# ALGEBRAIC STRING BRACKET AS A POISSON BRACKET

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ABSTRACT. We give a non-trivial Lie algebra map from the negative cyclic cohomology of an associative algebra with trace, to the space of functions on its Maurer-Cartan solutions. This is a generalized algebraic version of a theorem in [AZ] extending Goldman's result to the equivariant homology of the free loop space.

## CONTENTS

## 1. INTRODUCTION

Goldman original work [Go] on the Lie algebra of free homotopy classes of oriented closed curves on an oriented surface was extensively generalized through the introduction of String Topology by Chas and Sullivan [CS]. In particular, they generalized this Lie bracket to one on the equivariant homology of the free loop space of a compact and oriented manifold $M$. From the beginning, it was clear that this bracket had a deep relation to the holonomy map on a vector bundle; see [Go, CFP, CCR, CR]. This relation was the subject of a paper, [AZ], by the first and third author. It was shown there that using Chen's iterated integral one obtains a map of Lie algebras from the equivariant homology of the free loop space to the space of functions on a space of generalized flat connections.

Algebraic analogues of string topology Lie algebra have also been considered in recent years. Jones [J] had shown that for a simply connected topological space $X$ the equivariant homology of the free loop space is isomorphic to the negative cyclic cohomology of the algebra of cochains on $X$. Using this, and Connes long exact sequence relating negative cyclic cohomology and Hochschild cohomology, together with the BV algebra on Hochschild cohomology, Menichi [Men] deduced a Lie bracket on the negative cyclic cohomology in a way similar to the one in string topology [CS, Section 6].

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The starting point for this work was to obtain a generalization of the the results in [AZ] and place it in a more algebraic setting where the equivariant homology of the loop space is replaced by negative cyclic cohomology. A suitable setting for this is to consider a unital differential graded algebra $A$ over a field $k = \mathbb{R}$ or $\mathbb{C}$, with a reasonable trace $\mathrm{Tr} : A \to k$. Using the results of [T], the above assumptions imply an isomorphism of the Hochschild cohomologies of $A$ with values in $A$ and its dual $A^*$, $HH^\bullet(A, A) \cong HH^\bullet(A, A^*)$, such that the cup product on $HH^\bullet(A, A)$ and the dual of Connes $B$-operator on $HH^\bullet(A, A^*)$ make these spaces into a BV algebra. This BV algebra, together with a Connes long exact sequence between the Hochschild cohomology $HH^\bullet(A, A^*)$ and negative cyclic cohomology $HC^\bullet_-(A)$, imply a Lie algebra structure on $HC^\bullet_-(A)$ by a theorem of Menichi's [Men, Proposition 7.1], which is based on a similar marking/erasing result of Chas and Sullivan [CS, Theorem 6.1].

Now, using work of Gan and Ginzburg in [GG], we may look at the moduli space of Maurer-Cartan solutions,
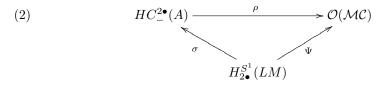
$$(1) \qquad\qquad \mathcal{MC} = \{a \in A^{odd} \mid da + a \cdot a = 0\}/\sim$$

Since we only consider odd elements, the trace induces a symplectic structure $\omega$ on $\mathcal{MC}$, and thus one can define a Poisson bracket on the function ring $\mathcal{O}(\mathcal{MC})$ of $\mathcal{MC}$. More details of this construction will be given in Section 3.

We may connect the two sides of the above discussion via a canonical map $\{a \in A^{odd} \mid da + a \cdot a = 0\} \to HC^-_\bullet(A), a \mapsto \sum_{n \geq 0} 1 \otimes a^{\otimes n}$, and dualizing this gives a map $\rho : HC^\bullet_-(A) \to \mathcal{O}(\mathcal{MC})$. We may now compare the two Lie algebras from above. Our main result then states, that the brackets are indeed preserved.

**Theorem 1.** $\rho : HC^{2\bullet}_-(A) \to \mathcal{O}(\mathcal{MC})$ *is a map of Lie algebras.*

In a special case considered in [AZ] this map becomes the generalized holonomy map from the equivariant homology of the free loop space of $M$ to the space of functions on the moduli space of generalized flat connections on a vector bundle $E \to M$. In fact one has a commutative diagram,

$$(2) \qquad\qquad \begin{array}{ccc} HC^{2\bullet}_-(A) & \xrightarrow{\quad\rho\quad} & \mathcal{O}(\mathcal{MC}) \\ & \nwarrow{\scriptstyle \sigma} \qquad \nearrow{\scriptstyle \Psi} & \\ & H^{S^1}_{2\bullet}(LM) & \end{array}$$

where $\Psi$ is the generalized holonomy dicussed in [AZ] and $\sigma$ comes from Chen's iterated integral map, as described in Section 5. In particular, for $\dim M = 2$, this recovers Goldman's results on the space of flat connections on a surface.

Another motivation of this work is to study string topology in a holomorphic setting via the moduli stack of the holomorphic structure on a fixed complex bundle $E \to M$, where $M$ is a complex manifold. Algebraically, this will correspond to the choice of the algebra $A = \Omega^{0,*}(M, End(E))$, with the Dolbeault differential $\bar{\partial}$. This discussion, once done at the chain level, relates to the algebraic structure of the B-model.

Finally, we remark, that the above discussion generalizes in a straight forward way to the case of a cyclic $A_\infty$ algebra $A$. This will be the topic of the last Section 6. In fact, by the same reasoning as above, we obtain the Lie bracket on the negative cyclic cohomology $HC^\bullet_-(A)$. Also, by symmetrization we may associate an $L_\infty$

algebra to $A$, which induces a Maurer-Cartan space similar to (1). We find, that the canonical map $\rho$ is still well-defined, such that Theorem 1 also remains valid in this generalized setting.

**Notation:** For a map $F$ of complexes, $F_\bullet$ (resp. $F^\bullet$) denotes the induced map in homology (resp. cohomology).

## 2. The Lie algebra $HC_-^\bullet(A)$

In this section, we recall the Lie algebra structure of the negative cyclic cohomology $HC_-^\bullet(A)$, for a dga $(A, d, \cdot)$ with a trace $\mathrm{Tr} : A \to k$. The Lie bracket comes from the long exact sequence that relates negative cyclic (co-)homology to Hochschild (co-)homology. For simplicity, we will work in the normalized setting.

**Definition 2.** Let $(A = \bigoplus_{i\in\mathbb{Z}} A^i, d : A^i \to A^{i+1}, \cdot)$ be a differential graded associative algebra over a field $k$, and let $M = \bigoplus_{i\in\mathbb{Z}} M^i$ be a differential graded $A$-bimodule. The (normalized) Hochschild chain complex defined as,

$$(3) \qquad \bar{C}_\bullet(A, M) := \prod_{n\geq 0} M \otimes \bar{A}^{\otimes n},$$

where $\bar{A} = A/k$, and $s$ denotes shifting down by one. The boundary $\delta : \bar{C}_\bullet(A, M) \to \bar{C}_{\bullet+1}(A, M)$ is defined by,

$$\delta(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n} (-1)^{\epsilon_i} a_0 \otimes \cdots \otimes d(a_i) \otimes \cdots \otimes a_n$$

$$+ \sum_{i=0}^{n-1} (-1)^{\epsilon_i} a_0 \otimes \cdots \otimes (a_i \cdot a_{i+1}) \otimes \cdots \otimes a_n - (-1)^{\epsilon'_n}(a_n \cdot a_0) \otimes a_1 \otimes \cdots \otimes a_{n-1},$$

where $a_0 \in M$, $a_1, \cdots, a_n \in A$, $\epsilon_0 = |a_0|$, $\epsilon_i = (|a_0| + \cdots + |a_{i-1}| + i - 1)$, and $\epsilon'_n = (|a_n| + 1) \cdot (|a_0| + \cdots + |a_{n-1}| + n - 1)$. Note that the differential is well defined; see [L]. Similarly, the (normalized) Hochschild cochain complex is defined by,

$$(4) \quad \bar{C}^n(A, M) := \left\{ f : s\bar{A}^{\otimes n} \to M \; \middle| \; f(a_1 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_n) = 0, \text{if } a_i = 1 \right\},$$

where the differential $\delta^* : \bar{C}^\bullet(A, M) \to \bar{C}^{\bullet-1}(A, M)$ is given by,

$$(\delta^* f)(a_1 \otimes \cdots \otimes a_n) := \sum_{i=1}^{n} (-1)^{|f| + \epsilon_i} f(a_1 \otimes \cdots \otimes da_i \otimes \cdots \otimes a_n)$$

$$+ d(f(a_1 \otimes \cdots \otimes a_n)) + \sum_{i=1}^{n-1} (-1)^{|f| + \epsilon_i} f(a_1 \otimes \cdots \otimes (a_i \cdot a_{i+1}) \otimes \cdots \otimes a_n)$$

$$+ (-1)^{|f|(|a_1|+1)} a_1 \cdot f(a_1 \otimes \cdots \otimes a_n) + (-1)^{|f| + \epsilon_n} f(a_1 \otimes \cdots \otimes a_{n-1}) \cdot a_n.$$

The respective (co-)homology theories are denoted by

$$H_\bullet(A, M) = H(\bar{C}_\bullet(A, M), \delta), \quad H^\bullet(A, M) = H(\bar{C}^\bullet(A, M), \delta^*).$$

Denoting by $A^* = Hom(A, k)$ the graded dual of $A$, we see that the dual of $\bar{C}_\bullet(A, A)$ is given by $\bar{C}^\bullet(A, A^*)$. Recall furthermore, that there is a cup product $\cup$ on $\bar{C}^\bullet(A, A)$ defined by

$$(f \cup g)(a_1 \otimes \cdots \otimes a_{m+n}) := f(a_1 \otimes \cdots \otimes a_m) \cdot g(a_{m+1} \otimes \cdots \otimes a_{m+n}).$$

Next, we define the (normalized) negative cyclic chains $\overline{CC}_\bullet^-(A)$ of $A$ to be the vector space $\bar{C}_\bullet(A, A)[[u]]$, where $u$ is of degree $+2$, and with differential $\delta + uB$, where $B : \bar{C}_\bullet(A, A) \to \bar{C}_{\bullet-1}(A, A)$ is Connes operator,

$$(5) \quad B(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n} (-1)^{\epsilon_i} 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1},$$

$$\text{where } \epsilon_i = (|a_i| + \cdots + |a_n| + n - i + 1)(|a_0| + \cdots + |a_{i-1}| + i - 1).$$

Thus, every element of $\overline{CC}_n^-(A)$ is an infinte sum $\sum_{i=0}^{\infty} a_i u^i \in \bar{C}_\bullet(A, A)[[u]]$, where $a_i \in \bar{C}_{n-2i}(A, A)$, $\delta$ acts on $a_i \in \bar{C}_\bullet(A, A)$, and $uB$ acts as

$$(6) \quad \cdots \overset{uB}{\longleftarrow} \bar{C}_\bullet(A, A) \cdot u^2 \overset{uB}{\longleftarrow} \bar{C}_\bullet(A, A) \cdot u \overset{uB}{\longleftarrow} \bar{C}_\bullet(A, A).$$

Dually, define the (normalized) negative cyclic cochains $\overline{CC}_-^\bullet(A)$ of $A$ by taking $\overline{CC}_-^\bullet(A) = \bar{C}^\bullet(A, A^*) \otimes k[v, v^{-1}]/vk[v]$, where $v$ is an element of degree $-2$. Explicitly, the degree $n$ part $\overline{CC}_-^n(A)$ is represented by finite sums $\sum_{i=0}^{k} a_i v^{-i}$ where $a_i \in \bar{C}^{n-2i}(A, A^*)$. The differential is given by $\delta^* + vB^*$, where $\delta^*$ acts on $\bar{C}^\bullet(A, A^*)$, and $vB^*$ acts as follows.

$$\cdots \overset{vB^*}{\longrightarrow} \bar{C}^\bullet(A, A^*) \cdot v^{-2} \overset{vB^*}{\longrightarrow} \bar{C}^\bullet(A, A^*) \cdot v^{-1} \overset{vB^*}{\longrightarrow} \bar{C}^\bullet(A, A^*).$$

Note, that if $C_\bullet(A, A)$ is finite dimensional in each degree, then the graded dual of $\overline{CC}_-^n(A)$ is isomorphic to the chain complex $\overline{CC}_n^-(A) = Hom(\overline{CC}_-^n(A), k)$, see also [HL, Lemma 3.7]. It is easy to see that $B^2 = \delta B + B\delta = 0$, and we define the associated (co-)homology theories by,

$$HC_\bullet^-(A) = H(\overline{CC}_\bullet^-(A), \delta + uB), \quad HC_-^\bullet(A) = H(\overline{CC}_-^\bullet(A), \delta^* + vB^*).$$

**Lemma 3.** *If $H_\bullet(A, A)$ is bounded from below, then both $\bar{C}_\bullet(A, A)[u]$ and $\bar{C}_\bullet(A, A)[[u]]$ with differential $\delta + uB$ calculate negative cyclic homology $HC_\bullet^-(A)$.*

This lemma follows from a spectral sequence argument for the inclusion $\bar{C}_\bullet(A, A)[u] \hookrightarrow \bar{C}_\bullet(A, A)[[u]]$, similarly to [HL, Lemma 3.6]. Note, that our sign convention is opposite to the one from [HL], but in agreement with [GJP], since our differential $\delta : \bar{C}_\bullet(A, A) \to \bar{C}_{\bullet+1}(A, A)$ is of degree $+1$.

From now on, we additionally assume, that we also have a suitable trace map.

**Definition 4.** Let $\text{Tr} : A \to k$ be a trace map, satisfying $\text{Tr}(da) = 0$ and $\text{Tr}(ab) = -(-1)^{|a| \cdot |b|} \text{Tr}(ba)$, for all $a, b \in A$. Assume furthermore that the map $\omega : A \to A^*$, $\omega(a)(b) := \text{Tr}(ab)$ is a bimodule map, which induces an isomorphism on homology $H(A) \to H(A^*)$. By abuse of language, we will also view $\omega$ as a map $\omega : A \otimes A \to k, \omega(a, b) = \text{Tr}(ab)$. In this case, $A$ is also called a *symmetric algebra*.

Notice that $\omega : A \to A^*$ induces a morphism of the Hochschild complexes $\omega_\sharp : \bar{C}^\bullet(A, A) \to \bar{C}^\bullet(A, A^*)$ via composition $\omega_\sharp(f) := \omega \circ f$, which is an isomorphism on

homology $\omega_\sharp^\bullet : H^\bullet(A, A) \to H^\bullet(A, A^*)$. We may thus transfer the cup product $\cup$ on $H^\bullet(A, A)$ to a product $\sqcup$ on $HH^\bullet(A, A^*)$, by setting $f \sqcup g := \omega_\sharp^\bullet((\omega_\sharp^\bullet)^{-1}f \cup (\omega_\sharp^\bullet)^{-1}g)$. Define furthermore the operator $\Delta : HH^\bullet(A, A^*) \to HH^\bullet(A, A^*)$ as the dual of $B$ on homology. Then we assume, that $(HH^\bullet(A, A^*), \sqcup, \Delta)$ is a BV-algebra, *i.e.* $\sqcup$ is a graded associative, commutative product, $\Delta^2 = 0$, and the bracket $\{a, b\} := (-1)^{|a|}\Delta(a \sqcup b) - (-1)^{|a|}\Delta(a) \sqcup b - a \sqcup \Delta(b)$ is a derivation in each variable.

Recall from Menichi [Men] that this BV-algebra induces a Lie algebra on the negative cyclic cohomology $HC_-^\bullet(A)$ using the long exact sequences of Hochschild and negative cyclic cohomology. The inclusion $\overline{CC}_\bullet^-(A) \xrightarrow{\times u} \overline{CC}_\bullet^-(A)$ given by multiplication by $u$ has cokernel $\bar{C}_\bullet(A, A)$. We thus obtain a short exact sequence

$$(7) \qquad 0 \to \overline{CC}_\bullet^-(A) \xrightarrow{\times u} \overline{CC}_\bullet^-(A) \to \bar{C}_\bullet(A, A) \to 0,$$

which induces Connes long exact sequence of homology groups.

$$(8) \qquad \cdots \to HH_n(A, A) \xrightarrow{\mathcal{B}_\bullet} HC_{n-1}^-(A) \to HC_{n+1}^-(A) \xrightarrow{I_\bullet} HH_{n+1}(A, A) \xrightarrow{\mathcal{B}_\bullet} \cdots.$$

Here, the projection to the $u^0$ term $I : \overline{CC}_\bullet^-(A) \to \bar{C}_\bullet(A, A)$ induces the map $I_\bullet$, and the connecting map $\mathcal{B}_\bullet$ is induced by the composition $\bar{C}_\bullet(A, A) \xrightarrow{B} \bar{C}_\bullet(A, A) \xrightarrow{inc} \overline{CC}_\bullet^-(A)$. Note, that unlike $inc \circ B : \bar{C}_\bullet(A, A) \to \overline{CC}_\bullet^-(A)$, the inclusion $inc : \bar{C}_\bullet(A, A) \to \overline{CC}_\bullet^-(A)$ is not a chain map.

Dually, we have the short exact sequence

$$0 \to \bar{C}^\bullet(A, A) \to \overline{CC}_-^\bullet(A) \to \overline{CC}_-^\bullet(A) \to 0,$$

inducing Connes long exact sequence of cohomology groups

$$(9) \qquad \cdots \to HH^n(A, A^*) \xrightarrow{I^\bullet} HC_-^n(A) \to HC_-^{n-2}(A) \xrightarrow{\mathcal{B}^\bullet} HH^{n-1}(A, A^*) \xrightarrow{I^\bullet} \cdots.$$

Notice that the composition

$$(10) \qquad\qquad\qquad B^\bullet = \mathcal{B}^\bullet \circ I^\bullet$$

is exactly the $\Delta$ operator of our BV-algebra on $HH^\bullet(A, A^*)$, so that we may obtain an induced Lie algebra from [Men, Lemma 7.2], much like the marking/erasing situation in [CS].

**Proposition 5** (L. Menichi [Men]). *The bracket $\{a, b\} := I^\bullet(\mathcal{B}^\bullet(a) \sqcup \mathcal{B}^\bullet(b))$ induces a Lie algebra structure on $HC_-^\bullet(A)$.*

We end this section with some examples of the above definitions.

**Examples 6.** Let $M$ be a smooth, compact and oriented Riemannian manifold.

- A first example is obtained by taking $A = \Omega^\bullet(M)$ the De Rham forms on $M$, $d = d_{DR}$ the exterior derivative on $A$, and $\text{Tr}(a) := \int_M a$.
- More generally, if $E \to M$ is a finite dimensional complex vector bundle over $M$, with a flat connection $\nabla$, then we may take $A = \Omega^\bullet(M, End(E))$ with the usual differential $d_\nabla$. Similarly, the trace is given by a combination of integration and trace in $End(E)$. The cyclic property of the trace guarantees that this induces an injective bimodule map $\omega : A \to A^*$ that is a quasi-isomorphism.
- Both of the above examples are special cases of ellitpic Calabi-Yau space as defined in [C]. By definition, this means that we have a bundle of finite dimensional associative $\mathbb{C}$ algebras over $M$, whose algebra of sections is

denoted by $A$. Furthermore, there is a differential operator $d : A \to A$, which is an odd derivative with $d^2 = 0$ making $A$ into an ellitpic complex, a $\mathbb{C}$ linear trace $\text{Tr} : A \to \mathbb{C}$, a hermitian metric $A \otimes A \to \mathbb{C}$, and a complex antilinear, $C^\infty(M, \mathbb{R})$ linear operator $* : A \to A$, satisfying certain natural conditions. It can be seen that this example satisfies the above assumptions. The details and other examples of elliptic Calabi-Yau spaces can be found in [C] and [DT].

## 3. Maurer-Cartan solutions

In this section we define the moduli space of Maurer Cartan solutions for a symmetric algebra $A = \oplus_{i \geq 0} A^i$, and then explain its symplectic nature. The main reference for this section is the paper [GG] by Gan-Ginzburg, together with Section 4 of [AZ]. Let us assume $k = \mathbb{R}$ or $\mathbb{C}$.

For $a, b \in A$ define the Lie bracket $[a, b] := a \cdot b - (-1)^{|a| \cdot |b|} b \cdot a$ and the bilinear form $\omega(a, b) := \text{Tr}(ab)$. The first remark is that $(A = A^{odd} \oplus A^{even}, d, [\cdot, \cdot], \omega)$ is a *cyclic differential graded Lie algebra* as it is defined in Section 4 of [AZ], therefore all results in [GG] applies here to define the Maurer-Cartan solutions.

**Definition 7.** We define the Maurer-Cartan moduli stack as

$$MC \quad := \quad \{a \in A^{odd} \mid da + \frac{1}{2}[a, a] = da + a \cdot a = 0\}, \text{ and}$$
$$\mathcal{MC} \quad := \quad MC/\sim,$$

where the equivalence is generated by the infinitesimal action of $A^0$ on $A$, where for $a \in A^0$, the vector field $\xi_x$ on $A$ is defined by,

$$\xi_x(a) = [x, a] - dx.$$

Recall that $\omega$ is a symplectic form and the infinitesimal action is Hamiltonian. Moreover, the map $\mu : a \mapsto \phi_a \in (A^{even})^*$, where

$$\phi(x) = \omega(da + \frac{1}{2}[a, a], x),$$

is the moment map corresponding to the Hamiltonian action above. One should think of the tangent space $T_{[a]}\mathcal{MC}$ at a class $[a]$ as the 3-term complex

(11)
$$T_{[a]}\mathcal{MC} : \quad T_{[a]}^{-1}\mathcal{MC} := A^{even} \xrightarrow{\xi(a)} T_{[a]}^0\mathcal{MC} := T_a A^{odd} = A^{odd} \xrightarrow{\mu'_a} T_{[a]}^1\mathcal{MC} := A^{even*},$$

graded by -1, 0 and 1. Here $\xi(a)$ is the map $x \mapsto \xi_x(a)$. The ker $\mu'$ is the Zarisky tangent space to $MC$ and the image of $\xi(a)$ accounts for the tangent space of the action orbit. Ideally, when 0 is a regular value for $\mu$ and the infinitesimal action of $A^{even}$ on $MC = \mu^{-1}(0)$ is free, this compex is concentrated in degree zero and the Zarisky tangent space to $\mathcal{MC}$ at $[a]$ is the cohomology group $H^0(T_{[a]}\mathcal{MC}) = H^*(A^{odd}, d_a)$ where $d_a b = db + [a, b]$.

Note that 3-term complex (11) is self-dual where the self-duality at the middle term is given by the symplectic form

(12)
$$\omega(X_a, Y_a) := \text{Tr}(X_a \cdot Y_a) \in k.$$

By assumption from the previous section, $\omega$ is non-degenerate. This gives rise to an isomorphism $T_{[a]}\mathcal{MC} \xrightarrow{\cong} (T_{[a]}\mathcal{MC})^*$ and equips $(T_{[a]}\mathcal{MC})$ with a symplectic form

given by (12). In the case of a nonsingular point $[a]$ this is the usual pairing on $H^0(T_{[a]}\mathcal{MC}) = H(A^{odd}, d_a)$ induced by $\omega$.

The function space $\mathcal{O}(\mathcal{MC})$ is defined to be the subspace of $\mathcal{O}(MC)$ invariant by the infinitesimal action. The symplectic form allows us to define the Hamiltonian vector field $X^\psi$ of a function $\psi \in \mathcal{O}(\mathcal{MC})$ via

$$\omega(X_a^\psi, Y_a) = \mathrm{d}\psi_a(Y_a) := \lim_{t \to 0} \frac{d}{dt}\psi(a + tY_a), \qquad \forall\, Y_a \in T_{[a]}^1 \mathcal{MC}.$$

We then define the Poisson bracket on $\mathcal{O}(\mathcal{MC})$ by,

$$\{\psi, \chi\} := \omega(X^\psi, X^\chi) = \mathrm{Tr}(X^\psi \cdot X^\chi).$$

## 4. The induced Lie map

In this section, we define a map $\rho : HC_-^{2\bullet}(A) \to \mathcal{O}(\mathcal{MC})$, and prove it respects the brackets. We start by defining a map $P : MC \to \bar{C}_\bullet(A, A)$, and in turn the map $R : MC \to \overline{CC}_\bullet^-(A)$ which factors through $P$. Dualizing $R$ will induce the wanted map $\rho$.

**Definition 8.** Recall that $MC = \{a \in A^{odd} \mid da + a \cdot a = 0\}$ and $\bar{C}_\bullet(A, A) = \prod_{n \geq 0} A \otimes \bar{A}^{\otimes n}$. Then, let $P : MC \to \bar{C}_\bullet(A, A)$ be given by the expression,

$$P(a) := \sum_{i \geq 0} 1 \otimes a^{\otimes i} = (1 \otimes 1) + (1 \otimes a) + (1 \otimes a \otimes a) + \cdots.$$

Notice that for $a \in MC$, it is $\delta(P(a)) = \sum 1 \otimes a \otimes \cdots \otimes da \otimes \cdots \otimes a + \sum 1 \otimes a \otimes \cdots \otimes (a \cdot a) \otimes \cdots \otimes a = 0$, due to the relation $da + a \cdot a = 0$ in $MC$. Thus, we obtain in fact a Hochschild homology class $[P(a)] \in HH_\bullet(A, A)$.

Next, define the map $R := inc \circ P$ as the composition $R : MC \xrightarrow{P} \bar{C}_\bullet(A, A) \xrightarrow{inc} \overline{CC}_\bullet^-(A)$. Just as above, we have that $\delta(R(a)) = 0$, and since we are in the normalized setting, we see that $B(R(a)) = 0$, so that $(\delta + uB)(R(a)) = 0$. The induced negative cyclic homology class is again denoted by $[R(a)] \in HC_\bullet^-(A)$. It is immediate to see that under the long exact sequence (8), we have that $I(R(a)) = P(a)$.

Using the pairing between between negative cyclic homology and negative cyclic cohomology, $\langle \cdot, \cdot \rangle : HC_-^\bullet(A) \otimes HC_\bullet^-(A) \to k$, we define the map $\rho$ by

$$\rho : HC_-^\bullet(A) \quad \to \quad \mathcal{O}(\mathcal{MC}),$$
$$\rho([\alpha])([a]) \quad := \quad \langle [\alpha], [R(a)] \rangle = \langle \alpha, R(a) \rangle, \qquad \text{for } [\alpha] \in HC_-^\bullet(A), [a] \in \mathcal{MC}.$$

To simplify notation, we will also write $\rho(\alpha)$ instead of $\rho([\alpha])$.

**Lemma 9.** $\rho$ is well-defined.

*Proof.* We need to show that the value $\rho([\alpha])([a]) = \langle \alpha, R(a) \rangle$ is independent of the choice of the representative $[a] \in \{x \in A^{odd} \mid dx + x \cdot x = 0\}/ \sim$. Infinitesimally, this amounts to showing that $L_{X(b)}\rho(\alpha)(a) = 0$, where $L_{X(b)}$ is the Lie derivative along a vector field in the direction $X(b)_a = db + [a, b] \in T_{[a]}MC$, for any $b \in A^{even}$. To see this, note that

$$\begin{aligned} L_{X(b)}\rho(\alpha)(a) &= (i_{X(b)} \circ d + d \circ i_{X(b)})\rho(\alpha)(a) \\ &= i_{X(b)} \circ d(\rho(\alpha))(a) \\ &= \langle \alpha, \frac{d}{dt}\Big|_{t=0} R(a + tX(b)_a) \rangle \end{aligned}$$

Now, for any $Y_a \in T_{[a]}MC$, we have

$$(13) \qquad \frac{d}{dt}\Big|_{t=0} R(a + tY_a) = 1 \otimes Y_a + 1 \otimes Y_a \otimes a + 1 \otimes a \otimes Y_a + \cdots$$
$$= \mathcal{B}(Y_a + (Y_a \otimes a) + (Y_a \otimes a \otimes a) + \cdots),$$

where we used Connes operator $\mathcal{B} : \bar{C}_\bullet(A, A) \to \overline{CC}_\bullet^-(A)$ from in the long exact sequence (8) applied to $Y_a + (Y_a \otimes a) + (Y_a \otimes a \otimes a) \in \bar{C}_\bullet(A, A)$. Thus, setting $Y_a = X(b)_a = db + [a, b]$ in the above expression, we obtain

$$\begin{aligned} L_{X(b)}\rho(\alpha)(a) &= \langle \alpha, \mathcal{B}\big(db + [a, b] + db \otimes a + [a, b] \otimes a \\ &\qquad\qquad + db \otimes a \otimes a + [a, b] \otimes a \otimes a + \cdots\big)\rangle \\ &= \langle \alpha, \mathcal{B} \circ \delta(b + (b \otimes a) + (b \otimes a \otimes a) + \cdots)\rangle \\ &= \langle \alpha, \delta \circ \mathcal{B}(b + (b \otimes a) + (b \otimes a \otimes a) + \cdots)\rangle \\ &= \langle \delta^* \alpha, \mathcal{B}(b + (b \otimes a) + (b \otimes a \otimes a) + \cdots)\rangle \\ &= 0. \end{aligned}$$

$\square$

We are now ready to prove our main theorem.

**Theorem 1.** $\rho : HC_-^{2\bullet}(A) \to \mathcal{O}(\mathcal{MC})$ *is a map of Lie algebras.*

*Proof.* We saw in (13) that $\frac{d}{dt}\big|_{t=0}R(a + tY_a) = \mathcal{B}(Y_a + (Y_a \otimes a) + (Y_a \otimes a \otimes a) + \cdots) \in \overline{CC}_\bullet^-(A)$, where $(Y_a + (Y_a \otimes a) + (Y_a \otimes a \otimes a) + \cdots) \in \bar{C}_\bullet(A, A)$ for $Y_a \in T_{[a]}\mathcal{MC}$. Therefore,

$$\begin{aligned} (d\rho(\alpha))_a(Y_a) &= \langle \alpha, \frac{d}{dt}\big|_{t=0} R(a + tY_a)\rangle \\ &= \langle \alpha, \mathcal{B}(Y_a + (Y_a \otimes a) + (Y_a \otimes a \otimes a) + \cdots)\rangle \\ &= \langle \mathcal{B}^* \alpha, Y_a + (Y_a \otimes a) + (Y_a \otimes a \otimes a) + \cdots\rangle \\ &= (\mathcal{B}^* \alpha)(1 + a + a \otimes a + \cdots)(Y_a), \end{aligned}$$

where $\alpha \in \overline{CC}_-^\bullet(A), \mathcal{B}^*\alpha \in \bar{C}^\bullet(A, A^*)$, and thus $(\mathcal{B}^*\alpha)(\sum_{i \geq 0} a^{\otimes i}) \in A^*$. Now, using the isomorphism $\omega_\sharp^\bullet : HH^\bullet(A, A) \to HH^\bullet(A, A^*)$ from definition 4, we apply its inverse to obtain an element $[f_\alpha] := (\omega_\sharp^\bullet)^{-1}\mathcal{B}^\bullet[\alpha] \in HH^\bullet(A, A)$. We then claim that the Hamiltonian vector field $X_a^{\rho(\alpha)}$ may be expressed as

$$(14) \qquad X_a^{\rho(\alpha)} = f_\alpha\Big(\sum_{i \geq 0} a^{\otimes i}\Big) \quad \in T_{[a]}\mathcal{MC}.$$

This should be compared with [AZ, Lemma 7.2] and [Go, Proposition 3.7]. To this end, first note, that the relation $0 = (\delta^* f)(\sum_{i \geq 0} a^{\otimes i}) = d_a(f(\sum_{i \geq 0} a^{\otimes i}))$, for $f \in \bar{C}^\bullet(A, A)$, shows that $X_a^{\rho(\alpha)}$ given by equation (14), represents a well-defined class in $T_{[a]}\mathcal{MC}$. We show (14), by applying the non-degeneracy of $\omega$ in the following equation, which is valid for any $Y_a \in T_{[a]}\mathcal{MC}$,

$$\begin{aligned} \omega(f_\alpha(\sum a^{\otimes i}), Y_a) &= \mathrm{Tr}(f_\alpha(\sum a^{\otimes i}) \cdot Y_a) = (\omega_\sharp f_\alpha)(\sum a^{\otimes i})(Y_a) \\ &= (\mathcal{B}^* \alpha)(\sum a^{\otimes i})(Y_a) = (d\rho(\alpha))_a(Y_a) = \omega(X_a^{\rho(\alpha)}, Y_a). \end{aligned}$$

Now, calculating the Lie bracket gives

$$
\begin{aligned}
\rho(\{\alpha,\beta\})(a) &= \langle\{[\alpha],[\beta]\},[R(a)]\rangle \\
&= \langle I^\bullet(\mathcal{B}^\bullet[\alpha] \sqcup \mathcal{B}^\bullet[\beta]),[R(a)]\rangle \\
&= \langle I^\bullet \omega_\sharp^\bullet((\omega_\sharp^\bullet)^{-1}\mathcal{B}^\bullet[\alpha] \cup (\omega_\sharp^\bullet)^{-1}\mathcal{B}^\bullet[\beta]),[R(a)]\rangle \\
&= \langle \omega_\sharp^\bullet([f_\alpha] \cup [f_\beta]), I_\bullet[R(a)]\rangle \\
&= \langle \omega_\sharp^\bullet([f_\alpha] \cup [f_\beta]),[P(a)]\rangle.
\end{aligned}
$$

To evaluate this expression, note that for $f_\alpha : \bar{A}^{\otimes m} \to A$ and $f_\beta : \bar{A}^{\otimes n} \to A$, $\omega_\sharp^\bullet([f_\alpha] \cup [f_\beta])$ is represented by the composition

$$
\bar{A}^{\otimes m+n} \xrightarrow{f_\alpha \otimes f_\beta} A \otimes A \xrightarrow{\cdot} A \xrightarrow{\omega} A^*.
$$

The first arrow with $f_\alpha \otimes f_\beta$ applied to $P(a) = 1 + (1 \otimes a) + (1 \otimes a \otimes a) + \cdots \in \prod_{i\geq 0} A \otimes \bar{A}^{\otimes i}$ then gives an expression, where we apply $a$ to all possible inputs in $\bar{A}^{\otimes n+m}$. To this, we then apply the product in $A$, and apply $\omega$ with input $1 \in A$, since $P(a) = 1 \otimes (\cdots)$. We thus obtain

$$
\begin{aligned}
\rho(\{\alpha,\beta\})(a) &= \mathrm{Tr}\Big(f_\alpha(1 + a + a \otimes a + \cdots) \cdot f_\beta(1 + a + a \otimes a + \cdots) \cdot 1\Big) \\
&\overset{(14)}{=} \mathrm{Tr}(X_a^{\rho(\alpha)} \cdot X_a^{\rho(\beta)}) = \omega(X_a^{\rho(\alpha)}, X_a^{\rho(\beta)}) = \{\rho(\alpha),\rho(\beta)\}(a).
\end{aligned}
$$

This is the claim of the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

## 5. Comparison with generalized holonomy

In this section we compare the map $\rho$ with the generalized holonomy map $\Psi$ studied in [AZ]. The relationship may be summarized in the diagram (2). This shows how a special case the result of this paper relates to the main theorem of [AZ]. The map $\mathrm{Tr} : A \to \mathbb{C}$ is induced by the trace function on $\mathfrak{g} \subseteq \mathrm{GL}(n,\mathbb{C})$ and integration of forms on $M$; see Example 6.

Our model of $S^1$-equivariant de Rham forms of $LM$ is $(\Omega(LM)[u], d+u\Delta)$ where $u$ is a generator of degree 2 and $\Delta : \Omega^\bullet(LM) \to \Omega^{\bullet-1}(LM)$ is the map induced by the $S^1$ action on $LM$; see [GJP]. This model is quasi-isomorphic to the small Cartan model $(\Omega_{inv}(LM)[u], d+i_X u)$ for the $S^1$ action, where $X$ is the fundamental vector field generated by the natural action of $S^1$. The quasi-isomorphism is given by the averaging map $\Omega^\bullet(LM) \to \Omega^\bullet_{inv}(LM)$. More explicitly, for $\omega \in \Omega^\bullet(LM)$, $\Delta(\omega)$ is given by,

$$
\Delta(\omega) = \int_{fibre} ev^*(\omega) \in \Omega^{\bullet-1}(LM)
$$

(15)
$$
\begin{array}{c}
S^1 \times LM \xrightarrow{ev} LM \\
\downarrow{\scriptstyle\pi} \\
LM
\end{array}
$$

Chen's iterated integral map and the trace map on $\mathfrak{g}$ (see (6.3) [AZ], and Theorem A in [GJP]) yields a map, which we denote by,

$$
S : (\bar{C}_\bullet(A,A), \delta) \to (\Omega^\bullet(LM), d).
$$

$S$ induces the map $S^{HH} : HH_\bullet(A, A) \longrightarrow H^\bullet(LM)$ on homology, and, after applying the pairing between homology and cohomology groups, we get,

$$H_\bullet(LM) \xrightarrow{\sigma^{HH}} HH^\bullet(A, A^*).$$

Extending $S$ by $u$-linearity, we obtain a map, which we denote by abuse of notation by the same letter,

$$S : (\bar{C}_\bullet(A, A)[u], \delta + uB) \to (\Omega^\bullet(LM)[u], d + u\Delta).$$

Since, by Lemma 3, $(\bar{C}_\bullet(A, A)[u], \delta + uB)$ and $(\bar{C}_\bullet(A, A)[[u]], \delta + uB)$ are quasi-isomorphic in our setting, we obtain the induced map $S^{HC} : HC_\bullet^-(A) \longrightarrow H_{S^1}^\bullet(LM)$ on homology. Composing $S^{HC}$ with the map $R : MC \to \overline{CC}_\bullet^-(A) = \bar{C}_\bullet(A, A)[u]$ from Section 4, we get,
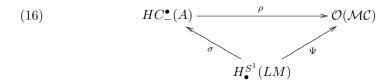
$$MC \xrightarrow{R} HC_\bullet^-(A) \xrightarrow{S^{HC}} H_{S^1}^\bullet(LM).$$

Thus by duality, and using Lemma 9, we have,

$$H_\bullet^{S^1}(LM) \xrightarrow{\sigma = \sigma^{HC}} HC_-^\bullet(A) \xrightarrow{\rho} \mathcal{O}(\mathcal{MC}).$$

The composition $\rho \circ \sigma$ is the generalized holonomy map $\Psi$ discussed in [AZ].

(16)

$$
\begin{array}{ccc}
HC_-^\bullet(A) & \xrightarrow{\quad\rho\quad} & \mathcal{O}(\mathcal{MC}) \\
& \nwarrow_{\sigma} \qquad \nearrow_{\Psi} & \\
& H_\bullet^{S^1}(LM) &
\end{array}
$$

It was proved in [AZ], that $\Psi$ is the morphism of Lie algebras. We will shortly see how this is a consequence of Theorem 1. We first recall the following theorem.

**Theorem 10** (S. Merkulov [Mer]). *The Chen integral induces a map of algebras* $(H_\bullet(LM), \bullet) \to (HH^\bullet(A, A), \cup)$.

Thus, by definition, $\sigma^{HH} : (H_\bullet(LM), \bullet) \to (HH^\bullet(A, A^*), \sqcup)$ is also a map of algebras. With this, we can now prove the following statement.

**Theorem 11.** *The map induced by the Chen iterated integrals* $\sigma : (H_\bullet^{S^1}(LM), \{\cdot, \cdot\}) \to (HC_-^\bullet(A), \{\cdot, \cdot\})$ *is a map of Lie algebras. Here, the first bracket is the string bracket and the second one is defined in the statement of Proposition 5.*

*Proof.* The brackets on $H_\bullet^{S^1}(LM)$ and $HC_-^\bullet(A)$ are determined by the products on $H_\bullet(LM)$ and $HH^\bullet(A, A^*)$, together with the maps in the corrsponding Gysin long exact sequences. By Theorem 10, it thus remains to show that the long exact sequences correspond to each other, *i.e.* that the following diagrams commute,

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & H_\bullet^{S^1}(LM) & \xrightarrow{m_\bullet} & H_{\bullet+1}(LM) & \xrightarrow{e_\bullet} & H_{\bullet+1}^{S^1}(LM) & \longrightarrow & H_{\bullet-1}^{S^1}(LM) & \longrightarrow \cdots \\
& \downarrow{\sigma} & & \downarrow{\sigma^{HH}} & & \downarrow{\sigma} & & \downarrow{\sigma} & \\
\cdots \longrightarrow & HC_-^\bullet(A) & \xrightarrow{\mathcal{B}^\bullet} & HH^{\bullet+1}(A, A^*) & \xrightarrow{I^\bullet} & HC_-^{\bullet+1}(A) & \longrightarrow & HC_-^{\bullet-1}(A) & \longrightarrow \cdots
\end{array}
$$

Equivalently, we need to show the commutativity of the following dual sequence,

$$\cdots \longrightarrow HC_\bullet^-(A) \xrightarrow{I_\bullet} HH_\bullet(A,A) \xrightarrow{\mathcal{B}_\bullet} HC_{\bullet-1}^-(A) \longrightarrow HC_{\bullet-1}^-(A) \longrightarrow \cdots$$

$$\downarrow S^{HC} \qquad\qquad \downarrow S^{HC} \qquad\qquad \downarrow S^{HH} \qquad\qquad \downarrow S^{HC}$$

$$\cdots \longrightarrow H_{S^1}^\bullet(LM) \xrightarrow{e^\bullet} H^\bullet(LM) \xrightarrow{m^\bullet} H_{S^1}^{\bullet-1}(LM) \longrightarrow H_{S^1}^{\bullet-1}(LM) \longrightarrow \cdots$$

The top long exact sequence is induced by the short exact sequence (7) while the bottom one is induced by the short exact sequence

$$(17) \qquad 0 \to (\Omega^\bullet(M)[u], d+u\Delta) \xrightarrow{\times u} (\Omega^\bullet(M)[u], d+u\Delta) \xrightarrow{j} \Omega^\bullet(M) \to 0,$$

where $j(\sum a_i u^i) = a_0$, *cf.* [GS, Ma]. In this picture, $m^\bullet$ corresponds to the connecting map of the long exact sequence (17). By a diagram chasing argument one finds that $m^\bullet = (i \circ \Delta)^\bullet$ where $i : \Omega^\bullet(M) \hookrightarrow \Omega^\bullet(M)[u]$ corresponds to $\mathcal{B}^\bullet = (inc \circ B)^\bullet$ using Chen iterated integrals as corollary of Thoerem A in [GJP]. Note that $i$ is not a chain map, whereas $i \circ \Delta$ is a chain map, since $\Delta d = d\Delta$ and $\Delta^2 = 0$, (*cf.* [GJP]). $\qquad\square$

## 6. $A_\infty$ GENERALIZATION

The previous sections, given for the case of dgas $(A, d, \cdot)$ with invariant inner product $\omega : A \otimes A \to k$, generalize in a straightforward way to the setting of cyclic $A_\infty$ algebras. In this section, we recall the relevant definitions (*cf.* [T]), and adopt the above to this situation.

**Definition 12.** An $A_\infty$ algebra on $A$ consists of a sequence of maps $\{\mu_n\}_{n\geq 1}$, where $\mu_n : A^{\otimes n} \to A$ is of degree $(2-n)$, such that

$$\forall n \geq 1 : \sum_{\substack{k+l=n+1 \\ r=0,\cdots,n-l}} (-1)^{\epsilon_l^r} \cdot \mu_k(a_1 \otimes \cdots \otimes \mu_l(a_{r+1} \otimes \cdots \otimes a_{r+l}) \otimes \cdots \otimes a_n) = 0,$$

where $\epsilon_l^r = (l-1) \cdot (|a_1| + \cdots + |a_r| - r)$. A unit is an element $1 \in k \subset A^0$ such that $\mu_2(a, 1) = \mu_2(1, a) = a$, and $\mu_n(\cdots \otimes 1 \otimes \cdots) = 0$ for $n \neq 2$. Again, we write $\bar{A} = A/k$. We define the Hochschild chain complex of $A$ with values in $A$ or $A^*$ to be the vector spaces $\bar{C}_\bullet(A, A)$ and $\bar{C}_\bullet(A, A^*)$ from equation (3) with the differentials modified as follows,

$$\delta : \bar{C}_\bullet(A, A) \to \bar{C}_\bullet(A, A), \delta(a_0 \otimes \cdots \otimes a_n) = \sum \pm a_0 \otimes \cdots \otimes \mu_k(\cdots) \otimes \cdots \otimes a_n$$

$$+ \sum \pm \mu_k(a_s \otimes \cdots \otimes a_0 \otimes \cdots \otimes a_r) \otimes a_{r+1} \otimes \cdots \otimes a_{s-1},$$

$$\delta : \bar{C}_\bullet(A, A^*) \to \bar{C}_\bullet(A, A^*), \delta(a_0^* \otimes \cdots \otimes a_n) = \sum \pm a_0^* \otimes \cdots \otimes \mu_k(\cdots) \otimes \cdots \otimes a_n$$

$$+ \sum \pm \mu_k^*(a_s \otimes \cdots \otimes a_0^* \otimes \cdots \otimes a_r) \otimes a_{r+1} \otimes \cdots \otimes a_{s-1},$$

where $\mu_k^*(a_s \otimes \cdots \otimes a_0^* \otimes \cdots \otimes a_r) \in A^*$ is given by

$$\mu_k^*(a_s \otimes \cdots \otimes a_n \otimes a_0^* \otimes a_1 \otimes \cdots \otimes a_r)(a) := \pm a_0^*(\mu_k(a_1 \otimes \cdots \otimes a_r \otimes a \otimes a_s \otimes \cdots \otimes a_n)).$$

Here, the signs are given by the usual Koszul rule, where we a factor of $(-1)^{\epsilon\epsilon'}$ is introduced, whenever elements of degree $\epsilon$ and $\epsilon'$ are being commuted. For an

explicit discussion of the signs, see *e.g.* [T]. Similarly, $\bar{C}^\bullet(A, A)$ and $\bar{C}^\bullet(A, A^*)$ are defined by the spaces from (4) with the modified differentials

$$\delta^* : \bar{C}^\bullet(A, A) \to \bar{C}^\bullet(A, A), \quad \delta^* f(a_1 \otimes \cdots \otimes a_n)$$
$$= \sum \pm f(a_1 \otimes \cdots \otimes \mu_k(\cdots) \otimes \cdots \otimes a_n) + \sum \pm \mu_k(a_1 \otimes \cdots \otimes f(\cdots) \otimes \cdots \otimes a_n),$$

$$\delta^* : \bar{C}^\bullet(A, A^*) \to \bar{C}^\bullet(A, A^*), \quad \delta^* f(a_1 \otimes \cdots \otimes a_n)$$
$$= \sum \pm f(a_1 \otimes \cdots \otimes \mu_k(\cdots) \otimes \cdots \otimes a_n) + \sum \pm \mu_k^*(a_1 \otimes \cdots \otimes f(\cdots) \otimes \cdots \otimes a_n).$$

Since $\delta^2 = 0, (\delta^*)^2 = 0$ in all the above cases, we obtain the associated homologies and cohomologies $H_\bullet(A, A), H_\bullet(A, A^*), H^\bullet(A, A)$, and $H^\bullet(A, A^*)$.

There is a generalized cup product $\cup$ on $H^\bullet(A, A)$ induced by,

$$(f \cup g)(a_1 \otimes \cdots \otimes a_n) := \sum_{k \geq 2} \pm \mu_k(a_1 \otimes \cdots \otimes f(\cdots) \otimes \cdots \otimes g(\cdots) \otimes \cdots \otimes a_n).$$

Furthermore, equation (5) defines an operator $B : \bar{C}_\bullet(A, A) \to \bar{C}_\bullet(A, A)$ with $B^2 = \delta B + B\delta = 0$. We define the negative cyclic chains $\overline{CC}_\bullet^-(A)$ of $A$ to be the vector space $\bar{C}_\bullet(A, A)[[u]]$ with differential $\delta + uB$, and denote the negative cyclic homology by $HC_\bullet^-(A)$. Dualizing $\overline{CC}_\bullet^-(A)$, we obtain $\overline{CC}_-^\bullet(A)$ with dual differential and denote the negative cyclic cohomology by $HC_-^\bullet(A)$. For the same reasons as in Section 2, we obtain the long exact sequences (8) and (9).

Finally, assume we have a trace $\text{Tr} : A \to k$, such that the associated map $\omega : A \otimes A \to k, \omega(a, b) = \text{Tr}(\mu_2(a \otimes b))$ is a quasi-isomorphism, which satisfies for $n \geq 1$,

$$(18) \qquad \omega(\mu_n(a_1 \otimes \cdots \otimes a_n), a_{n+1}) = \pm \omega(\mu_n(a_{n+1} \otimes a_1 \otimes \cdots \otimes a_{n-1}), a_n),$$

In this case, $\omega : A \to A^*$ induces a map of the Hochschild cohomologies $H^\bullet(A, A) \to H^\bullet(A, A^*), \omega_\sharp^\bullet(f) = \omega \circ f$, which we assume to be an isomorphism. Thus, we may transfer the product $\cup$ on $H^\bullet(A, A)$ to a product $\sqcup$ on $H^\bullet(A, A^*)$. $(HH^\bullet(A, A^*), \sqcup, \Delta = B^*)$ is a BV-algebra, *cf.* [T], so that we obtain the Lie bracket $\{a, b\} := I^\bullet(\mathcal{B}^\bullet(a) \sqcup \mathcal{B}^\bullet(b))$ on $HC_-^\bullet(A)$ just as in Proposition 5.

Using this setup, we may now also generalize Section 3.

**Definition 13.** Recall that there are maps from the the $n^{th}$ symmetric power of a vector space to the $n^{th}$ tensor power $S^n : A^{\wedge n} \to A^{\otimes n}$, where $S^n(a_1 \wedge \cdots \wedge a_n) = \sum_{\sigma \in \Sigma_n} (-1)^{\epsilon_\sigma}(a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)})$. Defining $\nu_n : A^{\wedge n} \to A$ as $\nu_n := \mu_n \circ S^n$, we obtain an $L_\infty$ algebra on $A$, *cf.* [LM, Theorem 3.1]. Furthermore, from (18), it is immediate to see that we have for $n \geq 1$,

$$\omega(\nu_n(a_1 \wedge \cdots \wedge a_n), a_{n+1}) = \pm \cdot \omega(\nu_n(a_{n+1} \wedge a_1 \wedge \cdots \wedge a_{n-1}), a_n).$$

For this $L_\infty$ algebra, recall from [GG, Section 2] that the Maurer-Cartan solutions are defined by,

$$MC := \left\{ a \in A^1 \mid \nu_1(a) + \frac{1}{2!}\nu_2(a \wedge a) + \frac{1}{3!}\nu_3(a \wedge a \wedge a) + \cdots = 0 \right\}, \text{ and}$$
$$\mathcal{MC} := MC/\sim,$$

where the equivalence is again generated by the infinitesimal action of $A^0$ on $A^1$, where for $a \in A^0$, the vector field $\xi_x$ on $A^1$ is defined by,

$$\xi_x(a) = \nu_1(x) + \nu_2(a \wedge x) + \frac{1}{2!}\nu_3(a \wedge a \wedge x) + \cdots .$$

Note, that under the above assumptions the tangent space to $\mathcal{MC}$ at $[a]$ is the self-dual 3-term complex,

$$(19) \quad T_{[a]}\mathcal{MC} : \quad T_{[a]}^{-1}\mathcal{MC} := A^0 \xrightarrow{\xi(a)} T_{[a]}^0\mathcal{MC} := T_aA^1 = A^1 \xrightarrow{\mu_a'} T_{[a]}^1\mathcal{MC} := A^{0*},$$

where

$$\mu_a'(b) = \nu_1(b) + \nu_2(a \wedge b) + \frac{1}{2!}\nu_3(a \wedge a \wedge b) + \cdots .$$

The self-duality at the middle term is given by the symplectic form

$$\omega(X_a, Y_a) = \mathrm{Tr}(\mu_2(X_a \otimes Y_a)) \in k.$$

This can be used to define the Hamiltonian vector field $X^\psi$ associated to a function $\psi \in \mathcal{O}(\mathcal{MC})$, and thus the Lie bracket on $\mathcal{O}(\mathcal{MC})$ via the usual formula $\{\psi, \chi\} = \omega(X^\psi, X^\chi)$.

We may now define the map $P : MC \to \bar{C}_\bullet(A, A)$ by

$$P(a) := \sum_{i \geq 0} 1 \otimes a^{\otimes i} = (1 \otimes 1_{\bar{A}^{\otimes 0}}) + (1 \otimes a) + (1 \otimes a \otimes a) + \cdots ,$$

and $R = inc \circ P : MC \to \overline{CC}_\bullet^-(A)$. As in definition 8, we may again see, that $\delta(P(a)) = 0$, and $(\delta + uB)(R(a)) = 0$, and we define

$$\rho : HC_-^{2\bullet}(A) \quad \to \quad \mathcal{O}(\mathcal{MC}),$$
$$\rho([\alpha])([a]) \quad := \quad \langle [\alpha], [R(a)] \rangle = \langle \alpha, R(a) \rangle, \qquad \text{for } [\alpha] \in HC^\bullet(A), [a] \in \mathcal{MC}.$$

With this, we have the same theorem as in the previous sections.

**Theorem 14.** *The map $\rho$ is a well-defined map of Lie algebras.*

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