LIFTING ABELIAN VARIETIES WITH ADDITIONAL STRUCTURE

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Introduction

This note is to investigate the moduli spaces of polarized O_B-abelian varieties with a determinant condition (see Section 1 for the definition). The main result is that when p is unramified in B and greater than 2, the moduli space is a complete intersection relative to the mixed characteristic base. See Theorem 4.5 and Section 4 for details. The main tools are Dieudonné modules and Grothendieck-Messing's deformation theory [G][Me].

The main point of this work, same as that in [N], is to deal with inseparable polarizations. The result of this work concludes: the naive moduli problem provides a integral model of a Shimura varieties $Sh_K(G,X)$ of PEL type, which is flat and the reduction is a locally complete intersection at the places where G is unramified and the level subgroup K is maximal. In the case that p is ramified in B, G. Pappas [P] showed that there are some examples (unitary Shimura varieties) in which the integral models constructed by the naive modular interpretation are not flat over the base.

Section 1 contains the definition of the moduli spaces we considered. Section 2 contains some well known classical results. Section 3 provides a certain degree classification of the Dieudonné modules in question. Section 4 uses Grothendieck-Messing and Fontaine's theory to lift and deform abelian varieties with additional structure.

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1. Moduli spaces

Let B be a semi-simple $\mathbb{Q}$-algebra with a positive involution $*$, O_B a maximal order stable under the involution $*$. Fix a $\mathbb{Q}$-valued skew-Hermitian B-module (V,ψ) with $\mathbb{Q}$-dimension $2g$ and an $\mathbb{R}$-linear representation h of $\mathbb{C}$ on $V_{\mathbb{R}}$ such that the symmetric bilinear form $(x,y)\to\psi(x,h(i)y)$ is positive definite. Let E be the field of the definition of the isomorphic class of the complex representation of B on $V_1 := V_{\mathbb{C}}^{-1,0}$, on which $h(z)$ acts as z. In fact, $E = \mathbb{Q}(\mathrm{Tr}(b;V_1))_{b\in B}$. We denote the representation V_1 by $\Phi(h)$. We will define a condition on the Lie algebra $\mathrm{Lie}(A)_S$ for an O_B-abelian scheme A over a base O_E-scheme S. This condition will be called the Kottwitz determinant condition defined by $\Phi(h)$, [K].

Let $\{e_i\}$ be a $\mathbb{Z}$-basis of O_B and X_i be some indeterminants. Define $f_h(X) := \det(\sum e_iX_i;V_1)$. The polynomial f_h is independent of the choice of basis and can be shown to have coefficients in O_E. For an O_B-abelian scheme A over an O_E-scheme S, we say that the Lie algebra $\mathrm{Lie}(A)_S$ satisfies the Kottwitz determinant condition if $\det(\sum e_iX_i;\mathrm{Lie}(A)) = f_h(X)$ in $\mathcal{O}_S[X]$. Now we can formulate the moduli problems:

Let $\mathcal{A}^{O_B}_{g,d,\Phi(h)}$ be the moduli stack over $\operatorname{Spec} O_E$ whose objects are (S, A, λ, ι), where:

- S is a locally noetherian O_E-scheme.
- A is an abelian scheme of dimension g over S.
- $\lambda : A \to A^t$ is a polarization with $\deg \lambda = d^2$.
- $\iota : O_B \to \operatorname{End}_S(A)$ is a ring mono-morphism such that $\lambda\, \iota(b^*) = \iota(b)^t \lambda$.
- $\operatorname{Lie}(A)_S$ satisfies the Kottwitz determinant condition defined by $\Phi(h)$.

The morphisms between two objects are isomorphisms as polarized O_B-abelian schemes over S. Let $\mathcal{A}^{O_B}_{g,d}$ be the moduli stack over $\operatorname{Spec}\mathbb{Z}$ parameterizing polarized O_B-abelian schemes $(A, \lambda, \iota)_S$ of relative dimension g with $\deg \lambda = d^2$.

Proposition 1.1. *(1) The forgetful morphism $\mathcal{A}^{O_B}_{g,d} \to \mathcal{A}_{g,d}$ over $\operatorname{Spec}\mathbb{Z}$ is representable, finite and unramified.*

(2)The forgetful morphism $\mathcal{A}^{O_B}_{g,d,\Phi(h)} \to \mathcal{A}^{O_B}_{g,d}$ over $\operatorname{Spec}\mathcal{O}_E$ is representable and closed immersion.

Therefore $\mathcal{A}^{O_B}_{g,d,\Phi(h)}$ and $\mathcal{A}^{O_B}_{g,d}$ are separated (Deligne-Mumford) algebraic stacks over $\operatorname{Spec} O_E$ and $\operatorname{Spec}\mathbb{Z}$, respectively.

Proof. (1) The relative representability follows from a consequence of Hilbert schemes: the functor $T \to \operatorname{End}_T(A_T)$ is representable by a disjoint union of projective schemes over S. Quasi-finiteness is due to the *following* lemma. Properness comes from the Néron mapping properties. Unramfiedness is due to the rigidity of the endomorphisms of abelian schemes.

(2) It is clear that to have the Kottwitz condition is a closed condition and the representability is standard. □

Lemma 1.2. *Let $(B, *), (E, ')$ be semi-simple $\mathbb{Q}$-algebras with positive involutions, and O_B, O_E orders which are stable under the involutions $*$ and $'$, respectively. Then the set $\operatorname{Hom}_{*-alg}(O_B, O_E)$ of the ring homomorphisms compatible with the involutions is finite.*

Proof. Let n be the number such that $n \operatorname{Tr}_{B/\mathbb{Q}}(1) = \operatorname{Tr}_{E/\mathbb{Q}}(1)$. For any $\mathbb{Q}$-algebra homomorphism $\varphi : B \to E$, we have $n \operatorname{Tr}_{B/\mathbb{Q}}(b) = \operatorname{Tr}_{E/\mathbb{Q}}(\varphi(b))$ for all $b \in B$. To see this, by tensoring with $\mathbb{C}$, we get a product of complex matrix rings mapping into a product of complex matrix rings. Since there is a unique irreducible representation of $M_n(\mathbb{C})$, up to conjugation there is only one algebra homomorphism from $M_n(\mathbb{C})$ to $M_{mn}(\mathbb{C})$, the diagonal embedding. Therefore, $\operatorname{Tr}_{E/\mathbb{Q}}(\varphi(b))$ is a multiple of $\operatorname{Tr}_{B/\mathbb{Q}}(b)$. Define a metric on $B \otimes \mathbb{R}$, $E \otimes \mathbb{R}$ by $(x, y) \to n \operatorname{Tr}_{B/\mathbb{Q}}(xy^*)$, $\operatorname{Tr}_{E/\mathbb{Q}}(xy')$, respectively. Then any $*$-homomorphism preserves the metrics.

Since any algebra homomorphism φ is determined by the image of a finite set of generators $\{\xi_i\}$, it suffices to show that the choice of each $\varphi(\xi_i)$ is finite. If φ preserves involutions, the image $\varphi(\xi_i)$ must lie in the set $O_E \cap \{x \in E \otimes \mathbb{R} \,|\, |x| = |\xi_i|\}$ which is discrete and compact, hence finite. □

We can choose an isomorphism $B \simeq \prod_i M_{n_i}(D_i)$ where D_i are division algebras with positive involutions $*_i$ such that 1) the involution $*$ induces the conjugate transpose: $(d_{kl})_i \to (d^{*_i}_{lk})_i$ and 2) O_B is sent to $\prod_i M_{n_i}(O_{D_i})$ with O_{D_i} maximal and stable under involutions. Let $(A, \lambda, \iota)_S$ be an object of $\mathcal{A}^{O_B}_{g,d,\Phi}$. Since all idempotents of O_B are fixed by the involution $*$, we can write $(A, \lambda, \iota)_S = \prod_i (A_i, \lambda_i, \iota_i)^{n_i}_S$, where $(A_i, \lambda_i, \iota_i)$ is a polarized O_{D_i}-abelian scheme over S with a certain condition Φ_i on the Lie algebras. So we have $\mathcal{A}^{O_B}_{g,d,\Phi} = \prod_i \mathcal{A}^{O_{D_i}}_{g_i,d_i,\Phi_i}$. Therefore, in practice it is enough to consider that case that B is a division algebra.

Remark. 1) The assumption that the order O_B is maximal and stable under the involution $*$ is unnecessary for $\mathcal{A}^{O_B}_{g,d,\Phi}$ to be a separated algebraic stack. However, we are mainly interested in such cases. While studying the reduction of $\mathcal{A}^{O_B}_{g,d,\Phi}$ at one prime over p, we may choose a weaker assumption: $O_B \otimes \mathbb{Z}_p$ is maximal and stable under the involution $*$.

2) The generic fiber of $\mathcal{A}^{O_B}_{g,d,\Phi}$ in general is not the canonical model of a single Shimura variety of PEL type. Rather the generic fiber of $\mathcal{A}^{O_B}_{g,d,\Phi}$ is the disjoint union of a finite number of those due to elementary divisors and failure of Hasse principle. We refer the reader to the works of de Jong [dJ], Kottwitz [K], and Rapoport-Zink [RZ] for definitions and detailed discussions.

3) The generic fiber $\mathcal{A}^{O_B}_{g,d} \otimes \mathbb{Q} = \coprod_\Phi \mathcal{A}^{O_B}_{g,d,\Phi} \otimes E_\Phi$, where Φ runs through all possible representation type $\mathrm{Lie}(A)$ of polarized O_B-abelian varieties of dimension g over $\mathbb{C}$. However, it is not true that $\mathcal{A}^{O_B}_{g,d} = \coprod_\Phi \mathcal{A}^{O_B}_{g,d,\Phi}$. There are examples in characteristic p in which the Lie algebra does not satisfy any determinant condition. Moreover, such an example can form a continuous family. Therefore it is certainly not true that $\mathcal{A}^{O_B}_{g,d}$ is flat over $\operatorname{Spec}\mathbb{Z}$, because X_R is flat if and only if the generic fiber X_K is scheme-theoretically dense in X_R, where R is a DVR.

2. The classical results

Let B be a division algebra with a positive involution $*$. Denote by F the center $Z(B)$ of B if the involution $*$ is of the first kind. If the involution $*$ is of second kind, denote by K the center $Z(B)$ of B and let F be the subfield of K fixed by σ, where $\sigma \in \mathrm{Gal}(K/F)$ is the element induced by the involution $*$. It is well known that F is a totally real field and K/F is a CM extension.

According to Albert's classification [Mu], they are divided into 4 cases:

(C) Type I. $B = F$.

(C) Type II. B is a totally indefinite quaternion algebra over F. Let $\bar{x} = \mathrm{tr}\, x - x$ be the canonical involution of B. Then any positive involution has the form $a\, \bar{x}\, a^{-1}$ for some $a \in B$, $a^2 \in F$ and totally negative. We can also choose an isomorphism

$$\mathbb{R} \otimes_{\mathbb{Q}} B \simeq M_2(\mathbb{R}) \times \cdots \times M_2(\mathbb{R})$$

such that the involution extends to the product of transpositions on the right side.

(D) Type III. B is a totally definite quaternion algebra over F. The canonical involution $x \to \mathrm{tr}\, x - x$ is positive and the unique positive one. There is an isomorphism

$$\mathbb{R} \otimes_{\mathbb{Q}} B \simeq \mathbb{H} \times \cdots \times \mathbb{H}$$

carrying the involution to the product of the standard involutions of $\mathbb{H}$.

(A) Type IV. B is a central simple algebra over K and it will satisfy the following properties

(i) if v is a finite place fixed by σ, $\mathrm{Inv}_v(B) = 0$, and

(ii) for any finite place v of K, $\mathrm{Inv}_v(B) + \mathrm{Inv}_{\sigma v}(B) = 0$.

For any positive involution $'$, there is an isomorphism

$$\mathbb{R} \otimes_{\mathbb{Q}} B \simeq M_q(\mathbb{C}) \times \cdots \times M_q(\mathbb{C})$$

which carries $'$ to the product of the complex conjugate transpositions. If $*$ is another positive involution, then $x^* = a\,x'\,a^{-1}$ for some a in B whose image in the right side is $(A_1, \dots, A_f)$ with A_i Hermitian positive definite.

Let V be as in Section 1. Let $2g := \dim_{\mathbb{Q}} V$, $f := [F : \mathbb{Q}]$, $q^2 := [B : K]$. Let r be the number in the following cases:

$$r := \begin{cases} \frac{g}{f} & \text{(Type I)} \\ \frac{g}{2f} & \text{(Type II or III)} \\ \frac{g}{q^2 f} & \text{(Type IV)} \end{cases}.$$

From [S], we know that for Type C or D, V_1 is a multiple of the reduced representation of B over $\mathbb{Q}$. For Type A, there are a few possibilities for V_1: suppose that $\{\tau_1, \dots, \tau_f\}$ is a CM type of K and let $n_i, \bar{n}_i$ be the multiplicity of the standard representation of $B \otimes_{K,\tau_i} \mathbb{C}$ and $B \otimes_{K,\sigma\tau_i} \mathbb{C}$ in V_1, respectively. Then $n_i + \bar{n}_i = qr$ for all $i = 1, \dots, f$.

We have the corresponding Hermitian symmetric domains:

Type C $\mathfrak{S} = \mathfrak{H}_r \times \cdots \times \mathfrak{H}_r$. (f copies)
Type D $\mathfrak{S} = \mathfrak{H}'_r \times \cdots \times \mathfrak{H}'_r$. (f copies)
Type A $\mathfrak{S} = \mathfrak{H}_{n_1,\bar{n}_1} \times \cdots \times \mathfrak{H}_{n_f,\bar{n}_f}$.

where $\mathfrak{H}_r$ is the Siegel space of degree r; $\mathfrak{H}'_r$ is the space of all complex alternating matrices z of size r such that $1 - \bar{z}^t z$ is positive hermitian; and $\mathfrak{H}_{r,s}$ is the space of complex matrices z with r rows and s columns such that $1 - \bar{z}^t z$ is positive hermitian.

We conclude that

1. For Type C or D, there is only one Kottwitz determinant condition on Lie algebras, so $E = \mathbb{Q}$. For Type A, all the possible Kottwitz determinant conditions on Lie algebras correspond to the set $\{\, (n_i, \bar{n}_i) \,|\, n_i + \bar{n}_i = rq,\ i = 1 \dots f \,\}$ and E is the reflex field of these pairs.
2. The dimension of $\mathcal{A}^{O_B}_{g,d,\Phi} \otimes \mathbb{C}$ is (C) $\frac{1}{2} r(r+1)f$, (D) $\frac{1}{2} r(r-1)f$, and (A) $\sum_{i=1}^{f} n_i \bar{n}_i$.

Classically, the moduli space attached to a division algebra of Type I is called a Hilbert-Siegel modular variety. Quaternionic Shimura varieties of PEL type are corresponding to the division algebras of Type II or Type III. Picard modular varieties are of Type IV.

3. Preparation on Dieudonné modules

In this section, k will be a perfect field of characteristic $p > 0$ containing the residue fields of O_K (over p). Denote by $W := W(k)$ the Witt vectors over k and $B(k)$ its fraction field. Let $W(1) := \mathbf{D}^*(\hat{G}_m)$, where $\mathbf{D}^*$ is the classical contravariant Dieudonné functor. For any abelian variety A over k, define $\mathbf{D}_*(A) := \mathrm{Hom}_W(\mathbf{D}^*(A[p^\infty]), W(1))$. This is not usual covariant Dieudonné module defined by p-typical curves in Cartier theory, e.g. in [Z]. The reason we do this linear dual simply is that it is easier to handle when more and more structure comes in. For example, the classical contravariant Dieudonné functor will give us a Dieudonné module with (left) $O_{B^{\mathrm{opp}}}$ action instead of O_B action, also the polarization will induces an pairing on M^t (weight -1) instead of on M. Keep in mind that we still have the crystalline theory, it just behaves in the covariant way instead. For example, we have $\mathbf{D}_*(A)\otimes R = H_1^{cris}(A/R)$ for any nilpotent PD-extension R of k.

We shall classify the Dieudonné module attached to a polarized O_B-abelian variety by putting it in a normal form. Assume that B is a division algebra, *p is unramified in* B and $p \neq 2$ if it is of type D. To each polarized O_B-abelian variety (A, λ, ι) over k, we associate a quasi-polarized $O_B \otimes \mathbb{Z}_p$-Dieudonné module M. Let $v_1, v_2, \dots, v_s$ be the primes of F lying over p. Then $M = \oplus M_v$, where the M_v's are quasi-polarized $O_{B,v}$-Dieudonné modules. Thus we can focus on each piece, drop the subscript v, and replace $O_B, O_K, O_F, \dots$ by their completions at v. Redefine g, f, q, r as in Section 2.

Fix an embedding $\nu : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$. We index the set of embeddings $\{\sigma_i\} = \mathrm{Hom}(O_F, W)$ in a way that $\sigma_{i+1} = \sigma\sigma_i$, for $i \in \mathbb{Z}/f\mathbb{Z}$, where σ is the Frobenius map on W. The chosen embeddings $\{\tau_i\}$ of the global CM field, denoted by K as in Section 2, into $\bar{\mathbb{Q}}$ associates a set of embeddings (again denoted by) $\{\tau_i\} \subset \mathrm{Hom}(O_K, W)$ with respect to ν, if we are in the case of type A.

Let $\{\tau_i\} \subset \mathrm{Hom}(O_K, W)$ be embeddings associated to the chosen CM type $\{\tau_i\}$ (in Section 2), by abuse of language, with respect to ν. Set $M^i := \{x \in M \,|ax = \sigma_i(a)x, a \in O_F\}$ and $N^i, \bar{N}^i$ the eigenspaces on which O_K acts via $\tau_i, \bar{\tau}_i = \sigma^f\tau_i$, resp. Corresponding to the classical classification of the division algebra B, we have the following cases:

(C) (I) $O_B = O_F$, $M = \oplus M^i$ is a quasi-polarized O_F-Dieudonné module such that
- $F : M^i \to M^{i+1}$, $\;V : M^i \to M^{i-1}$, $(M^f = M^0,\; M^{-1} = M^{f-1})$
- $\langle M^i, M^j\rangle = 0$ if $i \neq j$, and
- $\langle ax, y\rangle = \langle x, ay\rangle \;\; \forall x, y \in M,\; a \in O_F$.

(II) $O_B \simeq M_2(O_F)$, the involution $* : (a_{ij}) \to (a_{ji})$, $M = \oplus M^i$. Everything is the same as (I) except

$$\langle ax, y\rangle = \langle x, a^* y\rangle \;\; \forall\, x, y \in M,\; a \in O_B.$$

(D) $O_B \simeq M_2(O_F)$, $*$ is the standard adjoint involution and everything is the same as (II).

(A) $O_B \simeq M_q(O_K)$, the involution $* : (a_{ij}) \to (\bar{a}_{ji})$, $M = \oplus(N^i \oplus \bar{N}^i)$. The nonzero pairing only occurs in $\langle N^i, \bar{N}^i\rangle$ for all i. The operators F and V act on M of degree 1 and -1, respectively. We have 2 cases:
(split) $O_K \simeq W(\mathbb{F}_{p^f})^2$; $N^f = N^0$, $\bar{N}^f = \bar{N}^0$ $N^{-1} = N^{f-1}$, $\bar{N}^{-1} = \bar{N}^{f-1}$.
(inert) $O_K \simeq W(\mathbb{F}_{p^{2f}})$; $N^f = \bar{N}^0$, $\bar{N}^f = N^0$, $N^{-1} = \bar{N}^{f-1}$, $\bar{N}^{-1} = N^{f-1}$.

Observe that the idempotents of O_B in type A and C are fixed by the involution $*$. Using Morita's equivalence, it reduces to considering O_F or O_K-Dieudonné modules. For type D, consider the symmetric form $\langle x, y\rangle' := \langle x, Cy\rangle$, where C is the Weyl element. Now Morita's equivalence works well for this symmetric pairing and it reduces to consider anti-polarized O_F-Dieudonné modules. Combining with the determinant condition on the Lie algebra, we conclude that (reduced version):

(C) $M = \oplus M^i$ is a $\mathbb{Z}/f\mathbb{Z}$-graded quasi-polarized O_F-Dieudonné module with $\mathrm{rank}_W M^i = 2r$ and $\dim_k (M/VM)^i = r$.

(D) $M = \oplus M^i$ is a $\mathbb{Z}/f\mathbb{Z}$-graded quasi-anti-polarized O_F-Dieudonné module with $\mathrm{rank}_W M^i = 2r$ and $\dim_k (M/VM)^i = r$.

(A) (split) $M = \oplus(N^i \oplus \bar{N}^i)$ is a $\mathbb{Z}/f\mathbb{Z}$-graded quasi-polarized O_K-Dieudonné module with $\mathrm{rank}_W N^i = \mathrm{rank}_W \bar{N}^i = qr$ and $\dim_k (N/VN)^i = n_i$ (and with bar). The nonzero pairing only occurs in $\langle N^i, \bar{N}^i\rangle$.

(inert) $M = \oplus(N^i \oplus \bar{N}^i)$ is a $\mathbb{Z}/2f\mathbb{Z}$-graded quasi-polarized O_K-Dieudonné module with $\mathrm{rank}_W N^i = \mathrm{rank}_W \bar{N}^i = qr$ and $\dim_k (N/VN)^i = n_i$. The nonzero pairing only occurs in $\langle N^i, \bar{N}^i \rangle$.

We will do normal forms for type C and A together, and then deal with the case type D separately.

Lemma 3.1. *Let M be W-modules of rank $2g$ with a non-degenerate alternating form $\langle \cdot, \cdot \rangle$, and N a W-submodule of M with $pM \subset N$. Then there exist W-bases $\{e_i\}, \{f_i\}$ for N, M respectively such that*

- $e_i \in \{f_i, p\, f_i\}$.
- $\langle e_i, e_j \rangle \neq 0 \iff |i - j| = g$.

Proof. Set $\mathrm{ord}\langle x, y \rangle := \mathrm{ord}_p \langle x, y \rangle$ and $\mathrm{ord}\langle S_1, S_2 \rangle := \min_{x \in S_1, y \in S_2} \mathrm{ord}\langle x, y \rangle$, and denote by $[x, y]$ the W-submodule generated by x, y. Note that

$$M = [x, y] \oplus [x, y]^\perp \iff \mathrm{ord}\langle x, y \rangle = \mathrm{ord}\langle x, M \rangle = \mathrm{ord}\langle M, y \rangle.$$

Choose $f_1, f_{g+1} \in M$ s.t. $\mathrm{ord}\langle f_1, f_{g+1} \rangle = \mathrm{ord}\langle M, M \rangle$. We have three cases:

(i) If $f_1 \in N, f_{g+1} \in N$, then set $e_1 = f_1, e_{g+1} = f_{g+1}$. We have

$$\mathrm{ord}\langle e_1, e_{g+1} \rangle = \mathrm{ord}\langle N, N \rangle$$

this implies

$$N = [e_1, e_{g+1}] \oplus [e_1, e_{g+1}]^\perp, \ M = [f_1, f_{g+1}] \oplus [f_1, f_{g+1}]^\perp$$

and

$$p\,[f_1, f_{g+1}]^\perp \subset [e_1, e_{g+1}]^\perp.$$

This reduces to the lower rank case and by induction one can find symplectic bases.

(ii) If $f_1 \in N,\ f_{g+1} \notin N$, then set $e_1 := f_1, e_{g+1} := p\, f_{g+1}$. If $\mathrm{ord}\langle e_1, e_{g+1} \rangle = \mathrm{ord}\langle e_1, N \rangle = \mathrm{ord}\langle e_{g+1}, N \rangle$, then one can decompose N, M simultaneously as in case (i). Suppose not.

(a) If $\exists\, e_1' \in N$ s.t. $\mathrm{ord}\langle e_1', e_{g+1} \rangle < \mathrm{ord}\langle e_1, e_{g+1} \rangle$, then

$$\mathrm{ord}\langle e_1', f_{g+1} \rangle < \mathrm{ord}\langle f_1, f_{g+1} \rangle, \text{ contradiction.}$$

(b) If $\exists\, e_{g+1}' \in N$ s.t. $\mathrm{ord}\langle e_1, e_{g+1}' \rangle < \mathrm{ord}\langle e_1, e_{g+1} \rangle$, then

$$\mathrm{ord}\langle e_1, e_{g+1}' \rangle = \mathrm{ord}\langle f_1, f_{g+1} \rangle. \quad \because \mathrm{ord}\langle f_1, f_{g+1} \rangle \le \mathrm{ord}\langle e_1, e_{g+1}' \rangle$$

Replacing f_{g+1} by e_{g+1}', we are back to case (i).

(iii) If $f_1 \notin N,\ f_{g+1} \notin N$, then set $e_1 := p\, f_1, e_{g+1} := p\, f_{g+1}$. If $\mathrm{ord}\langle e_1, e_{g+1} \rangle = \mathrm{ord}\langle e_1, N \rangle = \mathrm{ord}\langle e_{g+1}, N \rangle$, then one can decompose N, M simultaneously as in case (i). Suppose not. Then $\exists\, e_1' \in N$ s.t.

$$\begin{aligned} & \mathrm{ord}\langle e_1', e_{g+1} \rangle < \mathrm{ord}\langle e_1, e_{g+1} \rangle \\ \Rightarrow\ & \mathrm{ord}\langle e_1', f_{g+1} \rangle + 1 < \mathrm{ord}\langle f_1, f_{g+1} \rangle + 2 \\ \Rightarrow\ & \mathrm{ord}\langle e_1', f_{g+1} \rangle \le \mathrm{ord}\langle f_1, f_{g+1} \rangle \\ \Rightarrow\ & \mathrm{ord}\langle e_1', f_{g+1} \rangle = \mathrm{ord}\langle f_1, f_{g+1} \rangle. \end{aligned}$$

Replacing f_1 by e_1', we are back to case (ii).

□

Apply this lemma to $N = FM^{i-1},\ M = M^i$ in the case type C and A (in the reduced version). We get normal forms:

Corollary 3.2.
Type C. There exist 2 W-bases $\{e_k^i\},\ \{f_k^i\}$ of M^i for each $i = 1, \dots, f$ such that

- $Fe_k^i \in \{f_k^{i+1}, pf_k^{i+1}\}$.
- $\{e_k^i, e_{r+k}^i\}_{k=1\dots r}$ *is a symplectic basis for* M^i.

Type A. There exist 2 W-bases $\{e_k^i\},\ \{f_k^i\}$ of N^i and $\{\bar{e}_k^i\},\ \{\bar{f}_k^i\}$ of $\bar{N}^i$ for each $i = 1, \dots, f$ such that

- $Fe_k^i \in \{f_k^{i+1}, pf_k^{i+1}\},\ F\bar{e}_k^i \in \{\bar{f}_k^{i+1}, p\bar{f}_k^{i+1}\}$.
- $\{e_k^i, \bar{e}_k^i\}_{k=1\dots qr}$ *is a symplectic basis for* $N^i \oplus \bar{N}^i$.

Now we are in type D. For a W-module M, denote by $[e_1, e_2, \dots, e_n]$ the W-submodule generated by the elements $\{e_i\}$.

Lemma 3.3. *Let M be a W-module with a non-degenerate symmetric form $\langle\cdot,\cdot\rangle$, N a W-submodule of M such that $pM \subset N$. Then we can decompose M and N w.r.t. $\langle\cdot,\cdot\rangle$:*

$$M = [f_1, f_{s+1}] \oplus [f_2, f_{s+2}] \oplus \cdots \oplus [f_{2s+1}] \oplus \cdots \oplus [f_n]$$

$$N = [e_1, e_{s+1}] \oplus [e_2, e_{s+2}] \oplus \cdots \oplus [e_{2s+1}] \oplus \cdots \oplus [e_n]$$

such that

- $e_k \in \{f_k, pf_k\}$ *for* $k \geq 2s+1$; $(e_k, e_{s+k}) = (pf_k, f_{s+k})$ *for* $k \leq s$;
- $\langle e_k, e_k\rangle = \langle e_{s+k}, e_{s+k}\rangle = 0$ *for* $k \leq s$.

Proof. We may assume that $\operatorname{ord}\langle N, N\rangle > \operatorname{ord}\langle M, M\rangle =: n$, otherwise there exists $e_1 \in N$ s.t. $\operatorname{ord}\langle e_1, e_1\rangle = n$, then $M = [e_1] \oplus [e_1]^\perp,\ N = [e_1] \oplus [e_1]^\perp$ and we are done.

Let $f_1 \in M$ s.t. $\operatorname{ord}\langle f_1, f_1\rangle = n$, and set $e_1 := pf_1$. If $\operatorname{ord}\langle e_1, e_1\rangle = \operatorname{ord}\langle e_1, N\rangle$, then $M = [f_1] \oplus [f_1]^\perp,\ N = [e_1] \oplus [e_1]^\perp$ and we are done.

Suppose not, i.e. $\exists\, e_2 \in N$, s.t. $\operatorname{ord}\langle e_1, e_2\rangle < \operatorname{ord}\langle e_1, e_1\rangle$, then $\operatorname{ord}\langle e_1, e_2\rangle$=1+n. Set $f_2 := e_2$, then we have $\operatorname{ord}\langle f_1, f_2\rangle = n$, which implies that $\operatorname{ord}\langle e_1, e_2\rangle = 1 + n = \operatorname{ord}\langle N, N\rangle$. Therefore, $M = [f_1, f_2] \oplus [f_1, f_2]^\perp,\ N = [e_1, e_2] \oplus [e_1, e_2]^\perp$. Moreover, the matrix $(\langle f_i, f_j\rangle)$ has the form

$$\begin{pmatrix} p^n u_1 & p^n \\ p^n & p^{n+b} u_2 \end{pmatrix},$$

but we may assume that it is $\begin{pmatrix} 0 & p^n \\ p^n & 0 \end{pmatrix}$ by the next lemma. This completes the proof. $\square$

Lemma 3.4. *Suppose $\{x_1, x_2\}$ is a basis of M such that the matrix $(\langle x_i, x_j\rangle)$ is*

$$\begin{pmatrix} p^n u_1 & p^n \\ p^n & p^{n+b} u_2 \end{pmatrix},$$

where the u_i's are units in W and b is a positive integer. Then we can choose a basis $\{y_1, y_2\}$ of M such that

- *the subspace spanned by $\bar{y}_2$ and that spanned by $\bar{x}_2$ in M/pM are the same, and*
- *the matrix $(\langle y_i, y_j\rangle)$ is* $\begin{pmatrix} 0 & p^n \\ p^n & 0 \end{pmatrix}$.

Proof. Set $y_2 := x_2 + tx_1$, then the condition for $\langle y_2, y_2\rangle = 0$ is

$$t^2u_1 + 2t + p^bu_2 = 0. \tag{1}$$

Let $s := tu_1$ and $u := u_1u_2$. One yields

$$s^2 + 2s + p^bu = 0 \tag{2}$$

By Hensel's Lemma, s has a solution in pW to (2), and so as t to (1).

Set $y_1 := x_1 + ay_2$, then we can choose a in W such that $\langle y_1, y_1\rangle = 0$. Modifying y_1 by a multiple of y_1, one may assume $\langle y_1, y_2\rangle = p^n$. We are done. □

Apply Lemma 3.3 to $N = FM^{i-1}$, $M = M^i$. We have a normal form for type D (in the reduced version).

Corollary 3.5.
Type D. There are an integer s_i and 2 W-bases $\{e^i_k\}$, $\{f^i_k\}$ of M^i for each $i = 1\dots f$ such that

- $Fe^i_k \in \{f^{i+1}_k, pf^{i+1}_k\}$ *for* $k > 2s_i$;
- $(Fe^i_k, Fe^i_{s_i+k}) = (f^{i+1}_k, pf^{i+1}_{s_i+k})$ *for* $k \le s_i$;
- $\langle e^i_j, e^i_k\rangle \ne 0 \iff (j = k > 2s_i)$ *or* $(j, k < 2s_i$ *and* $|j-k| = s_i)$.

4. On Liftings and Deformations

In this section, we use Grothendieck-Messing's deformation theory to lift polarized O_B-abelian varieties and to investigate their local moduli spaces. Let (B, O_B, E, O_E, Φ) be the setting in Section 1, Δ be the product of rational primes at which B is ramified. The ground field k is perfect in characteristic $p > 0$. Assume that p *does not divide* Δ and $p \ne 2$ if B contains a factor of type D. See Theorem 4.1 and 4.5 for the main results.

Theorem 4.1. *Let k be an algebraically closed field of characteristic $p > 2$. The reduction map $\mathcal{A}^{O_B}_{g,d,\Phi}(W[\sqrt{p}]) \to \mathcal{A}^{O_B}_{g,d,\Phi}(k)$ is surjective.*

Proof. Let (A, λ, ι) be a point of $\mathcal{A}^{O_B}_{g,d,\Phi}(k)$ and M its covariant Fontaine's Dieudonné module, the linear dual of the Dieudonné module in the sense of Fontaine (see [F], p.218; also [N], sec. 1). In order to lift (A, λ, ι) to $R := W[\sqrt{p}]$, we need to construct a free direct summand Fil of $M \otimes_W R$, as an R-module, such that

- $Fil \otimes R/\mathfrak{m}_R = VM/pM$;
- Fil is stable under the action of O_B;
- $\langle Fil, Fil\rangle = 0$.

Since p is unramified in B, we only need to consider the reduced version of types A,C and D. Note that $H_1(A/R)$ and any such an 'admissible' lifting Fil have natural decompositions H^i and Fil^i into eigenspaces with respect to the embeddings of O_F into W. Any lifting of $(VM/pM)^i$ in H^i will be stable under the action of O_F or O_K, and $\langle H^i, H^j\rangle = 0$ if $i \ne j$.With these, it is enough to construct Fil^i in H^i for each i such that

- $Fil^i \otimes R/\mathfrak{m}_R = (VM/pM)^i$;
- $\langle Fil^i, Fil^i\rangle = 0$.

Therefore we may drop the index i.

By the corollaries in Section 3, we find a solution for Fil by cases:

Type A and C. M is generated, as a W-module, by $\{X_i, Y_i\}_{i=1..s}$, $\{X_i, X_i', Y_i, Y_i'\}_{i=s+1..s+t}$ (W-basis) s.t.

- $Y_i, Y_i' \in VM$ for all i;
- $\begin{cases} \langle X_i, Y_i\rangle = p^{a_i} & \text{for } 1 \le i \le s. \\ \langle Y_i, Y_i'\rangle = p\langle X_i, X_i'\rangle = p^{a_i+1} & \text{for } i > s. \\ \text{all other pairings are zero.} \end{cases}$

Take Fil to be the R-submodule of $M \otimes R$ generated by $\{Y_i\}_{i=1..s}$, $\{Y_i + \pi X_i, Y_i' - \pi X_i'\}_{i=s+1..s+t}$, where $\pi := \sqrt{p}$.

Type D. M is generated, as a W-module, by $\{X_i, Y_i\}_{i=1..s}$, $\{X_i, Y_i\}_{i=s+1..s+t}$ s.t.

- $Y_i \in VM$ for all i;
- $\begin{cases} \langle X_i, Y_i\rangle = p^{a_i} & \text{for } 1 \le i \le s. \\ \langle Y_i, Y_i\rangle = -p\langle X_i, X_i\rangle = p^{a_i+1} & \text{for } i > s. \\ \text{all other pairings are zero.} \end{cases}$

Take Fil to be the R-submodule of $M \otimes R$ generated by $\{Y_i\}_{i=1..s}$, $\{Y_i + \pi X_i\}_{i=s+1..s+t}$.

This completes the proof. $\square$

Note that we only need k to be perfect in type A and C. In type D one needs some finite extension of k to have such a normal form. We make this stronger assumption of k to make the result simpler. We also need to use Fontaine's theory in the proof since Grothendieck-Messing's theory requires *topologically nilpotent* PD-structure on the base ring R, which limits $p > e+1=3$.

Lemma 4.2. *Let $(A_0, \lambda_0, \iota_0)$ be a polarized O_B-abelian variety over k (not necessary admitting a determinant condition defined by some Φ). The deformation functor $D := \mathrm{Def}[A_0, \iota_0]$ (without the polarization) is pro-presentable by $W[[x_1, \dots, x_t]]$, where t is the dimension of the k-vector space $\mathrm{Hom}_{O_B}(H^0(A_0^t, \Omega^1_{A_0^t}), H^1(A_0^t, \mathcal{O}_{A_0^t}))$.*

Proof. D is pro-representable because $\mathrm{Def}[A_0]$ is pro-representable. We identify $M/pM := \mathbf{D}_*(A_0) \otimes k$ with $H^1_{DR}(A_o^t/k)$, then $VM/pM = H^0(A_0^t, \Omega^1_{A_0^t})$ and $M/VM = H^1(A_o^t, \mathcal{O}_{A_o^t})$. Then one can see that $D(k[\epsilon]) = \mathrm{Hom}_{O_B}(VM/pM, M/VM) = Hom_{O_B}(H^0(\Omega^1_{A_0^t}), H^1(\mathcal{O}_{A_0^t}))$.

The smoothness of D is clear and left to the reader. $\square$

We need some notation to state the following proposition. Let V be a k-vector space with a k-linear O_B action. Denote by $\bigwedge^2_{O_B} V$ the k-vector space of $\bigwedge^2_k V$ modulo the subspace spanned by elements $(b\,x \wedge y - x \wedge b^* y)$ for all $b \in O_B$, x, $y \in V$.

Proposition 4.3. *Let $(A_0, \lambda_0, \iota_0)$ be a polarized O_B-abelian variety over k. Let Obs be the space $\bigwedge^2_{O_B}(VM/pM)$, where M is the covariant Dieudonné module of A_0. Let $\mathcal{O} := W[[x_1, \dots, x_t]]$ and $o := \dim_k Obs$. Then there exist elements $a_1, \dots, a_o$ of $\mathcal{O}$ such that $\mathrm{Def}[A_0, \lambda_0, \iota_0]$ is pro-representable by $\mathrm{Spf}\, \mathcal{O}/(a_1, \dots, a_o)$.*

Proof. First of all, we may assume that B is a division algebra by the same idempotent reduction in Section 1.

By the rigidity on the endomorphisms of abelian varieties, hence on the polarizations, $\mathrm{Def}[A_0, \lambda_0, \iota_0]$ is pro-representable by a closed formal subscheme of $\mathrm{Spf}\, \mathcal{O}$, say $\mathrm{Spf}\, \mathcal{O}/\mathfrak{a}$.

By Artin-Rees lemma, there exists an $n>0$ such that

$$\mathfrak{m}^n\cap\mathfrak{a}=\mathfrak{m}(\mathfrak{m}^{n-1}\cap\mathfrak{a})=\mathfrak{m}^n\cap\mathfrak{m}\mathfrak{a}$$

then we have a short exact sequence

$$0\to I\to R\to R'\to 0$$

where $R:=\mathcal{O}/\mathfrak{m}\mathfrak{a}+\mathfrak{m}^n$, $R':=\mathcal{O}/\mathfrak{a}+\mathfrak{m}^n$ and $I:=\mathfrak{a}/\mathfrak{m}\mathfrak{a}$ is an ideal of R with $I^2=0$, $\mathfrak{m}I=0$.

The canonical homomorphisms

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & \mathcal{O}/\mathfrak{a} \\ \downarrow & & \downarrow \\ R & \longrightarrow & R' \end{array}$$

give a O_B-abelian scheme (A',ι') over R' and a polarized O_B-abelian scheme (A,λ,ι) over R along with $(A,\iota)\otimes R'\simeq(A',\iota')$. We apply Morita's equivalence on $H_1^{dR}(A/R)$ and $H_1^{dR}(A'/R')$ and reduce to modules with O_K or O_F action. Denote by Fil and $H_1(R)$ (resp. Fil' and $H_1(R')$) be the free R (resp. R')-modules corresponding to $\mathrm{Lie}(A)_R$ and $H_1^{dR}(A/R)$ (resp. $\mathrm{Lie}(A)'_R$ and $H_1^{DR}(A'/R')$). We have natural isomorphism $H_1(R)\otimes R'\simeq H_1(R')$, and the polarization λ' induces a symplectic (resp. symmetric) form on $H_1(R)$ in types A and C (resp. in type D).

Morita's equivalence also gives

$$\dim_k\wedge^2_{O_B}(VM/pM)=\mathrm{rank}_R\wedge^2_{O_K}(Fil)\qquad\text{(A)}$$

$$\dim_k\wedge^2_{O_B}(VM/pM)=\mathrm{rank}_R\wedge^2_{O_F}(Fil)\qquad\text{(C)}$$

$$\dim_k Sym^2_{O_B}(VM/pM)=\mathrm{rank}_R Sym^2_{O_F}(Fil)\quad\text{(D)}$$

and the right hand sides are

$$\mathrm{rank}_R\oplus_{i=1}^f Fil^i\otimes_R\overline{Fil}^i\quad\text{(A)}$$

$$\mathrm{rank}_R\oplus_{i=1}^f\wedge^2_R Fil^i\qquad\text{(C)}$$

$$\mathrm{rank}_R\oplus_{i=1}^f Sym^2_R Fil^i\quad\text{(D)}$$

(A) Let $\{x_1^i,x_2^i,\dots,x_{n_i}^i\}$ and $\{\bar{x}_1^i,\bar{x}_2^i,\dots,\bar{x}_{\bar{n}_i}^i\}$ be R-bases of Fil^i and $\overline{Fil}^i$ resp.
(CD) Let $\{x_1,x_2,\dots,x_{r_i}\}$ be a R-basis of Fil^i.

Set

$$\bar{b}^i_{jk}:=\begin{cases}\langle x_j^i,\bar{x}_k^i\rangle & \text{(A)}\\ \langle x_j^i,x_k^i\rangle & \text{(CD)}\end{cases}$$

Since $\langle Fil',Fil'\rangle=0$ in R', we have $\bar{b}^i_{jk}\in I$. Let b^i_{jk} be an element of $\mathfrak{a}$ such that $b^i_{jk}\equiv\bar{b}^i_{jk}$ mod $\mathfrak{m}\mathfrak{a}$ for all i,j,k, and $\mathfrak{b}$ be the ideal of $\mathcal{O}$ generated by all b^i_{jk}.

Set $R'':=\mathcal{O}/\mathfrak{b}+\mathfrak{m}\mathfrak{a}+\mathfrak{m}^n$. By the construction of R'', $\langle Fil,Fil\rangle=0$ in R''. Hence the polarization λ' lifts to R'' and there is a map $\mathcal{O}/\mathfrak{a}\to R''$. This yields $\mathfrak{a}\subset\mathfrak{b}+\mathfrak{m}\mathfrak{a}+\mathfrak{m}^n$, therefore $\mathfrak{b}=\mathfrak{a}$. This completes the proof. $\square$

Lemma 4.4. *Let $(R, K, k = R/(\pi))$ be a complete discrete valuation ring of mixed characteristic $(0, p)$. Let A be a finitely generated commutative R-algebra.*
(1) If for any homomorphism $f : A \to L$ with L algebraically closed of characteristic p, there exists a lifting $\tilde{f} : A \to W(L)[\sqrt{p}]$ such that the following diagram commutes

$$\begin{tikzcd} & W(L)[\sqrt{p}] \arrow[d] \\ A \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & L \end{tikzcd}$$

Then $\dim A \otimes \kappa = \dim A \otimes K$.
(2) If $B = R[[X_1, \ldots, X_t]]/(a_1, \ldots, a_o)$ and $\dim B \otimes k = t - o$, then B is flat over R and is a complete intersection relative to R ([EGA],19.3.6).

Proof. (1) Let C be the image of the map $A \to A \otimes K$. The closed subscheme $\operatorname{Spec} C$ is the scheme-theoretic image of the morphism $j : \operatorname{Spec} A \otimes K \to \operatorname{Spec} A$, hence C is flat over R and the underlying topological space of $\operatorname{Spec} C$ is the Zariski closure of the image of j. We will show that the image of j is a Zariski dense open subset of $\operatorname{Spec} A$. Then $\operatorname{Spec} C \to \operatorname{Spec} A$ is a homeomorphism, hence $\dim A \otimes k = \dim C \otimes k = \dim C \otimes K = \dim A \otimes K$.

For every $\mathfrak{p} \in \operatorname{Spec} A \otimes k$, we have the following diagram

$$\begin{tikzcd} & R' \arrow[r] \arrow[d] & W(\overline{k(\mathfrak{p})})[\sqrt{p}] \arrow[d] \\ A \arrow[ur, "\tilde{\pi}"] \arrow[urr, "\tilde{\pi}"] \arrow[r, "\pi"'] & A/\mathfrak{p} \arrow[r] & \overline{k(\mathfrak{p})} \end{tikzcd}$$

where R' is the image of the lifting $\tilde{\pi} : A \to W(\overline{k(\mathfrak{p})})[\sqrt{p}]$. Let $\mathfrak{P}_K$ be the kernel of the morphism $\tilde{\pi}_K : A \otimes K \to R' \otimes K$. We have that $\mathfrak{P} := j(\mathfrak{P}_K)$ is the kernel of $\tilde{\pi}$, whose Zariski closure contains $\mathfrak{p}$, as $\mathfrak{P} \subset \mathfrak{p}$. We are done.

(2) The dimension of B is $t - o + 1$ since $\dim B \geq t - o + 1$ and $\dim B/(\pi) = t - o$. Since $R[[X_1, \ldots, X_t]]$ is Cohen-Macaulay, $\{a_1, \ldots, a_o, \pi\}$ is a regular sequence in $R[[X_1, \ldots, X_t]]$ and π is not a zero divisor of B. Therefore B is flat over R and is a complete intersection relative to R. $\square$

Theorem 4.5. *The structure morphism $\mathcal{A}^{O_B}_{g,d,\Phi} \to \operatorname{Spec} O_E[1/2\Delta]$ is flat and a complete intersection morphism ([EGA], 19.3.6).*

Proof. The problem is at least local in the base, so we may replace $O_E[1/2\Delta]$ by one of its local rings. Call this local ring R and its fraction field K. Choosing a suitable level structure, we may assume $X := \mathcal{A}^{O_B}_{g,d,n,\Phi} \otimes R$ is a scheme and may further assume that the residue field k is an algebraic closure of a finite field and R is complete by faithfully flat base change. By Theorem 4.1 and Lemma 4.4 (1), we have $\dim X_k = \dim X_K$, i.e. every geometric fiber outside 2Δ has the same dimension.

By Proposition 4.3 and Lemma 4.4 (2), it is enough to show that $\dim X_k = t - o$. We may assume that B is a division algebra. From Section 2, the dimensions of these geometric fibers are (A) $\sum_{i=1}^f n_i \bar{n}_i$, (C) $\frac{1}{2}r(r+1)f$, and (D) $\frac{1}{2}r(r-1)f$. On the other hand, we compute in the proof of Proposition 4.3 that (A) $t = \sum_{i=1}^f 2n_i\bar{n}_i$, $o = \sum_{i=1}^f n_i\bar{n}_i$; (C) $t = r^2 f$, $o = \frac{1}{2}r(r-1)f$; and (D) $t = r^2 f$, $o = \frac{1}{2}r(r+1)f$. One verifies that $\dim X_k = t - o$ in each case. This completes the proof. $\square$

Remark. Proposition 4.3 is inspired by a theorem of Mumford ([O], Theorem 2.3.3). We modified the proof there by the crystalline theory. The original proof could work by describing the space of the obstruction classes of the polarization.

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