ON SEVERAL VARIETIES OF CACTI AND THEIR RELATIONS

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Abstract. We consider four operads which are different varieties of the cacti operad. These are cacti without local zeros (or spines) and cacti proper as well as both varieties with fixed constant size one of the constituting loops. We show that both types of cacti are a semi–direct products as operads of their normalized versions with a re–scaling operad based on $\mathbb{R}_{>0}$ which are homotopic to direct products. Furthermore, we introduce the notion of bi–crossed operads and show that the cacti proper are a bi–crossed product of the operad of cacti without spines and the operad based on the monoid given by the circle group S^1. We also prove that this particular bi–crossed operad product is homotopy equivalent to the semi–direct product of the spineless cacti with the group S^1. Lastly we recall how to realize the cacti operads as suboperads of the Arc operad and show that the cacti are generated as a suboperad by the cacti without spines and a S^1 twist whose corresponding cycle represents the BV operator.

Introduction

The cacti operad was introduced by Voronov [V] to give an operadic interpretation of the string bracket and Batalin–Vilkovisky structure found by Chas and Sullivan [CS]. In [CS] the BV structure comes from the overall S^1 equivariant setting while [V] shows that this structure can be further understood by giving a homotopy equivalence of the cacti operad with the framed little discs operad which governs BV algebras as demonstrated by Getzler [Ge]. Moreover there is a nice formulation of the action of cacti using a push–forward or "Umkehr" map in homology by using a Thom–Pontrjagin construction [V,CJ]. Taking this point of view one obtains a spectra version of the construction [CJ].

In [KLP] we constructed an embedding of an operad which is close to cacti, has less structure, into a more general operad composed of surfaces with embedded weighted arcs and gluing of surfaces with these

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arcs and showed that this operad generates a Gerstenhaber bracket, which becomes BV when another operation inherent in the Arc operad –the Fenchel–Nielsen type twist– is also taken into consideration. This should and can be compared to the relationship between framed little discs and little discs, as we show below. Moreover in [KLP] we had to use certain homotopies to realize the key homologies. These explicit homologies had one more restriction, that of a normalization, in common which is not stable under composition.

This prompts us to define different species of cacti and to study their relations. These different species consist of the original cacti, cacti without additionally marked local zeros which we call cacti without spines and lastly for both versions their normalized counterparts, which are made up of circles of radius 1. It is actually a little surprising that one can go through the whole theory with normalized cacti. Here the gluing rules are a little different, but for instance yield an associative product already on the level of chains, a fact that can be compared to Moore loops.

The exact statement is that the unnormalized versions of cacti are equivalent as operads to the operadic semi–direct product of the normalized version with a scaling operad, which we define on the spaces $\mathbb{R}^n_{>0}$, that controls the radii of the circles. We also show that this semi–direct product is homotopic to the direct product. This makes the two operads homotopy equivalent as spaces, but furthermore the products are also compatible up to homotopy, so that the two operads are homotopy equivalent.

The relationship between framed little discs and little discs is that the framed little discs are a semi–direct product of the little discs with the operad built on the circle group S^1 [SW]. Actually for this construction which we review below one only needs a monoid.

The relationship for the cacti is a little more involved. For this we define the notion of bi–crossed products of operads which is an extension of the bi–crossed product of groups [Ka,Tak]. We show that cacti are a bi–crossed product of spineless cacti with an operad built on S^1. By analyzing the construction of the operads built on monoids in a symmetric tensor category where the tensor product is a product, we can relate the bi–crossed structure to the semi–direct product. To be precise we show that the particular bi–crossed product giving rise to cacti is homotopy equivalent to the semi–direct product of cacti without spines and the group operad built on S^1 thus showing that cacti without spines are homotopy equivalent to the little discs operad.

Along the way we give several other pictorial realizations of the various types of cacti including trees and chord diagrams, which might be useful to relate this theory to other parts of mathematics. In particular the trees with grafting are reminiscent of the Connes-Kreimer operads [CK] while the chord diagram approach is close to Kontsevich's graph realization of the Chern–Simons theory (cf. e.g. [BN]).

Turning back to the Arc operad, we can see that the bi–crossed product corresponding to cacti is realized as the suboperad generated by the suboperad of cacti without spines and the Fenchel–Nielsen type twist. We would also like to point out that many of the other features of cacti such as the gluing, the chord diagram description and the perimeter used in the gluing are natural in the Arc operad. The symmetries and asymmetries are also clearly visible, as e.g. the distinction of one boundary component, which breaks the cyclic structure of the Arc operad in the cacti suboperad. Lastly, we would like to remark that the Arc operad has a natural generalization to any genus and number of punctures thus giving a straightforward generalization of cacti to higher genus and to punctured surfaces.

The paper is organized as follows.

In §1 we introduce the notation and all the varieties of cacti we wish to consider. In addition, we provide several pictorial realizations of these objects.

In the second paragraph §2, we start by introducing an operad called operad of spaces which can be defined in any symmetric tensor category with products such as topological spaces with Cartesian product. This operad lends itself to the description of the semi–direct product with a monoid whose construction we also review. More generally we define semi–direct products for operads and lastly we introduce the new notion of bi–crossed product for operads which is modeled on the bi–crossed product for groups.

In section §3 we apply the theory of §2 to cacti. First we prove that the unnormalized versions of the (spineless) cacti operads are the semi–direct products of their normalized version and a rescaling operad built on $\mathbb{R}^n_{>0}$. Moreover we show that this semi–direct product is homotopy equivalent to a direct product. Section §3 also contains the main result that is that the cacti are a bi–crossed product of cacti without spines and the operad built on the group S^1 which is turn is homotopic to the semi–direct product of these operads. As a corollary, we obtain that the cacti without spines are homotopy equivalent to the little discs operad.

In section §4, we briefly consider the Arc operad and the suboperads corresponding to the various cacti operads and point out that the bi–crossed product is naturally realized in the arc operad as the suboperad generated by the image of cacti without spines and the Fenchel–Nielsen type twist. Incidentally it is these twists which provide the cycle corresponding to the BV operator in cohomology.

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1. Cacti

1.1. General setup.

1.1.1. **Notation.** By an S^1 in the plane we will mean by abuse of notation the image of the standard $S^1 \subset \mathbb{R}^2$ with the induced metric and orientation under an orientation preserving embedding $f : S^1 \to \mathbb{R}^2$. A configuration of S^1's in the plane is the union of the images of several of these maps.

By an S^1_r in the plane we will mean by abuse of notation the image of the standard circle of radius r: $S^1_r \subset \mathbb{R}^2$ with the induced metric and orientation under an orientation preserving embedding $f : S^1_r \to \mathbb{R}^2$.

A configuration of S^1s respectively S^1_rs in the plane is the union of the images of several of these maps.

1.1.2. **Reparametrizations.** Notice that a circle in a plane comes with a natural parameter. This is inherited from the natural parameter θ_r of the standard parameterization of $S^1_r : (r\cos(\theta_r), r\sin(\theta_r))$. Sometimes we have to reparameterize a circle, so that its length changes. We of course do not use arbitrary reparameterizations, but only the type which preserves the relative position of intersection and marked points. To be precise let $f : S^1_r \to \mathbb{R}^2$ be a parameterization of a circle in the plane. Then $f_R : S^1_R \to \mathbb{R}^2$ called the reparameterization to length R is defined to be the map $f_R = f \circ rep^R_r$ with $rep^R_r : S^1_R \to S^1_r$ given by $\theta_R \mapsto \theta_r$.

1.1.3. **Dual black and white graph.** Given a configuration of S^1s in the plane with at most finitely many intersection points we can associate to it a dual graph in the plane. This is a graph with two types of vertices, white and black. The first set of vertices is given by replacing each circle by a white vertex. The second set of vertices is given by replacing the intersection points with black vertices. The edges run only from white to black vertices, where we join two such vertices if the intersection point corresponding to the black vertex lies on the circle represented by the white vertex.

We remark that all the S^1s are pointed by the image of $0 = (0, 1)$. On any given S^1 in the plane we will call this point zero or base point.

If we are dealing with circles of radii different from one we will label the vertices of the trees by the radius of the respective circle.

1.1.4. **Cacti and trees.** The configurations corresponding to cacti will all have trees as their dual graphs. We would like to point out that these trees are planar trees, i.e. they are realized in the plane or equivalently they have a cyclic order.

Moreover we would like to consider rooted trees, i.e. trees with one marked vertex called root. Recall that specifying a root induces a natural orientation for the tree and a height function on vertices. The orientation of edges is toward the root and the height of a vertex is the number of edges traversed by the unique shortest path from the vertex to the root. Due to the orientation we can speak of incoming and outgoing edges, where the outgoing edge is unique and points toward the root. Naturally the edges point from the higher vertices to the lower vertices.

We call a configuration of S^1s in the plane rooted if one of the circles is marked by a point, we also call this point the global zero. In this case, we include a black vertex in the dual graph for this marked point and make this the root, so that the dual graph of a rooted configuration is a rooted tree.

1.1.5. **The perimeter or outside circle and the global zero.** Given a rooted configuration of S^1s in the plane whose dual graph is a rooted tree there is a surjective map of S^1 to this configuration which is a local embedding. This map is given by starting at the marked point or global zero of the root of the configuration going around this circle in the positive sense until one hits the first intersection point and then starting to go around the next circle in the cyclic order again in the positive direction until the next intersection point and so on until one again reaches the zero of the root.

We will call this map and, by abuse of notation, its image the perimeter or the outside circle of the configuration. We will also call the zero of the perimeter the global zero.

1.2. Normalized cacti without spines.

1.2.1. **Definition.** We define *normalized cacti without spines* to be labelled rooted collections of parameterized S^1s grafted together in a tree like fashion with the gluing points being the zeros of the S^1. More precisely we set:

$Cact^1(n)$:= {rooted configurations of n labelled S^1s in the plane with at most finitely many points of intersection whose dual black and white graph is a connected tree such that at the points of intersection the circles of greater height all have zero as their point of intersection }/ isometric isotopies preserving the incidence conditions.

This space has an obvious topology by moving the intersection points. There is an obvious action of $\mathbb{S}_n$ by permuting the labels.

We would like to point out that there is only one zero which is not necessarily an intersection point, namely that of the root. It can however also be an intersection point. Moreover, one could rewrite the condition of having a dual black and white graph that is a tree in the form: given any two circles their intersection is at most one point.

1.2.2. **Gluing.** We define the following operations

$$\circ_i : Cact^1(n) \times Cact^1(m) \to Cact^1(n+m-1) \tag{1.1}$$

by the following procedure: given two normalized cacti without spines we reparameterize the i–th component circle of the first cactus to have length m and glue in the second cactus by identifying the outside circle of the second cactus with the i–th circle of the first cactus.

The following is easy to check:

1.2.3. **Proposition.** The gluings make the spaces $Cact^1(n)$ into a topological operad.

1.3. **Cacti without spines.**

1.3.1. **Definition.** We define *cacti without spines* by the same procedure only taking S^1_rs, i.e. circles of different radii.

$Cact(n)$:= {rooted configurations of n labelled S^1_rs in the plane with at most finitely many points of intersection points whose dual black and white graph is a connected tree such that at the points of intersection the circles of greater height all have zero as their point of intersection }/ isometric isotopies preserving the incidence conditions.

Again $\mathbb{S}_n$ acts via permuting the labels.

1.3.2. **Gluing.** We define the following operations

$$\circ_i : Cact(n) \times Cact(m) \to Cact(n+m-1) \tag{1.2}$$

by the following procedure which differs slightly from the above: given two cacti without spines we reparameterize the outside circle of the second cactus to have length r_i which is the length of the i-th circle of the first cactus. Then glue in the second cactus by identifying the outside circle of the second cactus with the i–th circle of the first cactus.

The following Proposition is straightforward.

1.3.3. **Proposition.** The gluing endows the spaces $Cact(n)$ with the structure of a topological operad.

1.3.4. **Remark.** We could also define a cactus without spines abstractly by circles (without base–points), incidence conditions, one marked point on the configuration and a map of S^1 giving the perimeter. If one would wish to add base-points one can do so by setting the first point at which the perimeter map reaches a circle to be the base-point of that circle.

1.3.5. **Remark.** There is an obvious map from normalized spineless cacti to spineless cacti. This map is not a map of operads, since the gluing procedures differ. The gluings, however, are equivariant up to homotopy under this map, so that the homology operads of normalized cacti without spines and cacti without spines coincide.

1.4. **Pictorial realizations.**

1.4.1. **The dual tree of a cactus without spines.** In the case that the configuration of circles is a cactus without spines there is a dual tree that we can associate to it that is a regular tree, which is not black and white.

This is done as follows. Vertices just correspond to circles. They will be labelled by the radius of the respective circle. We will draw an edge between two vertices if the circles have a common point and if one circle is higher than the other in the height of the dual graph. We will label the edge by the length of the arc on the lower circle between the intersection point and the previous intersection point. Here we consider the global zero also as an intersection point. In this procedure we give the edges the cyclic order that is dictated by the perimeter. Notice that the length of the arcs is allowed to be zero i.e. the labels on the edges are in $\mathbb{R}_{\geq 0}$. Using this structure we can actually see the compactification of normalized cacti. On the open part with only double points the weight on the edges are restricted by the equations $0 < \sum w_i < 1$. Allowing intersections of more than two components at a time leads to the compactification.

If we do not want to use the height function of the black and white tree we can still define a height function via the outside circle. Start at height zero for the root. If the perimeter hits a component for the first time, increase the height by one and assign this height to the component. Each time you return to a component decrease the height by one.

Given such a labelled tree it is clear that it gives a prescription on how to grow a cactus. Start at the root and draw a based loop of length given by the label of the root. For the first edge mark the point at the distance given by the label of the edge along the loop. Then mark a second point by travelling the distance of the label of the second edge and so on. Now at the next level of the tree draw a loop based at the marked point of the previous level and again mark points on it according to the outgoing edges. This will produce a cactus without spines.

Lastly, we wish to point out that now the composition looks like the grafting of trees into vertices as in the Connes–Kreimer [CK] tree operads.

1.4.2. **The chord diagram of a cactus.** There is yet another representation of a cactus. If one regards the outside loop, then this can be viewed as a collection of points on an S^1 with an identification of these points, plus a marked point corresponding to the global zero. We can represent this identification scheme by drawing one chord for each two points being identified as the beginning and end of a circle. This chord diagram comes equipped with a decoration of its arcs by their length or alternatively can be thought of as embedded in $\mathbb{R}^2$. To obtain a cactus from such a diagram, one simply has to collapse the chords. This type of chord diagram of the outside circle is explicit in the embedding of cacti into the arc operad where the perimeter is indeed the outside circle, as explained below and in [KLP].

This kind of representation is reminiscent of Kontsevich's formalism of chord diagrams (cf. eg. [BN]) as well as the shuffle algebras and diagrams of Goncharov [Go]. We wish to point out that although the multiplication is similar to Kontsevich's and also could be interpreted as cutting the circle at the global zero resp. the local zero, it is not quite the same. However, the exact relationship and the co–product deserve further study.

A representation of a cactus without spines in all possible ways including its image in the Arc operad can be found in figure 1.

The following definition is the original definition of cacti due to Voronov.

1.5. **Cacti with spines.**

1.5.1. **Definition [V].** Define *Voronov cacti or cacti with spines or simply cacti* in the same fashion as cacti without spines, but without

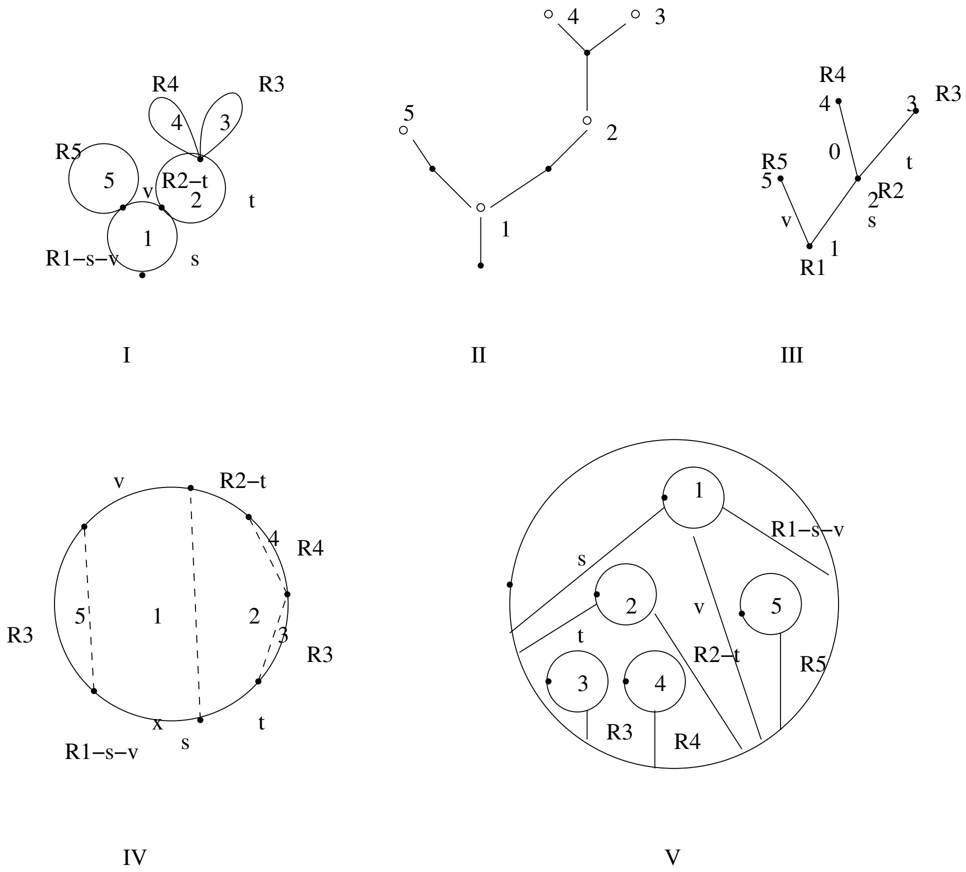

FIGURE 1. I. A cactus without spines; II Its black and white tree; III Its dual tree; IV Its chord diagram; V Its image in Arc

requiring that the zeros be the intersection points. In addition, and this is key, we add a global zero/base point to the configuration, which means that we mark a circle and a point on that circle. The circle with the global base point will be the root. We call the n–th component of this operad $Cacti(n)$.

The perimeter or outside circle will be given by the same procedure as 1.1.5 by starting at the global zero.

1.5.2. **Gluing.** We define the following operations

$$\circ_i : Cacti(n) \times Cacti(m) \to Cacti(n+m-1) \tag{1.3}$$

by the following procedure which differs slightly from the above: given two cacti without spines we reparameterize the outside circle of the second cactus to have length r_i which is the length of the i-th circle of the first cactus. Then glue in the second cactus by identifying the outside circle of the second cactus with the i–th circle of the first cactus. We remark that now the local zero of the i–th circle which is identified with the global zero, viz. the starting point of the outside circle which now is not necessarily the intersection point.

1.5.3. **Proposition [V].** The cacti form an operad.

1.6. **Normalized cacti.**

1.6.1. **Definition.** We define *normalized cacti* to be cacti with the restriction that all circles have radius one. We denote the respective spaces by $Cacti^1(n)$.

1.6.2. **Remark.** The contents of Remark 1.3.5 applies analogously in the cacti situation.

1.7. **Remark.** There are natural forgetful morphisms from cacti to cacti without spines forgetting all the local zeros. We arrange the map in such a way, that the global zero becomes the base–point of the spineless cactus. This works for the normalized version as well. These maps are not maps of operads, the relationship between the different varieties is given by a bi–crossed product, see below. There is an embedding of spineless cacti into cacti as a suboperad, however, by considering the global zero to be the zero of the root and by making the first intersection point at which the perimeter reaches a lobe of the cactus the local zero of that circle. This even realizes cacti without spines as a suboperad of cacti.

1.8. **Pictorial realizations.**

1.8.1. **The tree of a cactus.** The missing information of a cactus without spines relative to a cactus proper is the location of the local zeros. We just add this information as a second label on each vertex. Notice that the local zero of the root component then need not be the global zero. The label we associate to the root is the position of the local zero with respect to the global zero.

1.8.2. **The chord diagram.** The chord diagram of a cactus again is the chord diagram of a cactus without spines, where the location of the spines is additionally marked on the S^1. Here there is a choice if the local zero coincides with an intersection point. Just to fix notation we well mark the first occurrence of the endpoint of a chord, where first means in the natural orientation starting at the global zero.

A representation of a cactus (with spines) in all possible ways including its image in Arc can be found in figure 2.

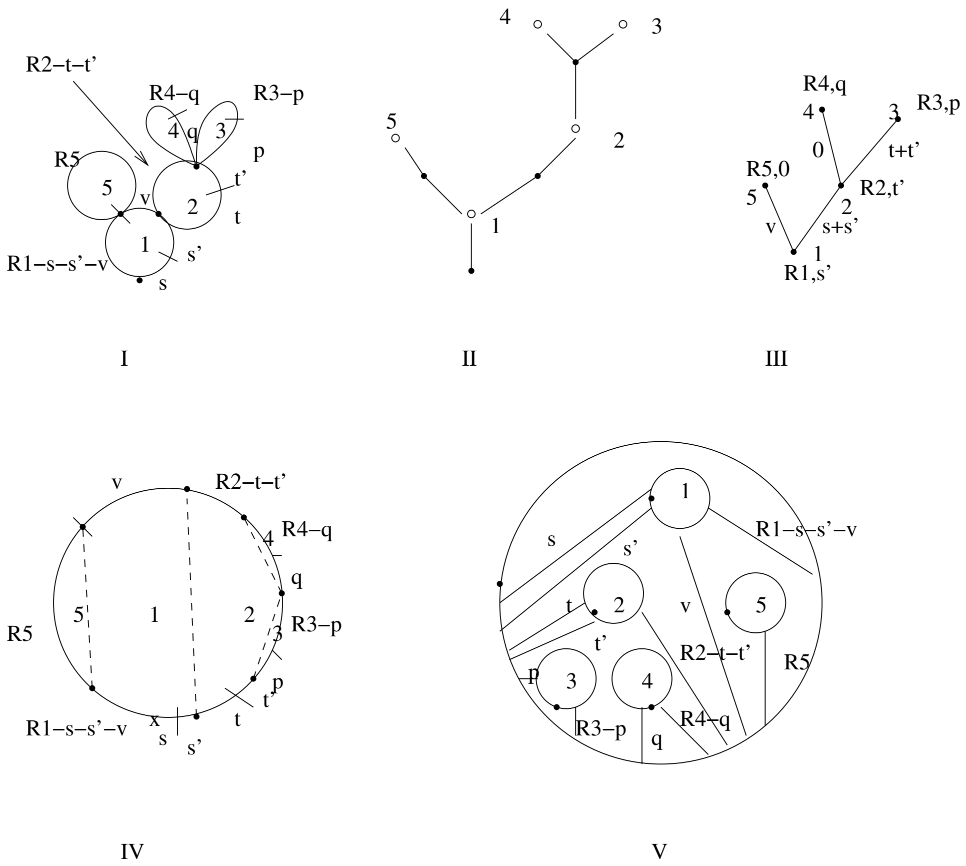

FIGURE 2. I. A cactus (with spines); II Its black and white tree; III Its dual tree; IV Its chord diagram; V Its image in Arc

2. SEMIDIRECT AND BI–CROSSED PRODUCTS

2.1. **Operads of spaces.** The following procedure is motivated by topological spaces with Cartesian product, by actually works in any strict symmetric monoidal category where the monoidal product is a product (i.e. we have projection maps).

Let X be a topological space, then we can form the iterated Cartesian product $X \times \cdots \times X$. We simply denote the n–fold product by $X(n)$. This space has an action of $\mathbb{S}_n$ by permutation of the factors. We denote the corresponding morphism also by elements of $\mathbb{S}_n$. Given a subset $I \subset \{1, \ldots, n\}$ we denote the projection $\pi_I : X(n) \to X(I) = \times_{i \in I} X$.

$$\bar{\circ}_i : X(n) \times X(m) \overset{\pi_{\{1,\dots,n\}\setminus i}}{\to} X(n-1) \times X(m) \to X(m+n-1) \overset{\sigma}{\to} X(m+n-1) \tag{2.1}$$

where $\sigma \in \mathbb{S}_{m+n-1}$ is the permutation that shuffles the last m factors into the place i. This means on elements

$$((x_1, \dots, x_n)\bar{\circ}_i(x'_1, \dots, x'_m)) = (x_1, \dots x_{i-1}, x'_1, \dots, x'_m, x_{i+1}, \dots, x_n) \tag{2.2}$$

2.2. **The cyclic version.** Using $X((n)) = X(n+1)$ with the $\mathbb{S}_{n+1}$ action and the gluing above, one obtains a cyclic version of the construction.

2.3. **Operads built on monoids.** Let S be a monoid with associative multiplication $\mu : S \times S \to S$. For simplicity we will denote this multiplication just by juxtaposition: $s, s' \in S; ss' := \mu(s, s')$. We will take S to be an object in a strict symmetric monoidal category.

We set $S(n) := S^{\times n}$ and endow it with the permutation action.

2.3.1. **The standard operad.** We consider the following maps:

$$\begin{array}{rcl} \circ_i : S(n) \times S(m) & \to & S(n+m-1) \\ ((s_1, \dots, s_n), (s'_1, \dots, s'_m)) & \mapsto & (s_1, \dots s_{i-1}, s_i s'_1, \dots, s_i s'_m, s_{i+1}, \dots, s_n) \end{array}$$

It is easy to check that these maps define an operad in the same category as S. This operad is unital if S has a unit.

2.3.2. **Examples.**

1) One standard example is that of a Lie group in topological spaces.
2) Another nice example is that of a field k. Then $k(n) = k^n$ and the gluing is plugging in vectors into vectors scaled by scalar multiplication. There are $\mathbb{Z}$–graded and super versions of this given by including the standard supersign for the permutation action where in the $\mathbb{Z}$–graded version one uses the induced $\mathbb{Z}/2\mathbb{Z}$–grading.
3) The example S^1 is particularly nice. In this case (see e.g. [KLP]) we can see that $H_*(S(*))$ is the operad built on $\mathbb{Z}/2\mathbb{Z}$.

2.3.3. **Remark.** There are several other natural versions of operads and cyclic operads which can be defined analogously to the operads built on circles which are presented in [KLP].

2.4. **Semi–direct products with monoids.** We now turn to the situation where the monoid S acts on all the components of an operad.

I.e. Let $Op(n)$ be an operad in a symmetric monoidal category $\mathcal{C}$ and let S be a monoid in the same category such that S acts on $Op(n)$.

$$S \times Op(n) \xrightarrow{\rho} Op(n)$$

s.t. the following diagram is commutative

$$\begin{array}{ccc} S \times S \times Op(n) & \xrightarrow{\mu \times id} & S \times Op(n) \\ {\scriptstyle id\times\rho}\uparrow & & \downarrow{\scriptstyle \rho} \\ S \times Op(n) & \xrightarrow{\rho} & Op(n) \end{array}$$

We can use such an operation to twist the concatenations i.e. by using the universal map

$$Op(n) \times S \times Op(m) \to Op(n+m-1)$$

More precisely for a given $s \in S$ we can define:

$$\circ_i^s : Op(n) \times Op(m) \to Op(n+m-1)$$

by

$$O \circ_i^s O' = (O \circ_i \rho(s, O'))$$

2.4.1. **Definition.** If the twisted multiplications $\circ_i^{\rtimes}$ below yield an operad we define the semi–direct product $Op \rtimes S$ of an operad Op with a monoid S in the same category to be given by the spaces

$$(Op \rtimes S)(n) := Op(n) \times S(n)$$

with the composition

$$\begin{aligned} \circ_i^{\rtimes} : (Op \rtimes S)(n) \times (Op \rtimes S)(m) &\to (Op \rtimes S)(n+m-1) \\ ((O, s), (O', s')) &:= (O \circ_i^s O', s \circ_i s') \end{aligned} \tag{2.3}$$

2.4.2. **Example.** It is well known that the operad of framed little discs which is the operad governing BV algebras is the semi–direct product of the little discs operad that governs Gerstenhaber algebras with the operad based on the circle group S^1.

2.4.3. **The action.** If we are in a category that satisfies the conditions of 2.1, we can break down the operad structure based on a monoid into two parts. The first is the structure of operads of spaces and the second is the diagonal action.

More precisely let $\Delta : S \to S(n)$ be the diagonal and $\mu : S \times S \to S$ be the multiplication

$$\rho_\Delta : S \times S(n) \stackrel{\Delta}{\to} S(n) \times S^n \stackrel{\mu^n}{\to} S(n)$$

Here we denote my μ^n the diagonal multiplication:

$$\mu^n((s_1, \dots, s_n), (s'_1, \dots, s'_n)) = (s_1 s'_1, \dots, s_n s'_n)$$

$$\begin{aligned} \circ_i : S(n) \times S(m) &\stackrel{(id\times\pi_i)(\Delta)\times id}{\longrightarrow} S(n) \times S \times S(m) \\ &\stackrel{id\times\rho_\Delta}{\longrightarrow} S(n) \times S(m) \stackrel{\bar{\circ}_i}{\to} S(n+m-1) \end{aligned} \tag{2.4}$$

where $\bar{\circ}_i$ is the operation of the operad of spaces and

$$(id \times \pi_i)(\Delta) : S(n) \stackrel{\Delta}{\to} S(n) \times S(n) \stackrel{id\times\pi_i}{\to} S(n) \times S$$

2.5. **Semi–direct products.** Consider two operads $C(n)$ and $D(n)$ together with a collection of maps:

$$\circ_i^D : C(n) \times D(n) \times C(m) \quad \to \quad C(n+m-1) \quad i = 1, \dots n \tag{2.5}$$

—we used the superscripts to indicate that we view the dependence of the map on D as a perturbation of the original operad structure.

We define products on the product, i.e. the collection $(C \times D)(n) := C(n) \times D(n)$ via

$$\begin{aligned} \circ_i : (C \times D)(n) \times (C \times D)(m) &= C(n) \times D(n) \times C(m) \times D(m) \\ &\longrightarrow C(n+m-1) \\ (c, d) \circ_i (c', d') &= (c \circ_i^d c', d \circ_i d') \end{aligned} \tag{2.6}$$

where we use the upper index on the operations to show that we use the universal maps (2.5) with fixed middle argument.

2.5.1. **Definition.** We call the operad C *normal* with respect to D if there are maps of the type (2.5) such that the maps (2.6) give an operad structure to the product $C \times D$. Given two such operads we call the product together with the operad structure (2.6) the *semi–direct product* of C and D which we denote $C \rtimes D$.

In the case we are dealing with unital operads we will also require that the matchings are such that $1_C \times 1_D$ is a unit in the obvious notation, and that $\circ_i^{1_D} = \circ_i$.

2.5.2. **Examples.** Examples of this structure are usually derived if the operad D acts on the operad C and the twisted operations are defined by first applying this action. The semi–direct product with a monoid is such an example as are the semi–direct products of [KLP] and the (spineless) cacti with respect to their normalized versions (see below).

2.5.3. **The right semi–direct product.** There is a right version of the semi–direct products using maps

$$\circ_i^C : D(n) \times C(m) \times D(m) \quad \to \quad D(n+m-1) \quad i = 1, \dots n \tag{2.7}$$

which again define products on the products $(C \times D)(n) := C(n) \times D(n)$ via

$$\begin{aligned} \circ_i^C : (C \times D)(n) \times (C \times D)(m) &= C(n) \times D(n) \times C(m) \times D(m) \\ &\longrightarrow C(n+m-1) \\ (c,d) \circ_i (c',d') &= (c \circ_i c', d \circ_i^{c'} d') \end{aligned} \tag{2.8}$$

We call the operad D *normal* with respect to C if there are maps of the type (2.7) such that the maps (2.8) give an operad structure to the product $C \times D$. Given two such operads we call the product together with the operad structure (2.6) the *right semi–direct product* of C and D which we denote $C \ltimes D$.

This structure typically appears when one has a right action of the operad C on the operad D.

2.6. **Bi–crossed products.**

Consider two operads $C(n)$ and $D(n)$ together with a collection of maps:

$$\begin{aligned} \circ_i^D : C(n) \times D(n) \times C(m) &\to C(n+m-1) \quad i = 1, \dots n \\ \circ_i^C : D(n) \times C(m) \times D(m) &\to C(n+m-1) \quad i = 1, \dots n \end{aligned} \tag{2.9}$$

—we used the superscripts to indicate that we view the dependence on the other operad as a perturbation of the original operad structure.

We define products on the product, i.e. the collection $(C \times D)(n) := C(n) \times D(n)$ via

$$\begin{aligned} \circ_i (C \times D)(n) \times (C \times D)(m) &= C(n) \times D(n) \times C(m) \times D(m) \\ &\longrightarrow C(n+m-1) \\ (c,d) \circ_i (c',d') &= (c \circ_i^d c', d \circ_i^{c'} d') \end{aligned} \tag{2.10}$$

where we use the upper index on the operations to show that we use the universal maps (2.9) with fixed middle argument.

2.6.1. **Definition.** We call two operads C and D *matched* if there are maps of the type (2.9) called matchings such that the maps (2.10) give an operad structure to the product $C \times D$. Given two matched operads we call the product together with the operad structure (2.10) the *bi–crossed product* of C and D which we denote $C \bowtie D$.

In the case we are dealing with unital operads we will also require that the matchings are such that $1_C \times 1_D$ is a unit in the obvious notation, and that $\circ_i^{1_C} = \circ_i$ and $\circ_i^{1_D} = \circ_i$.

2.6.2. **Examples.**

1) Factoring the multiplication maps through the first and third projection and using the structure maps of the two operads we obtain the direct product.
2) If D is the operad based on a monoid and choosing the maps $\circ_i^C$ to be unperturbed and defining the maps $\circ_i^D$ as in (2.3) we obtain the semi–direct product.
3) If C and D are concentrated in degree 1 and happen to be groups then the bi–crossed product is that of matched groups [Ka,Tak]. Otherwise we obtain that of matched monoids.
4) Below we will show that the cacti operad is a bi–crossed product of the cactus operad and the operad built on S^1 as a monoid.

3. The relations of the cacti operads

3.1. The relation between normalized (spineless) cacti and (spineless) cacti.

3.1.1. **The scaling operad.** We define the scaling operad $\mathcal{R}_{>0}$ to be given by the spaces $\mathcal{R}_{>0}(n) := \mathbb{R}^n_{>0}$ with the permutation action by $\mathbb{S}_n$ and the following products

$$(r_1, \dots, r_n) \circ_i (r'_1, \dots, r'_m) = (r_1, \dots r_{i-1}, \frac{r_i}{R} r'_1, \dots, \frac{r_i}{R} r'_m, r_{i+1}, \dots r_n)$$

where $R = \sum_{k=1}^m r'_k$. It is easy to check that this indeed defines an operad.

3.1.2. **The action of the scaling operad on cacti.** Fix an element $\vec{r} := (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$ and set $R = \sum_i r_i$. Then $\vec{r}$ defines a map $\rho : S^1 \to S^1$ via

$$\rho : S^1 \xrightarrow{rep^1_R} S^1_R \xrightarrow{cont_{\vec{r}}} S^1_n \xrightarrow{rep^n_1} S^1 \tag{3.1}$$

Where $cont_{\vec{r}}$ acts on S^1_R in the following way. Decompose S^1_R into a linear sequence of arcs of length r_i starting at zero. Now contract the i–th arc homogeneously with a scaling factor $\frac{1}{r_i}$ to length one.

Using this map on the i–lobe of a normalized (spineless) cactus which we think of as an S^1 with marked points, where the marked points are the intersection points, the local zero and if applicable the global zero.

$$\begin{aligned} \rho_i : Cact^1(n) \times \mathcal{R}_{>0}(m) &\to Cact^1(n) \\ \rho_i : Cacti^1(n) \times \mathcal{R}_{>0}(m) &\to Cacti^1(n) \end{aligned} \tag{3.2}$$

Notice we would like to take a right action.

3.2. **The perturbed multiplication.** We define the perturbed multiplication via

$$\begin{aligned} \circ_i^{\mathcal{R}_{>0}} : Cacti^1(n) \times \mathcal{R}_{>0}(m) \times Cacti^1(m) \\ \xrightarrow{\rho_i \times id} Cacti^1(n) \times Cacti^1(m) \xrightarrow{\circ_i} Cacti^1(n+m-1) \end{aligned} \tag{3.3}$$

We also use the analogous perturbed action for $Cact^1$

3.2.1. **Proposition.** Cacti (without spines) are homotopic to normalized cacti without spines. Their operad structure is given by a semi–direct product of their normalized version with the scaling operad that is homotopic to the direct product.

$$\begin{aligned} \rho_i : Cact^1(n) \times \mathbb{R}^n_{>0} &\simeq Cact(n) \\ \rho_i : Cacti^1(n) \times \mathbb{R}^n_{>0} &\simeq Cacti(n) \end{aligned} \tag{3.4}$$

and furthermore

$$\begin{aligned} Cact &\simeq \mathcal{R}_{>0} \ltimes Cact^1 \sim Cact^1 \times \mathcal{R}_{>0} \\ Cacti &\simeq \mathcal{R}_{>0} \ltimes Cacti^1 \sim Cacti^1 \times \mathcal{R}_{>0} \end{aligned} \tag{3.5}$$

as operads. where the operadic compositions are given by

$$(\vec{r}, C) \circ_i (\vec{r'}, C') = (\vec{r} \circ_i \vec{r'}, C \circ_i^{\vec{r'}} C',) \tag{3.6}$$

Proof. By §1 $Cact(n)$ can be given by planar rooted trees with n vertices and edges labelled with non–negative numbers s.t. the sum of the incoming edges is less or equal to the label of the vertex. The space $Cact^1(n)$ is on the other hand just given by planar rooted trees with n vertices labelled with 1 and the edges labelled in such a way that the sum of labels on incoming edges is less or equal to 1. Now we see that denoting the vector of weights of incoming edges by $(w_1, \dots, w_k)$ we have an action of $\mathbb{R}_{>0}$ on the weights by $(\lambda, (w_1, \dots, w_k)) \mapsto (\lambda w_1, \dots, \lambda w_k)$ and an action on the label of the vertex by $r \mapsto \lambda r$. It is easy to check that this induces the homeomorphism of the first claim. For the second claim we notice that the global incidence conditions are shifted under the scaling in a way that is compensated by perturbed multiplication. The behavior of the radii under gluing is easily seen to be that given by the scaling operad. Lastly the perturbed multiplications are easily seen to be homotopic the the unperturbed multiplications, the homotopy being given by a path $\vec{r} \to (1, \dots, 1)$.

The analogous arguments hold for $Cacti$.

3.3. **The relation between spineless cacti and cacti.** We would now like to specialize the monoid of §2 to $S = S^1$. We already showed that the normalized and non–normalized versions of the different species of cacti are homotopic and moreover that they are related by taking the direct product with the scaling operad. Below we will see that cacti with spines are a bi–crossed product of the cacti without spines and the operad built on S^1. Furthermore we show that this bi–crossed product is homotopic to the semi–direct product. Thus we see that the relation of cacti with and without spines is analogous to the relation of framed little discs and little discs.

3.3.1. **The action of S^1 and the twisted gluing.** There is an action of S^1 on $Cact(n)$ given by rotating the base point *clockwise* around the perimeter. We denote this action by

$$\rho^{S^1} : S^1 \times Cact(n) \to Cact(n)$$

With this action we can define the twisted gluing

$$\begin{aligned} \circ_i^{S^1} : Cact(n) \times S^1 \times Cact(m) &\to Cact(n+m-1) \\ (C, \theta, C') &\mapsto C \circ \rho^{S^1}(\theta, C') =: C \circ_i^{\theta} C' \end{aligned} \tag{3.7}$$

3.3.2. **The homotopy diagonal defined by a spineless cactus.** Given a cactus without spines $C \in Cact(n)$ the orientation reversed perimeter (i.e. going around the outer circle *clockwise*) gives a map

$$\Delta_C : S^1 \to S^n \tag{3.8}$$

As one goes around the perimeter the map goes around each circle once and thus the image is homotopic to the diagonal

$$\Delta_C(S^1) \sim \Delta(S^1) \tag{3.9}$$

A picture of the image of Δ_C for a two component cactus is depicted in figure 3.

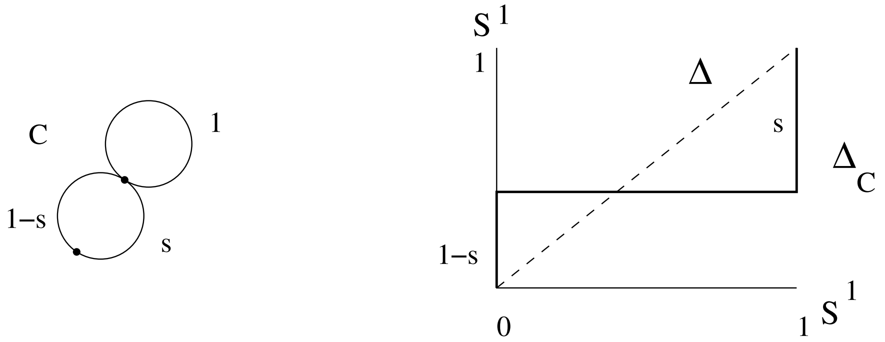

FIGURE 3. The image of Δ_C for a two component cactus

3.3.3. **The action based on a cactus.** We can use the map Δ_C to give an action of S^1 and $(S^1)^{\times n}$.

$$\rho^C : S^1 \times (S^1)^{\times n} \stackrel{\Delta_C}{\to} (S^1)^{\times n} \times (S^1)^{\times n} \stackrel{\mu^n}{\to} (S^1)^{\times n} \tag{3.10}$$

And furthermore using concatenations with projections we can define maps

$$\begin{aligned} \circ^C : (S^1)^{\times n} \times (S^1)^{\times m} &\stackrel{(id \times \pi_i)(\Delta)\times id}{\longrightarrow} (S^1)^{\times n} \times S^1 \times (S^1)^{\times m} \\ &\stackrel{id \times \rho^C}{\longrightarrow} (S^1)^{\times n} \times (S^1)^{\times m} \stackrel{\bar{\circ}_i}{\longrightarrow} (S^1)^{\times n+m-1} \end{aligned} \tag{3.11}$$

3.3.4. **Theorem.** The operad of (normalized) cacti is the bi–crossed product of the operad of (normalized) spineless cacti with the operad $\mathcal{S}^1$ based on S^1 and furthermore this bi–crossed product is homotopic to the semi–direct product of the operad of cacti without spines with the circle group S^1.

$$Cacti \simeq Cact \bowtie \mathcal{S}^1 \sim Cact \rtimes \mathcal{S}^1 \tag{3.12}$$

with respect to the operations of (3.7) and (3.11).

Similarly

$$Cacti^1 \simeq Cact^1 \bowtie \mathcal{S}^1 \sim Cact \rtimes \mathcal{S}^1 \tag{3.13}$$

Proof. As spaces the operad of cacti is the direct product of spineless cacti and the operad built on the monoid S^1.

$$Cacti(n) = Cact(n) \times S^1(n)$$

where for the identification we use intersection points and the global zero to define the parameterizations of the S^1s constituting the cactus. Then the marked points (or local zeros) are specified by their coordinate, i.e. all zeros are fixed by a point on $S^1(n)$.

The product on the cacti operad in this identification is given by

$$(C,\theta)\circ_i(C',\theta') = (C\circ_i^{\theta_i} C', \theta\circ_i^{C'}\theta') \tag{3.14}$$

where we used the operations of (3.7) and (3.11).

This comes from the observation that gluing the global zero to the local zero is the same as first using the S^1 action that rotates the global zero around the perimeter in a clockwise fashion by the amount which is given by the coordinate of the local zero on the cactus to be glued in and then using the standard gluing. During the movement of the global zero the coordinates of the local zeros change according to the map ρ_C.

This proves the first assertion. That the perturbed multiplications are indeed matched can easily be checked by hand or by remarking that the above gluing procedure is that of the cactus operad which is indeed an operad.

Now, in the semi–direct product the multiplication is given by

$$(C,\theta)\circ_i(C',\theta') = (C\circ_i^{\theta_i} C', \theta\circ_i\theta') \tag{3.15}$$

Now by (3.9) and (3.11) $\circ_i^C \sim \circ_i$ are homotopic and $\circ_i^C$ depends continuously on C.

This proves the second statement. To obtain the version of the theorem in the normalized situation, we remark that the bi–crossed structure does not depend on the size of the lobes.

Using the statement of [V] that the cacti operad is homotopy equivalent to framed little discs and using the recognition principle of [SW] we obtain:

3.3.5. **Corollary.** The operad of (normalized) cacti without spines is homotopy equivalent to the little discs operad.

3.4. **The explicit presentation of the operations in normalized spineless cacti.** Parallel to [CS,V], we can give explicit generators for the operations yielding the Gerstenhaber structure on the homology spineless cacti (see figure 4).

We would like to emphasize that the product $\cdot$ is *associative* on the nose and not just up to homotopy. As usual, the multiplication $*$ defines the bracket via the odd commutator. Its iterations are given

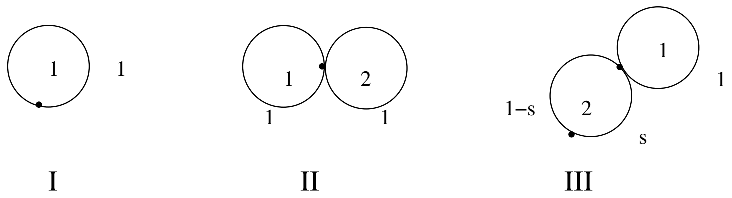

FIGURE 4. I. The identity II. The product $\cdot$. III. The operation $*$.

in figure 5 from which one can also read off the associator (pre–Lie) relation.

All the other homologies can be translated to normalized cacti (with or without spines) in a straightforward fashion from [KLP].

4. The suboperads of Arc defined by the cacti operads

We would like to recall that the cacti without spines and cacti can be embedded into the Arc operad up to an overall scaling factor as defined in [KLP]. Moreover there is an S^1 action given by the twist operator δ. For the complete details, we refer to [KLP].

The embedding is basically constructed as follows: start by decomposing the cactus into the arcs of its perimeter, where the break point of a cactus with spines are the intersection points, the global zero and the local zeros. Then one runs an arc from each arc to an outside pointed circle which is to be drawn around the cactus configuration. The arcs should be embedded starting in a counterclockwise fashion around the perimeter of the circle. The marked points on the inside circles which are the lobes of the cactus are the local zeros for the cactus with spines and the global zero and the first intersection point for a cactus without spines.

For more orientation, we include two figures: the figure 6 shows the framing i.e. embedding of two cacti without spines and a cactus with spines into arc and figure 7 shows the identity in arc and the family of weighted arcs corresponding to the twist which yields the BV operator. Compare also the figures 1 V and 2 V for more elaborate examples.

The main result of the arc picture is

4.1. **Theorem.** There is an embedding of the (spineless) cacti operad into Arc (up to a projective factor). Furthermore the suboperad in Arc generated by the Fenchel–Nielsen type twist δ and the spineless cacti is equal to the embedding of the cacti operad which in turn is given by all arc families on genus zero surfaces with a marked component where

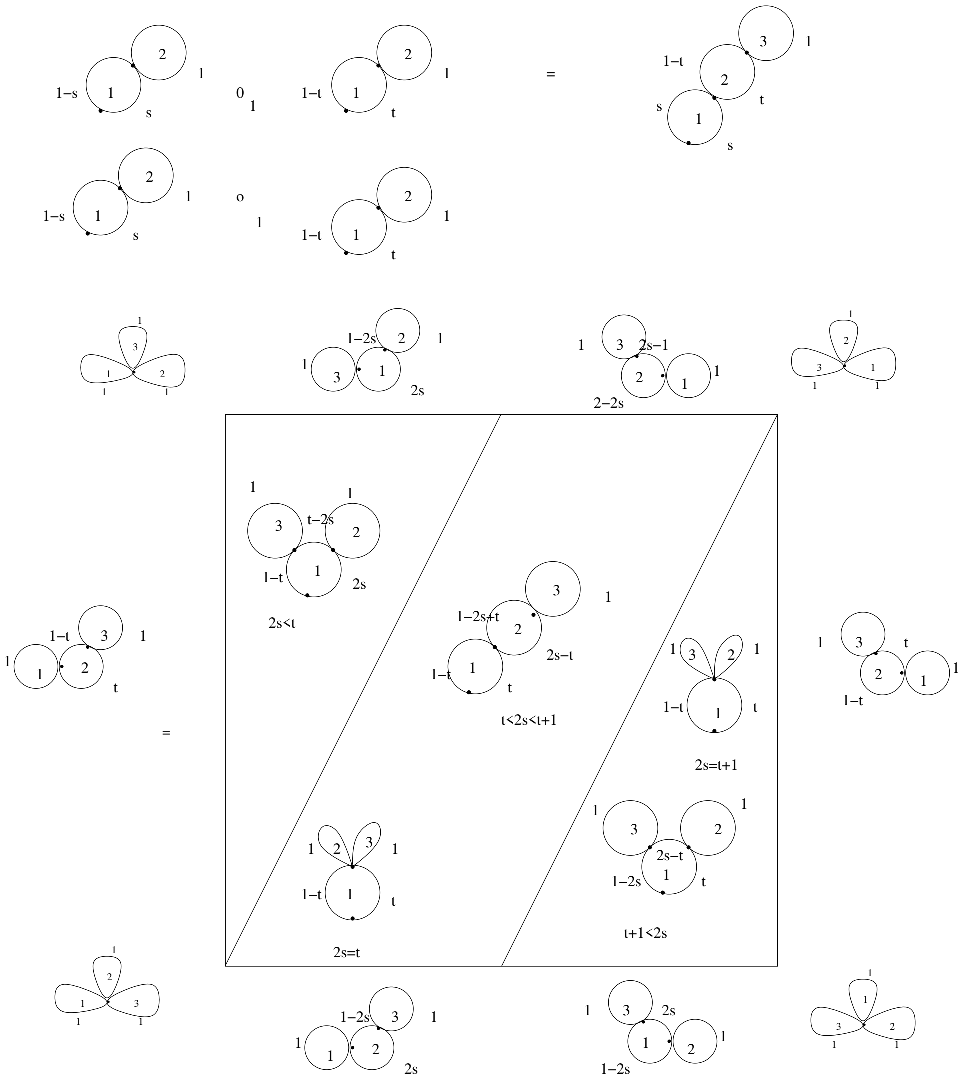

FIGURE 5. The associator in normalized spineless cacti

arcs run only between this marked component and one of the other components.

Similarly there is a map of the normalized (spineless) cacti operad into Arc which is a map of operads up to homotopy.

Proof. We refer to [KLP] for the proof of the first part. That this relationship also holds for the normalized versions of cacti can be seen by factoring the map to the Arc operad through (spineless) cacti.

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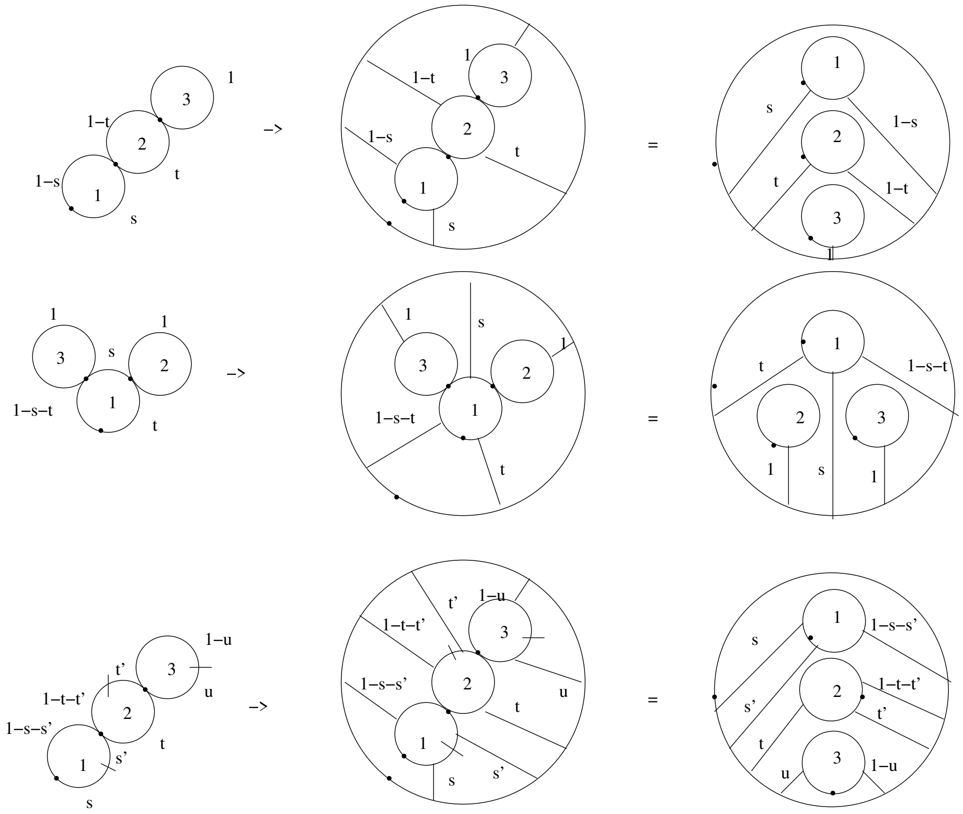

FIGURE 6. The embedding of cacti into arc

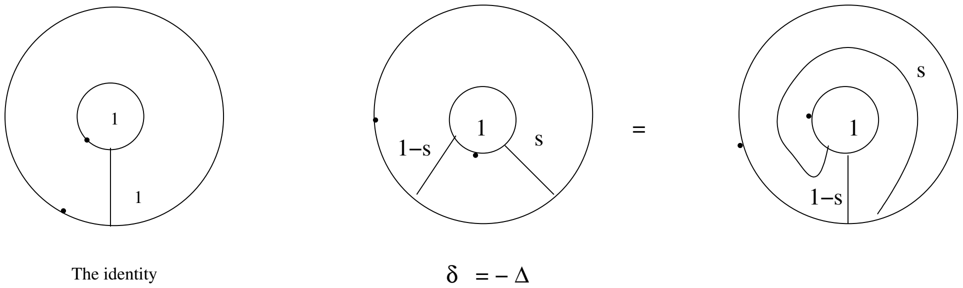

FIGURE 7. The identity and the twist yielding the operator

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