CURVATURE DECOMPOSITION OF G_2 MANIFOLDS

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Abstract. Explicit formulas for the G_2-components of the Riemannian curvature tensor on a manifold with a G_2 structure are given and the norm of the Riemannian curvature is related to the norms of Ricci-type contractions. An equation for the most general symmetric tensor derived from the Riemannian curvature in terms of torsion components and their derivatives with respect to the canonical G_2 connection is presented. A topological obstruction for the existence of a closed G_2-structure on compact 7-manifold is obtained in terms of the integral norms of the curvature components. Integral inequalities for compact closed G_2 manifold are produced and limiting cases are investigated. A study is made of warped products and cohomogeneity one G_2 manifolds with non-trivial torsion and few or no non-zero curvature components. As a consequence every Fernandéz-Gray type of G_2 structure for which vanishing of the scalar curvature is possible may be realized for some G_2 structure so that the associated metric has holonomy contained in G_2.

1. Introduction

A 7-dimensional Riemannian manifold is called a G_2 manifold if its structure group reduces to the exceptional Lie group G_2. The existence of a G_2 structure is equivalent to the existence of a positive, non-degenerate three-form on the manifold, sometimes called the fundamental form of the G_2 manifold. From the purely topological point of view, a 7-dimensional paracompact manifold is a G_2 manifold if and only if it is an oriented spin manifold admitting a nowhere vanishing spinor field [36].

In [22], Fernández and Gray divide G_2 manifolds into 16 classes according to how the covariant derivative of the fundamental three-form behaves with respect to its decomposition into G_2 irreducible components, see also [13]. If the fundamental form is parallel with respect to the Levi-Civita connection then the Riemannian holonomy group is contained in G_2, we will say that the G_2 manifold or the G_2 structure on the manifold is *parallel*. In this case the induced metric on the G_2 manifold is Ricci-flat, a fact first observed by Bonan [4]. It was shown by Gray [29] (see also [22, 6, 38]) that a G_2 manifold is parallel precisely when the fundamental form is harmonic. The first examples of complete parallel G_2 manifolds were constructed by Bryant and Salamon [9]. Compact examples of parallel G_2 manifolds were obtained first by Joyce [33, 32] and by Kovalev [35]. Examples of G_2 manifolds in other Fernández-Gray classes may be found in [20, 10, 8].

The geometry of G_2 structures has also attracted much attention from physicists. One reason is that G_2 structures preserve a spinor field which may then play the rôle of a supersymmetry in string theory [28, 27, 37].

Based on the general theory of Cartan, Robert Bryant observes in [8] that, for a G_2 structure, the diffeomorpism invariants, polynomial in derivatives up to second order of the defining three-form, are sections of a vector bundle of rank 392. This includes the Riemannian curvature tensor of the underlying metric. In the same paper Robert Bryant describes the G_2 invariant splitting of this bundle into eleven

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irreducible components. As a particular case the G_2-irreducible components of the Riemannian curvature are given.

In the present note we study the links between vanishing of curvature components with the first order invariants of the G_2 structure up and vice versa.

In our first main result we describe algebraically the curvature components in terms of the Ricci tensor, $\star$-Ricci tensor introduced in [16] (here re-baptized "the ϕ-Ricci tensor" to emphasize the dependency on the three-form ϕ rather than the metric) and the scalar curvature, Theorem 4.3. We express the intrinsic torsion, which is the obstruction to the Levi-Civita connection of the induced Riemannian metric to be a G_2-connection, in terms of the exterior derivative and co-derivative of the fundamental three form and vice-versa. This allows us to connect explicitly the canonical connection, i.e., the G_2-connection with minimal torsion, to the Levi-Civita connection. Consequently, we determine four of the curvature components in terms of the intrinsic torsion and its derivatives. We consider linear combinations of the Ricci and ϕ-Ricci tensors, the most general symmetric 2-tensor derived from the Riemannian curvature, and give a formula in terms of the intrinsic torsion and its covariant derivatives with respect to the canonical connection in Lemma 4.6.

All these facts help us attack the problem how each of the curvature components depends on the intrinsic torsion and whether its vanishing determines the G_2 structure. We apply this to nearly parallel G_2 manifolds (for which the results are well-known), as well as to G_2 manifolds with closed fundamental three form. These two cases have the peculiar property that the full space of second order diffeomorphism invariants is determined by the components of the Riemannian curvature of the underlying metric. This is, in general, not true for a G_2 structure [8] and Remark 4.4.

We show that three of the curvature components of a nearly parallel G_2 structure vanish. This was observed by Ramon Reyes Carrion, see [12]. A few more details may be found in [14].

A special case of G_2 structures with closed fundamental form (closed G_2 structures for short) is made. We find a topological obstruction to the existence of closed G_2 structures on compact manifolds in terms of the integral norms of the curvature components. This constitutes our second main observation, Theorem 6.11.

It is known [16] (see also [8]) that any compact Einstein (i.e. the trace-free Ricci curvature component is zero) closed G_2 manifold is parallel. In Corollary 6.6, we observe that the vanishing of the 27-dimensional curvature component on a compact closed G_2 manifold also implies the G_2 structure is parallel. We show that the 64-dimensional component of the Riemannian curvature vanishes exactly when the 64-dimensional component of the covariant derivative of the intrinsic torsion with respect to the canonical connection is zero. The concept of extremally Ricci pinched (extremal for short) closed G_2 structures was introduced and studied by Robert Bryant in [8]. We demonstrate that extremal closed G_2 structures are precisely those for which the remaining component of the covariant derivative of the intrinsic torsion is zero. This allows us to characterize compact extremal closed G_2 manifold with an integral equality in Corollary 6.12. Moreover, we obtain that closed G_2 manifolds for which intrinsic torsion is parallel with respect to the canonical connection are precisely the extremal ones with vanishing 64-dimensional curvature component. Our third main result is Theorem 6.21. We prove that all extremal G_2 spaces with such parallel torsion are locally isometric to the example constructed by Robert Bryant in [8]. We conjecture that all compact extremal closed G_2 manifolds are of that kind.

In the last section we present some examples. Warped products and cohomogeneity one G_2 manifolds with exactly one non-vanishing curvature component and non-trivial torsion are given. As a consequence every Fernandéz-Gray type of G_2 structure for which the scalar curvature can vanish is possible may be realized for some G_2 structure so that the associated metric has holonomy contained in G_2. Examples of manifolds admitting both parallel G_2 structures and G_2 structures with torsion were given in [1] where a family of three-forms and metrics were shown to contain G_2-structure both parallel and with non-trivial torsion. The examples we give here clearly also admit both compatible parallel G_2 structures and ones with torsion. One important difference between the types of structures considered by Agricola et al and those we give here is that, for our part, the examples have trivial and non-trivial torsion for G_2-structures compatible the same metric, while the metric associated to the G_2 structures in [1] deform with the G_2-structure.

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2. The fundamental three-form

Let $(V, \langle \cdot, \cdot \rangle)$ be an n-dimensional Euclidean vector space. Say that an n-form vol on V is a volume form for the given inner product if $|\mathrm{vol}| = 1$ with respect to the inner product induced on the exterior algebra $\Lambda^* V^*$. Write $i_v \colon \Lambda^p V^* \to \Lambda^{p-1} V^*$ for the interior product. Suppose now that $n = 7$ and that ϕ is a three-form on V such that

$$i_u\phi \wedge i_v\phi \wedge \phi = 6 \langle u, v \rangle \mathrm{vol} \tag{2.1}$$

for some positive definite inner product $\langle \cdot, \cdot \rangle$ and volume form vol. Then ϕ is non-degenerate in the sense that $X \mapsto \Lambda^2 V^*$ is injective. It follows from this that the isotropy group of ϕ is the simple Lie group G_2, see e.g. [30]. Fix a unit vector $e \in V^*$ and let V' be the orthogonal complement of e in V^*. Then $\phi = \omega \wedge e + \psi^+$ for some $\omega \in \Lambda^2 V'$, $\psi^+ \in \Lambda^3 V'$ and $\mathrm{vol} = \mathrm{vol}' \wedge e$ where vol' is a volume form of the inner product restricted to V'. Since $|e| = 1$ follows that $\omega^3 = 6$. As the isotropy group of any vector in V is isomorphic to $SU(3)$ it follows that ψ^+ is the real part of some complex volume form normalized so that $2\omega^3 = -3\psi^+ \wedge J\psi^+$, where J is the complex structure defined by ω and ψ_+, see e.g. [13]. A basis e^i for V' may then be chosen so that $Je^1 = e^2$ and so on. Then $\omega = e^{12} + e^{34} + e^{56}$ and $\psi^+ = e^{135} - e^{245} - e^{146} - e^{236}$. Setting $e = e^7$ we may write

$$\phi = e^{127} + e^{347} + e^{567} + e^{135} - e^{245} - e^{146} - e^{236}, \tag{2.2}$$

and

$$*\phi = e^{1234} + e^{3456} + e^{5612} - e^{2467} + e^{1367} + e^{2357} + e^{1457}.$$

From equation (2.1) it is clear that G_2 is a closed subgroup of $SO(7)$. Via the inner product the Lie algebra $\mathfrak{g}_2$ of G_2 may be identified with the 14 dimensional subspace of $\Lambda^2 V^*$ complementary to the span of $i_u\phi \colon u \in V$. Say that a three-form satisfying equation (2.1) for some positive inner product and volume form on V is a G_2 *three-form* or *fundamental three-form of* G_2. The inner product and volume form defined so are said to be *associated to the three-form.* Alternatively, fixing an inner product and volume form, any three-form satisfying the relation (2.1) is called *compatible* with the metric and given orientation. A basis of one-forms $\{e^i\}$

over a vector space V for which a G_2 three-form ϕ has the expression (2.2) is called G_2 *adapted*.

A G_2 three-form ϕ gives a splitting of the exterior algebra Λ^*V. We have equivariant projections $p^r_d\colon \Lambda^r V^* \to \Lambda^r V^*$ given by

$$\begin{aligned} p^2_7(\alpha) &= \tfrac{1}{3}(\alpha + *(\alpha\wedge\phi)),\\ p^2_{14}(\alpha) &= \tfrac{1}{3}(2\alpha - *(\alpha\wedge\phi)),\\ p^3_1(\beta) &= \tfrac{1}{7}*(*\phi\wedge\beta)\phi,\\ p^3_7(\beta) &= \tfrac{1}{4}*(*(\phi\wedge\beta)\wedge\phi),\\ p^3_{27}(\beta) &= \beta - (p^3_1 + p^3_7)(\beta). \end{aligned} \tag{2.3}$$

Subscript d where d is $1, 7, 14$, or 27 indicates the dimension of the image of the relevant projection, denoted by Λ^r_d. These are irreducible representations of G_2 for each of the projections given above. Projection for $r > 3$ are obtained by composing with the Hodge star operator $*\colon \Lambda^r \to \Lambda^{7-r}$, $p^r_d := * \circ p^{7-r} *$.

Suppose a is a non-zero constant. The G_2 three-form $\bar\phi := a^3\phi$ has associated metric and volume $\bar g = a^2 g$, $\bar{\mathrm{vol}} := a^7\mathrm{vol}$. Since the Hodge operator $\bar *$ of $\bar g$ satisfies $\bar * = \lambda^{7-2p}*$ one easily verifies that the associated projections satisfy $\overline{p^r_d} = p^r_d$ and so are invariant under rescaling of the G_2 structure.

On several occasions in what follows we shall come across representations that do not occur as subspaces in the exterior algebra of the standard representation. We fix notation for these as follows. Choose a system of positive roots for $\mathfrak{g}_2$ such that the standard representations has highest weight $(1, 0)$ and the adjoint representation $(0, 1)$. Then we will write $V^{(\mu_1,\mu_2)}_d$ for (the isomorphism class of) the irreducible representation with highest weight (μ_1, μ_2) and dimension d. This means for instance that the standard representation is $V^{(1,0)}_7$, the adjoint representation $\mathfrak{g}_2$ is $V^{(0,1)}_{14}$, while the space of trace-less symmetric tensors is $V^{(2,0)}_{27}$. When the dimension is sufficient to identify the representation the superscript will be dropped.

The 27 dimensional subspace Λ^3_{27} is isomorphic the space of traceless symmetric tensors over V. This isomorphism may be given explicitly as the restriction of

$$\lambda_3(e\otimes e) := e\wedge(e\lrcorner\phi) \tag{2.4}$$

to trace-free tensors. A map in the opposite direction is given by contracting an arbitrary three-form with the fundamental form over two indices

$$\sigma(\alpha)(u, v) = \langle i_u\phi, i_v\alpha\rangle\,.$$

The two-tensor $\sigma(\alpha)$ is only a symmetric tensor when $p^3_7(\alpha)$ vanishes and trace-free only if $p^3_1(\alpha) = 0$. Note that $\lambda_3(g) = 3\phi$. For a symmetric tensor h with zero trace one has the simple relation

$$|\lambda_3(h)|^2 = 2\,\|h\|^2\,. \tag{2.5}$$

A few words on how identities such as (2.5) and (2.3) are verified as these techniques are well-established. All identities here are relations between maps to or from an irreducible representation. Then Schur's Lemma ensures that any two such maps must be equal up to a constant multiple. It is then sufficient to calculate left- and right-hand side on a *test element*. As a service to the reader we provide a few samples. Choose a G_2 adapted basis e^i of V^* such that the G_2 three-form may be written as $\phi = \omega\wedge e^7 + \psi^+$ with ω and ψ^+ as above. Then $\omega \in \Lambda^2_7$, $e^{12} - e^{34} \in \Lambda^2_{14}$ are test elements in Λ^2V^*. In degree 3 we have $\phi \in \Lambda^3_1$ while $\psi^- := -e^{246} + e^{136} + e^{235} + e^{145} \in \Lambda^3_7$ and $4\omega\wedge e^7 - 3\psi^+ \in \Lambda^3_{27}$. We give more test elements and an example of how these are used in the proof of the following Lemma.

Lemma 2.1. *Let V denote $\mathbb{R}^7$ equipped with G_2 three-form ϕ (2.2) and associated metric g. Let $\wedge_3\colon V^* \otimes \Lambda^2 V^* \to \Lambda^3 V^*$ be the map given by the wedge product $\wedge_3(\alpha \otimes \beta) = \alpha \wedge \beta$. The tensor product $V^* \otimes \Lambda^2_{14}$ decomposes as*

$$V^* \otimes \Lambda^2_{14} \cong V_{64}^{(1,1)} + V_{27}^{(2,0)} + V_7^{(1,0)}$$

and the restriction $\wedge_3|\colon V^* \otimes \Lambda^2_{14} \to \Lambda^3 V^*$ has kernel $V_{64}^{(1,1)}$ and cokernel Λ^3_1. Moreover, the identity

$$7\left\|\gamma\right\|^2 = \left\|\wedge_3(\gamma)\right\|^2$$

holds for every γ in the 27 dimensional irreducible submodule of $V^* \otimes \Lambda^2_{14}$.

Proof. First, since V^* and Λ^2_{14} are non-isomorphic representation the decomposition of their tensor product contains no trivial summand. So it is clear that the cokernel of $\wedge_3\colon V^* \otimes \Lambda^2_{14} \to \Lambda^3 V^*$ must contain Λ^3_1. Furthermore, as $V_{27}^{(2,0)}$ is real and irreducible, there is, up to scale, precisely one invariant map $S^2(V_{27}^{(2,0)}) \to \mathbb{R}$. By Schur's Lemma, there is a constant c so that the relation $c\left\|\gamma\right\|^2 = \left\|\wedge_3(\gamma)\right\|$ holds for all $\gamma \in V_{27}^{(2,0)}$.

Let $\{e^i\}$ be a G_2 adapted basis. Write $\pi\colon V^* \otimes \Lambda^2 V^* \to V^* \otimes \Lambda^2_{14}$ for the orthogonal projection $\alpha \otimes \beta \mapsto \alpha \otimes p^2_{14}(\beta)$. Then $\pi(e^i \otimes e^{i7})$ provides a test element in a submodule of $V^* \otimes \Lambda^2_{14}$ isomorphic to $V^* \cong V_7^{(1,0)}$ and one may calculate $\wedge_3(\pi(e^i \otimes e^{i7})) = -\psi^- \in \Lambda^3_7$. This shows that $\operatorname{coker}(\wedge_3|)$ can be at most $\Lambda^3_{27} + \Lambda^3_1$.

Set $\gamma' := e^7 \otimes (e^{12} - e^{34}) \in V^* \otimes \Lambda^2_{14}$. Then $\wedge_3(\gamma') = e^{127} - e^{347} \in \Lambda^3_{27}$ which proves that $\operatorname{coker}(\wedge_3|) = \Lambda^3_1$ and also shows that $V^* \otimes \Lambda^2_{14}$ contains irreducible submodules isomorphic to V_{27} and V. The decomposition now by noting that the dimension of the Cartan product $V^{(1,1)}$ of $V^* \otimes \Lambda^2_{14}$ is 64. It is then clear that $\ker(\wedge_3|) \cong V_{64}^{(1,1)}$.

All we now need is to find a test element in $V_{64}^{(1,1)} \subset V^* \otimes \Lambda^2_{14}$. Since $p^3_7(\wedge_3(\gamma')) = 0$, γ' itself must lie the submodule isomorphic to $V_{64}^{(1,1)} + V_{27}^{(2,0)}$. Composing inclusion $i\colon \Lambda^3 V^* \hookrightarrow V^* \otimes \Lambda^2 V^*$ with the projection π we obtain $\gamma'' := \pi(i(\wedge_3(\gamma'))) - \gamma'$. It is easy to check that

$$\langle \gamma'', \gamma' \rangle = 0, \qquad \left\|\gamma''\right\|^2 = \tfrac{16}{3}, \qquad \left\|\gamma'\right\|^2 = 4,$$

in the tensor-norm, and,

$$\wedge_3(\gamma'') = \tfrac{4}{3} \wedge_3 (\gamma').$$

We then have $4\gamma'' - 3\gamma' \in V_{64}^{(1,1)}$ and $\gamma := \gamma'' + \gamma' \in V_{27}^{(2,0)}$. Evaluating the norms of γ and $\wedge_3(\gamma)$ completes the proof. $\square$

When maps are not between irreducible modules, but are still G_2 equivariant, calculations on the components of the tensors are facilitated by the fact that ϕ generates the space of invariant tensors. This guarantees relations between contractions of metric g, ϕ and $*\phi$ spelled out in [8] and also exploited in [16][1]. For ease of reference we give these identities here. Let ϕ be a G_2-three-form and $*\phi$ its dual four-form via the associated metric and orientation. Write ϕ_{ijk} for the components of ϕ and ϕ_{ijkl} for the components of $*\phi$ with respect to a basis e^i of

[1] We should warn that the choice of orientation in [16] is the opposite of the one here so translations should be made with due care wherever $*$ appears.

one-forms on V^*. Then one has the identities, see [8]

$$\phi_{ipq}\phi_{pqj} = 6\delta_{ij}, \tag{2.6}$$

$$\phi_{ijp}\phi_{pkl} = \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} + \phi_{ijkl}, \tag{2.7}$$

$$\phi_{ijpq}\phi_{pqkl} = 4(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}) + 2\phi_{ijkl}, \tag{2.8}$$

$$\phi_{ipq}\phi_{pqjk} = 4\phi_{ijk}, \tag{2.9}$$

$$\phi_{ijp}\phi_{pklm} = \delta_{ik}\phi_{jlm} - \delta_{jk}\phi_{ilm} + \delta_{il}\phi_{jmk} - \delta_{jl}\phi_{imk} + \delta_{im}\phi_{jkl} - \delta_{jm}\phi_{ikl}, \tag{2.10}$$

where repeated indices here and below means summation is taking place. The identities listed here are valid only when the basis chosen is orthonormal. For a general basis one must replace δ_{pq}s with the components g_{pq} of the associated metric in this basis and simple summations must be replaced with contractions with the inverse metric. This means, for instance, that the first identity becomes $\phi_{ipr}g^{pq}g^{rs}\phi_{qsj} = 6g_{ij}$. To prove these identities one simply notes the symmetries of the left hand side, for instance $\phi_{ijpq}\phi_{pqkl}$ must be the components of a tensor in $S^2(\Lambda^2V^*)$. The space of G_2 invariant elements of $S^2(\Lambda^2V^*)$ is generated by $g \owedge g$, where $\owedge$ is the Kulkarni-Nomizu product (see equation (4.18) below) and $*\phi$. Therefore an identity $\phi_{ijpq}\phi_{pqkl} = c_1(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}) + c_2\phi_{ijkl}$, for constants c_1, c_2, must hold for the components of $*\phi$ taken with respect to an orthonormal basis. The constants are found easily by evaluating for two sets of index values, say $ijlk = 1212$ and $ijkl = 1234$ where components are taken with respect to a G_2 adapted basis.

When we speak of a calculation in a G_2 adapted or orthonormal basis, we refer to a calculation exploiting the identities (2.6)- (2.10).

Remark 2.2. Other standard forms of the G_2 three-form may be used. Taking $\omega = \sum_{i\in\mathbb{Z}/7\mathbb{Z}} e^{i(i+1)(i+3)}$ to be the standard three-form and $e^{1234567}$ the volume form results in the opposite orientation of V to the one indicated by equation (2.1). One consequence of taking this different convention is that certain signs in the formulas corresponding to (2.6)- (2.10) change, see [16]. Therefore translations of results from one convention to another should be made with due care wherever $*$ appears. On the other hand, the choices of standard form of the three-forms here and in for instance [8] are equivalent in the sense that one may be obtained from the other by an *even* permutation of indices in (2.2).

The 'standard' form chosen here is particularly apt when dealing with examples build over 6 dimensional geometries as we shall be in section 7, see also [13, 17].

3. Torsion of a G_2 structure

3.1. **Canonical connection and intrinsic torsion.** First a few generalities. Suppose $P \subset F(M,g)$ is a G structure on a Riemannian manifold (M,g), i.e., a reduction of the bundle of oriented orthonormal frames over a Riemannian manifold (M,g) with structure group $G \subset SO(n)$. Write V for the induced representation of G on $\mathbb{R}^n$. Then the tangent bundle TM may be identified with the associated bundle $P \times_G V$. Similarly, if $\mathbb{W} \subset T^{(p,q)}M$ is any bundle of tensors for which the induced representation of G on fibers is W then $\mathbb{W} = P \times_G W$. Tensor fields $\gamma \in \Gamma(\mathbb{W})$ may be identified with equivariant functions $\gamma\colon P \to V$, $\gamma(pg) = g^{-1}\gamma(p)$. We adopt the convenient but somewhat abusive notation $\gamma \in W$, meaning γ is a section of the bundle associated to W. Note that splitting W into G invariant subspaces $W = W' + W''$ induces a corresponding splitting of associated bundles and tensors. The G equivariant maps between representations give rise to bundle maps on the associated bundles.

The *structure function* $\sigma(P,g)$ of P may be defined as follows. Let $\mathfrak{g}$ be the Lie algebra of G. The isomorphism $\delta\colon V^* \otimes \mathfrak{so}(n) \to \Lambda^2V^* \otimes V$ given by $(\delta\alpha)(X,Y) :=$

$-\alpha(X)Y+\alpha(Y)X$ sends $V\otimes\mathfrak{g}$ onto an isomorphic subspace of $\Lambda^2V^*\otimes V$. Let $p^\perp$ be the projection to the orthogonal complement with respect to the inner product induced by g. Pick any G connection on TM and let T be its torsion. Then the structure function is $\sigma(P,g):=p^\perp(T)$, see [38]. The structure function vanishes identically if and only if the Levi-Civita connection ∇^g is a G connection. Let $\xi\in V^*\otimes\mathfrak{g}^\perp$ be defined by $\delta\xi=\sigma(P,g)$. This is the *intrinsic torsion* of the G structure. The *canonical connection* $\overline{\overline{\nabla}}:=\nabla^g-\xi$ is then the unique G connection associated to P whose torsion $\overline{T}$ obeys $\overline{T}=p^\perp(\overline{T})$. Splitting the intrinsic torsion into its G irreducible components allows one to classify G structures according to their torsion type á la Gray, Hervella and Fernandéz.

Suppose that Φ is a tensor on M parallel with respect to some G connection ∇ on M. Then Φ is in particular invariant by the action of G on tangent spaces. So contraction with Φ gives G-equivariant bundle maps $c^\Phi\colon W\to W'$ between G-invariant bundles W and W' of tensors over M. Considered as a section of the bundle $Hom(W,W')$, c^Φ is then again parallel with respect to ∇. This implies that

$$\nabla_X(c^\Phi(\Psi))=c^\Phi(\nabla_X\Psi),\qquad X\in\Gamma(TM),\ \Psi\in\Gamma(W),\tag{3.11}$$

independent of the specific bundles and contractions involved. We will use this in several places below.

For G_2 structures the complement $\mathfrak{g}_2^\perp$ of $\mathfrak{g}_2$ in $\mathfrak{so}(7)$ is isomorphic to the standard representation $V=V_7$ of G_2. So the decomposition of the intrinsic torsion ξ follows from the splitting

$$V\otimes V=S^2V+\Lambda^2V=S^2_0V+\mathbb{R}+\Lambda^2_{14}+\Lambda^2_7.\tag{3.12}$$

We write ξ_d for the projection of ξ to the d-dimensional subspace of $V^*\otimes\mathfrak{g}_2^\perp$ so corresponding to the decomposition (3.12)

$$\xi=\xi_{27}+\xi_1+\xi_{14}+\xi_7.$$

We shall use . to denote the induced action of a lie algebra $\mathfrak{g}$ on tensor product of a representation V of G. Extending this to associated bundles we apply the short-hand $\nabla^g\gamma=\overline{\overline{\nabla}}\gamma+\xi.\gamma$. With the convention that $\wedge_p\colon V^*\otimes\Lambda^pV^*\to\Lambda^{p+1}V^*$ is the map $\alpha\otimes\beta\to\alpha\wedge\beta$ we define

$$d^{\overline{\overline{\nabla}}}\beta:=\wedge_p(\overline{\overline{\nabla}}\beta)=d\beta-\wedge_p(\xi.\beta)$$

for p-forms β on M.

3.2. **Manifolds with G_2 structure.** A *G_2 structure or G_2 three-form* on a 7 dimensional manifold M is a three-form ϕ such that for any two vector fields X,Y

$$i_X\phi\wedge i_Y\phi\wedge\phi=6g(X,Y)\mathrm{vol}(g),\tag{3.13}$$

where g is a Riemannian metric and $\mathrm{vol}(g)$ is a volume element for g. Fixing the three-form, we say that g and $\mathrm{vol}(g)$ satisfying (3.13) are the metric and orientation *associated* to ϕ. A different viewpoint is offered by fixing a metric g and a orientation. Then a three-form ϕ is called *compatible* with this choice when (3.13) holds. Given a G_2 structure local orthonormal frame fields e^i may be chosen such that ϕ in this basis takes on the standard form (2.2). Such a frame field is called a *G_2 adapted frame.*

3.3. **Derivatives of the fundamental three-form.** The *torsion components* of a G_2 structure are differential forms $\tau_p\in\Omega^p(M)$ such that [8]

$$\begin{aligned}d\phi&=\tau_0{*\phi}+3\tau_1\wedge\phi+{*\tau_3},\\ d{*\phi}&=4\tau_1\wedge{*\phi}+\tau_2\wedge\phi.\end{aligned}\tag{3.14}$$

This pair of equations are the *structure equations* for the G_2 form ϕ.

The torsion type or Fernandéz-Gray class of a G_2 structure is determined by vanishing of torsion components. We use the original notation so that $\tau_0 \leftrightarrow 1$, $\tau_1 \leftrightarrow 4$, $\tau_2 \leftrightarrow 2$, and $\tau_0 \leftrightarrow 3$. For instance, a three-form has type $1+3$ if $\tau_2 = 0 = \tau_1$ and strict, or proper, type $1+3$ if $\tau_2 = 0 = \tau_1$, and $\tau_0 \not\equiv 0$ and $\tau_3 \not\equiv 0$. If all components are zero we say that ϕ is parallel.

The torsion τ and intrinsic torsion may be related explicitly as follows. Let $\bar{\xi}$ denote the two-tensor obtained through the isomorphism $\mathfrak{g}_2^\perp \to V^*$ given by composition $\mathfrak{g}_2^\perp \simeq \Lambda^2_7 \to \Lambda^1_7$. In an orthonormal frame e_i this is the contraction

$$\bar{\xi} = \xi_{ipq}\phi_{pqj} e^i \otimes e^j, \tag{3.15}$$

where $\xi_{ijk} = g(\xi_{e_i} e_j, e_k)$. The components of the intrinsic torsion may be recovered from $\bar{\xi}$ by the relation

$$\xi_{ijk} = \tfrac{1}{6}\bar{\xi}_{ip}\phi_{pjk}.$$

In fact, still working in an orthonormal frame e_i one then has the pleasant looking expression for the covariant derivative of ϕ: $(\nabla^g{}_i\phi)_{jkl} = -\frac{1}{2}\bar{\xi}_{ip}\phi_{pjkl}$ which leads to the relations

$$\begin{aligned} \bar{\xi}_1 &= -\tfrac{1}{2}\tau_0 g, & \bar{\xi}_7 &= 2{*}(\tau_1 \wedge {*}\phi), \\ \bar{\xi}_{14} &= \tau_2, & \bar{\xi}_{27} &= \sigma(\tau_3). \end{aligned} \tag{3.16}$$

These may also be found in [34]. An example of an application of the identity (3.11) is the component-wise relation of covariant derivatives of $\bar{\xi}$ and ξ

$$\overline{\nabla}_i \xi_{jkl} = \overline{\nabla}_i \bar{\xi}_{jp}\phi_{pkl}.$$

4. Curvature of G_2 manifolds

For a G structure on (M, g) the Riemannian curvature tensor may be given the following expression in terms of the canonical connection $\overline{\nabla}$ and the intrinsic torsion ξ:

$$\mathrm{R}^g = \overline{\mathrm{R}} + (\overline{\nabla}\xi) + (\xi^2).$$

Here, $\overline{\mathrm{R}} \in \Lambda^2 \otimes \mathfrak{g}$ is the curvature of the canonical connection, $(\overline{\nabla}\xi) \in \Lambda^2 V^* \otimes \mathfrak{g}^\perp$ is $(\overline{\nabla}\xi)_{X,Y} := (\overline{\nabla}_X\xi)_Y - (\overline{\nabla}_Y\xi)_X$ and $(\xi^2) \in \Lambda^2 V^* \otimes \mathfrak{so}(7)$ is the tensor

$$(\xi^2)_{X,Y} Z = \xi_{(\delta\xi)(X,Y)} Z + [\xi_X, \xi_Y] Z.$$

Alternatively, using the metric g we view R^g as a tensor with values in the bundle $S^2(\mathfrak{so}(n))$ and decompose $\mathfrak{so}(n) = \mathfrak{g} + \mathfrak{g}^\perp$. For vector spaces V and W we define $V \odot W := (V \otimes W + W \otimes V) \cap S^2(V + W)$. Then

$$S^2(\mathfrak{so}(n)) = S^2(\mathfrak{g}) + \mathfrak{g} \odot \mathfrak{g}^\perp + S^2(\mathfrak{g}^\perp).$$

We may refine this a little further by introducing the map $\mathrm{b}\colon \Lambda^2 V^* \otimes \mathit{End}(V) \to \Lambda^3 V^* \otimes V$ given by $(\mathrm{b}r)(X,Y,Z) := r(X,Y)Z + r(Y,Z)X + r(Z,X)Y$. Then the Bianchi identity says $\mathrm{b}\mathrm{R}^g = 0$, so $\mathrm{R}^g \in \mathcal{K} := S^2(\mathfrak{so}(n)) \cap \ker(\mathrm{b})$, the space of algebraic curvature tensors. Let $\mathcal{K}(\mathfrak{g}) := S^2(\mathfrak{g}) \cap \mathcal{K} = (\Lambda^2 V^* \otimes \mathfrak{g}) \cap \mathcal{K}$ where the last equality is the well-known fact that a tensor in $\Lambda^2 V^* \otimes \mathfrak{g}$ that satisfies the Bianchi identity may be viewed as a symmetric endomorphism of $\mathfrak{g}$. We may then simply decompose $\mathcal{K}$ into $\mathcal{K}(\mathfrak{g})$ and it orthogonal complement

$$\mathcal{K} = \mathcal{K}(\mathfrak{g}) + \mathcal{K}(\mathfrak{g})^\perp.$$

Proposition 4.1. *Let P be a G structure on a Riemannian manifold M with metric g. Then the components of the Riemannian curvature in $\mathcal{K}(\mathfrak{g})^\perp$ are determined by the components of the covariant derivative of the intrinsic torsion $\overline{\nabla}\xi$ and the tensor (ξ^2) through $\mathrm{b}|^{-1}_{\mathcal{K}(\mathfrak{g})^\perp}$.*

Remark 4.2. A proof of Proposition 4.1 may be found in [18], but the argument is well-known from the theory of holonomy groups, see for instance [38] and references. Note that one may equally well express the Riemannian curvature as

$$\mathrm{R}^g = \overline{\mathrm{R}} + (\nabla^g \xi) - [\xi^2],$$

where $[\xi^2]_{X,Y} := [\xi_X, \xi_Y]$. In the almost Hermitian case $\mathfrak{g} = \mathfrak{u}(m)$ and $\mathfrak{g}^\perp = [\![\Lambda^{(2,0)}]\!]$. It then follows that $[\xi^2] \in \Lambda^2 V^* \otimes (([\![\Lambda^{(2,0)}]\!] \otimes [\![\Lambda^{(2,0)}]\!]) \cap \mathfrak{so}(2m)) \subset \Lambda^2 V^* \otimes \mathfrak{u}(m)$. Based on this it was argued in [19] that the components of Riemannian curvature in $\mathcal{K}^\perp$ are determined by the components of $\nabla^g \xi$. However, this does not carry over to the G_2 setting simply because $[\mathfrak{g}_2^\perp, \mathfrak{g}_2^\perp] \not\subset \mathfrak{g}_2$ as one easily verifies.

For G_2 structures the decomposition of $\mathcal{K}$ is easily obtained. Note that

$$\begin{aligned} S^2(\mathfrak{g}_2^\perp) &\cong S^2 V \cong V_{27} + V_1, \qquad && S^2(\mathfrak{g}_2^\perp) \cap \mathcal{K} = 0, \\ S^2(\mathfrak{g}_2) &= V_{77}^{(2,0)} + V_{27} + V_1, && S^2(\mathfrak{g}_2) \cap \mathcal{K} = V_{77}^{(2,0)}, \\ \mathfrak{g}_2 \odot \mathfrak{g}_2^\perp &\cong V \otimes \mathfrak{g}_2 = V_{64}^{(1,1)} + V_{27} + V_7, && (\mathfrak{g}_2 \odot \mathfrak{g}_2^\perp) \cap \mathcal{K} = V_{64}^{(1,1)}. \end{aligned} \tag{4.17}$$

This may be proven by the standard method. Taking sample elements in each of the spaces on the left one sees that their images under b, generically, has components in Λ^4_{27} and Λ^4_1. This suffices for the first line. The two last lines follow by identifying the highest weight of the product representation (in the cases considered, the sums of the highest weights of the factors) and calculating dimensions.

The decomposition of $S^2(\mathfrak{so}(7))$ follows from this. Using that b restricted to $S^2(\mathfrak{so}(7))$ maps surjectively on to Λ^4 one obtains [8]

$$\mathcal{K} = V_{77}^{(2,0)} + V_{64}^{(1,1)} + 2V_{27} + V_1.$$

Comparing to the $\mathfrak{so}(7)$ decomposition of $\mathcal{K}$, see e.g.,[3]

$$\mathcal{K} = \mathcal{W} + \mathcal{R}_0 + \mathcal{S}, \qquad \mathcal{R}_0 \cong S^2_0 \mathbb{R}^7, \qquad \mathcal{S} \cong \mathbb{R},$$

we see that the space of algebraic Weyl tensors $\mathcal{W}$ on a G_2 manifold decomposes as

$$\mathcal{W} = \mathcal{W}_{77} + \mathcal{W}_{64} + \mathcal{W}_{27},$$

where $\mathcal{W}_{77} := \mathcal{K} \cap S^2(\mathfrak{g}_2) \cong V_{77}^{(2,0)}$ while $\mathcal{W}_{64} := \mathcal{K} \cap (\mathfrak{g}_2 \odot \mathfrak{g}_2^\perp) \cong V_{64}^{(1,1)}$ and $\mathcal{W}_{27} := \mathcal{W}/(\mathcal{W}_{77} + \mathcal{W}_{64}) \cong \Lambda^3_{27} \cong S^2_0 V_7$. This, at least from the point of view of splitting the space of algebraic curvature tensors $\mathcal{K}$ in to G_2 irreducible subspaces, gives the *decomposition of the Riemannian curvature tensor of a G_2 manifold.* However, it is sometimes useful to have a more explicit description of these submodules. To gain this, we first need to do a little more linear algebra.

So let for the moment ϕ, g be the standard G_2 structure on $V_7 = V = \mathbb{R}^7$. Let r_g be the usual Kulkarni-Nomizu product viewed as an $SO(7)$ equivariant map $S^2 V^* \to S^2(\Lambda^2 V^*)$,

$$\begin{aligned} r_g(h)(x,y,z,w) &:= (h \owedge g)(x,y,z,w) = \\ &h(y,z)g(x,w) - h(x,z)g(y,w) + h(x,w)g(y,z) - h(y,w)g(x,z). \end{aligned} \tag{4.18}$$

This of course actually takes values in $\mathcal{K}$, as one may easily verify. A G_2 equivariant map r_ϕ also from $S^2 V^*$ to $\mathcal{K}$ can be given as

$$r_\phi(a_1 \odot a_2) := (a_1 \lrcorner \phi) \odot (a_2 \lrcorner \phi) - \tfrac{1}{3} \mathrm{b}\left((a_1 \lrcorner \phi) \odot (a_2 \lrcorner \phi)\right).$$

Here and elsewhere, $a \odot b := a \otimes b + b \otimes a$. The Bianchi map must of course be composed with the proper musical morphisms. Contractions going in the opposite direction may be given as

$$c^g(r)(u,v) := r(u, e_i, e_i, v),$$

where e_i is an orthonormal basis. This is just the usual Ricci contraction. Using the isomorphism $S^2(\Lambda^2 V) \cong_g S^2(\Lambda^2 V^*)$ we set

$$c^\phi(r)(u,v) := 4r(u \lrcorner \phi, v \lrcorner \phi).$$

The first equations of (4.19) and (4.20) below, correspond to the result $c^g(h \owedge g) = (n-2)h + \mathrm{tr}_g(h)g$ of taking the Ricci contraction of a Kulkarni-Nomizu product, see [3]. These and the remaining equations may be verified by a calculation in an orthonormal basis.

$$\begin{aligned} (c^g \circ r_g)|_{S^2_0V^*} &= 5, \quad (c^\phi \circ r_g)|_{S^2_0V^*} = 4,\\ (c^g \circ r_\phi)|_{S^2_0V^*} &= 1, \quad (c^\phi \circ r_\phi)|_{S^2_0V^*} = \tfrac{92}{3}, \end{aligned} \tag{4.19}$$

$$(c^g \circ r_g)(g) = 12g, \qquad (c^\phi \circ r_g)(g) = -24g. \tag{4.20}$$

So in analogy to the characterization $\mathcal{R}_0 = \{r_g(h) \colon h \in S^2_0V^*\}$, the space $\mathcal{W}_{27}$ may be described as

$$\mathcal{W}_{27} := \left\{ r_g(h) - 5r_\phi(h) \colon h \in S^2_0V^* \right\}.$$

The last elements from the linear algebra of V are the projections

$$P^{\mathfrak{g}_2} \colon S^2(\mathfrak{so}(7)) \to S^2(\mathfrak{g}_2), \qquad P^{\odot} \colon S^2(\mathfrak{so}(7)) \to \mathfrak{g}_2 \odot \mathfrak{g}_2^{\perp}.$$

These maps may be given closed form expression in terms of the projections $p^2_d,\ d = 7,\ 14$, (2.3).

Now let (M,ϕ) be a G_2 manifold with associated metric g. As discussed above, all maps extend to smooth bundle maps on the associated vector bundles. Define the *ϕ-Ricci tensor* as [2]

$$\mathrm{Ric}^\phi(X,Y) := c^\phi(\mathrm{R}^g), \tag{4.21}$$

and write $\mathrm{Ric}^g = c^g(\mathrm{R}^g)$ and $s_g = \mathrm{tr}_g(\mathrm{Ric}^g)$ for the Ricci- and scalar curvature of g. The identities (4.20) show that $\mathrm{tr}_g(\mathrm{Ric}^\phi) = -2s_g$ [16]. As usual, a subscript 0 indicates the trace-less part of a symmetric tensor. We shall also introduce the $\mathcal{W}$-Ricci-tensor:

$$\mathrm{Ric}^{\mathcal{W}} := \tfrac{1}{20}\left(4\mathrm{Ric}^g_0 - 5\mathrm{Ric}^\phi_0\right).$$

Its relevance is clear from

Theorem 4.3. *Let (M,ϕ) be a G_2 manifold with associated metric g. Then the space of algebraic curvature tensors has the orthogonal splitting:*

$$\mathcal{K} = \mathcal{W}_{77} + \mathcal{W}_{64} + \mathcal{W}_{27} + \mathcal{R}_0 + \mathcal{S},$$

where the space of algebraic Weyl curvatures is $\mathcal{W} = \mathcal{W}_{77} + \mathcal{W}_{64} + \mathcal{W}_{27}$. In terms of the scalar curvature, Ricci curvature and ϕ-Ricci curvature the orthogonal projections to these subspaces are

$$\begin{aligned} S &= \tfrac{1}{84} s_g r_g(g),\\ R_0 &= \tfrac{1}{5} r_g(\mathrm{Ric}^g_0),\\ W_{27} &= \tfrac{3}{112}(r_g - 5r_\phi)(\mathrm{Ric}^{\mathcal{W}}),\\ W_{64} &= P^{\odot}(W - W_{27}),\\ W_{77} &= P^{\mathfrak{g}_2}(W - W_{27}). \end{aligned} \tag{4.22}$$

The three G_2 invariant components of the Weyl curvature are conformal invariants of the associated metric.

In particular, $W_{27} = 0$ exactly when $\mathrm{Ric}^{\mathcal{W}} = 0$ and the norm of the Riemannian curvature satisfies the equality

$$\|\mathrm{R}^g\|^2 = \|W_{77}\|^2 + \|W_{64}\|^2 + \tfrac{15}{28}\|\mathrm{Ric}^{\mathcal{W}}\|^2 + \tfrac{4}{5}\|\mathrm{Ric}^g_0\|^2 + \tfrac{1}{21}s_g^2.$$

[2]This is denoted the $*$-Ricci tensor in [16]

Tensor	Components in $\mathcal{K}$				Other components
	$V^{77}_{(0,2)}$	$V^{64}_{(1,1)}$	$V^{27}_{(2,0)}$	$V^{1}_{(0,0)}$	
$\bar{\bar{\nabla}}\xi_1$					V^7
$\bar{\bar{\nabla}}\xi_7$			✓	✓	$V^7,\ V^{14}$
$\bar{\bar{\nabla}}\xi_{14}$		✓	✓		V^7
$\bar{\bar{\nabla}}\xi_{27}$		✓	✓		$V^7,\ V^{14}, V^{77}_{(3,0)}$
$\xi_1 \otimes \xi_1$				✓	
$\xi_7 \otimes \xi_7$			✓	✓	
$\xi_{14} \otimes \xi_{14}$	✓		✓	✓	
$\xi_{27} \otimes \xi_{27}$	✓	✓	2×✓	✓	$V^{182}_{(4,0)}$
$\xi_1 \odot \xi_7$					V^7
$\xi_1 \odot \xi_{14}$					V^{14}
$\xi_1 \odot \xi_{27}$			✓		
$\xi_7 \odot \xi_{14}$		✓	✓		V^7
$\xi_7 \odot \xi_{27}$		✓	✓		$V^7,\ V^{14}, V^{77}_{(3,0)}$
$\xi_{14} \odot \xi_{27}$		✓	✓		$V^7,\ V^{14},\ V^{189}_{(2,1)}, V^{77}_{(3,0)}$
$\bar{\mathrm{R}}$	✓	✓	2×✓	✓	$V^7,\ V^{14}, V^{77}_{(3,0)}$

TABLE 1. G_2-irreducible components of tensors contributing to curvature.

Proof. The first statement, which is clear from the discussion above, is also contained in [8]. Write the Riemannian curvature as

$$\mathrm{R}^g = W_{77} + W_{64} + (r_g(h_1) - 5r_\phi(h_1)) + r_g(h_2) + Kr_g(g),$$

where h_1 and h_2 are trace-less symmetric two-tensors and apply the contractions c^g and c^ϕ to obtain the first three equations in (4.22). The last two relations in (4.22) follow from the decompositions (4.17). The expression for the norm of the Riemannian curvature is now a matter of applying the following relations

$$\|r_g(h)\|^2 = 20\,\|h\|^2\,, \qquad \|r_\phi(h)\|^2 = \tfrac{92}{3}\,\|h\|^2\,, \qquad \langle r_\phi(h), r_g(h)\rangle = 4\,\|h\|^2\,,$$

valid for any trace-less symmetric two-tensor h, while $\|r_g(g)\|^2 = 336$.

Conformal invariance of the three components W_{77}, W_{64} and W_{27} follows from the conformal invariance of $\mathrm{Ric}^{\mathcal{W}}$, see Corollary 4.7 below, and invariance of the projections p^2_7 and p^2_{14} under rescaling of the G_2 three-form. □

Remark 4.4. Note that the components of $\bar{\bar{\nabla}}\xi$ and $\xi \odot \xi$ given in the right-most column of Table 1 correspond to the second order diffeomorphism invariants of (M, ϕ) not captured by the Riemannian curvature, see [8]. Another interesting point made in [8] (also made for almost Hermitian structures in [11]) is that the first order identities for the torsion that derives from $d^2\phi = 0 = d^2{*\phi}$ are encoded in a subspace isomorphic to $2V_7 + V_{14}$. For G_2 these relations between invariants necessarily take their values in the complement of the space of algebraic curvature tensors inside the much larger space of diffeomorphism invariants polynomial in derivatives of ϕ up to order two. Andrew Swann has pointed out to us that these first order constraints on the torsion also may be seen as coming from the fact that cokernel of the restriction $\mathrm{b}\colon \Lambda^2V^* \otimes \mathfrak{g}_2 \to \Lambda^3V^* \otimes V$ is isomorphic to precisely $2V_7 + V_{14}$, and the restriction $\mathrm{b}\colon \Lambda^2V^* \otimes \mathfrak{g}_2^\perp \to \Lambda^3V^* \otimes V$ being injective.

For the examples we shall be considering in section 5 and 6 all diffeomorphism invariants up to order 2 are encoded in the Riemannian curvature tensor and the exterior derivatives of the torsion components. This appears to be a distinguishing feature of the wider class of G_2-structures of type 1+2+4. We thank Robert Bryant for reminding us of the importance of the full space of second order diffeomorphism invariants of a G_2 structure.

4.1. **Ricci curvatures of G_2 manifolds.** The isomorphism (2.4) has striking consequences. Most importantly, the covariant derivatives of ξ_7, ξ_{14} and ξ_{27} each have precisely one component in a 27 dimensional irreducible subspace of $V^* \otimes V^* \otimes \mathfrak{g}_2^{\perp}$. Each of these may be identified with corresponding 27 dimensional components of suitable exterior derivatives. Similarly, each 27 dimensional component of the 'algebraic' curvature components $\xi_d \odot \xi_{d'}$ has an equivalent expression in the exterior algebra. This was used in [8] to obtain an expression for the Ricci curvature of a G_2 manifold. An expression of the Ricci tensor when $\tau_2 = 0$ is given in terms of covariant derivatives of the skew-symmetric torsion in [25] and a formula for the Ricci tensor in terms of the covariant derivatives of the intrinsic torsion was very recently presented in [34]. We give a generalization to more general trace-less symmetric two-tensor formed G_2 equivariantly from the Riemannian curvature.

For $k = (k_1, k_2) \in \mathbb{R}^2$ write

$$\mathrm{Ric}_0^k := k_1 \mathrm{Ric}_0^g + k_2 \mathrm{Ric}_0^\phi. \tag{4.23}$$

We shall call this tensor the *generalized Ricci tensor* and say that a G_2 manifold (M, ϕ) is *generalized Einstein* if $\mathrm{Ric}_0^k = 0$ for some $k = (k_1, k_2)$.

Remark 4.5. Since rescaling of Ric_0^k does not affect the generalized Einstein equation there is of course only really an $\mathbb{R}P(1)$ worth of such constraints.

Since $S_0^2 V$ has multiplicity two in $\mathcal{K}$ any trace-less symmetric two tensor on a G_2 manifold must equal Ric_0^k for some $k \in \mathbb{R}^2$. Before given the formula we need to give expressions for the way the component $\tau_2 \odot \tau_3$ and $\tau_3 \otimes \tau_3$ determine three-forms. Choose local G_2 adapted frames $(e_1, \dots, e_7)$. For a two-form α and three-form β set

$$[\alpha \odot \beta] := \sum_k i_{e_k}\alpha \wedge i_{e_k}\beta, \qquad [\beta^2]^A := \sum_k *(i_{e_k}\beta \wedge i_{e_k}\beta),$$
$$[\beta^2]^B := \sum_k ((i_{e_k}\phi) \lrcorner \beta) \wedge i_{e_k}\beta.$$

Then this is independent of the chosen frame and so extend to smooth contractions on M. The first two contractions are in fact $SO(7)$ equivariant. Since V_7 does not occur in the decomposition of $S^2(V_{27})$ guarantees that for $\beta \in \Lambda^3_{27}$ $[\beta^2]^A, [\beta^2]^B \in \Lambda^3_1 + \Lambda^3_{27}$. Since there is summand isomorphic to V_7 in $\Lambda^2_{14} \otimes \Lambda^3_{27}$ it is not quite obvious that $[\alpha \odot \beta] \in \Lambda^3_1 + \Lambda^3_{27}$ should hold for $\alpha \in \Lambda^2_{14}$ and $\beta \in \Lambda^3_{27}$ but this is none-the-less true.

Lemma 4.6. *Let (M, ϕ) be a G_2 manifold. Then*

$$\begin{aligned} \lambda_3\big(\mathrm{Ric}_0^k\big) = \big(&-(5k_1+4k_2)d*(\tau_1 \wedge *\phi) + 2(5k_1+4k_2)\tau_1 \wedge *(\tau_1 \wedge *\phi) \\ &- (k_1-4k_2)d\tau_2 + \tfrac{1}{2}(k_1+2k_2)*(\tau_2 \wedge \tau_2) + (k_1+4k_2)*d\tau_3 + k_2[\tau_3^2]^A + \tfrac{1}{2}k_1[\tau_3^2]^B \\ &- \tfrac{1}{2}(k_1-4k_2)\tau_0\tau_3 + (k_1-4k_2)\tau_1 \wedge \tau_2 + (3k_1-4k_2)*(\tau_1 \wedge \tau_3) + 2k_2[\tau_2 \odot \tau_3]\big)_{27}. \end{aligned} \tag{4.24}$$

Proof. Theorem 4.1 and the remarks given above show that any symmetric two-tensor formed from R^g contracting with ϕ and g must have an expression as a linear combination of the terms on the right hand side where coefficients are determined entirely in terms of the linear algebra of G_2 and so in particular are independent of the underlying manifold. Obtaining the given expression is then a matter of evaluating the left- and right-hand side on examples. □

A similar argument may be used to obtain the scalar curvature of a G_2 manifold, see [8, 26, 34]

$$s^g = \tfrac{21}{8}\tau_0^2 + 12\delta\tau_1 + 30\,|\tau_1|^2 - \tfrac{1}{2}\,|\tau_2|^2 - \tfrac{1}{2}\,|\tau_3|^2\,. \tag{4.25}$$

Lemma 4.6 gives

Corollary 4.7. *The symmetric trace-less tensor* $\mathrm{Ric}^{\mathcal{W}}$ *is a conformal invariant of a* G_2 *structure.*

Proof. Under a conformal change $\phi \to \tilde{\phi} = e^3 f\phi$ the torsion transforms as $\tau = (\tau_0, \ldots, \tau_3) \to \tilde{\tau} = (e^{-f}\tau_0, \tau_1 + df, e^f\tau_2, e^{2f}\tau_3)$ - this much is well-known and easy to check. Putting $\tilde{\tau}$ in to formula (4.24) a simple calculation verifies that with $k_1 = 4,\ k_2 = -5$ the only change of the generalized Ricci tensor is a rescaling. □

Remark 4.8. The subscript 27 at the end of equation (4.24) means the projection p^3_{27} has been applied. It is rather pleasing to note that all possible contributions are realized. This seems to confirms a 'general principle' for G-structures: if representation theory tells us that a tensor may contribute in an expression as above, then it does. Since there is a two parameter family of generalized Ricci tensors and a two parameter family of contributions from $\tau_3 \otimes \tau_3$ it is not surprising that contributions from the chosen representatives A and B vanish for certain values k. That those are precisely $(1, 0)$ and $(0, 1)$ appear to be pure coincidence.

As noted in [8], a generic G_2 structure has a two-parameter family of 'canonical' G_2 connections $\nabla^{(s,t)}$. The generalized Ricci-curvature defined above should therefore have an interpretation as the symmetric part of the contraction $c_g(R^{\nabla^{(s,t)}})$ of the curvature $R^{\nabla^{(s,t)}}$ of $\nabla^{(s,t)}$. One may obtain different formulas for the Ricci-curvatures by expressing the exterior derivatives of the torsion components in terms of $d^{\nabla^{(s,t)}}$ instead. We do this below for $\overline{\overline{\nabla}}$.

A rather long but straight-forward computation on test elements shows that

$$\begin{aligned} d{*}(\tau_1 \wedge *\phi) = d^{\overline{\overline{\nabla}}}{*}(\tau_1 \wedge *\phi) + \tfrac{1}{2}{*}(\tau_0\tau_1 \wedge \phi) + \tfrac{8}{3}\tau_1 \wedge *(\tau_1 \wedge *\phi) - 2\,|\tau_1|^2\,\phi \\ + \tfrac{1}{3}\tau_1 \wedge \tau_2 + \tfrac{4}{3}(\tau_1 \wedge \tau_2)_7 + \tfrac{2}{3}{*}(\tau_1 \wedge \tau_3) - \tfrac{4}{3}{*}(\tau_1 \wedge \tau_3)_7, \end{aligned}$$

$$\begin{aligned} d\tau_2 = d^{\overline{\overline{\nabla}}}\tau_2 + \tfrac{2}{3}\tau_1 \wedge \tau_2 + \tfrac{8}{3}(\tau_1 \wedge \tau_2)_7 + \tfrac{1}{6}{*}(\tau_2 \wedge \tau_2) + \tfrac{1}{6}\,|\tau_2|^2\,\phi \\ - \tfrac{1}{6}[\tau_2 \odot \tau_3] + \tfrac{1}{6}{*}((\tau_2 \lrcorner \tau_3) \wedge \phi), \end{aligned} \tag{4.26}$$

and,

$$\begin{aligned} d\tau_3 = d^{\overline{\overline{\nabla}}}\tau_3 - \tfrac{1}{6}{*}(\tau_0\tau_3) + \tau_1 \wedge \tau_3 - \tfrac{8}{3}{*}({*}(\tau_1 \wedge \tau_3)_7) - \tfrac{1}{6}(\tau_2 \lrcorner \tau_3) \wedge \phi \\ - \tfrac{1}{6}{*}[\tau_3^2]^A - \tfrac{1}{6}{*}[\tau_3^2]^B + \tfrac{1}{6}\,|\tau_3|^2\,{*}\phi. \end{aligned}$$

It is somewhat surprising that there is no summand $[\tau_2 \odot \tau_3]$ in the last expression. More surprises are in store when this is used in formula (4.24). First, for $\beta \in \Lambda^3 V^*$ define

$$[\beta^2]^C := [\beta^2]^A - 2[\beta^2]^B.$$

Then Lemma 4.6 is reformulated as

Tensor	Ric^g_0	$\mathrm{Ric}^{\mathcal{W}}$	Ric^{ϕ}
$\overline{\overline{\nabla}}\xi_7$	✓		✓
$\overline{\overline{\nabla}}\xi_{14}$	✓	✓	✓
$\overline{\overline{\nabla}}\xi_{27}$	✓	✓	✓
$\xi_7 \otimes \xi_7$	✓		✓
$\xi_{14} \otimes \xi_{14}$	✓	✓	✓
$(\xi_{27} \otimes \xi_{27})^C$	✓	✓	✓
$\xi_1 \odot \xi_{27}$	✓	✓	✓
$\xi_7 \odot \xi_{14}$	✓	✓	✓
$\xi_7 \odot \xi_{27}$	✓	✓	✓
$\xi_{14} \odot \xi_{27}$		✓	✓

Table 2. G_2-irreducible components of tensors contributing to Ricci curvatures.

Lemma 4.9. *Let (M, ϕ) be a G_2 manifold. Then*

$$\begin{aligned}(4.27)\quad \lambda_3\big(\mathrm{Ric}^k_0\big) = \big(&-(5k_1+4k_2)d^{\overline{\overline{\nabla}}}*(\tau_1\wedge *\phi) - \tfrac{2}{3}(5k_1+4k_2)\tau_1\wedge *(\tau_1\wedge *\phi)\\ &-(k_1-4k_2)d^{\overline{\overline{\nabla}}}\tau_2 + \tfrac{1}{3}(k_1+5k_2)*(\tau_2\wedge\tau_2) + (k_1+4k_2)*d^{\overline{\overline{\nabla}}}\tau_3 - \tfrac{1}{6}(k_1-2k_2)[\tau_3^2]^C\\ &-\tfrac{2}{3}(k_1-2k_2)\tau_0\tau_3 - \tfrac{4}{3}(k_1+2k_2)\tau_1\wedge\tau_2 + \tfrac{2}{3}(k_1-4k_2)*(\tau_1\wedge\tau_3) + \tfrac{1}{6}(k_1+8k_2)[\tau_2\odot\tau_3]\big)_{27}.\end{aligned}$$

Corollary 4.10. *The components of the covariant derivative $\overline{\overline{\nabla}}\xi$ and symmetric products $\xi_d \odot \xi_{d'}$ contribute to the traceless symmetric tensors* $\mathrm{Ric}^g, \mathrm{Ric}^{\phi}$ *and* $\mathrm{Ric}^{\mathcal{W}}$ *according to the ticks in Table 2.*

Remark 4.11. There is a number of interesting features of this equation. The most important is that only one combination of a two parameter family of possible contributions from $\tau_3 \otimes \tau_3$ are realized. This appears to break the principle referred to in remark 4.8.

A classification of G_2 structures with $\tau_1 = \tau_2 = 0$ for which the torsion is parallel with respect to the unique G_2 *characteristic* connection determined by having three-form torsion: $T(X,Y)Z = -T(X,Z)Y$ was made in [23]. One corollary of this classification is that structures with torsion $\tau = \tau_3$, parallel with respect to the characteristic connection are never Einstein, at least when the stabilizer of the torsion is not Abelian.

Note that when $d^{\overline{\overline{\nabla}}}\tau_3 = 0$ the Einstein equation for a G_2 structure of type $1+3$ reduces to a quadratic equation $[\tau_3^2]^C + 4\tau_0\tau_3 = 0$ for a torsion tensor $\tau = (\tau_0, \tau_3)$. Moreover, if such a structure is Einstein then it is generalized Einstein for any value of $k = (k_1, k_2)$. It is very likely that this system has non-trivial solutions.

Remark 4.12. The three-form $\rho^\star$ introduced in [16] corresponds to $k_1 = -1/6$ and $k_2 = 1/24$.

5. Curvature of G_2 structures of type $1+4$

The results of this section are well known. We take this class as a first example to demonstrate how the results of the previous sections should be used as everything is straight-forward.

We first analyze the structure equations. With $\tau_2 = \tau_3 = 0$ equations (3.14) are:

$$d\phi = \tau_0 *\phi + 3\tau_1\wedge\phi, \qquad d*\phi = 4\tau_1\wedge *\phi.$$

Taking the differential again yields $d\tau_1 = 0$, $d\tau_0 + \tau_0\tau_1 = 0$. An easy argument now shows that either $\tau_0 \equiv 0$ and τ_1 is closed or τ_0 is never zero and τ_1 is exact: $\tau_1 = -d\ln(\tau_0)$. Therefore the class $1+4$ of G_2-structures consists in G_2 structures ϕ locally conformally equivalent to a parallel structure and those globally conformally equivalent to a nearly parallel structure, see [15] for details.

5.1. **Curvature.** As the kernel $\mathrm{b}\colon S^2(\mathfrak{g}_2) \to \Lambda^4 V^*$ is precisely $\mathcal{W}_{77}$ and the torsion $\tau = (\tau_0, \tau_1, \tau_2, \tau_3)$ vanishes identically for a parallel G_2 structure it is clear that the only non-trivial component of the Riemannian curvature is precisely W_{77} for such a structure.

By the conformal invariance of the components of the Weyl tensor, see Theorem4.3, the components of the Riemannian curvature for a G_2 structure with $\tau = \tau_1$ satisfy

$$W_{64} = 0 = W_{27}, \qquad s_g = 12\delta\tau_1 + 30\,|\tau_1|^2$$

$$\lambda_3\,(\mathrm{Ric}^g_0) = \left(-5d^{\overline{\nabla}}{*}(\tau_1 \wedge {*}\phi) - \tfrac{10}{3}\tau_1 \wedge {*}(\tau_1 \wedge {*}\phi)\right)_{27}.$$

Compact manifolds with G_2 structure locally conformal to a parallel G_2 structure were described in [31, 40].

Suppose now that $\tau = \tau_0$. Then (M, ϕ) is nearly parallel and the structure equations give $d\tau_0 = 0$. It is well-known that the associated metric of a nearly parallel G_2 structure is Einstein – this is also obvious from formula (4.24), in fact $\mathrm{Ric}^k_0 = 0$ for all k. Furthermore, checking Table 1 one sees that $W_{64} = 0$. Thus we obtain (c.f. [12, 14])

Lemma 5.1. *The Riemannian curvature tensor of the metric associated to a nearly parallel G_2 structure has the form*

$$\mathrm{R}^g = W + \tfrac{1}{32}\tau_0^2 r_g(g),$$

where $W \in \mathcal{W}_{77}$.

For the strict class $1+4$ one then again has $W_{64} = 0 = W_{27}$ by the conformal invariance of these components, Theorem 4.3. Moreover, if g is complete and Einstein then (M, g) is conformally equivalent to the standard metric on the 7-sphere, so $W_{77} = 0$. We will see these conformal changes of the unique (see [24]) nearly parallel G_2 structure on S^7 turn up again when we consider examples in section 7.

6. Curvature of closed G_2 structures

A closed G_2 structure is by definition given by a closed G_2 three-form ϕ. Let us first examine the consequences of the structure equations (3.14). When $d\phi = 0$ the torsion τ has only one component $\tau = \tau_2$:

$$d{*}\phi = \tau \wedge \phi, \tag{6.28}$$

whence $\delta^g\phi = \tau = \bar{\xi}$. This is equivalent to the equation $\wedge_3(\xi) = 0$ for the intrinsic torsion ξ viewed as a tensor in $\Lambda^2 V^* \otimes V^*$. Expressed in terms of the components of ξ with respect to an orthonormal frame this is just the identity

$$\xi_{ijk} + \xi_{jki} + \xi_{kij} = 0. \tag{6.29}$$

Differentiating and applying Hodge star to the structure equation (6.28) yields

$$d\tau \wedge \phi = 0, \qquad \text{and} \qquad \delta^g\tau = 0. \tag{6.30}$$

The first equation is equivalent to $(d\tau)_7 = 0$ while the second may be interpreted as $\nabla^g\tau$ having no component in the 7 dimensional irreducible $SO(7)$-submodule of $V^* \otimes \Lambda^2 V^*$. Using Table 1 and Lemma 2.1 either of the equations (6.30) can be seen to be equivalent to statement of the following Lemma.

Lemma 6.1. *Suppose $d\phi = 0$. Then $\overline{\nabla}\tau \in V_{64} + V_{27} \subset V^* \otimes \Lambda^2_{14}$.* □

Equation (4.26) then becomes

$$d^{\overline{\nabla}}\tau = d\tau - \tfrac{1}{6}*(\tau\wedge\tau) - \tfrac{1}{6}|\tau|^2\,\phi, \tag{6.31}$$

after rearranging. Lemma 6.1 shows that $d^{\overline{\nabla}}\tau \in \Lambda^3_{27}$. This gives the important observation:

$$\tfrac{1}{3}*d(\tau^3) = \langle d\tau, *(\tau\wedge\tau)\rangle = \langle d^{\overline{\nabla}}\tau, *(\tau\wedge\tau)_{27}\rangle. \tag{6.32}$$

6.1. **The Ricci curvature of a closed G_2 structure.**

Remark 6.2. Many of the results we give below have appeared in slightly different form and for certain special cases in [8] and [16]. The main difference between the results given here and those that have appeared earlier is the interpretation in terms of the components of the covariant derivative $\overline{\nabla}\tau$.

When the G_2 three-form is closed the formula for the generalized Ricci curvature (4.27) may be written

$$\lambda_3\big(\mathrm{Ric}^k_0\big) = -(k_1-4k_2)d^{\overline{\nabla}}\tau + \tfrac{1}{3}(k_1+5k_2)*(\tau\wedge\tau)_{27}. \tag{6.33}$$

This leads to

$$\begin{aligned}\|\mathrm{Ric}^k_0\|^2 = \tfrac{1}{2}(k_1-4k_2)^2|d^{\overline{\nabla}}\tau|^2 &+ \tfrac{1}{21}(k_1+5k_2)^2\,|\tau|^4 \\ &- \tfrac{1}{3}(k_1+5k_2)(k_1-4k_2)\langle d^{\overline{\nabla}}\tau, *(\tau\wedge\tau)_{27}\rangle\end{aligned} \tag{6.34}$$

Here we have used equation (2.5) and the following relations, valid for any two-form $\tau \in \Lambda^2_{14}$, see [8].

$$*(\tau\wedge\tau\wedge\phi) = -|\tau|^2, \qquad |\tau\wedge\tau|^2 = |\tau|^4, \qquad \text{and,} \qquad |(\tau\wedge\tau)_{27}|^2 = \tfrac{6}{7}|\tau|^4.$$

All these may be verified by observing that, since the trivial representation occurs in the symmetric powers $S^2\Lambda^2_{14}$ and $S^4\Lambda^2_{14}$ with multiplicity one, left- and right-hand side of the two equation must be proportional. The constant of proportionality is then found by evaluating on a sample element. Alternatively, the last equation follows from the first two by projecting $(\tau\wedge\tau)_{27} = \tau\wedge\tau + \frac{1}{7}|\tau|^2 *\phi$ and taking the norm squared.

Remark 6.3. Identity (6.34) has some easy applications. For instance, integral identities such as

$$\int_M \Big(36\,\|\mathrm{Ric}^g_0\|^2 - 25\|\mathrm{Ric}^{\mathcal{W}}\|^2 - \tfrac{45}{28}s_g^2\Big)\,dV_g = 0. \tag{6.35}$$

hold on a compact manifold M with closed G_2 structure.

In particular, one gets the result established in [16], see also [8], that a compact Einstein closed G_2 manifold is parallel.

Keep in mind that the scalar curvature of the metric associated to a closed G_2 structure is $s_g = -\frac{1}{2}|\tau|^2$. Equation (6.32) and Stokes' Theorem applied to integration of (6.34) then give

Proposition 6.4. *Suppose (M,ϕ) is a compact G_2 manifold with $d\phi = 0$. Then*

$$\int_M \|\mathrm{Ric}^k_0\|^2 dV_g \geqslant \tfrac{4}{21}(k_1+5k_2)^2\int_M s_g^2 dV_g,$$

where equality holds if and only if $k_1 = 4k_2$ or $d^{\overline{\nabla}}\tau = 0$.

For $k = (1,0)$ one gets the inequality established by Robert Bryant in [8].

Closely related statements are obtained by applying the Cauchy-Schwartz inequality to the last summand of equation (6.34).

Proposition 6.5. *Suppose M is a manifold equipped with a closed G_2 structure ϕ with torsion τ. Let k_1, k_2 be real numbers and set $K_1 := |k_1 - 4k_2|$, $K_2 := |k_1 + 5k_2|$. The inequality*

$$\tag{6.36} \tfrac{1}{2}\big(K_1|d^{\overline{\nabla}}\tau| - \sqrt{\tfrac{2}{21}}K_2\,|\tau|^2\big)^2 \leqslant \|\mathrm{Ric}^k_0\|^2 \leqslant \tfrac{1}{2}\big(K_1|d^{\overline{\nabla}}\tau| + \sqrt{\tfrac{2}{21}}K_2\,|\tau|^2\big)^2$$

then holds everywhere in M.

As a particular instance we apply Theorem 4.3 to obtain,

Corollary 6.6. *Suppose (M, ϕ) is a compact manifold with closed G_2 structure ϕ. If the 27-dimensional curvature component W_{27} is zero then ϕ is parallel.*

Corollary 6.7. *Let (M, ϕ) be a G_2 manifold with $d\phi = 0$. If the torsion satisfies $d^{\overline{\nabla}}\tau = 0$ then*

$$\tag{*} \|\mathrm{Ric}^k_0\|^2 \leqslant \tfrac{4}{21}(k_1 + 5k_2)^2 s_g^2,$$

and, in fact, equality must hold everywhere and for all values of k_1 and k_2.

If M is compact and real numbers k_1, k_2 exist so that the inequality (*) holds everywhere in M then $(k_1 - 4k_2)d^{\overline{\nabla}}\tau = 0$.

Proof. Suppose ϕ is a closed G_2 three-form with torsion such that $d^{\overline{\nabla}}\tau = 0$. Then the inequality (*) holds by the second inequality of (6.36) and taking the first inequality in to account, equality must hold for any k_1 and k_2. This proves the first part of Corollary 6.7. The second statement follows by using the inequality (*) in formula (6.34) to obtain

$$\tag{**} \tfrac{1}{2}(k_1 - 4k_2)^2|d^{\overline{\nabla}}\tau|^2 - \tfrac{1}{3}(k_1 + 5k_2)(k_1 - 4k_2)\langle d^{\overline{\nabla}}\tau, *(\tau\wedge\tau)_{27}\rangle \leqslant 0$$

When M compact we may integrate the inequality (**). Using relation (6.32) we obtain

$$\tfrac{1}{2}(k_1 - 4k_2)^2\int_M |d^{\overline{\nabla}}\tau|^2 dV_g \leqslant 0.$$

∎

Motivated by Proposition 6.4 and Corollary 6.7, we shall say that a closed G_2 structure ϕ is *extremal or extremally pinched* if the torsion τ satisfies $d^{\overline{\nabla}}\tau = 0$.

Remark 6.8. Equivalently, ϕ has *extremally pinched Ricci curvature*, the original term used by Robert Bryant in [8].

Corollary 6.9. *Suppose ϕ is a closed G_2 three-form on a manifold M. If $\mathrm{Ric}^k_0 = 0$ for some $k = (k_1, k_2)$ then*

$$|k_1 - 4k_2|\,|d^{\overline{\nabla}}\tau| = \sqrt{\tfrac{2}{21}}\,|k_1 + 5k_2|\,|\tau|^2$$

If M is compact then the associated Riemannian metric is generalized Einstein for $k = (k_1, k_2)$ if and only if either

(1) *$k_1 + 5k_2 = 0$ and ϕ is extremal, or,*
(2) *ϕ is parallel.*

6.2. **The curvature components W_{77} and W_{64} and topology of closed G_2 structures.**

Lemma 6.10. *Suppose ϕ is a closed G_2 three-form with associated metric g. Let $S\colon \mathfrak{so}(n)\otimes\mathfrak{so}(n) \to \mathfrak{so}(n)\otimes\mathfrak{so}(n)$ be defined by $S(\alpha\otimes\beta) = \beta\otimes\alpha$. Let $\overline{\nabla}$ be the canonical connection and ξ be the intrinsic torsion of ϕ. Write $(\overline{\nabla}\xi)_{64}$ for the orthogonal projection of $(\overline{\nabla}\xi)$ to the irreducible 64 dimensional submodule of*

$\Lambda^2 V^* \otimes \mathfrak{g}_2^\perp$ and $\overline{\mathrm{R}}_{64}$ for the projection $\overline{\mathrm{R}}$ to the irreducible 64 dimensional submodule of $\Lambda^2 V^* \otimes \mathfrak{g}_2$. Then the component W_{64} of the Riemannian curvature R^g satisfies

$$W_{64} = S((\overline{\nabla}\xi_{64}) + (\overline{\nabla}\xi)_{64} = S(\overline{\mathrm{R}}_{64}) + \overline{\mathrm{R}}_{64}.$$

Let τ be the torsion of ϕ and write $\overline{\nabla}\tau_{64}$ and $\overline{\nabla}\tau_{27}$ for the respective components of $\overline{\nabla}\tau$ in the 64 and 7 dimensional subspaces of $V^* \otimes \Lambda^2_{14}$. Then

$$\|W_{64}\|^2 = 2\|(\overline{\nabla}\xi)_{64}\|^2 = 2\|\overline{\mathrm{R}}_{64}\|^2 = \tfrac{1}{3}\|\overline{\nabla}\tau_{64}\|^2.$$

Proof. Note that $W_{64} \in S^2(\mathfrak{so}(7))$ so $S(W_{64}) = W_{64}$. Since $\xi \otimes \xi$ lies in a submodule of $S^2(V^* \otimes \mathfrak{g}_2^\perp)$ isomorphic to $S^2(\mathfrak{g}_2)$ the tensor (ξ^2) does not contribute to W_{64} and so $W_{64} = \overline{\mathrm{R}}_{64} + (\overline{\nabla}\xi)_{64} = S(\overline{\mathrm{R}}_{64} + (\overline{\nabla}\xi)_{64})$. However, $\overline{\mathrm{R}}_{64} \in \mathfrak{g}_2^\perp \otimes \mathfrak{g}_2$ while $(\overline{\nabla}\xi)_{64} \in \mathfrak{g}_2 \otimes \mathfrak{g}_2^\perp$, so

$$\mathfrak{g}_2^\perp \otimes \mathfrak{g}_2 \ni \overline{\mathrm{R}}_{64} - S((\overline{\nabla}\xi)_{64}) = (\overline{\nabla}\xi)_{64} - S(\overline{\mathrm{R}}_{64}) \in \mathfrak{g}_2 \otimes \mathfrak{g}_2^\perp\,.$$

Therefore $\overline{\mathrm{R}}_{64} = S((\overline{\nabla}\xi)_{64})$ and $W_{64} = S((\overline{\nabla}\xi)_{64}) + (\overline{\nabla}\xi)_{64} = S(\overline{\mathrm{R}}_{64}) + \overline{\mathrm{R}}_{64}$.

Let e_i be a local orthonormal frame. We write $\overline{\nabla}_i\tau_{jk} = (\overline{\nabla}_{e_i}\tau)(e_j, e_k)$ and so on. Using equation (3.15) and (3.16) we get

$$\begin{aligned}
(\overline{\nabla}\xi)_{ijkl} &= \overline{\nabla}_i\xi_{jkl} - \overline{\nabla}_j\xi_{ikl} \\
&= \tfrac{1}{6}\big(\overline{\nabla}_i\tau_{jp} - \overline{\nabla}_j\tau_{ip}\big)\phi_{pkl}, \\
&= \tfrac{1}{6}\big((d^{\overline{\nabla}}\tau)_{ijp} - \overline{\nabla}_p\tau_{ij}\big)\phi_{pkl} \\
&= \tfrac{1}{6}\big((d^{\overline{\nabla}}\tau)_{ijp} - (\overline{\nabla}\tau_{27})_{pij} - (\overline{\nabla}\tau_{64})_{pij}\big)\phi_{pkl}.
\end{aligned}$$

Here we make explicit use of the principle given by equation (3.11). Since $d^{\overline{\nabla}}\tau \in \Lambda^3_{27}$ by Lemma 6.1 projection gives:

$$(\overline{\nabla}\xi_{64})_{ijkl} = -\tfrac{1}{6}(\overline{\nabla}\tau_{64})_{pij}\phi_{pkl}.$$

Take the tensor norm and use equation (2.6)) to get

$$\begin{aligned}
\|(\overline{\nabla}\xi_{64})\|^2 &= \tfrac{1}{36}(\overline{\nabla}\tau_{64})_{pij}\phi_{pkl}(\overline{\nabla}\tau_{64})_{qij}\phi_{qkl}, \\
&= \tfrac{1}{6}(\overline{\nabla}\tau_{64})_{pij}(\overline{\nabla}\tau_{64})_{pij}, \\
&= \tfrac{1}{6}\|\overline{\nabla}\tau_{64}\|^2.
\end{aligned}$$

The final equation of the Lemma follows from this. □

Closed G_2 structures are distinguished also by having certain topological data naturally associated.

Theorem 6.11. *Suppose M is a compact 7 dimensional manifold with a closed fundamental three-form ϕ. Let g be the associated metric. Then*

(6.37)
$$\langle p_1(M) \cup [\phi], [M]\rangle = -\frac{1}{8\pi^2}\int_M \Big\{\|W_{77}\|^2 - \tfrac{1}{2}\|W_{64}\|^2 - \tfrac{9}{7}\|\mathrm{Ric}^g_0\|^2 + \tfrac{45}{28^2}s_g^2\Big\}\,dV_g,$$

where $p_1(M)$ is the first Pontrjagin class of M.

This generalizes Proposition 10.2.7. of [32].

Proof. We shall be working in a local orthonormal frame e_i. First, using equations (2.6) and (6.29) the expression

$$\begin{aligned}
\mathrm{R}^g_{ijab} = \overline{\mathrm{R}}_{ijab} + \tfrac{1}{6}(\overline{\nabla}_i\tau_{jp} - \overline{\nabla}_j\tau_{ip})\phi_{pab} + \tfrac{1}{36}\tau_{pq}\tau_{pr}\phi_{qij}\phi_{rab} \\
+ \tfrac{1}{36}(\tau_{ia}\tau_{jb} - \tau_{ib}\tau_{ja}) - \tfrac{1}{18}\tau_{ip}\tau_{jq}\phi_{pqab}
\end{aligned}$$

is obtained. This relates the Riemannian curvature R^g of the metric g associated to ϕ with the curvature $\overline{\mathrm{R}}$ of the canonical connection $\overline{\nabla}$ of ϕ with the the torsion expressed in terms of τ rather than intrinsic torsion ξ.

Note that from Chern-Weil theory $p_1(M)$ may be represented by the 4-form

$$\tfrac{1}{8\pi^2}\operatorname{tr}(\mathrm{R}^g\wedge\mathrm{R}^g)=\tfrac{1}{16\pi^2}\mathrm{R}^g_{ijab}\mathrm{R}^g_{klab}e^{ijkl}.$$

Now,

$$\begin{aligned}8\pi^2\langle p_1(M)\cup[\phi],[M]\rangle &= \int_M \operatorname{tr}(\mathrm{R}^g\wedge\mathrm{R}^g)\wedge\phi\\ &= \int_M \langle \operatorname{tr}(\mathrm{R}^g\wedge\mathrm{R}^g),*\phi\rangle\, dV_g,\\ &= \tfrac{1}{2}\int_M \mathrm{R}^g_{abij}\mathrm{R}^g_{cdij}\phi_{abcd}dV_g\\ &= -\int_M \|\mathrm{R}^g\|^2\, dV_g+\tfrac{1}{2}\int_M(\mathrm{R}^g_{ijab}\phi_{abt})(\mathrm{R}^g_{ijcd}\phi_{cdt})dV_g.\end{aligned}$$

The contraction $\mathrm{R}^g_{ijab}\phi_{abt}$ gives

$$\begin{aligned}\mathrm{R}^g_{ijab}\phi_{abt} &= \tfrac{1}{6}(\overline{\nabla}_i\tau_{jp}-\overline{\nabla}_j\tau_{ip})\phi_{pab}\phi_{abt}+\tfrac{1}{36}\tau_{pq}\tau_{pr}\phi_{qij}\phi_{rab}\phi_{abt}\\ &\quad+\tfrac{1}{36}(\tau_{ia}\tau_{jb}-\tau_{ib}\tau_{ja})\phi_{abt}-\tfrac{1}{18}\tau_{ip}\tau_{jq}\phi_{pqab}\phi_{abt}\\ &= \overline{\nabla}_i\tau_{jt}-\overline{\nabla}_j\tau_{it}+\tfrac{1}{6}\tau_{pq}\tau_{pt}\phi_{qij}-\tfrac{1}{6}\tau_{ip}\tau_{jq}\phi_{pqt}.\end{aligned}$$

where the identities (2.6)- (2.10) are applied. By definition $\wedge_3(\overline{\nabla}\tau)=: d^{\overline{\nabla}}\tau$ so

$$\mathrm{R}^g_{ijab}\phi_{abt}=\big((d^{\overline{\nabla}}\tau)_{ijt}-\overline{\nabla}_t\tau_{ij}\big)+\tfrac{1}{6}(\tau_{pq}\tau_{pt}\phi_{qij}-\tau_{ip}\tau_{jq}\phi_{pqt})\tag{6.38}$$

The evaluation of the integrand $(\mathrm{R}^g_{ijab}\phi_{abt})(\mathrm{R}^g_{ijcd}\phi_{cdt})$ is now reduced to evaluating 9 different contractions. Some of these are easy, for instance

$$\begin{gathered}[(d^{\overline{\nabla}}\tau)_{ijt}-\overline{\nabla}_t\tau_{ij}][(d^{\overline{\nabla}}\tau)_{ijt}-\overline{\nabla}_t\tau_{ij}]=2|d^{\overline{\nabla}}\tau|^2+\|\overline{\nabla}\tau\|^2,\\ \tau_{pq}\tau_{pt}\phi_{qij}\tau_{rs}\tau_{rt}\phi_{sij}=6\tau_{pq}\tau_{pt}\tau_{rq}\tau_{rt}=6\,|\tau|^4\end{gathered}$$

The last equality is obtained by the standard method: observe that the right hand side is a fourth order homogeneous G_2 invariant polynomial in $\tau\in\Lambda^2_{14}$. So up to scale it must equal $|\tau|^4$. The constant of proportionality is found by evaluating on a test element. In the same way one obtains $\tau_{ip}\tau_{jq}\phi_{pqt}\tau_{ir}\tau_{js}\phi_{rst}=3\,|\tau|^4$, and, $\tau_{pq}\tau_{pt}\phi_{qij}\tau_{ir}\tau_{js}\phi_{rst}=0$. To evaluate the remaining terms it is useful to first note that

$$\begin{aligned}(*(\tau\wedge\tau)+|\tau|^2\,\phi)_{ijt} &= \tau_{pq}(\tau_{pt}\phi_{qij}+\tau_{pi}\phi_{qjt}+\tau_{pj}\phi_{qti})\\ &= 2(\tau_{ip}\tau_{jq}\phi_{pqt}+\tau_{jp}\tau_{tq}\phi_{pqi}+\tau_{tp}\tau_{iq}\phi_{pqj}).\end{aligned}$$

Lemma 6.1 and relation (6.32) then imply that $(d^{\overline{\nabla}}\tau)_{ijt}\tau_{pq}\tau_{pt}\phi_{qij}=\frac{2}{3}*d(\tau^3)$, and, $(d^{\overline{\nabla}}\tau)_{ijt}\tau_{ip}\tau_{jq}\phi_{pqt}=\frac{1}{3}*d(\tau^3)$. Note that the contraction $\overline{\nabla}_k\tau_{ij}\phi_{ijl}$ is zero for all k and l as $\overline{\nabla}\tau\in V^*\otimes\Lambda^2_{14}$. So $\overline{\nabla}_t\tau_{ij}\tau_{pq}\tau_{pt}\phi_{qij}=0$. Using this identity twice, applying $\wedge_3(\xi)=0$ three times and juggling indices along the way we get

$$\begin{aligned}\overline{\nabla}_t\tau_{ij}\tau_{ip}\tau_{jq}\phi_{pqt} &= -\overline{\nabla}_t\tau_{ij}\tau_{ip}(\tau_{pq}\phi_{tqj}+\tau_{tq}\phi_{jqp})\\ &= \overline{\nabla}_t\tau_{ij}\tau_{pq}\tau_{pi}\phi_{qjt}+\overline{\nabla}_t\tau_{ij}\tau_{tq}(\tau_{jp}\phi_{qip}+\tau_{qp}\phi_{ijp})\\ &= \overline{\nabla}_t\tau_{ij}\tau_{pq}\tau_{pi}\phi_{qjt}-\overline{\nabla}_t\tau_{ij}\tau_{jp}(\tau_{iq}\phi_{qpt}+\tau_{pq}\phi_{qti}),\\ &= \overline{\nabla}_t\tau_{ij}\tau_{pq}(\tau_{pi}\phi_{qjt}+\tau_{pj}\phi_{qti}+\tau_{pt}\phi_{qij})+\overline{\nabla}_t\tau_{ij}\tau_{ip}\tau_{jq}\phi_{pqt}.\end{aligned}$$

Thus,

$$\overline{\overline{\nabla}}_t\tau_{ij}\tau_{ip}\tau_{jq}\phi_{pqt} = \tfrac{1}{3}{*d}(\tau^3).$$

It is curious that the inner product of the two summands of equation (6.38) only contributes to the norm squared through divergence. Moreover, the two terms in the second summand are orthogonal. The net result is

$$(\mathrm{R}^g_{ijab}\phi_{abt})(\mathrm{R}^g_{ijcd}\phi_{cdt}) = 2|d^{\overline{\overline{\nabla}}}\tau|^2 + \|\overline{\overline{\nabla}}\tau\|^2 + \tfrac{1}{4}\,|\tau|^4 + \tfrac{2}{9}{*d}(\tau^3).$$

Using equation (6.34) with $k = (1,0)$, Lemmata 2.1 and 6.10 this may be reformulated as

$$(\mathrm{R}^g_{ijab}\phi_{abt})(\mathrm{R}^g_{ijcd}\phi_{cdt}) = 3\,\|W_{64}\|^2 + \tfrac{40}{7}\|\mathrm{Ric}^g_0\|^2 - \tfrac{13}{147}s_g^2 + \tfrac{34}{63}{*d}(\tau^3).$$

Integrating this and applying Theorem 4.3 and remark 6.3 we arrive at the stated identity. □

Corollary 6.12. *Suppose M is a compact 7-dimensional manifold equipped with a closed G_2 structure ϕ. Then*

$$\langle p_1(M)\cup[\phi],[M]\rangle \geqslant -\frac{1}{8\pi^2}\int_M \left\{\|W_{77}\|^2 - \tfrac{1}{2}\,\|W_{64}\|^2 - \tfrac{3}{16}s_g^2\right\} dV_g.$$

Equality holds if and only if ϕ is extremal.

Proof. Use Proposition 6.4 in equation (6.37). □

Corollary 6.13. *Let M be compact and ϕ an extremal G_2 three-form on M. Then*

$$\langle p_1(M)\cup[\phi],[M]\rangle \geqslant -\frac{1}{8\pi^2}\int_M \left\{\|W_{77}\|^2 - \tfrac{3}{16}s_g^2\right\} dV_g.$$

Equality holds if and only if M with its associated metric g is locally isometric to $G/\,\mathrm{U}(2)$, where G is the 11-dimensional Lie group described in Theorem 6.21.

Proof. Corollary 6.12 shows that for an extremal three-form ϕ the stated inequality holds if and only if $W_{64} = 0$. By Lemma 6.10 this holds if and only if $(\overline{\overline{\nabla}}\tau)_{64} = 0$. For an extremal closed G_2 three-form $d^{\overline{\overline{\nabla}}}\tau = 0$ so by Lemma 2.1 this implies $\overline{\overline{\nabla}}\tau = 0$. The stated result is now a consequence of Theorem 6.21 below. □

Corollary 6.14. *Suppose M is compact, equipped with an extremal closed G_2 three-form ϕ. Suppose furthermore that the component W_{77} of the Riemannian curvature is zero. Then*

$$\langle p_1(M)\cup[\phi],[M]\rangle \geqslant 0$$

and equality holds if and only if ϕ is parallel and the associated metric is flat.

Proof. From Corollary6.12 we get

$$\langle p_1(M)\cup[\phi],[M]\rangle \geqslant -\frac{1}{8\pi^2}\int_M \|W_{77}\|^2\, dV_g$$

with equality if and only if $s_g^2 = \frac{1}{4}\,|\tau|^2 = 0$ and $W_{64} = 0$. □

Lemma 6.15 (Integral Weitzenböck Formula for τ)**.** *Let τ be the torsion of a closed G_2 structure ϕ on a compact manifold M. Then*

$$\int_M 4\mathrm{R}^g(\tau,\tau)dV_g = \int_M (2|d^{\overline{\overline{\nabla}}}\tau|^2 - \|\overline{\overline{\nabla}}\tau\|^2 + \tfrac{1}{6}\,|\tau|^4)dV_g. \tag{6.39}$$

Proof. This may also be found in [16]. However, as conventions are different we give an outline. Take the covariant derivative of the one-form Θ with components $\mathrm{R}^g_{ijkl}\phi_{pij}\tau_{kl}$ (expressed in terms of a local orthonormal frame) with respect to the Levi-Civita connection and contract with the metric. The result is clearly a divergence and so, by Stokes' Theorem integrates to zero on the compact M. But we also have

$$\delta^g\Theta = -(\nabla^g{}_p\mathrm{R}^g_{ijkl})\phi_{pij}\tau_{kl} - \mathrm{R}^g_{ijkl}\nabla^g{}_p\phi_{pij}\tau_{kl} - \mathrm{R}^g_{ijkl}\phi_{pij}\nabla^g{}_p\tau_{kl}$$

Writing $\delta^g\phi = \tau$ and applying the second Bianchi identity to the factor $\nabla^g{}_p\mathrm{R}^g_{ijkl}\phi_{pij}$ gives

$$\delta^g\Theta = \mathrm{R}^g_{ijkl}\tau_{ij}\tau_{kl} - \mathrm{R}^g_{ijkl}\phi_{pij}\nabla^g{}_p\tau_{kl}.$$

The proof is finished by expressing $\nabla^g\tau$ as a covariant derivative with respect to the canonical connection and then using the identities derived in the course of the proof of equation (6.37). □

Remark 6.16. The name we have given this Lemma stems from the equivalence of equation (6.39) with the usual integral Weitzenböck formula, see [16]

$$\int_M |d\tau|^2\,dV_g = \int_M \tfrac{1}{2}\Big(\|\nabla^g\tau\|^2 + 2\mathrm{R}^g(\tau,\tau)\Big)\,dV_g.$$

This is special instance of an example considered in [39], namely the Weitzenböck formula for two forms in Λ^2_{14}.

The integrand $\mathrm{R}^g(\tau,\tau)$ may be evaluated using Theorem 4.3 while the norms of the derivatives and τ may be expressed in terms of curvature components using formula (6.34) and Lemma 6.10. This gives

Proposition 6.17. *Let (M,ϕ) be a compact G_2 manifold M with closed G_2 structure ϕ and torsion τ. Then the integral identity holds*

$$\int_M (4W_{77}(\tau,\tau) + 3\,\|W_{64}\|^2)dV_g = \int_M \left(\tfrac{16}{7}\|\mathrm{Ric}^g_0\|^2 + \tfrac{127}{198}s_g^2\right)dV_g.$$

Proof. A straightforward computation in an orthogonal frame shows that for any symmetric two-tensor h

$$4r_g(h)(\tau,\tau) = -4h_{pq}\tau_{pr}\tau_{qr}, \qquad 4r_g(g)(\tau,\tau) = -8\,|\tau|^2 = 16 s_g,$$
$$4r_\phi(h) = -\tfrac{8}{3}h_{pq}\tau_{pr}\tau_{qr} + \tfrac{4}{3}\operatorname{tr}_g(h)\,|\tau|^2\,.$$

This gives

$$4\mathrm{R}^g(\tau,\tau) = 4W_{77}(\tau,\tau) - \tfrac{1}{4}\big(\mathrm{Ric}_0^{(3,1/4)}\big)_{pq}\tau_{pr}\tau_{qr} + \tfrac{4}{21}s_g^2. \tag{6.40}$$

The contraction $\tau_{pr}\tau_{qr}$ gives the components of the tensor $\tau\otimes_g\tau := (e_r\lrcorner\tau)\otimes(e_r\lrcorner\tau) = \frac{1}{2}\tau_{pr}\tau_{qr}e^p\odot e^q$. It is easy to check that

$$\lambda_3(\tau\otimes_g\tau) = *(\tau\wedge\tau) + |\tau|^2\,\phi,$$
$$\lambda_3\big(\mathrm{Ric}_0^{(3,1/4)}\big) = -2d^{\overline{\nabla}}\tau + \tfrac{17}{12}*(\tau\wedge\tau)_{27}.$$

Equations (2.5) and (6.32) then imply that

$$\begin{aligned}\big(\mathrm{Ric}_0^{(3,1/4)}\big)_{pq}\tau_{pr}\tau_{qr} &= \tfrac{1}{2}\left\langle -2d^{\overline{\nabla}}\tau + \tfrac{17}{12}*(\tau\wedge\tau)_{27}, *(\tau\wedge\tau)_{27}\right\rangle \\ &= -\tfrac{1}{3}*d(\tau^3) - \tfrac{17}{7}s_g^2,\end{aligned} \tag{6.41}$$

whence

$$4\mathrm{R}^g(\tau,\tau) = 4W_{77}(\tau,\tau) - \tfrac{5}{12}s_g^2 + \tfrac{1}{12}*d(\tau^3),$$

and clearly

$$\int_M 4\mathrm{R}^g(\tau,\tau)dV_g = \int_M (4W_{77}(\tau,\tau) - \tfrac{5}{12}s_g^2)dV_g. \tag{6.42}$$

On the other hand,

$$\begin{aligned}2|d^{\overline{\nabla}}\tau|^2 - \|\overline{\nabla}\tau\|^2 + \tfrac{1}{6}|\tau|^4 &= -\|(\overline{\nabla}\tau)_{64}\|^2 + (2-\tfrac{6}{7})\big(2\|\mathrm{Ric}_0^g\|^2 - \tfrac{8}{21}s_g^2 + \tfrac{2}{9}{*d}(\tau^3)\big) + \tfrac{2}{3}s_g^2 \\ &= -3\|W_{64}\|^2 + \tfrac{16}{7}\|\mathrm{Ric}_0^g\|^2 + \tfrac{2\cdot 17}{3\cdot 49}s_g^2 + \tfrac{16}{63}d(\tau^3).\end{aligned}$$

This gives the following alternative expression for the Weitzenböck formula

$$\int_M 4\mathrm{R}^g(\tau,\tau) = \int_M (-3\|W_{64}\|^2 + \tfrac{16}{7}\|\mathrm{Ric}_0^g\|^2 + \tfrac{2\cdot 17}{3\cdot 49}s_g^2).$$

We compare this expression to equation (6.42) and rearrange to get

$$\begin{aligned}\int_M (4W_{77}(\tau,\tau) + 3\|W_{64}\|^2)dV_g &= \int_M \left(\tfrac{16}{7}\|\mathrm{Ric}_0^g\|^2 + \tfrac{127}{198}s_g^2\right)dV_g \\ &= \int_M \left(\tfrac{16}{7}\|\mathrm{Ric}^g\|^2 + \tfrac{9}{28}s_g^2\right)dV_g.\end{aligned}$$

□

Corollary 6.18. *Let M be a compact G_2 manifold with closed G_2 structure ϕ and torsion τ. If the curvature components W_{77} and W_{64} are identically zero then M is parallel and flat.*

Remark 6.19. A computation similar to (6.41) of the proof of Proposition6.17 shows that

$$\tfrac{1}{4}r_g(\mathrm{Ric}^g)(\tau,\tau) = \mathrm{Ric}^g_{pq}\tau_{pr}\tau_{qr} = -\tfrac{1}{6}{*d}(\tau^3).$$

and so, for a G_2 structure ϕ with torsion $\tau \in \Lambda^2_{14}$,

$$\int_M r_g(\mathrm{Ric}^g)(\tau,\tau)dV_g = 0.$$

6.3. **Closed fundamental three-forms with parallel torsion.** We give a brief description of the only known example of an extremal closed G_2 structure on a compact manifold due to Robert Bryant [8]. Let G be the space of affine transformations of $\mathbb{C}^2$ preserving the canonical complex volume form. Then $SU(2)$ is a subgroup in G and $M = G/SU(2)$ is a 7-dimensional homogeneous space, diffeomorphic to $\mathbb{R}^7$ admitting an invariant extremal closed G_2 structure ϕ as well as a free and properly discontinuous action of a discrete subgroup $\Gamma \subset G$ for which ϕ is invariant. So ϕ descends to an extremal closed G_2 structure $\tilde\phi$ on the compact quotient $\tilde M := \Gamma \setminus M$.

Andrew Swann made us aware of the following alternative description of Bryant's example. Note that $SL(2,\mathbb{C}) = SU(2).Sol_3$ where Sol_3 is the space of complex upper diagonal 2×2 matrices $\left(\begin{smallmatrix} e^t & z \\ 0 & e^{-t}\end{smallmatrix}\right)$, with t real and z a complex number (by Iwasawa decomposition, or simply applying the Gram-Schmidt process to the column vectors of elements of $SL(2,\mathbb{C})$). This gives the alternative description of M as the Lie group $Sol_3 \rtimes \mathbb{C}^2$. Taking any basis of left-invariant one-forms $e = (e^i)$ on M thought of as a Lie group gives a G_2 three-form ϕ by requiring that e is a G_2 adapted frame field. Below we shall show that, up to isometries, despite the availability of the construction of a multitude of invariant fundamental three-forms on M there is only one extremal three-form.

Even better we shall show that any manifold M' with extremal G_2 structure ϕ' such that the torsion τ is parallel with respect to $\overline{\nabla}$, locally is isometric to (M,ϕ). Lemma 6.1 and Proposition 6.10 give

Lemma 6.20. *A closed G_2 form ϕ is extremal if and only if its torsion τ satisfies $(\overline{\nabla}\tau)_{27} = 0$. The torsion of a closed G_2 form ϕ is parallel with respect to the canonical connection if and only if ϕ is extremal and the component W_{64} of the Riemannian curvature is zero.*

Theorem 6.21. *Let (M, ϕ) be a G_2 manifold such that $d\phi = 0$, $d{*}\phi = \tau \wedge \phi$ and $\overline{\nabla}\tau = 0$, where $\overline{\nabla}$ is the canonical connection of the G_2 connection. Then (M, g) has extremally pinched Ricci curvature. Furthermore, (M, ϕ) is locally isometric to a homogeneous space $G/\operatorname{U}(2)$ where G is the Lie group consisting of affine transformations of $\mathbb{C}^2$ which preserve the norm of the complex volume form on $\mathbb{C}^2$.*

Proof. If $\overline{\nabla}\tau = 0$ then (M, ϕ) is extremal and the norm of the torsion is constant. For an extremal three-form $\tau^3 = 0$ if the torsion tensor has constant norm, see [8]. Then τ has constant rank 4 with stabilizer $U(2) \subset G_2$ and the holonomy of $\overline{\nabla}$ reduces to a $U(2) \subset G_2$. Which one of the two possible $U(2)$'s (up to conjugation) is determined as follows. The subalgebras of $\mathfrak{g}_2$ of dimension 4 are contained in the maximal $SO(4)$ and are (up to a finite quotient) homomorphic to $U(2)$. The action of $SO(4)$ on $\mathbb{R}^7$ may be written in terms of the standard representations V_+, V_- of $\mathfrak{so}(4) = \mathfrak{su}_+(2) + \mathfrak{su}_-(2)$ as $\mathbb{R}^7 = S^2V_+ + V_+ \otimes V_-$ (modulo complexifications). Take a basis e_1, e_2, e_3, e_4 of V_+V_- and e_5, e_6, e_7 of S^2V_+. Note that under the action of $SO(4)$ the space of two-forms is

$$\Lambda^2 = 2S^2V_+ + (S^3V_+ + V_+)V_- + S^2V_-,$$

so

$$\Lambda^2_{14} = S^2V_+ + S^2V_- + S^3V_+V_-.$$

We may give explicit bases of the subspaces $S^2V_\pm$. They are $e^{12} + e^{34} - 2e^{56}, e^{13} + e^{42} - 2e^{67}, e^{14} + e^{23} - 2e^{57}$ for S^2V_+ and $e^{12} - e^{34}$ and so on for S^2V_-. Note that elements of S^2V_+ always have rank 6. Therefore the stabilizer of a rank 4 element of $\mathfrak{g}_2$ is $\mathfrak{u}_+(2) = \mathbb{R}_- + \mathfrak{su}_+(2) \subset \mathfrak{so}(4) \subset \mathfrak{g}_2$. Its action on $\mathbb{R}^7$ is $S^2V_+ + (L + \bar{L})V_+$ and on Λ^2

$$2S^2V_+ + (L^2 + \mathbb{R} + \bar{L}^2) + (L + \bar{L})(S^3V_+ + V_+).$$

We now fix $U(2) \subset G_2$ as the stabilizer of $\tau := 6(e^{12} - e^{34})$, where $U(2)$ acts on $\mathbb{R}^7$ as $\mathfrak{su}(2) \oplus \mathbb{C}^2$. The Berger algebra of $\mathfrak{u}(2)$ with respect to this representation is trivial, see [18] for details. So the curvature $\overline{\mathrm{R}}$ of the canonical connection $\overline{\nabla}$ is determined algebraically by the torsion squared through the Bianchi identity and therefore $\overline{\nabla}\overline{\mathrm{R}} = 0$. A computation yields,

$$\begin{aligned}\overline{\mathrm{R}} = -4(e^{12} - e^{34}) \otimes (e^{12} - e^{34}) \\ - 2(e^{12} + e^{34} - e^{56}) \otimes (e^{12} + e^{34} - 2e^{56}) \\ - 2(e^{13} + e^{42} - e^{67}) \otimes (e^{13} + e^{42} - 2e^{67}) \\ - 2(e^{14} + e^{23} - e^{57}) \otimes (e^{14} + e^{23} - 2e^{57}))\end{aligned}$$

The bracket on $\mathfrak{u}(2) \oplus \mathbb{R}^7$ given by $[A + x, B + y] = Ay - Bx - \xi_x y + \xi_y x - \overline{\mathrm{R}}_{x,y}$ may now be computed. It is a Lie bracket on an 11-dimensional algebra with group G as stated. $\square$

Remark 6.22. Note that this clearly does *not* imply uniqueness of homogeneous spaces with closed G_2 three-form. First of all, many examples of homogeneous spaces (with compact quotients) admitting invariant closed G_2-structures are known, see [20, 21]. These, generically, do not admit an extremal three-form, by Theorem 6.21. In fact, through equation (6.31), the extremallity condition may be seen as a non-degeneracy condition on the derivative $d\tau$, which is, again generally speaking, never satisfied on examples built over nilpotent and solvable Lie algebras.

Even the space $\tilde{M} = G/\,U(2)$ with its extremal, invariant three-form may be described in several different ways, as we have already seen. Apart from the descriptions given above, $\tilde{M}$ may be described as the $U(1)$ quotient of the semi-direct product of the space of matrices $\left(\begin{smallmatrix} re^{i\theta} & z \\ 0 & r^{-1}e^{i\theta} \end{smallmatrix}\right)$ with $r > 0$, θ real and z complex with $\mathbb{C}^2$. As the isotropy group becomes smaller the number of invariant G_2 three-forms increase but, up to isometry, there is only one extremally pinched structure. It may also be possible to enlarge the group G to the full space of isometries, but we note here that G is the maximal group acting on $\tilde{M}$ leaving the extremal form invariant.

In the light of the evidence given here we make the following conjecture

Conjecture. *Suppose M is compact and ϕ is an extremal closed G_2 structure on M. Then the universal covering space $\tilde{M}$ of M is isometric to $Sol_3 \rtimes \mathbb{C}^2$ with its unique invariant, extremal three-form.*

7. Examples

The examples we will be considering here are all, locally, of the form $M = I \times M^*$ where M^* is a 6-dimensional manifold carrying a one-parameter family of $SU(3)$ structures given by $(\omega_t, \psi_t^+)_{t\in I}$. This gives G_2 structures $\phi := dt \wedge \omega_t + \psi_t^+$ with associated metric $g = dt^2 + g_t$, where g_t is the metric on M^* determined by $\omega_t,\ \psi_t^+$. Strictly speaking, pull-backs such as $g = \pi_1^*(dt^2) + \pi_2^*(g_t)$ where π_i is projection to the i'th factor ought to be included, but to simplify notation we set $dt := \pi_1^*(dt)$ and so.

The examples all admit parallel or nearly parallel G_2 structures. Some are flat, and some has constant sectional curvature. In particular, all have curvature form $W_{77} + S$, but for some $W_{77} = 0$ (constant sectional curvature) and some (parallel) $Scal = 0$. The compatible three-forms, however, appear to range over pretty much any torsion type not obstructed by the value of the scalar curvature.

7.1. **Warped products.** Let M^* be an $n-1$ dimensional manifold with metric g^*. Write R^{g^*}, Ric^* and $(n-1)(n-2)\rho^* = s^*$ for the Riemannian, Ricci and scalar curvatures of g^*. Set $M := I \times M^*$, where I is an interval (i.e., an open, connected subset of $\mathbb{R}$) with warped product metric $g := dt^2 + f^2 g^*$ and curvatures R^g, Ric, $n(n-1)\rho = s$. An elementary calculation shows that the curvatures of g are related to (the pull-backs through $M \to M^*$ of) those of g^* via

$$\mathrm{R}^g = f^2\mathrm{R}^{g^*} - \tfrac{1}{2}(ff')^2 g^* \owedge g^* - ff'' dt^2 \owedge g^*, \tag{7.43}$$

It follows straight from equation (7.43) that

Lemma 7.1. [3] *The warped product metric $g = dt^2 + f^2 g^*$ is Einstein if and only if the conditions* (1) *and* (2) *are satisfied.*

(1) *g^* is Einstein,*
(2) *$(f')^2 + \rho f^2 = \rho^*$.*

If $g = dt^2 + f^2 g^*$ is Einstein, then $f'' + \rho f = 0$, and

$$\mathrm{R}^g = f^2 W^* + \tfrac{1}{2}\rho g \owedge g, \tag{7.44}$$

where R^g is the Riemannian curvature of g and W^* is the Weyl curvature of W^* considered as $(4,0)$ *tensors.* ∎

7.2. **Curvature of nearly Kähler and Calabi-Yau 3-folds.** Let M^* be a 6-dimensional manifold equipped with an $SU(3)$ structure, i.e., with data $g^*, J, \omega, \psi^+, \psi^-$ where g^* is a Riemannian metric, J is an almost complex structure, ω is a non-degenerate form, and $\psi^+ + i\psi^-$ is a complex volume form (of type $(3,0)$). The normalization conditions

$$\omega = g \circ J, \qquad 2\omega^3 = 3\psi^+ \wedge \psi^-, \tag{7.45}$$

may be imposed to give relations

$$*\psi^+ = \psi^-, \qquad J\psi^+ = -\psi^-,$$

and so on. When M^* is Calabi-Yau or nearly Kähler structure it may be assumed that the differentials of the forms above satisfy

$$d\omega = 3\sigma\psi^+, \qquad d\psi^- = -2\sigma\omega^2, \tag{7.46}$$

where σ is a non-negative constant related to scalar curvature s^* and normalized scalar curvature ρ^* through $\sigma^2 = s^*/30 = \rho^*$, see e.g. [2]. We fix terminology, by saying that M^* is nearly Kähler if (7.46) holds for some constant σ, Calabi-Yau if $\sigma = 0$ and strict nearly Kähler if $\sigma > 0$. For nearly Kähler 6-dimensional manifold the Riemannian curvature tensor takes the form similar to the one of type 1 and parallel G_2 structure

$$\mathrm{R}^{g^*} = K^* + \tfrac{1}{2}\rho^*(g^* \owedge g^*),$$

where $K^*(\omega) = 0$, $K^*(\iota_X\psi^\pm) = 0$ for all $X \in \Gamma(M^*)$, see [12]. We shall say that a special almost Hermitian manifold (M^*, ω, ψ^+) with curvature of this form is of curvature type $\mathcal{NK}$.

Proposition 7.2. *Suppose $(M^*, g^*, \omega, \psi^+, \psi^-)$ is an $SU(3)$ manifold with curvature type $\mathcal{NK}$. Then any warped product $M = I \times M^*$, $g = dt^2 + f^2 g^*$ has $\mathrm{Ric}^{\mathcal{W}} = 0$. Any Einstein warped product $M = I \times M^*$, $g = dt^2 + f^2 g^*$ has $\mathrm{Ric}_0^g = 0 = \mathrm{Ric}_0^\phi$.*

Proof. This is an easy consequence of the forms of the curvature tensors of type $\mathcal{NK}$ and $\mathcal{NP}$ and Einsteinian warped product metrics. □

7.3. **Warped Products over nearly Kähler 3-folds.** Let $I \subset R$ be an open interval and set $M = M^* \times I$ where M^* has an $SU(3)$ structure $(g^*, J, \omega, \psi^+, \psi^-)$. Define

$$\begin{aligned} \omega_t &= f^2\omega, \\ \psi_t^+ &= f^3(\cos\theta\psi^+ - \sin\theta\psi^-), \\ \psi_t^- &= f^3(\sin\theta\psi^+ + \cos\theta\psi^-), \end{aligned}$$

for smooth functions $f, \theta\colon I \to \mathbb{R}$ with $f > 0$. A *warped* G_2-fundamental three-form ϕ is defined on M by

$$\phi = \omega_t \wedge dt + \psi_t^+. \tag{7.47}$$

This is compatible with the warped product metric

$$g = dt^2 + f^2 g^*,$$

and has

$$*\phi = \tfrac{1}{2}\omega_t^2 + \psi_t^- \wedge dt.$$

Suppose $(M^*, g^*, J, \omega, \psi^+, \psi^-)$ is nearly Kähler, normalized according to equation (7.45) with structure equations (7.46). We define the warped G_2 structure as in equation (7.47). Then

$$\begin{aligned} d\omega_t &= 2f^{-1}f'dt \wedge \omega_t + 3f^{-1}\sigma(\cos\theta\psi_t^+ + \sin\theta\psi_t^-), \\ d\psi_t^+ &= 3f^{-1}f'dt \wedge \psi_t^+ - d\theta \wedge \psi_t^- + 2f^{-1}\sigma\sin\theta\omega_t^2, \\ d\psi_t^- &= 3f^{-1}f'dt \wedge \psi_t^- + d\theta \wedge \psi_t^+ - 2f^{-1}\sigma\cos\theta\omega_t^2, \end{aligned}$$

and the differentials of the fundamental forms are

$$\begin{aligned} d\phi = -3f^{-1}\left(f' - \sigma\cos\theta\right)\psi_t^+ \wedge dt \\ + \left(\theta' + 3f^{-1}\sigma\sin\theta\right)\psi_t^- \wedge dt + 2f^{-1}\sigma\sin\theta\omega_t^2, \end{aligned}$$

and

$$d{*}\phi = 2f^{-1}\left(f' - \sigma\cos\theta\right)dt \wedge \omega_t^2.$$

¿From this the torsion components $\tau_p \in \Omega^p(M)$ are obtained. We have

$$\begin{array}{ll} \tau_0 = \frac{4}{7}\left(\theta' + 6f^{-1}\sigma\sin\theta\right), & \tau_1 = f^{-1}\left(f' - \sigma\cos\theta\right)dt, \\ \tau_2 = 0, & \tau_3 = -\frac{1}{7}\left(\theta' - f^{-1}\sigma\sin\theta\right)\left(4\omega_t \wedge dt - 3\psi_t^+\right). \end{array}$$

The following elementary fact is recorded here for ease of reference

Lemma 7.3. *Suppose b is a non-zero continuous function. Then the solutions to the equation*

$$\theta' = b\sin\theta \tag{7.48}$$

may be given as: either $\sin\theta = 0 = \theta'$, or

$$\cos\theta = \frac{1-a^2}{1+a^2}, \qquad \sin\theta = \pm\frac{2a}{1+a^2},$$

where $a(t) = \exp\int^t b(s)ds$. □

This Lemma ensures that given any function f, the torsion components τ_0 and τ_3 may be made to vanish either simultaneously, with $\sin\theta = 0 = \theta'$ or, when $\sigma \neq 0$, separately, by choosing an appropriate solution θ to equation (7.48). Proposition 7.2 and a little book-keeping then proves

Proposition 7.4. *Suppose ϕ is a warped G_2 three-form over a 6 dimensional nearly Kähler manifold M^*.*

1. *If the associated metric has holonomy contained in G_2 then ϕ is either parallel or of strict type $1+3$ if M^* is Calabi-Yau, or parallel, of type $4,\ 1+4,\ 3+4$, or $1+3+4$ if M^* is strict nearly Kähler.*
2. *If the metric associated to ϕ is Einstein with non-zero scalar curvature then ϕ is of type 4 or $1+3+4$ and g has negative scalar curvature if M^* is Calabi-Yau. If M^* has positive scalar curvature then the type 1 clearly for ϕ clearly requires that g has positive scalar curvature. The classes $4,\ 1+4$, $3+4$ and $1+3+4$ may be realized for any sign of the scalar curvature of the metric associated to ϕ. The proper class $1+3$ is not realized.*

7.4. **Cohomogeneity one examples.** A variation on this theme is obtained by considering M^* homogeneous $M^* = G/H$ with $H \subset SU(3)$. To be concrete we take $G = SU(3)$, $H = T^2$. On M^* there are invariant forms $\omega_i \in \Lambda^2$, $i = 1, 2, 3$, $\psi^+, \psi^- \in \Lambda^3$ such that

$$\omega_i \wedge \psi^\pm = 0, \quad \omega_i^2 = 0, \quad \mathrm{vol}_0 := \omega_1\omega_2\omega_3 = \tfrac{1}{4}\psi^+ \wedge \psi^-,$$
$$d\omega_i = \tfrac{1}{2}\psi^+, \qquad d\psi^- = -2\sum_{i<j}\omega_i\omega_j.$$

On $M = I \times M^*$ we set

$$\omega_{i,t} := f_i^2\omega_i, \qquad \omega_t := \sum_i \omega_{i,t},$$
$$\psi_t^+ := f_1f_2f_3(\cos\theta\psi^+ - \sin\theta\psi_-),$$
$$\psi_t^- := f_1f_2f_3(\sin\theta\psi^+ + \cos\theta\psi_-),$$
$$\mathrm{vol}_t = \tfrac{1}{4}\psi_t^+ \wedge \psi_t^- = \tfrac{1}{6}\omega_t,$$
$$\Omega_t := f_1f_2f_3\sum_{i<j}\omega_i\omega_j = \sum_{i<j}\tfrac{f_k}{f_if_j}\omega_{i,t}\omega_{j,t}.$$

Anytime indices i, j, k occur as above this should be understood as $\{i, j, k\} = \{1, 2, 3\}$. An easy calculation gives

$$d\omega_{i,t} = \ln(f_i^2)'dt \wedge \omega_{i,t} + \tfrac{1}{2}\tfrac{f_i}{f_jf_k}(\cos\theta\psi_t^+ + \sin\theta\psi_t^-),$$
$$d\omega_t = \sum_i \ln(f_i^2)'dt \wedge \omega_{i,t} + h(\cos\theta\psi_t^+ + \sin\theta\psi_t^-),$$
$$d\psi_t^+ = -g'\psi_t^+ \wedge dt + \theta'\psi_t^- \wedge dt + 2\sin\theta\Omega_t,$$
$$d\psi_t^- = -g'\psi_t^- \wedge dt - \theta'\psi_t^+ \wedge dt - 2\cos\theta\Omega_t,$$

where $g = \ln(f_1f_2f_3)$, $h = \frac{f_1^2+f_2^2+f_3^2}{2f_1f_2f_3}$. Now set

$$\phi := \omega_t \wedge dt + \psi_t^+.$$

Then ϕ is compatible with the metric and volume

$$g = dt^2 + \sum_i f_i^2 g_i, \qquad \mathrm{vol} = \mathrm{vol}_t \wedge dt,$$

and

$$*\phi = \tfrac{1}{2}\omega_t^2 + \psi_t^- \wedge dt.$$

Suppose that f_1, f_2, f_3 are such that $(f_if_j)' = f_k$ (there is a one-parameter family of such triples, see [17] for details). Then the metric g has holonomy contained in G_2. For such a triple one furthermore has $g' = h$. Taking this into account we calculate the torsion components of ϕ.

$$\tau_0 := \tfrac{4}{7}(\theta' + 2h\sin\theta),$$
$$\tau_1 := \tfrac{1}{3}h(1 - \cos\theta),$$
$$\tau_2 := -\frac{2(1-\cos\theta)}{3f_1f_2f_3}\sum_i(2f_i^2 - f_j^2 - f_k^2)\omega_{i,t},$$
$$\tau_3 := \tfrac{3}{7}\left(\theta' - \tfrac{1}{3}h\sin\theta\right)\psi_t^+ - \tfrac{4}{7}\sum_i\left(\theta' - \frac{5f_i^2 - 2(f_j^2+f_k^2)}{2f_1f_2f_3}\sin\theta\right)\omega_{i,t} \wedge dt.$$

First note that ϕ is parallel if and only if $\cos\theta = 1$. If $\cos\theta \not\equiv 1$ then $\tau_1 \not\equiv 0$. Taking $\cos\theta = -1$ gives a $2 + 4$ structure, which is a type 4 structure when $f_1 = f_2 = f_3$.

If $f_1 = f_2 = f_3 =: f$ holds then $\tau_2 \equiv 0$ and

$$\tau_3 = \left(\theta' - \tfrac{1}{3}h\sin\theta\right)\left(\tfrac{3}{7}\psi_t^+ - \tfrac{4}{7}\omega_t \wedge dt\right).$$

In general this gives a type $1+3+4$ structure, but using Lemma 7.3 again we may make either τ_0 or τ_3 vanish to obtain $1+4$ and $3+4$ structures, too.

Now suppose $f_3 \geqslant f_2 \geqslant f_1$ and $f_3 > f_1$. The generic type is $1+2+3+4$, but we may once again use Lemma 7.3 to eliminate τ_0 to get class $2+3+4$.

Proposition 7.5. *There are warped product and cohomogeneity one metrics g with $Hol(g) \subset G_2$ and three-forms ϕ compatible with g of every admissible type apart (possibly) from $1+2+3$.* □

A three-form of *admissible type* means not one of the proper types $1, 2, 3, 1+2$, and $2+3$. Type $1+2$ doesn't exist as a proper class and every other proper type in this list has either strictly positive or strictly negative scalar curvature, c.f equation (4.25).

Remark 7.6. The following was related to us by Robert Bryant [7]: for a fixed metric g, the exterior differential system corresponding to the equation $p_7^4(d\phi) = 0$ for a compatible G_2-three-form ϕ is involutive at points where the torsion $\tau \not\equiv 0$, with last nonzero Cartan character $s_6 = 6$, so that the general local solution depends on 6 functions of 6 variables.

This along with Proposition 7.5 has the corollary that

Theorem 7.7. *For every admissible Fernandéz-Gray type, there is a G_2 three-form ϕ of this type, such that the associated metric g has holonomy $Hol(g) \subset G_2$.*

Remark 7.8. Note that the type 1 structures on the cohomogeneity one space $I \times SU(3)/T^2$ are warped products over the unique nearly Kähler structure g^* on $SU(3)/T^2$, see [17]. Therefore the analysis carried out in section 7.3 applies with conclusion as in Proposition 7.4.

7.5. **Complete and compact examples.**

Example 7.9. Suppose (M^*, ω, ψ^+) is Calabi-Yau. Then for any smooth function $\theta\colon \mathbb{R} \to \mathbb{R}$, the three-form ϕ defined as above with f constant is smooth. The G_2 three-form is of strict type $1+3$ if θ is non-constant and parallel otherwise. The associated metric is the Riemannian product metric on $\mathbb{R} \times M^*$ and hence has holonomy contained in $SU(3) \subset G_2$. Compact examples may be obtained by the method of [3], Section 9.109.

For certain other choices of (M^*, g^*) and non-constant function $f\colon I \to \mathbb{R}$ the warped product metric on $I \times M^*$ also extends to a complete metric. This is so for

$$\begin{aligned}
g_{\mathbb{R}^7} &= dt^2 + t^2 g_{S^6}, & I &= \mathbb{R}^+,\\
g_{S^7} &= dt^2 + \sin^2(t) g_{S^6}, & I &= (0,\pi),\\
g_{H^7} &= \begin{cases} dt^2 + \sinh^2(t) g_{S^6}, & I = \mathbb{R}^+,\\ dt^2 + e^{2t} g_{\mathbb{R}^6}, & I = \mathbb{R}. \end{cases}
\end{aligned}$$

Here g_N refers to the constant sectional curvature metric of N with $|\rho|$ and ρ^* equal to 0 or 1.

Example 7.10. We consider the last case first. The structure on M^*, as the examples of 7.9, has $\sigma = 0$ and f strictly positive. The warped three-form ϕ is therefore smooth for any smooth function $\theta\colon \mathbb{R} \to \mathbb{R}$. The generic type is $1+3+4$ for non-constant θ and 4 for constant theta. Type $1+3$ three-forms require $e^t = \cos\theta$ and therefore are not complete.

The remaining three cases are all cohomogeneity one metrics over the isotropy irreducible space $M^* = S^6 = G_2/SU(3)$ equipped with its standard, homogeneous, nearly Kähler structure so that $\sigma = 1$. Structures with $\tau_1 = 0$ are: the flat G_2 structure on $\mathbb{R}^7$ $f = t,\ \theta = 0$ and the unique nearly parallel (type 1) structure on S^7, $f = \sin(t),\ \theta = t$. These are, of course, complete and smooth.

Type $1 + 4$ structures are given by the equation $f d\theta = \sin\theta dt$. Consequently, the metric and three-form may conveniently be re-written on the familiar form

$$\phi = (\theta')^{-3}(\sin^2\theta\omega \wedge d\theta + \sin^3\theta(\cos\theta\psi^+ - \sin\theta\psi^-)),$$
$$g = (\theta')^{-2}(d\theta^2 + \sin^2\theta g_{S^6}).$$

It is then clear that the three-forms of type $1 + 4$ arise as point-wise conformal changes of the standard nearly parallel structure on the 7-sphere. Examination of the solutions of Lemma 7.3 corresponding to $b = f^{-1}$ shows that constants A and B exist so that

$$\theta' = A\cos\theta + B, \quad \text{where} \quad \begin{cases} |B/A| > 1, & \text{for } f = \sin(t), \\ |B/A| < 1, & \text{for } f = \sinh(t), \\ |B/A| = 1, & \text{for } f = t. \end{cases}$$

The requirement that $|\theta'| > 0$ imposes restrictions in the two latter cases. It is clear that metric and three-form are smooth where $\theta' \neq 0$. In the first case we see that all type $1 + 4$ warped G_2 structures arise as global conformal changes of the standard nearly parallel structure on S^7, see [15, 5].

Example 7.11. The type 4 structures are also interesting. First consider three-forms ϕ compatible with the standard metric on $\mathbb{R}^7$. Since $\theta' = 0 = \sin\theta$, there are only two possibilities: either ϕ is parallel ($\cos\theta = 1$) or $\tau_1 = d\ln(t^2)$ ($\cos\theta = -1$). In the latter case one notes that

$$t^{-6}\phi = s^2\omega \wedge ds + s^3\psi^+ = \iota^*\phi,$$
$$t^{-4}g = ds^2 + s^2 g_{S^6} = \iota^* g,$$

where $s := -t^{-1}$ and ι is the map $\iota\colon \mathbb{R}^7 \setminus \{0\} \to \mathbb{R}^7 \setminus \{0\},\ x \mapsto -x/|x|^2$.

The standard metric of the 7 sphere is compatible with the three-form

$$\phi = \sin^2(t)\omega \wedge dt + \sin^3(t)\psi^+.$$

We note that this three-form satisfies the necessary conditions of [17]. However, the three-form has type 4 with torsion

$$\tau_1 = \frac{\cos(t) - 1}{\sin(t)} dt = -\tan(t/2)dt = -d\ln\cos^2(t/2).$$

This is clearly singular at one point. Setting $r = 2\cos^2(t/2),\ s = \tan(t/2)$ then $ds = \frac{1}{2}dt/\cos^2(t/2)$ and we may write

$$\phi = \frac{8}{(1+s^2)^3}(s^2\omega \wedge ds + s^3\psi^+), \qquad g_{S^7} = \frac{4}{(1+s^2)^2}(ds^2 + s^2 g_{S^6}) = r^2 g_{\mathbb{R}^7},$$

This transformation realizes the same structure smoothly on $\mathbb{R}^7$. The metric represented this way is not complete.

For hyperbolic space the same procedure may be followed and one obtains

$$\phi = \frac{8}{(1-s^2)^3}(s^2\omega \wedge ds + s^3\psi^+), \qquad g = \frac{4}{(1-s^2)^2}(ds^2 + s^2 g_{S^6}),$$

where $s = \tanh(t/2) \in (0, 1)$.

7.5.1. *Cohomogeneity one metrics over* $SU(3)/T^2$. For the instance of the cohomogeneity one metrics based on $M^* = SU(3)/T^2, Sp(2)/Sp(1)\,U(1)$ the analysis carried out in the final section of [17] applies here, too. Whenever θ is an odd function of t, f_1 is odd with $f_1'(0) = 1$ and f_2 is even with $f_2(0) > 0$ the metric g and three-form ϕ extend smoothly to the non-compact manifolds isomorphic to the bundles of anti-self-dual forms over $\mathbb{C}P(2)$ and S^4, respectively. These then carry smooth G_2 structures compatible with the holonomy G_2 structures which are either parallel type or of strict type $2+4$, $2+3+4$, or $1+2+3+4$.

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