An Equality of Cusp Invariants and Cusp Contributions to the Dimension Formula

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In this paper we prove an equality between two invariants of isolated cusp singularity (V, p) of even dimension n, namely, the contribution of the cusp $\chi_\infty(p)$ to the arithmetic genus and the signature defect $\sigma(p)$ of the cusp.

THEOREM. *When $n = 2k \geq 2$, we have*

$$2^n \chi_\infty(p) = \sigma(p).$$

The signature defect was defined by Hirzebruch[**H1**], which coincides with ours (see Section 1) in the case of Hilbert modular cusps. We can also find other generalizations by Morita[**Mo**] and Looijenga[**L**]. The invariant $\chi_\infty(p)$ of a Hilbert modular cusp is called the ψ-invariant by Ehlers[**E**]. Satake[**Sa**] defined $\chi_\infty(p)$ in general as the contribution of the cusp to the dimension formula of the space of cusp forms by means of Riemann-Roch Theorem.

The theorem above has been conjectured by Satake and Ogata (the cojecture C1 in [**SO**]), while in the case of Hibert modular cusps by Ehlers[E] (see also Kommentare **51** in [**H2**]).

Theorem combinning with the result of Atiyah, Donnelly and Singer [**ADS1**][**ADS2**] and Müller[**Mu**] gives a proof of the conjecture stated in p.95 [**HG**], which is generalized in [**SO**] as the conjecture(C3'). The equivalent conjecture(C3) for odd dimension was proved by Ogata [**O**]. We shall explain the original conjecture. Let (V, p) be a Hilbert modular cusp, that is, $V - \{p\}$ is isomorphic to the quotient $\mathbb{H}^n / S(M, V)$ of the product of n copies of the upper half plane $\mathbf{H}$ by the semidirect product $S(M, V)$ of a fractional ideal M of a totally real number field F and a finite index subgroup V of totally positive units in F preserving M. Then the contribution of the cusp (V, p) to the dimension formula of the space of cusp forms derived from the Selberg trace formula (cf. [**Sh**]) is given by

$$w_p := (-1)^{n/2} (2\pi)^{-n} d(M) \lim_{s \to 1} L(M, V, s),$$

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where $d(M)$ is the discriminant of M and $L(M,V,s)$ is defined by

$$L(M,V,s) = \sum_{\mu \in M-\{0\}/V} \frac{\mathrm{signNorm}(\mu)}{|\mathrm{Norm}(\mu)|^s} \quad \text{for} \quad \mathrm{Re}s > 1.$$

Atiyah, Donnelly and Singer[**ADS1**][**ADS2**] and Müller[**Mu**] showed that $\sigma(p) = 2^n w_p$.

COROLLARY. *Let (V,p) be a Hilbert modular cusp. In the above notation we have*

$$\chi_\infty(p) = w_p.$$

In other words, both contributions of the cusp to the dimension formula of the space of Hilbert modular cusp forms from Riemann-Roch Theorem and from the Selberg trace formula coincide.

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§1. Invariants of Cusps.

Let (V,p) be an isolated cusp singularity of dimension n, i.e., $V-\{p\}$ is a quotient of a tube domain $\mathbb{R}^n + \sqrt{-1}C$ with a nondegenerate open convex cone $C \subset \mathbb{R}^n$ by an action of a discrete group which is a semidirect product of a lattice N in $\mathbb{R}^n$ and $\Gamma \subset \mathrm{GL}(N) \cap \mathrm{Aut}C$ (see in detail [**T**]). Let $(U,X) \to (V,p)$ be a desingularization of the cusp by using of toroidal embeddings of Mumford[**KKMS**] so that X is a divisor with simple normal crossings. Hence X can be decomposed into the union of irreducible components $X = \cup_{i\in I} X_i$, and all X_i and intersections $X_J := \cap_{j \in J} X_j$ $(J \subset I)$ are nonsingular toric varieties in the sense of Oda[**Od**]. Let $\delta_i = [X_i]$ be the cohomology class of X_i in $H^2(U,\partial U;\mathbf{Z})$. Then by definition we have

$$\chi_\infty(p) = \left[\prod_{i\in I} \frac{\delta_i}{1-e^{-\delta_i}}\right]_n [U,\partial U],$$

which is given by evaluation after expanding as a formal power series. Here $[U,\partial U]$ is the fundamental class of $(U,\partial U)$ in $H_{2n}(U,\partial U;\mathbf{Z})$. And the signature defect of (V,p) is given by

$$\sigma(p) := \left[\prod_{i\in I} \delta_i \coth \delta_i\right]_n [U,\partial U] - \mathrm{sign}(U,\partial U),$$

where $\mathrm{sign}(U, \partial U)$ is the signature of the bilinear form on $H^n(U, \partial U; \mathbb{R})$ defined by cup product $H^n(U) \times H^n(U, \partial U) \to H^{2n}(U, \partial U)$.

LEMMA 1.1. *The definition of $\chi_\infty(p)$ and $\sigma(p)$ is independent of the choice of a toroidal desingularization $(U, X) \to (V, p)$.*

PROOF: Let Z be a nonsingular projective algebraic variety containing X. Then we have

$$\begin{aligned}\chi_\infty(p) &= \mathrm{Td}(T_Z) - \mathrm{Td}(T_Z(-\log X)) \\ &= \chi(\mathcal{O}_Z) - \mathrm{Td}(T_Z(-\log X)),\end{aligned}$$

and

$$\begin{aligned}\sigma(p) &= \{L_k(T_Z) - L_k(T_Z(-\log X))\} - \{\mathrm{sign}(Z) - \mathrm{sign}(Z - U, -\partial U)\} \\ &= \mathrm{sign}(Z - U, -\partial U) - L_k(T_Z(-\log X)),\end{aligned}$$

where Td is the Todd polynomial and L_k is the Hirzebruch L-polynomial. We must show that $\mathrm{Td}(T_Z(-\log X))$ and $L_k(T_Z(-\log X))$ are independent of the choice of a resolution of singularity. Let $\pi_i : (U_i, X_i) \to (V, p)$ $(i = 1, 2)$ be two desingularizations. Then we can find a desingularization $\pi_3 : (U_3, X_3) \to (V, p)$ so that π_3 factors through both π_1 and π_2. Hence it is enough to consider a desingularization $(W, Y) \to (V, p)$ factoring $(U, X) \to (V, p)$. Set $q : (W, Y) \to (U, X)$. We claim $q^*T_U(-\log X) \cong T_W(-\log Y)$. In fact, since the question is locall, if let $(\tilde{U}, \tilde{X}) \to (U, X)$ and $(\tilde{W}, \tilde{Y}) \to (W, Y)$ be the universal coverings with the covering transformation group Γ, then we have

$$\tilde{q}^* T_{\tilde{U}}(-\log \tilde{X}) \cong T_{\tilde{W}}(-\log \tilde{Y})$$

because of the isomorphism $T_{\tilde{W}}(-\log \tilde{Y}) \cong N \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{W}}$ (see [**Od**]), where $\tilde{q} : (\tilde{W}, \tilde{Y}) \to (\tilde{U}, \tilde{X})$. Lemma follows from the functoriality of Chern classes and the claim.

Let $\triangle := \{J \subset I; J \neq \emptyset, X_J \neq \emptyset\}$ and $\tau_\infty(p) := \left[\prod_{i \in I} \delta_i \coth \delta_i\right]_n [U, \partial U]$. Then we know from [**SO**]

$$\tau_\infty(p) - 2^n \chi_\infty(p) = \sum_{j=1}^{n} (-2)^{n-j\sharp} \Delta(j), \tag{1.2}$$

where $\Delta(j) := \{J \in \Delta; |J| = j\}$ for $1 \le j \le n$ and $\sharp$ denotes the cardinality.

§2. $\mathrm{sign}(U, \partial U)$.

In this section we represent $\mathrm{sing}(U, \partial U)$ by means of Δ. For $J \in \Delta(j)$ we consider a bilinear form B_J on $H^{n-2j}(X_J; \mathbb{R})$ defined by

$$B_J(u, v) := (u \cup v \cup c_j(N_{X_J}))[X_J],$$

where $c_j(N_{X_J})$ is the j-th Chern class of the normal bundle N_{X_J} of X_J in U. We see that X_J contributes to $\mathrm{sign}(U, \partial U)$ as $\mathrm{sign} B_J$ by taking acount of the following diagram.

$$\begin{array}{ccccc} H^n(U) & \times & H^n(U, \partial U) & \to & H^{2n}(U, \partial U) \\ \downarrow & & \uparrow & & \uparrow \\ H^n(X_J) & \times & H^{n-2j}(X_J) & \to & H^{2n-2j}(X_J) \end{array}$$

PROPOSITION 2.1. *There exists a following exact sequence.*

$$0 \to H^n(X) \to \oplus_{i \in \Delta(1)} H^n(X_i) \to \oplus_{J \in \Delta(2)} H^n(X_J) \to \dots$$
$$\to \oplus_{K \in \Delta(k)} H^n(X_K) \to 0.$$

PROOF: $H^j(X)$ carries a canonical mixed Hodge structure whose weight filtration is given by the spectral sequence degenerationg at E_2-terms

$$E_1^{pq} = H^q(\coprod_{J \in \Delta(p+1)} X_J; \mathbb{R}) \Rightarrow H^{p+q}(X; \mathbb{R}).$$

We claim that the canonical mixed Hodge structure on $H^j(X)$ is pure for $j \ge n$. In fact, let Z be a nonsingular projective algebraic variety containing X and consider the following commutative diagram of mixed Hodge structures (cf. [**St**]).

$$\begin{array}{ccc} H^j_X(U) & \xrightarrow{\alpha} & H^j(U) \\ \uparrow\beta & & \uparrow\gamma \\ H^j_X(Z) & \to & H^j(Z) \end{array}$$

Here the mixed Hodge structure on $H^j(U)$ is induced by the isomorphism $H^j(U) \xrightarrow{\sim} H^j(X)$. Since β is bijective and since α is surjective for $j \ge n$ (see [**GM**]), γ is surjective for $j \ge n$. Since $H^j(Z)$ has a natural pure Hodge structure, the mixed Hodge structure on $H^j(U)$ is pure for $j \ge n$. The claim says that $E_2^{pq} = 0$ for $p + q \ge n$ and $p \ge 1$. Proposition is the case $q = n$.

PROPOSITION 2.2. *We have*

$$\mathrm{sign}(U, \partial U) = \sum_{j=1}^{k} (-1)^{j+1} \sum_{J \in \Delta(j)} \mathrm{sign} B_J.$$

PROOF: It follows from the isomorphism $H^j(U) \xrightarrow{\sim} H^j(X)$ and Proposition 2.1.

We must compute $\mathrm{sign} B_J$. Since X_J is a complete intersection, we have $c_j(N_{X_J}) = \prod_{i \in J} c_1(N_{X_i}|_{X_J})$. We note that $c_1(N_{X_i})$ is negative because $X = \cup_{i \in I} X_i$ is contracted to a point. Hence we consider a bilinear form on $H^{n-2j}(X_J)$ of the form

$$B_J(\prod_{i=1}^{j} c_1(L_i))(u, v) := (u \cup v \cup \prod_{i=1}^{j} c_1(L_i))[X_J],$$

where $L_1, \ldots, L_j$ are ample line bundles on X_J.

PROPOSITION 2.3. *In the above notation we have*

$$\mathrm{sign} B_J(\prod_{i=1}^{j} c_1(L_i)) = \mathrm{sign} B_J(c_1(L_1)^j).$$

In order to prove Proposition 2.3 we need a lemma, which is implicitly in [**CKS**].

LEMMA 2.4. *Let Y be a projective manifold and let $c_i \in H^{1,1}(Y; \mathbb{R})$ ($i = 1, \ldots, s$) so that $\sum_{i=1}^{s} \mu_i c_i$ is positive for $\mu_i > 0$. Define Kähler operators $\mathbb{L}_i$ by taking cup product with c_i, respectively. Then for $l \leq \dim Y$ the kernel of the map*

$$\mathbb{L}_1 \ldots \mathbb{L}_s : H^{l-s}(Y) \to H^{l+s}(Y)$$

is contained in $\sum_{i=1}^{s} \mathrm{Ker} \prod_{j \neq i} \mathbb{L}_j$.

PROOF: It follows from the same line of the proof of Corollary(1.17) in [**CKS**] by taking $(s-1)$-th cohomology of the complex $B^{\cdot}(N_1, \ldots, N_s; V)$. Corollary(1.17) is the cese $s = 2$.

Proof of Proposition 2.3: Let $\lambda_i = (\lambda_{i1}, \dots, \lambda_{ij}) \in (\mathbb{R}_{\geq 0})^j - \{0\}$ for $i = 1, \dots, j$. If we take $c_i = c_i(\lambda_i) = \sum_{t=1}^{j} \lambda_{it} c_1(L_i)$, then Lemma 2.4 implies that the bilinear form $B_J(\prod_{i=1}^{j} c_i(\lambda_i))$ is nondegenerate for any $\lambda_i \in (\mathbb{R}_{\geq 0})^j - \{0\}$. Hence the signature is the same for all such λ_i.

Noting that $H^{n-2j}(X_J) \cong H^{k-j,k-j}(X_J)$ (see **[Od]**), we have

$$\begin{aligned}\operatorname{sign} B_J(c_1(L_1)^j) &= \sum_{i=0}^{k-j} (-1)^{k-j-i} \dim P^{k-j-i,k-j-i}(X_J) \\ &= (-1)^{k-j} h^{k-j,k-j}(X_J) + 2 \sum_{i=0}^{k-j-1} (-1)^i h^{i,i}(X_J),\end{aligned}$$

where $P^{i,i}(X_J)$ is the primitive part of $H^{i,i}(X_J)$ with respect to the Kähler form $c_1(L_1)$ and $h^{i,i}(X_J) = \dim H^{i,i}(X_J)$ (cf. **[W]**). Since $\operatorname{sign} B_J = (-1)^j \operatorname{sign} B_J(c_1(N^*_{X_i}|_{X_J})^j)$ for any $i \in J$ from Proposition 2.3, we have

$$\begin{aligned}\operatorname{sign}(U, \partial U) &= \sum_{j=1}^{k} (-1)^{j+1} \sum_{J \in \Delta(j)} (-1)^j \{(-1)^{k-j} h^{k-j,k-j}(X_J) \\ &\quad + 2 \sum_{i=0}^{k-j-1} (-1)^i h^{i,i}(X_J)\} \\ &= \sum_{j=1}^{k} (-1)^{k+j+1} \sum_{J \in \Delta(j)} h^{k-j,k-j}(X_J) \\ &\quad - 2 \sum_{j=1}^{k} \sum_{J \in \Delta(j)} \sum_{i=1}^{k-j} (-1)^{k-j-i} h^{k-j-i,k-j-i}(X_J).\end{aligned}$$

For $J \in \Delta(j)$ we denote $\Delta_J := \{K \in \Delta; J \subset K\}$ and $\Delta_J(k) := \{K \in \Delta_J; |K| = k\}$. Then we have

$$h^{i,i}(X_J) = \sum_{t=0}^{i} (-1)^{i-t} \binom{2k-j-t}{i-t} \sharp \Delta_J(t+j)$$

(see [**Od**]). Here we set $\operatorname{sign}(U, \partial U) = I_1 + I_2$. Then

$$\begin{aligned} I_1 &= \sum_{j=1}^{k} (-1)^{k+j+1} \sum_{J \in \Delta(j)} \sum_{t=0}^{k-j} (-1)^{k-j-t} \binom{2k-j-t}{k-j-t} \sharp \Delta_J(t+j) \\ &= \sum_{j=1}^{k} \sum_{t=0}^{k-j} (-1)^{t+1} \binom{2k-j-t}{k-j-t} \sum_{J \in \Delta(j)} \sharp \Delta_J(t+j), \\ I_2 &= -2 \sum_{j=1}^{k} \sum_{J \in \Delta(j)} \sum_{i=1}^{k-j} (-1)^{k-j-i} \\ &\quad \sum_{t=0}^{k-j-i} (-1)^{k-j-i-t} \binom{2k-j-t}{k-j-i-t} \sharp \Delta_J(t+j) \\ &= -2 \sum_{j=1}^{k} \sum_{i=1}^{k-j} \sum_{t=0}^{k-j-i} (-1)^{t} \binom{2k-j-t}{k-j-i-t} \sum_{J \in \Delta(j)} \sharp \Delta_J(t+j). \end{aligned}$$

Here we used the equality $\sum_{J \in \Delta(j)} \sharp \Delta_J(t+j) = \binom{t+j}{j} \sharp \Delta(t+j)$. By change of variables, we have

$$\begin{aligned} I_1 &= \sum_{j=1}^{k} \sum_{t=j}^{k} (-1)^{j+t+1} \binom{2k-t}{k-t} \sharp \Delta(t) \\ &= \sum_{t=1}^{k} \sum_{j=1}^{t} (-1)^{j+t+1} \binom{2k-t}{k-t} \sharp \Delta(t) \\ &= \sum_{t=1}^{k} (-1)^{t} \binom{2k-t}{k-t} \sharp \Delta(t), \end{aligned}$$

where we used the equality $\sum_{j=1}^{t} (-1)^j \binom{t}{j} = (1-1)^t - \binom{t}{0} = -1$. Also we have

$$
\begin{aligned}
I_2 &= -2\sum_{j=1}^{k}\sum_{i=1}^{k-j}\sum_{t=j}^{k-i}(-1)^{t+j}\binom{2k-t}{k-i-t}\binom{t}{j}{}^{\sharp}\Delta(t) \\
&= -2\sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\sum_{t=j}^{k-i}(-1)^{t+j}\binom{2k-t}{k-i-t}\binom{t}{j}{}^{\sharp}\Delta(t) \\
&= -2\sum_{i=1}^{k-1}\sum_{t=1}^{k-i}\sum_{j=1}^{t}(-1)^{t+j}\binom{2k-t}{k-i-t}\binom{t}{j}{}^{\sharp}\Delta(t) \\
&= 2\sum_{i=1}^{k-1}\sum_{t=1}^{k-i}(-1)^{t}\binom{2k-t}{k-i-t}{}^{\sharp}\Delta(t) \\
&= 2\sum_{t=1}^{k-1}\sum_{i=1}^{k-t}(-1)^{t}\binom{2k-t}{k-i-t}{}^{\sharp}\Delta(t).
\end{aligned}
$$

Thus we have

$$
I_1 + I_2 = (!)^{k\sharp}\Delta(k) + \sum_{t=1}^{k-1}(-1)^{t\sharp}\Delta(t)\left\{\binom{2k-t}{k-t} + 2\sum_{i=1}^{k-t}\binom{2k-t}{k-i-t}\right\}.
$$

Here we have

$$
\begin{aligned}
\binom{2k-t}{k-t} + 2\sum_{i=1}^{k-t}\binom{2k-t}{k-i-t} &= \binom{2k-t}{k-t} + \sum_{i=1}^{k-t}\binom{2k-t}{k-i-t} \\
&\quad + \sum_{i=1}^{k-t}\binom{2k-t}{k+i} \\
&= \sum_{i=0}^{k-t}\binom{2k-t}{i} + \sum_{i=k+1}^{2k-t}\binom{2k-t}{i} \\
&= 2^{2k-t} - \sum_{i=k-t+1}^{k}\binom{2k-t}{i}.
\end{aligned}
$$

Hence we have the following.

Proposition 2.5. *We have*

$$\operatorname{sign}(U,\partial U)=\sum_{j=1}^{k}(-1)^j\left\{2^{2k-j}-\sum_{i=k-j+1}^{k}\binom{2k-j}{i}\right\}{}^{\sharp}\Delta(j).$$

From the equality(1.2) and Proposition 2.5 we have the equality

$$\text{(2.6)}\quad \sigma(p)-2^n\chi_\infty(p)=\sum_{j=k+1}^{2k}(-2)^{2k-j\sharp}\Delta(j)+\sum_{i=k-j+1}^{k}\binom{2k-j}{i}{}^{\sharp}\Delta(j).$$

§3. Relations in Δ.

In this section we draw several relations in Δ, which we need in the next section.

Proposition 3.1. *We have the following relations.*

$$\text{(3.1.1)}\qquad \sum_{j=1}^{2k}(-1)^{j\sharp}\Delta(j)=0,$$

and for $1\le t\le 2k-2$

$$\text{(3.1.2)}\qquad (1-(-1)^t)^{\sharp}\Delta(t)=\sum_{j=t+1}^{2k}(-1)^j\binom{j}{t}{}^{\sharp}\Delta(j).$$

Proof: The equality (3.1.1) follows from the fact that the Euler number of a closed oriented topological minifold of odd dimension vanishes. For $J\in\Delta(t)$ the complex $\Delta_J-\{J\}$ induces a triangulation of $2k-t-1$ dimensional sphere. Hence we have

$$1-(-1)^{2k-t}=\sum_{j=t+1}^{2k}(-1)^{j-t-1\sharp}\Delta(j).$$

By taking summation over all $J\in\Delta(t)$ we have (3.1.2).

LEMMA 3.2. *For a positive integer l there exist integers $\alpha_0, \alpha_1, \dots, \alpha_l$ and $\beta_0, \beta_1, \dots, \beta_l$ with $\alpha_0 = \beta_0 = 1$ satisfying the following conditions:*

$$\binom{j}{2l-1} = \sum_{i=0}^{l} \alpha_i \binom{j-2l+i}{2l-1-i}, \tag{3.2.a}$$

$$\binom{j}{2l} = \sum_{i=0}^{l} \beta_i \binom{j-2l-1+i}{2l-i}. \tag{3.2.b}$$

PROOF: For a positive integer i we define a polynomial $\binom{x}{i}$ to be $\prod_{\nu=0}^{i-1}(x-\nu)/i!$ and $\binom{x}{0} = 1$. Then we note that if a polynomial $f(x)$ of degree n with values in integers at $x = 0, 1, \dots, n-1$, then we can write $f(x) = \sum_{i=0}^{n-1} a_i \binom{x}{i}$ with $a_i \in \mathbb{Z}$. Put $f(x) = \binom{x+2l+1}{2l}$. Then we have $f(x) = \sum_{i=0}^{n-1} a_i \binom{x}{i}$. By induction we have

$$\binom{x+2l+1}{2l} = \sum_{i=0}^{2l} \beta_i \binom{x}{2l-i}.$$

The polynomial $g(x) = \binom{x}{2l} - \sum_{i=0}^{l} \beta_i \binom{x-2l-1+i}{2l-i}$ has zeros at $x = l+1, l+2, \dots, 2l-1$. On the other hand, $g(x) = \sum_{i=l+1}^{2l} \beta_i \binom{x-2l-1+i}{2l-i}$, hence the equality $g(l+1) = 0$ implies $\beta_{2l} = 0$. By induction we can easily see that $g(x) = 0$. Since $\binom{x}{i}$ is a monic polynomial of degree i, we have $\beta_0 = 1$. This showes (3.2.b).

We can also prove (3.2.a) in a similar way.

Remark 3.3. From the conditions in Lemma 3.2 we have the relations of coefficients:

$$\alpha_{l-s} = \binom{2l+2s-1}{2l-1} - \sum_{i=l-s+1}^{l} \alpha_i \binom{2s+i-1}{2l-i-1}, \tag{3.3.a}$$

$$\beta_{l-s} = \binom{2l+2s+1}{2l} - \sum_{i=l-s+1}^{l} \beta_i \binom{2s+i}{2l-i}, \tag{3.3.b}$$

for $0 \le s \le l$. In particular, $\alpha_l = 2$ and $\beta_l = 2l+1$.

PROPOSITION 3.4. *For $0 \leq l \leq k-1$ we have*

$$\sum_{j=2l+1}^{2k} (-1)^j \binom{j-l-1}{l} {}^\sharp\Delta(j) = 0.$$

PROOF: Since $\binom{j-l-1}{l} = 0$ for $l+1 \leq j \leq 2l$, we may take summation over $l+1 \leq j \leq 2k$.

For $l = 0$, it is just the relation (3.1.1). Let $l > 0$. We shall prove it by induction. Firstly let $l = 2s-1$. Let $\alpha_0, \alpha_1, \dots, \alpha_s$ be the integers defined by Lemma 3.2 for $l = s$. Set

$$J_1 = \sum_{j=s+1}^{2k} (-1)^j \sum_{i=0}^{s-1} \alpha_i \binom{j-2s+i}{2s-1-i} {}^\sharp\Delta(j).$$

By changing order of summations, we have

$$\begin{aligned} J_1 &= \sum_{i=0}^{s-1} \alpha_i \sum_{j=2s-i}^{2k} (-1)^j \binom{j-2s+i}{2s-1-i} {}^\sharp\Delta(j) \\ &= \sum_{j=2s}^{2k} (-1)^j \binom{j-2s}{2s-1} {}^\sharp\Delta(j), \end{aligned}$$

where we used the assumption of induction and $\alpha_0 = 1$. On the other hand, from (3.2.a) we have

$$J_1 = \sum_{j=s+1}^{2k} (-1)^j \left\{ \binom{j}{2s-1} - 2\binom{j-s}{s-1} \right\} {}^\sharp\Delta(j).$$

Here we may take summation only over $2s \leq j \leq 2k$. Hence we have

$$\begin{aligned} J_1 &= \sum_{j=2s}^{2k} (-1)^j \binom{j}{2s-1} {}^\sharp\Delta(j) - 2 \sum_{j=2s}^{2k} (-1)^j \binom{j-s}{s-1} {}^\sharp\Delta(j) \\ &= \left\{ -2{}^\sharp\Delta(2s-1) + \sum_{j=2s}^{2k} (-1)^j \binom{j}{2s-1} {}^\sharp\Delta(j) \right\} \\ &\quad - 2 \sum_{j=2s-1}^{2k} (-1)^j \binom{j-s}{s-1} {}^\sharp\Delta(j). \end{aligned}$$

The first term is the relation (3.1.2), hence, vanishes and the second term vanishes by the assumption of induction.

Next let $l = 2s$. Let $\beta_0, \beta_1, \dots, \beta_s$ be the integers defined by Lemma 3.2 for $l = s$. Set

$$J_2 = \sum_{j=s+1}^{2k} (-1)^j \sum_{i=0}^{s-1} \beta_i \binom{j-2s-1+i}{2s-i} {}^\sharp\Delta(j).$$

Then we have

$$\begin{aligned} J_2 &= \sum_{i=0}^{s-1} \beta_i \sum_{j=2s-i+1}^{2k} (-1)^j \binom{j-2s-1+i}{2s-i} {}^\sharp\Delta(j) \\ &= \sum_{j=2s+1}^{2k} (-1)^j \binom{j-2s-1}{2s} {}^\sharp\Delta(j). \end{aligned}$$

On the other hand, from (3.2.b) we also have

$$\begin{aligned} J_2 &= \sum_{j=s+1}^{2k} (-1)^j \left\{ \binom{j}{2s} - (2s+1)\binom{j-s-1}{s} \right\} {}^\sharp\Delta(j) \\ &= \sum_{j=2s+1}^{2k} (-1)^j \binom{j}{2s} {}^\sharp\Delta(j) - (2s+1) \sum_{j=s+1}^{2k} (-1)^j \binom{j-s-1}{s} {}^\sharp\Delta(j). \end{aligned}$$

The first term is the relation (3.1.2), hence, vanishes and the second term vanishes by the assumption. Thus we proved the proposition.

§4. Proof of Theorem.

We define series of integers $\{a_j(l)\}_{j=1}^{2k}$ for $l \geq 0$ by

$$\sum_{j=1}^{2k} a_j(0)^\sharp\Delta(j) = \sum_{j=1}^{k} (-1)^j \sum_{i=k-j+1}^{k} \binom{2k-j}{i} {}^\sharp\Delta(j) + \sum_{j=k+1}^{2k} (-2)^{2k-j\sharp}\Delta(j),$$

and inductively by

$$\sum_{j=1}^{2k} a_j(l)^\sharp\Delta(j) = \sum_{j=1}^{2k} a_j(l-1)^\sharp\Delta(j) + a_{2l-1}(l-1) \sum_{j=2l-1}^{2k} (-1)^j \binom{j-l}{l-1} {}^\sharp\Delta(j).$$

From Proposition 3.4 we see that for all $l \geq 0$

$$\sum_{j=1}^{2k} a_j(l)^\sharp \Delta(j) = \sum_{j=1}^{2k} a_j(0)^\sharp \Delta(j).$$

We shall prove $a_j(k) = 0$ for all j. We need two lemmas.

LEMMA 4.1. *We have*

$$\binom{2k-1}{k} = \sum_{i=k-l+1}^{k} \binom{l-1}{k-i}\binom{2k-l}{i} \quad \text{for} \quad 1 \leq l \leq k,$$

and

$$\binom{2k-1}{k} = \sum_{i=0}^{2k-l} \binom{l-1}{k-i}\binom{2k-l}{i} \quad \text{for} \quad k+1 \leq l \leq 2k.$$

LEMMA 4.2. *We have*

$$1 - \sum_{i=0}^{l}(-1)^i \binom{j-2i-1}{l-i}\binom{j-i-1}{i} = 0.$$

PROOF: Since $\binom{j-2i-1}{l-i}\binom{j-i-1}{i} = \binom{l}{i}\binom{j-i-1}{l}$, we shall prove

$$1 - \sum_{i=0}^{l}(-1)^i \binom{l}{i}\binom{j-i-1}{l} = 0.$$

Set $f_l(x) = 1 - \sum_{i=0}^{l}(-1)^i\binom{l}{i}\binom{x-i}{l}$. Then we have $f_0(x) = 0$. Since $\binom{l}{i} = \binom{l-1}{i} + \binom{l-1}{i-1}$ for $i \geq 1$ and since $\binom{x-i}{l} - \binom{x-i-1}{l} = \binom{x-i-1}{l-1}$, we have

$$\begin{aligned}
f_l(x) &= 1 - \binom{x}{l} - \sum_{i=1}^{l}(-1)^i \left\{\binom{l-1}{i} + \binom{l-1}{i-1}\right\}\binom{x-i}{l} \\
&= 1 - \sum_{i=0}^{l-1}(-1)^i\binom{l-1}{i}\binom{x-i}{l} - \sum_{i=0}^{l-1}(-1)^{i+1}\binom{l-1}{i}\binom{x-i-1}{l} \\
&= 1 - \sum_{i=0}^{l-1}(-1)^i\binom{l-1}{i}\binom{x-i-1}{l-1} \\
&= f_{l-1}(x-1).
\end{aligned}$$

Hence we have $f_l(x) = f_0(x-l) = 0$.

PROPOSITION 4.3. *For $l \geq 0$ the number $a_j(l)$ vanishes for $j \leq 2l$, is equal to*

$$(-1)^j \sum_{i=k-j+1+l}^{k-l} \left\{1 - \sum_{t=0}^{l-1} (-1)^t \binom{j-2t-1}{k-i-t} \binom{j-t-1}{t}\right\} \binom{2k-j}{i}$$

for $2l+1 \leq j \leq k+l-1$, and

$$(-1)^j \sum_{i=0}^{2k-j} \left\{1 - \sum_{t=0}^{l-1} (-1)^t \binom{j-2t-1}{k-i-t} \binom{j-t-1}{t}\right\} \binom{2k-j}{i}$$

for $j \geq k+l$.

PROOF: For $l = 0$, it coincides with the definition of $a_j(0)$. Let $l > 0$. By definition $a_{2l-1}(l) = 0$. We assume that $a_j(l-1)$ for $2l-1 \leq j \leq k+l-2$ is equal to

$$(-1)^j \sum_{i=k-j+l}^{k-l+1} \left\{1 - \sum_{t=0}^{l-2} (-1)^t \binom{j-2t-1}{k-i-t} \binom{j-t-1}{t}\right\} \binom{2k-j}{i},$$

in particular, $a_{2l-1}(l-1) = (-1)^l \binom{2k-2l+1}{k-l+1}$. If $l \leq k-1$, then we have

$$\begin{aligned}
a_{2l}(l-1) &= \sum_{i=0}^{1} \left\{1 - \sum_{t=0}^{l-2} (-1)^t \binom{2l-2t-1}{l-i-t} \binom{2l-t-1}{t}\right\} \binom{2k-2l}{k-l+i} \\
&= \left\{1 - \sum_{t=0}^{l-2} (-1)^t \binom{2l-2t-1}{l-1-t} \binom{2l-t-1}{t}\right\} \\
&\quad \times \left\{\binom{2k-2l}{k-l} + \binom{2k-2l}{k-l+1}\right\} \\
&= \left\{1 - \sum_{t=0}^{l-2} (-1)^t \binom{2l-2t-1}{l-1-t} \binom{2l-t-1}{t}\right\} \binom{2k-2l+1}{k-l+1} \\
&= (-1)^{l-1} \binom{l}{l-1} \binom{2k-2l+1}{k-l+1}.
\end{aligned}$$

Here we used Lemma 4.2 for the last equality. Hence we have

$$a_{2l}(l) = a_{2l}(l-1) + a_{2l-1}(l-1)\binom{2l-l}{l-1} = 0.$$

Next noting that

$$\binom{2k-2l+1}{k-l+1} = \sum_{i=k+l-j}^{k-l+1} \binom{j-2l+1}{k-l+1-i}\binom{2k-j}{i}$$

for $2l-1 \leq j \leq k+l-1$ from Lemma 4.1, we have for $2l+1 \leq j \leq k+l-2$

$$\begin{aligned}
a_j(l) &= a_j(l-1) + (-1)^j a_{2l-1}(l-1)\binom{j-l}{l-1} \\
&= (-1)^j \sum_{i=k-j+l}^{k-l+1} \left\{1 - \sum_{t=0}^{l-1}(-1)^t \binom{j-2t-1}{k-i-t}\binom{j-t-1}{t}\right\}\binom{2k-j}{i}.
\end{aligned}$$

By applying Lemma 4.2 to $i = k-j+l$ and $i = k-l+1$ we get the desired expression of $a_j(l)$. In a similar way we also have the expression of $a_j(l)$ for $j \geq k+l-1$.

Finally we shall prove $a_{2k}(k) = 0$. We have

$$\begin{aligned}
a_{2k}(k-1) &= \sum_{i=0}^{1} \left\{1 - \sum_{t=0}^{k-2}(-1)^t \binom{2k-2l-1}{k-i-t}\binom{2k-t-1}{t}\right\}\binom{0}{i} \\
&= 1 - \sum_{t=0}^{k-2}(-1)^t \binom{2k-2l-1}{k-t}\binom{2k-t-1}{t} \\
&= \sum_{t=k-1}^{k} (-1)^t \binom{2k-2l-1}{k-t}\binom{2k-t-1}{t} \\
&= (-1)^{k-1}\binom{k}{k-1}.
\end{aligned}$$

Since $a_{2k-1}(k-1) = (-1)^k$, we have

$$a_{2k}(k) = a_{2k}(k-1) + a_{2k-1}(k-1)\binom{k}{k-1} = 0.$$

Proposition 4.3 implies $a_j(k) = 0$ for all j, hence we complete the proof of Theorem.

References

[ADS1] M. F. Atiyah, H. Donnelly and I. M. Singer, *Eta invariants, signature defects of cusps, and values of L-functions*, Ann. of Math. **118** (1983), 131–177.

[ADS2] M. F. Atiyah, H. Donnelly and I. M. Singer, *Signature defects of cusps, and values of L-functions: The nonsplit case*, Ann. of Math. **119** (1984), 131–177.

[CKS] E. Cattani, A. Kaplan and W. Schmid, *L^2 and intersection cohomologies for a polarizable variation of Hodge structure*, Invent. math. **97** (1987), 217–252.

[E] F. Ehlers, *Eine Klasse komplexer Manigfaltigkeiten und die Auflösung einiger isolierter Singularitäten*, Math. Ann. **218** (1975), 127–156.

[GM] M. Goresky and R. MacPherson, *On the topology of complex algebraic maps*, in "Algebraic Geometry," Lecture Notes in Math. **961**, Springer-Verlag, Berlin, Heidelberg and New York, 1982, pp. 119–129.

[H1] F. Hirzebruch, *Hilbert modular surfaces*, Enseign. Math. **19** (1974), 183–281.

[H2] F. Hirzebruch, "Gesammelte Abhandlungen II," Springer-Verlag, Berlin, Heidelberg and New York, 1987.

[HG] F. Hirzebruch and G. van der Geer, "Lectures on Hilbert Modular Surfaces," Presses de l'Universite de Montreal, Montreal, 1981.

[KKMS] G. Kempf, F. Knudson, D. Mumford and B. Saint-Donat, "Toroidal Embeddings I," Lecture Notes in Math. **339**, Springer-Verlag, Berlin, Heidelberg and New York, 1973.

[L] E. Looijenga, *Riemann-Roch and smoothings of singularities*, Topology **25** (1986), 293–302.

[Mo] S. Morita, *Almost complex manifolds and Hirzebruch invariants for isolated singularities in complex spaces*, Math. Ann. **211** (1974), 245–260.

[Mu] W. Müller, "Manifolds with Cusps of Rank One," Lecture Notes in Math.**1244**, Springer, Heidlberg, New York, 1987.

[O] S. Ogata, *Special Values of zeta functions associated to cusp singularities*, Tohoku Math. J. **37** (1985), 367–384.

[Od] T. Oda, "Convex Bodies and Algebraic Geometry," Springer-Verlag, Berlin, Heidelberg, New York, London, Paris and Tokyo, 1988.

[Sa] I. Satake, *On numerical invariants of arithmetic varieties of* **Q**-*rank one*, in "Automorphic Forms of Several Variables," Taniguchi Symposium, Katata,1983 (I. Satake and Y. Morita, eds.), Progress in Math. **46**, Birkhäuser, Basel, Boston and Stuttgart, 1984, pp. 353–369.

[Sh] H. Shimizu, *On discrete groups acting on the product of upper half planes*, Ann. of Math. **77** (1963), 33–71.

[SO] I. Satake and S. Ogata, *Zeta functions associated to cusps and their special values*, in "Automorphic Forms and Geometry of Arithmetic Varieties," (Y. Namikawa and K. Hashimoto eds.) Adv. Studies in Pure Math. **15**, Kinokuniya, Tokyo and North-Holland, Amsterdam, New York, Oxford, 1989, pp. 1–27.

[St] J. Steenbrink, *Mixed Hodge structures associated with isolated singularities*, Proc. Symp. Pure Math. **40-2** (1983), 513–536.

[T] H. Tsuchihashi, *Higher dimensional analogues of periodic continued fractions and cusp singularities*, Tohoku Math. J. **35** (1983), 607–639.
[W] A. Weil, "Variétés Kählériennes," Act. Sci. Ind. 1267, Hermann, Paris, 1958.

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