < C_2, \tag{92}$$

with constants C_1 and C_2 independent of $h \in (0, 1]$.

Proof. As above, let v_h be a normalized eigenform associated with the eigenvalue $\lambda_i(h)$:

$$L_h v_h = \lambda_i(h) v_h, \; \|v_h\| = 1. \tag{93}$$

By (90), we have

$$\lambda_i'(h) = ((2h\Delta_H + 4h^3\Delta_{-1,2} + K_1 + 2hK_2 + 3h^2K_3)v_h, v_h), \tag{94}$$

from where, using the positivity of operators Δ_H and $\Delta_{-1,2}$ in $L^2(M, \Lambda T^*M)$, and the estimates (35) and (39) (with $h = 1$), we obtain

$$\lambda_i'(h) \geq -C_1\|v_h\|^2_{0,1} - C_2h^2\|v_h\|^2_{1,0} - C_3. \tag{95}$$

The estimate (41) implies

$$C_1\|v_h\|^2_{0,1} + C_2h^2\|v_h\|^2_{1,0} \leq C_3\lambda_i(h) + C_4, h \in (0, 1]. \tag{96}$$

By (95) and (96), we conclude that

$$\lambda_i'(h) \geq -C_5\lambda_i(h) - C_6. \tag{97}$$

This estimate can be rewritten in the following way:

$$\frac{d}{dh}((\lambda_i(h) + \frac{C_6}{C_5})e^{C_5h}) \geq 0, \tag{98}$$

that means that the function $(\lambda_i(h)+\frac{C_6}{C_5})e^{C_5h}$ is increasing in h for h small enough. By the positivity of the operator L_h in $L^2(M, \Lambda T^*M)$, every eigenvalue $\lambda_i(h)$ is positive, so the function $(\lambda_i(h) + \frac{C_6}{C_5})e^{C_5h}$ semibounded from below near zero. Therefore, this function has a limit when h tends to zero, that, clearly, implies the existence of the limit for the function λ_i.

The second assertion of this Proposition is an immediate consequence of the first one and the estimate (96).

Proposition 6.1 allows us to introduce the limitting spectrum of the operator Δ_h^k as a set of all limitting values $\lambda^k_{\lim,i}$, given by (91):

$$\sigma_{\lim}(\Delta_h^k) = \{\lambda^k_{\lim,i} : i = 0, 1, \ldots\}. \tag{99}$$

By an analogy with the case of semiclassical asymptotics for Schrodinger operator, we may assume that the structure of the limitting spectrum $\sigma_{\lim}(\Delta_h^k)$ is defined in a big extent by a limitting value of the bottoms of spectrum of the operator Δ_h^k. So let

$$\lambda_0^k(h) = \min_{u\in C^\infty(M,\Lambda^k T^*M)} \frac{(\Delta_h^k u, u)}{\|u\|^2}, \tag{100}$$

and

$$\lambda_{\lim,0}^k = \lim_{h\to 0} \lambda_0^k(h). \tag{101}$$

There are two other quantities: the bottom $\lambda_{F,0}^k$ of the spectrum of the operator Δ_F^k in $L^2(M, \Lambda^k T^*M)$:

$$\lambda_{F,0}^k = \min_{u\in C^\infty(M,\Lambda^k T^*M)} \frac{(\Delta_F^k u, u)}{\|u\|^2}, \tag{102}$$

and the bottom $\lambda_{\mathcal{F},0}^k$ of the leafwise spectrum of the operator Δ_F^k in $L^2(L, \Lambda^k T^*M)$:

$$\lambda_{\mathcal{F},0}^k = \overline{\bigcup\{\sigma(\Delta_L^k) : L \in V/\mathcal{F}\}}, \tag{103}$$

where

$$\lambda_{L,0}^k = \min_{u\in C^\infty(L,\Lambda^k T^*M)} \frac{(\Delta_L^k u, u)}{\|u\|^2}, \tag{104}$$

the operator Δ_L^k is the restriction of the operator Δ_F^k on the leaf L.

Proposition 6.2 *Under current hypotheses, we have the following relations:*

$$\lambda_{F,0}^k \le \lambda_{\lim,0}^k \le \lambda_{\mathcal{F},0}^k, \ k = 1, \dots, n. \tag{105}$$

Proof. Let v_h be a normalized eigenform associated with the bottom eigenvalue $\lambda_0^k(h)$:

$$L_h^k v_h = \lambda_0^k(h) v_h, \ \|v_h\| = 1. \tag{106}$$

By the definition of $\lambda_{F,0}^k$, we have the estimate

$$(\Delta^k v_h, v_h) \ge \lambda_{F,0}^k. \tag{107}$$

By (38), we obtain

$$\lambda_0^k(h) \ge (1-h^2)\lambda_{F,0}^k + C_1 h^2 \|v_h\|_{1,0}^2 h(K_1 v_h, v_h) - C_2 h^2, \tag{108}$$

where C_1 and C_2 are positive constants. By (92), we have

$$\lim_{h\to 0} h(K_1 v_h, v_h) = 0. \tag{109}$$

Taking this into account, by (108), we immediately complete the proof of the inequality $\lambda^k_{F,0} \leq \lambda^k_{\lim,0}$.

Theorem 0.1 implies that $N^k_h(\lambda) > 0$ for any $\lambda > \lambda^k_{\mathcal{F},0}$ and h small enough, from where the desired inequality $\lambda^k_{\lim,0} \leq \lambda^k_{\mathcal{F},0}$ follows immediately.

We conclude this Section with some remarks and examples, concerning quantities $\lambda^k_{F,0}$, $\lambda^k_{\lim,0}$ and $\lambda^k_{\mathcal{F},0}$.

Remarks. (1) When the foliation $\mathcal{F}$ is a fibration or, more general, $\mathcal{F}$ is amenable in some sense (see also Section 7), relations (105) turns out to be identities [19].

(2) We don't know if the equality $\lambda^k_{F,0} = \lambda^k_{\lim,0}$ is always true. It is, clearly, so for $k = 0$:

$$\lambda^0_{F,0} = \lambda^0_{\lim,0} = 0. \tag{110}$$

Another remark is as follows. If the Betti number $b_k(M)$ is not zero, then $\lambda^k_0(h) = 0$ for all h, that also implies

$$\lambda^k_{F,0} = \lambda^k_{\lim,0} = 0.$$

(3) Here we give an example of the foliation such that the bottom $\lambda^0_{F,0} = 0$ of the operator Δ^0_F in $L^2(M)$ is a point of discrete spectrum.

Example. Let Γ be a discrete, finitely generated group such that

(a) Γ has property (T) of Kazhdan;

(b) Γ is be embedded in a compact Lie group G as a dense subgroup.

For definitions and examples of such groups, see, for instance,[14, 21].

Let us take a compact manifold X such that $\pi_1(X) = \Gamma$. Let $\tilde{X}$ be the universal covering of X equipped with a left action of Γ by deck transformations. We will assume that Γ acts on G by left translations. Let us consider the suspension foliation $\mathcal{F}$ on a compact manifold $M = \tilde{X} \times_\Gamma G$ (see, for instance, [5]). A choice of a left invariant metric on G provides a bundle-like metric on M, so $\mathcal{F}$ is a Riemannian foliation. We may assume that leafwise metric is chosen in such a way that any leaf of the foliation $\mathcal{F}$ is isometric to $\tilde{X}$.

There is defined a natural action of Γ on M and the operator Δ^0_F is invariant under this action. Let $E(0,\lambda), \lambda > 0$, denote the spectral projection of the operator Δ^0_F in $L^2(M)$, corresponding to the interval $(0,\Lambda)$, and $E(0,\lambda)L^2(M)$ be the corresponding Γ-invariant spectral subspace.

Claim. In this example, the bottom $\lambda^0_{F,0} = 0$ of leafwise Laplacian in $L^2(M)$ is a nondegenerate point of discrete spectrum of the operator Δ^0_F, that is, an isolated eigenvalue of the multitplicity 1.

From the contrary, let us assume that zero lies in the essential spectrum of the operator Δ^0_F in $L^2(M)$. Then, for any $\varepsilon > 0$ and $\lambda > 0$, there is a function $u_\varepsilon \in C^\infty(M)$ such that u_ε belongs to the space $E(0,\lambda)L^2(M)$, $\|u_\varepsilon\| = 1$ and

$$(\Delta_F u_\varepsilon, u_\varepsilon) = \|\nabla_F u_\varepsilon\| \le \varepsilon, \tag{111}$$

where ∇_F denotes the leafwise gradient. From (111), we can easily derive that the representation of the group Γ in $E(0,\lambda)L^2(M)$ has almost invariant vector, that, by the property (T), implies the existence of an invariant vector $v_0 \in E(0,\lambda)L^2(M)$.

Since Γ is dense in G, Γ-invariance of v_0 implies its G-invariance, that, in turn, implies that v_0 is a lift of some non-zero element $v \in C^\infty(X)$ via the natural projection $M \to X$. It can be easily checked that v belongs to the corresponding spectral space $E(0,\lambda)L^2(X)$ of the Laplace operator Δ_X in $L^2(X)$. From other hand, the operator Δ_X has a discrete spectrum, so zero is an isolated point in the spectrum of Δ_X, and $E(0,\lambda)L^2(X)$ is a trivial space if $\lambda > 0$ is small enough. So we get a contradiction, which imply that zero lies in the discrete spectrum of the operator Δ^0_F in $L^2(M)$.

(4) In the case of a fibration, we also have that zero is an isolated point in the spectrum of the operator Δ^0_F in $L^2(M)$, but, in this case, it is an eigenvalue of infinity multitplicity, so that it lies in the essential spectrum of the operator Δ^0_F in $L^2(M)$.

(5) Unlike the scalar case, it is not always the case that all of the semiaxis $[\lambda_{\text{lim},0}, +\infty)$ is contained in $\sigma_{lim}(\Delta_h)$. Indeed, let, as in Example of (3), $\lambda^0_{F,0} = 0$ is a nondegenerate point of discrete spectrum of the operator Δ^0_F. Then, by means of the perturbation theory of the discrete spectrum (see, for instance, [15]), we can state that, for $h > 0$ small enough, $\lambda^0(h) = 0$ is the only eigenvalue of the operator Δ^0_h near zero. So we conclude that there exists a $\lambda_1 > 0$ such that, for any $h > 0$ small enough,

$$\sigma_{lim}(\Delta_h) \bigcap [\lambda_1, +\infty) = 0. \tag{112}$$

7 Some remarks on the main asymptotical formula

In this Section, we discuss some aspects of the main asymptotical formula (8). We are, especially, interested in a discussion of the formula (11). We will make use of the notation of previous Sections.

So recall that the whole picture which we observe in the foliation case is the following. In a general case, for any $k = 0, 1, \ldots, n$, we have only that

$$\lambda^k_{F,0} \leq \lambda^k_{\lim,0} \leq \lambda^k_{\mathcal{F},0}, \tag{113}$$

and these relations turns into identities, if the foliation $\mathcal{F}$ is a fibration or, more general, is amenable in some sense (see Section 6 and [19] for discussion).

By (8), the function $N^k_h(\lambda)$ behaves as usual when λ is greater than the bottom of the leafwise spectrum of Δ^k_F:

$$N^k_h(\lambda) \sim Ch^{-q}, \lambda \geq \lambda^k_{\mathcal{F},0}, \tag{114}$$

but, if $\lambda^k_{F,0} < \lambda^k_{\mathcal{F},0}$, there might be limitting values for eigenvalues $\lambda^k_i(h)$ of the operator Δ^k_h, lying in the interval $(\lambda^k_{F,0}, \lambda^k_{\mathcal{F},0})$. So the function $N^k_h(\lambda)$ is nontrivial on the interval $(\lambda^k_{\lim,0}, \lambda^k_{\mathcal{F},0})$, but the fact mentioned above that the right-hand side of (8) depends only on leafwise spectral data of the operator Δ^k_F implies the formula

$$\lim_{h\to 0+} h^q N^k_h(\lambda) = 0,\ \ \lambda < \lambda^k_{\mathcal{F},0}. \tag{115}$$

It means that the set of eigenvalues of Δ^k_h in the interval $(\lambda^k_{\lim,0}, \lambda^k_{\mathcal{F},0})$ is "thin" in the whole set of eigenvalues of Δ_h. By analogy with [27], (115) in the case $k = 0$ may be called as a weak foliated version of "Riemann hypothesis".

This is quite different from what we have in the case of Schrodinger operator or in the fibration case. For instance, if H_h is the Schrodinger operator on a compact manifold M (we may consider M, being equipped with a trivial foliation $\mathcal{F}$ which leaves are points):

$$H_h = -h^2\Delta + V(x), x \in M. \tag{116}$$

we have

$$\lambda_{F,0} = \lambda_{\lim,0} = \lambda_{\mathcal{F},0} = \inf V_-, \tag{117}$$

where

$$V_-(x) = \min(V(x), 0), x \in M, \tag{118}$$

and the following asymptotical formula for spectrum distribution function $N_h(\lambda)$ in semiclassical limit:

$$N_h(\lambda) = (2\pi)^{-n}h^{-n}\int_{\{(x,\xi):\xi^2+V(x)\leq\lambda\}} dx d\xi + o(h^{-n}), h \to 0+. \tag{119}$$

So, if $h \to 0$. the picture is as follows:

$$N_h(\lambda) \sim Ch^{-n}, \lambda > \inf V_-, \tag{120}$$

where $n = \dim M$ and

$$N_h(\lambda) = 0, \lambda \leq \inf V_-. \tag{121}$$

It is worthwhile to note facts in spectral theory of coverings, which are very similar to ones in spectral theory of foliations mentioned above. Let us consider the case of Laplace-Beltrami operator on functions.

Let $\tilde{M} \to M$ be a normal covering with a covering group Γ. Recall that a tower of coverings is a set $\{M_i\}_{i=1}^{\infty}$ of finite-fold subcoverings of this covering with the corresponding covering groups Γ_i such that:

(1) for each i, Γ_i is a normal subgroup of finite index in Γ;

(2) for each i, Γ_{i+1} is contained in Γ_i;

(3) $\bigcap_i \Gamma_i = \{e\}$.

Let $\sigma(\Delta_{M_i})$ be a set of eigenvalues of the Laplacian on M_i, and $N_{M_i}(\lambda)$ be its distribution function. For any i, we have an embedding

$$\sigma(\Delta_{M_i}) \subset \sigma(\Delta_{M_{i+1}}), \tag{122}$$

and when i tends to the infinity the spectrum $\sigma(\Delta_{M_i})$ of a finite covering approaches to a limit

$$\sigma_{lim}(\Delta) = \bigcup_i \sigma(\Delta_{M_i}). \tag{123}$$

Then, the bottom $\lambda_{\mathrm{lim},0}$ of limitting spectra $\sigma_{\mathrm{lim}}(\Delta)$ and the bottom $\lambda_{M,0}$ of the spectrum $\sigma(\Delta_M)$ of the manifold M are, clearly, equal to 0. By [4], the bottom $\lambda_{\tilde{M},0}$ of the spectrum $\sigma(\Delta_{\tilde{M}})$ of the covering manifold is equal to $\lambda_{M,0}$:

$$\lambda_{\tilde{M},0} = \lambda_{M,0}, \tag{124}$$

if and only if the group Γ is amenable.

Moreover, by [10], for any function $f \in C_c^{\infty}(\mathbf{R})$, we have

$$\lim_{i\to\infty} (vol\ M_i)^{-1} tr\ f(\Delta_{M_i}) = tr_{\Gamma}\ f(\Delta_M), \tag{125}$$

where tr_Γ is von Neumann trace on the the algebra of Γ-invariant operators on $\tilde{M}$ [2]. In particular, if $N_i(\lambda)$ is the eigenvalue distribution function of the Laplace-Beltrami operator Δ_{M_i}, then

$$\lim_{i\to\infty}(vol\ M_i)^{-1}N_i(\lambda) = N_\Gamma(\lambda), \lambda \in \mathbf{R}, \tag{126}$$

$$\lim_{i\to\infty}(vol\ M_i)^{-1}N_i(\lambda) = 0, \lambda < \lambda_{\tilde{M},0}, \tag{127}$$

where $N_\Gamma(\lambda)$ is spectrum distribution function of the operator $\Delta_{\tilde{M}}$ constructed by means of the Γ-trace tr_Γ, $\lambda_{\tilde{M},0} = \inf \sigma(\Delta_{\tilde{M}})$

A little bit more general possibility to arrange finite-dimensional approximation of the spectrum of a covering, making use of sequences of finite-dimensional representations of a covering group Γ, converging to the left regular representations of Γ, is considered in [27]. Analogues of (8) and (115) can be also found in [27].

We may point out two common features of spectral theory for Laplacian on a covering and spectral theory for leafwise Laplacian on foliated manifold. From the tangential point of view, both of them can be treated as type II spectral problems in a sense of theory of operator algebras, and asymptotical spectral problems mentioned above can be considered as finite-dimensional (of type I) approximations to these spectral problems. Actually, some spectral characteristics related with such an approximation don't depend on a choice of a bundle-like metric on M, and, moreover, are invariants of quasi-isometry of metrics (coarse invariants in a sense of [26]). One of the simplest characteristics of such a kind which we have already met is the notion of amenability.

We can introduce some quantative spectral characteristics of the tangential Laplacian Δ_F^k related with adiabatic limits. For any λ, let $r_k(\lambda)$ be given as

$$r_k(\lambda) = -\limsup_{h\to 0} \ln N_h^k(\lambda)/\ln h. \tag{128}$$

Otherwise speaking, $r_k(\lambda)$ equals the least bound of all r such that

$$N_h^k(\lambda) \sim Ch^{-r}, h \to 0. \tag{129}$$

If $\lambda < \lambda_{\lim,0}^k$, we put $r_k(\lambda) = -\infty$.

Then we can easily state the following properties of the function $r_k(\lambda)$:

1. $0 \le r_k(\lambda) \le q$ for any $\lambda \ge \lambda_{\lim,0}^k$;
2. $r_k(\lambda)$ is not decreasing in λ;
3. $r_k(\lambda) = q$ if $\lambda > \lambda_{\mathcal{F},0}^k$.

4. if the foliation $\mathcal{F}$ is amenable, then:

$$r_k(\lambda) = q, \quad \lambda > \lambda^k_{\mathcal{F},0},$$
$$r_k(\lambda) = -\infty, \quad \lambda \leq \lambda^k_{\mathcal{F},0}.$$

5. $r_k(\lambda) = 0$ iff the interval $[0, \lambda]$ lies in the discrete spectrum of the operator Δ^k_F in $L^2(M, \Lambda^k T^* M)$. As we have seen in the previous Section, such situation can happen (a property (T) case).

Then we expect that some invariants of the function $r_k(\lambda)$ introduced above near $\lambda = 0$ might to be independent of the choice of metric on M (otherwise speaking, to be coarse invariants), and, moreover, be topological or homotopic invariants of foliated manifolds.

From transversal point of view, both of them are related with some sort of "non-commutative" fibration in sense of noncommutative differential geometry [7]. Here the relation (115) reflects a nontriviality of geometry of these "fibrations" in the nonamenable case.

Now we point out two facts in noncommutative spectral geometry of foliations, which are closely related with (115). When the foliation $\mathcal{F}$ is Riemannian, we can consider $M/\mathcal{F}$ as a noncommutative Riemannian manifold. More precise, we can define the corresponding spectral triple (in a sense of [8]) as follows:

1. An involutive algebra $\mathcal{A}$ is an algebra $C^\infty_c(G_\mathcal{F})$ of smooth, compactly supported functions on the holonomy groupoid $G_\mathcal{F}$ of the foliation $\mathcal{F}$;

2. A Hilbert space $\mathcal{H}$ is a space $L^2(M, \Lambda H^*)$ of the transversal differential forms, on which an element k of the algebra $\mathcal{A}$ is represented via a smoothing tangential operator with the tangential kernel k;

3. an operator D is the transverse signature operator $d_H + \delta_H$ of a bundle-like metric on M.

Let $C^*(G_\mathcal{F})$ ($C^*_r(G_\mathcal{F})$) be the full (reduced) C^*-algebra of the foliation respectively. There is the natural projection $\pi : C^*(G_\mathcal{F}) \to C^*_r(G_\mathcal{F})$. We say that the foliation $\mathcal{F}$ is amenable, if the projection $\pi : C^*(G_\mathcal{F}) \to C^*_r(G_\mathcal{F})$ is an isomorphism.

The first fact is that, in a case of the foliation $\mathcal{F}$ is nonamenable, this noncommutative Riemannian manifold has pieces of various dimension with the top dimension, being, certainly, equal to q in the following sense.

Let us consider subsets of $V/\mathcal{F}$ as involutive ideals in $C^*(G_\mathcal{F})$. We can speak about the top spectral dimension of the pieces of our space which are contained $\mathcal{I}$ in the following way (see [8] for details). We say that this bound is less than k, if for any $a \in \mathcal{I}$ the distributional zeta function

$$\zeta_a(z) = \text{tr } a|D|^{-z} \tag{130}$$

extends holomorphically to the halfplane $\{z \in \mathbf{C} : Re\ z > k\}$. By the Tauberian theorem, the top dimension of the subset in the space can be also detected by means of asymptotics of the distributional spectrum distribution function

$$N_a(\lambda) = tr(aE_\lambda(|D|)), a \in \mathcal{I}, \lambda \in \mathbf{R}, \tag{131}$$

where $E_\lambda(|D|)$ is the spectral projection of the operator $|D|$, corresponding to the semiaxis $(-\infty, \lambda)$, or the theta-function

$$\theta_a(t) = tr(ae^{-tD^2}), a \in \mathcal{I}, t > 0. \tag{132}$$

For instance, the top spectral dimensions of the pieces of our space which are contained $\mathcal{I}$ is less than k, if for any $a \in \mathcal{I}$ the distributional theta function $\theta_a(t)$ satisfies the estimate

$$\theta_a(t) \leq Ct^{-k/2}, 0 < t \leq 1. \tag{133}$$

Then we have (compare with Proposition 4.4 in [20]):

An involutive ideal $\mathcal{I}$ in $C^*(G_\mathcal{F})$ has the top dimension q iff $\mathcal{I} \bigcap \pi(C^*_r(G_\mathcal{F})) \neq \emptyset$. In particular, if $\pi(\mathcal{I}) = 0$, then the top spectral dimension of $\mathcal{I}$ is less than q.

The other fact is related with the support of the "noncommutative" integral, given by the Dixmier trace Tr_ω. Namely, it can be shown that in the case under consideration the Dixmier trace $Tr_\omega(k)$, corresponding to the spectral triple introduced above exists and doesn't depend on a choice of ω for any $k \in C^*(G_\mathcal{F})$. Then we have

$$Tr_\omega(k) = 0 \tag{134}$$

for any $k \in C^*(G_\mathcal{F}), \pi(k) = 0$. To relate these facts with the spectral theory of the tangential Laplace-Beltrami operator Δ_F, we have to note that, by [19]:

(1) the operator $f(\Delta_F)$ belongs to the C^*- algebra $C^*(G_\mathcal{F})$ for any $f \in C^\infty_c(\mathbf{R})$, and,

(2) by spectral theory, $\pi(f(\Delta_F)) = 0$ for any $f \in C_c^\infty(\mathbf{R})$ such that $supp\ f \subset (\lambda_{\lim,0}, \lambda_{\mathcal{F},0})$.

It seems also to be true that the function $r_k(\lambda)$ introduced above takes values in the spectrum dimension Sd of the noncommutative spectrum space in question (see [8]).

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