Wronskian addition formula and its applications

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Abstract

We associate with every positive integer m the same number of Wronskian determinants of the order m corresponding to particular sequences of sine functions. We show that all these determinants have a form of a product of integer powers of sine and cosine functions multiplied by some integer factor. This result can be interpreted as addition formula for sine functions. We consider various applications of this formula including Wronskian representations for the solutions of the Schrödinger operators with the DPT potentials and the solutions of the Cauchy initial problem for the KdV hierarchy with Darboux-Pöshl-Teller initial conditions.

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1 Wronskian addition formula for sine functions

Let n and $d > 0$ be some nonegative integers, $m = n + d$, [1] q and x_0 be some complex valued parameters , and $\varphi_p(x)$ are the functions defined as follows:

$$\varphi_p(x) := \sin q a_p(x - x_0),\ a_p = p, \quad \text{if} \quad 1 \le p \le m - n; \tag{1}$$

$$a_{m-n+j} = m - n + 2j, \quad j = 1, \ldots, n.$$

We use below the standard notation

$$W(f_1, f_2, \ldots, f_m) := \det ||\partial_x^{j-1} f_k(x)||; \quad j, k = 1, \ldots, m,$$

for Wronskian determinant of any m functions and the short notation $W_{mn} = W_{mn}(q, x, x_0) := W(\varphi_1, ..., \varphi_m)$, where φ_x were defined in (1).
With these notations we have the following formulas. [2]

$$W_{m0} = (-2q)^{\frac{m(m-1)}{2}} \prod_{k=0}^{m-1} k! \cdot \sin^{\frac{m(m+1)}{2}} q(x - x_0), \tag{2}$$

$$W_{mn} = c_{mn} \cdot \sin^{\frac{m(m+1)}{2}} q(x - x_0)\ \cos^{\frac{n(n+1)}{2}} q(x - x_0), \tag{3}$$

$$c_{mn} = (-2q)^{\frac{m(m-1)}{2}} \cdot 2^{n^2} \prod_{p=0}^{d-1} p! \prod_{k=1}^{n} \frac{(d + 2k - 1)!}{(2k - 1)!} (k - 1)!, \quad m \ge 2,\ d \ge 1$$

We note $\hat{W}_{mn}$ the determinant obtained from W_{mn} replacing sin functions $\varphi_j(x)$ by the sinh functions $g_j(x)$ of the same argument:

$$\hat{W}_{mn} := W(g_1, \ldots, g_m),$$
$$g_p(x) := \sinh q a_p(x - x_0),\ a_p = p, \quad \text{if} \quad 1 \le p \le m - n;$$
$$a_{m-n+j} = m - n + 2j, \quad j = 1, \ldots, n.$$

Then the following hyperbolic versions of (2) and (3) holds

$$\hat{W}_{m0} = (2q)^{\frac{m(m-1)}{2}} \prod_{k=0}^{m-1} k! \cdot \sinh^{\frac{m(m+1)}{2}} q(x - x_0), \tag{4}$$

[1] We shall not consider the case d=0, m=n, since it is equivalent to the case n=0, d=m modulo replacing q by $2q$ in the corresponding formulas

[2] product of factorials in this formulas can be also written as a product of powers:

$$1! \cdot 2! \cdots k! = 1^k \cdot 2^{k-1} \cdot 3^{k-2} \cdots k.$$

$$\hat{W}_{mn} = |c_{mn}| \cdot \sinh^{\frac{m(m+1)}{2}} q(x-x_0)\ \cosh^{\frac{n(n+1)}{2}} q(x-x_0), \tag{5}$$

where c_{mn} is the same as before.
Replacing (1) by the system of functions

$$\begin{aligned}
&h_p(x) := \cos q a_p(x-x_0),\ a_p = p, \quad \text{if}\, p \quad \text{is odd}\\
&h_p(x) := \sin q a_p(x-x_0),\ a_p = p, \quad \text{if}\, p \quad \text{is even},\\
&1 \le p \le m-n;\\
&a_{m-n+j} = m-n+2j, \quad j = 1, \dots, n.\\
&h_{m-n+j}(x) := \cos q a_{m-n+j}(x-x_0), \quad \text{if}\, m-n \quad \text{is odd}\\
&h_{m-n+j}(x) := \sin q a_{m-n+j}(x-x_0), \quad \text{if}\, m-n \quad \text{is even},
\end{aligned}$$

we get for their Wronskian

$$w_{mn} := W(h_1, \dots, h_m), \tag{6}$$

the following analog of (2) and (3):

$$w_{m0} = s_{m0}(2q)^{\frac{m(m-1)}{2}} \prod_{k=0}^{m-1} k! \cdot \cos^{\frac{m(m+1)}{2}} q(x-x_0), \tag{7}$$

$$w_{mn} = s_{mn}|c_{mn}| \cdot \cos^{\frac{m(m+1)}{2}} q(x-x_0)\ \sin^{\frac{n(n+1)}{2}} q(x-x_0), \tag{8}$$

where c_{mn} has the same meaning as before and the sign factor s_{mn} is given by the formula

$$s_{mn} = (-1)^{\frac{(m-1)((m-n)^2+n)}{2}}.$$

This sign factor depends only on the values of m and n modulo 4. In particular $s_{mn} = 1$ if $m = 1(mod\, 4)$, $s_{mn} = -1$ if $m = 3(mod\, 4)$, and

$$s_{00} = 1, s_{01} = -1, s_{02} = -1, s_{03} = 1, s_{20} = 1, s_{21} = -1, s_{22} = -1, s_{23} = 1,$$

where the values of the indices m and n are reduced modulo 4.

Finally, replacing in the latter formulas sin and cos functions $h_j(x)$ by sinh and cosh functions $l_j(x)$ of the same arguments and using the notation $\hat{w}_{mn}$ for their Wronskian we get the formulas:

$$\begin{aligned}
&\hat{w}_{mn} := W(l_1 \dots, l_m),\\
&l_p(x) := \cosh q a_p(x-x_0),\ a_p = p, \quad \text{if}\, p \quad \text{is odd}\\
&l_p(x) := \sinh q a_p(x-x_0),\ a_p = p, \quad \text{if}\, p \quad \text{is even},\\
&1 \le p \le m-n;\\
&a_{m-n+j} = m-n+2j, \quad j = 1, \dots, n.\\
&l_{m-n+j}(x) := \cosh q a_{m-n+j}(x-x_0), \quad \text{if}\, m-n \quad \text{is odd}\\
&l_{m-n+j}(x) := \sinh q a_{m-n+j}(x-x_0), \quad \text{if}\, m-n \quad \text{is even}
\end{aligned}$$

$$\hat{w}_{m0} = (2q)^{\frac{m(m-1)}{2}} \prod_{k=0}^{m-1} k! \cdot \cosh^{\frac{m(m+1)}{2}} q(x - x_0), \tag{9}$$

$$\hat{w}_{mn} = |c_{mn}| \cdot \cosh^{\frac{m(m+1)}{2}} q(x - x_0)\ \sinh^{\frac{n(n+1)}{2}} q(x - x_0). \tag{10}$$

Proof
The proof of (2),(3) ,(5) (8) and (39) is obtained by use of the following propositions concerning Wronskians of n smooth functions.
Proposition 1.

$$W(f \cdot f_1, \dots, f \cdot f_n) = f^n(x) W(f_1, f_2, \dots, f_n), \tag{11}$$

where f is any smooth function of x.
This statement was first proved by Frobenius [7].

Another well known result which we use below is a formula of change of independent variable in a Wronskian. Let $t = f(x)$ be a new variable so that $f_j(x) = \tilde{f}_j(t)$. Than we have the following statement.
Proposition 2.

$$W(f_1, \dots, f_n) = (f'(x))^{\frac{n(n-1)}{2}} \cdot W(\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_n). \tag{12}$$

In particular, this proposition exhibits the dependence of the RHS of (3) on the parameter q:

$$W_{mn}(q, x, x_0) = W_{mn}(1, q(x - x_0), 0) q^{\frac{m(m-1)}{2}}$$

Therefore it is enough to prove (2) and (3) in the case $q = 1, x_0 = 0$. We assume that this restriction is satisfied until the end of this section.

Formulas (5) (8) and (39) represent the trivial consequence of (2) and (3).We wrote them explicitely only by the reason of further applications to the differential equations considered below. It is enough to replace q by iq in order to obtain (5) from (3). Replacing x_0 by $x_0 + \pi/2q$ we switch from (3) to (8). Replacing in q by iq in (8) we obviously get (39).

The proof of (3) can be obtained now in two steps. First, using the expansion

$$\sin jx = \sin x[(2\cos x)^{j-1} - (j-2)\,(2\cos x)^{j-3} + \frac{(j-3)(j-4)}{2}(2\cos x)^{j-5} + \dots] =$$

$$= u_j(\cos x) \sin x, \tag{13}$$

where $u_j(t)$ is the Tchebyshev polynomial, we can drop out of the Wronskian in the RHS of (3) the factor $\sin^m x$ by virtue of Proposition 1.

Next, we make the change of variable $t = \cos x$ and use the formula (12). This leads us to appearance of the extra factor $(-\sin x)^{\frac{m(m-1)}{2}}$ and a new Wronskian W_c:

$$W_c := W(1, u_2(t), \dots, u_{m-n}(t), u_{m-n+2}(t), u_{m-n+4}(t), \dots u_{m+n}(t)), \quad (14)$$

where the functions $u_j(t)$ are Tchebyshev polynomials:

$$u_j(t) = \frac{\sin jx}{\sin x} =$$

$$= \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k C_{j-k-1}^k \cdot (2t)^{j-1-2k},\ t = \cos x.$$

Now W_{mn} can be written as

$$W_{mn} = \sin^m x \cdot (-\sin x)^{\frac{m(m-1)}{2}} W_c = -(-\sin x)^{\frac{m(m+1)}{2}} W_c, \quad (15)$$

It is easy to see that W_c is equal to the Wronskian obtained by replacing the Tchebyshev polynomials by their first terms i.e. by the terms containing highest powers of t. In the case $n = 0$ this leads to the following formula.

$$W_c = W_p, \quad W_p := W(1, 2t, (2t)^2, \dots, (2t)^{m-1}). \quad (16)$$

Wronskian matrix corresponding to W_p has (in this particular case) the upper triangular form and hence W_c is equal to the product of its diagonal elements. Therefore, in the case $n = 0$ we get the following expression for W_c:

$$W_c = 2^{\frac{m(m-1)}{2}} \cdot \prod_{j=1}^{m-1} j!$$

Substitution of this expression into (15) completes the proof of (3) for W_{m0}.

In the case when $m > n, n = 1$, W_c reduces to the following expression

$$W_c = W(1, 2t, (2t)^2, \dots, (2t)^{m-2}, (2t)^m) = t \cdot 2^{\frac{m(m-1)}{2}+1}\, m! \prod_{k=1}^{m-2} k!,$$

which is again in agreement with the general formula (3).

In the case $m > n > 1$ it is easy to show that the Wronskian (14) is the same as obtained by replacing all $u_j(t)$ by their terms containing highest powers of t i.e.

$$W_c = W_p, \quad W_p :=$$

$$= W(1, 2t, \dots, (2t)^{m-n-1}, (2t)^{m-n+1}, (2t)^{m-n+3}, \dots, (2t)^{m+n-1}). \quad (17)$$

Wronskian matrix corresponding to W_p is not upper triangular any more. Nevertheless it can be always reduced to the upper triangular form by the elementary transformations of the right columns or of the upper rows. This can be done in such a way that the diagonal elements of the reduced matrix are proportional to the same powers of t as the diagonal elements of the non reduced matrix. For instance, in the case $n = 2$ it is easy to obtain this way the formula

$$W_{m2} = 2^3 \cdot (-2)^{\frac{m(m-1)}{2}} \cdot \frac{(m-1)!(m+1)!}{3} \prod_{j=1}^{m-3} j! \sin^{\frac{m(m+1)}{2}} x \cdot \cos^3 x,$$

which is again in agreement with (3). The possibility of similar transformation of W_c reducing it to determinant of upper triangular matrix might be proved for any value of n, $m > n \geq 1$. The related procedure preserves the structure of the principal diagonal i.e. we have the same powers of t at the same places as before but with different constant coefficients. These coefficients are certainly different from zero since W_c is obviously different from zero everywhere except the point $t = 0$. The above arguments show that W_{mn} really has the same functional structure as the RHS of (3). To complete the proof it is necessary to check that the coefficient c_{nm} is really the same as in (3). As soon as the correctness of the functional structure of (3) is established, c_{nm} can be calculated as follows.

Both expressions in the LHS and RHS of (3) are the polynomials of e^{ix}. So to check that the constant factor is correct we have to compute the coefficients of the highest power of e^{ix} in both sides of (3) and compare them. The related calculation leads to the following formula for c_{mn}:

$$c_{nm} = 2^{\frac{m(m-1)+n(n+1)}{2}} (-1)^{\frac{m(m-1)}{2}} V(1, 2, \ldots, d, d+2, \ldots, d+2n), \;\; d = m - n. \tag{18}$$

In the last formula V is a following Vandermond determinant:

$$V = V(a_1, \ldots, a_m) = \prod_{m \geq j > k \geq 1} (a_j - a_k),$$

where a_j were defined by (1). The short calculation gives the following formula for V:

$$V = 2^{\frac{n(n-1)}{2}} \prod_{p=0}^{p=d-1} p! \prod_{k=1}^{k=n} \frac{(d+2k-1)!}{(2k-1)!} (k-1)!$$

Substitution of the last expression in (16) completes the calculation of c_{mn} and hence also the proof of (3) $\square$.

The following list of the first sixteen Wronskians W_{mn} with $m \leq 6$ illustrates well the behavior of the coefficients c_{mn}, especially that of $c_{m,m-1}$, when m increases.

$W(\sin x, \sin 2x) = -2\sin^3 x,$
$W(\sin x, \sin 3x) = -8\cos x\ \sin^3 x,$
$W(\sin x, \sin 2x, \sin 3x) = -2^4 \cdot \sin^6 x,$
$W(\sin x, \sin 2x, \sin 4x) = -2^5 \cdot 3\sin^6 x\ \cos x,$
$W(\sin x, \sin 3x, \sin 5x) = -2^{10} \cdot \sin^6 x\ \cos^3 x,$
$W(\sin x, \sin 2x, \sin 3x, \sin 4x) = 2^8 \cdot 3\sin^{10} x,$
$W(\sin x, \sin 2x, \sin 3x, \sin 5x) = 2^{11} \cdot 3 \cdot \cos x \sin^{10} x,$
$W(\sin x, \sin 2x, \sin 4x, \sin 6x) = 2^{13} \cdot 3 \cdot 5\ \sin^{10} x \cos^3 x,$
$W(\sin x, \sin 3x, \sin 5x, \sin 7x) = 2^{20} \cdot 3\sin^{10} x\ \cos^6 x,$
$W(\sin x, \sin 2x, \sin 3x, \sin 4x, \sin 5x) = 2^{15} \cdot 3^2 \cdot \sin^{15} x,$
$W(\sin x, \sin 2x, \sin 3x, \sin 4x, \sin 6x) = 2^{16} \cdot 3^2 \cdot 5\cos x\ \sin^{15} x,$
$W(\sin x, \sin 2x, \sin 3x, \sin 5x, \sin 7x) = 2^{21} \cdot 3^2 \cdot 5\sin^{15} x \cos^3 x,$
$= 94.371.840\sin^{15} x \cos^3 x,$
$W(\sin x, \sin 2x, \sin 4x, \sin 6x, \sin 8x) = 2^{24} \cdot 3^2 \cdot 5 \cdot 7\cos^6 x\ \sin^{15} x$
$= 5.284.823.040 \cdot \cos^6 x\ \sin^{15} x,$
$W(\sin x, \sin 3x, \sin 5x, \sin 7x, \sin 9x) = 2^{35} \cdot 9\cos^{10} x\ \sin^{15} x$
$= 309.237.645.312 \cdot \cos^{10} x\ \sin^{15} x,$
$W(\sin x, \sin 2x, \sin 3x, \sin 4x, \sin 5x, \sin 6x)$
$= -2^{23} \cdot 3^3 \cdot 5 \cdot \sin^{21} x = -1.132.462.080\ \sin^{21} x,$
$W(\sin x, \sin 2x, \sin 3x, \sin 4x, \sin 5x, \sin 7x) = -3^4 \cdot 5 \cdot 2^{25} \cdot \cos x\ \sin^{21} x =$
$= -13.589.544.960\cos x\ \sin^{21} x.$

2 Wronskian representations of the DPT potentials and beyond

Below we need the following two propositions.
Let $v(x)$ is any potential and $f_1(x, E_1), f_2(x, E_2), \ldots, f_m(x, E_m)$ are any solutions of the 1D Schrödinger (Sturm-Liouville) equation

$$-f'' + v(x)f = Ef, \tag{19}$$

corresponding to $E = E_1, E_2, \ldots, E_m$.
Proposition 3 *The solution ψ of the Schrödinger equation*

$$-\psi'' + u\psi = E\psi, u = v - 2\partial_x^2 \log W(f_1, f_2, \ldots, f_m)) \tag{20}$$

is given by the formula

$$\psi(x,E) = \frac{W(f_1, f_2, \dots, f_m, f(x))}{W(f_1, f_2, \dots, f_m)}. \tag{21}$$

In the case $m = 1$, this proposition was found and proved by Darboux [5], and in general case by Crum [4]. Remarkably, their result can be extended to the partial differential equations and their non Abelian and lattice versions, [15] which founds many applications in modern theory of solitons [16, 17]. Below we need this extension only for the particular case of the scalar evolution PDE of the following form:

$$\partial_t f = \sum_{j=0}^{N} v_m(x,t) \partial_x^j f(x,t). \tag{22}$$

Let $f_1, f_2, \dots, f_m$ be any m linearly independent solutions of (22) different from f. Then the following statement holds.
Proposition 4.[3] *The RHS of (21) satisfy to the following PDE*

$$\partial_t \psi = \sum_{j=0}^{N} u_m(x,t) \partial_x^j \psi(x,t), \tag{23}$$

where the coefficients $u_m(x,t)$ can be explicitely written in terms of v_j and $f_k(x,t)$. In particular, assuming that $v_N = const$ and $v_{N-1} = 0$ we obtain the following formulas for u_N, u_{N-1}, u_{N-2}:

$$u_N = v_N, u_{N-1} = v_{N-1} = 0, u_{N-2} = v_{N-2} + N v_N \partial_x^2 \log W(f_1, f_2, \dots, f_m)). \tag{24}$$

The RHS of (21) represents the action of the differential operator of order N on the function f :

$$\psi = D(m) f := (\partial_x^m + \sum_{k=0}^{m-1} g_k(x,t) \partial_x^k) f.$$

$D(m)$ can be also represented in a form of a product of the first order differential operators in many different ways. Let $J = (j_1, j_2, \dots, j_m)$ be any permutation of the numbers $(1, 2, \dots, m)$. To each permutation corresponds the following factorization of $D(m)$:

$$D(m) = D_{j_m} \cdot D_{j_{m-1}} \cdots D_{j_1}, \qquad D_{j_k} = \partial_x - \sigma_{j_k},$$

[3] See [16, 17] for the proof and further details

$$\sigma_{j_k} := \partial_x \log \frac{W(f_{j_1}, f_{j_2}, \ldots, f_{j_k})}{W(f_{j_1}, f_{j_2}, \ldots, f_{j_{k-1}})}. \tag{25}$$

Assume now that $J = (1, 2, ..., m)$, $v = 0$, q and x_0 are real, and $f_j = \varphi_j$, where φ_j are defined in (1). Combining formula (3) with Proposition 3 we get the following result. The function ψ corresponding to the case $f_j = \varphi_j, v = 0$, solves the Schrödinger equation with real singular periodic potential u_{mn}:

$$-\psi'' + U_{mn}(x)\psi = E\psi, \tag{26}$$

where

$$U_{m,n} = -\partial_x^2 \log W_{mn} = \frac{q^2 m(m+1)}{\sin^2 q(x-x_0)} + \frac{q^2 n(n+1)}{\cos^2 q(x-x_0)}, \qquad m > n. \tag{27}$$

Replacing W_{mn} by $\hat{W}_{mn}$ we obtain the real potential $\tilde{U}_{m,n}$ exponentially decreasing at infinity:

$$\tilde{U}_{m,n} = -\partial_x^2 \log \hat{W}_{mn} = \frac{q^2 m(m+1)}{\sinh^2 q(x-x_0)} - \frac{q^2 n(n+1)}{\cosh^2 q(x-x_0)}, \qquad m > n. \tag{28}$$

General solution of the related Schrödinger equation (26) is obtained from (21) by choosing $f_j(x) = g_j(x)$. Quite similarly we have

$$u_{mn} := -2\partial_x^2 \log w_{mn} = \frac{q^2 m(m+1)}{\cos^2 q(x-x_0)} + \frac{q^2 n(n+1)}{\sin^2 q(x-x_0)}, \qquad m > n. \tag{29}$$

where w_{mn} was defined in (6) and the solution $\psi(x, E)$ of the related Schrödinger equation can be obtained by choosing $f_j = h_j$ in (21).

It is clear as well that

$$\hat{u}_{mn} := 2\partial_x^2 \log \hat{w}_{mn} = -\frac{q^2 m(m+1)}{\cosh^2 q(x-x_0)} + \frac{q^2 n(n+1)}{\sinh^2 q(x-x_0)}, \qquad m > n, \tag{30}$$

and the solutions of the related Schrödinger equation are obtained from (21) by choosing $f_j(x) = l_j(x)$. In the case $n = 0$, $\hat{u}_{mn}$ is a nonsingular real reflectionless potential and the discrete negative eigenvalues of the related Schrödinger operator on the real line are $E_j = -q^2 j^2$.

The integrability of (26) with potential (27) with $q = 1, x_0 = 0$ was discovered first time by Darboux in 1882 and rediscovered for its hyperbolic version (28) by Pöshl and Teller in 1933. Therefore, it is reasonable to call (27) , (28), (29) and (30) Darboux-Pöshl-Teller or shortly DPT potentials.

It seems, that the Wronskian determinant representations for DPT potentials and the related solutions of the Schrödinger equations written in Wronskian form were missing in the literature.

The solution of (26) found by Darboux [5] corresponds to particular factorization (25) with $J = (1, 2, \ldots, m)$ i.e. the related function ψ can be written as

$$\psi = D_m \cdot D_{m-1} \cdots D_1 f, \qquad D_k = \partial_x - \sigma_k.$$

The related functions σ_k now can be computed combining (3) and (25), where, of course, the constant factor in the RHS of (3) can be ignored.

$$\sigma_j = qj \cot q(x - x_0), j \leq m - n \quad (31)$$

$$\sigma_{m-n+k} = (m - n + k) q \cot q(x - x_0) + kq \tan q(x - x_0), k = 1, ..., n. \quad (32)$$

Original derivation of (31) by Darboux [5] was based on the observation that the functions $\sin^m(x - x_0) \cos^n q(x - x_0)$ represent the particular solutions of the Schrödinger equation (26) with potential $U_{m-1,n-1}$ corresponding to $E = q^2(m+n)^2$ and was not using the general Crum's formulas as we did.

The advantage of Crum representation for the DPT potentials is that it allows to describe immediately the action of KdV and higher flows of the KdV hierarchy on those potentials taken as initial data by using Proposition 4. The j-th equation of the KdV hierarchy can be obtained as a compatibility condition of (19) and the following evolution equation

$$\frac{\partial f}{\partial t_j} = c_j \frac{\partial^{2j+1} f}{\partial x^{2j+1}} - \frac{(2j+1)c_j}{2} v(x, t_j) \partial_x^{2j-1} f + \sum_{p=0}^{2j-2} v_p(x, t) \partial_x^p f. \quad (33)$$

In particular for $j = 1$ and $c_j = -4$ compatibility of (19) and (33) implies that v satisfies the KdV equation

$$v_t = 6\, v\, v_x - v_{xxx} \quad (34)$$

In general the choice of c_j is arbitrary : it fixes normalization of the KdV hierarchy. Now according to the Proposition 4 for any given solution $v(x, t_j)$ of j-th KdV equation the sequence $f_1, \ldots, f_m$ of the common solutions of (19) and (33), generates the new solution $u(x, t_j)$ of the same equation:

$$u(x, t_j) = v(x, t_j) - 2\partial_x^2 \log W(f_1, f_2, \ldots, f_m). \quad (35)$$

In particular, in the case

$$v=0, f_p=\sin(qp(x-x_0)+(-1)^j c_j q^{2j+1}p^{2j+1}t_j), p=1,\dots,d, \quad d=m-n,$$

$$f_{d+l}=\sin[q(d+2l)(x-x_0)+(-1)^j c_j q^{2j+1}(d+2l)^{2j+1}t_j], l=1,\dots,n, \quad m\geq n,$$

the RHS of (35) is the solution of the j-th KdV equation with the initial data equal to the DPT potential (27). Therefore, we obtained a very simple description of the isospectral deformations of the DPT potentials generated by the KdV hierarchy. In particular the solution of (34) with the initial condition (27) is given by the formula (35) with $v=0$ and

$$f_p=\sin(qp(x-x_0)+4q^3p^3t), p=1,\dots,d, \quad d=m-n$$

$$f_{d+l}=\sin[q(d+2l)(x-x_0)+4q^3(d+2l)^3t], l=1,\dots,n, \quad m\geq n.$$

Similar formulas for the action of KdV and higher KdV flows on the potentials (29),(28),(30) are also obvious from the previous discussion. For instance, the solutions of the KdV equation (34) with the initial condition $v(x,0)=\hat{u}_{mn}(x)$ defined by (30) is given by the formula

$$v(x,t)=-2\partial_x^2\log W(L_1,\dots,L_m), \tag{36}$$

where the functions $L_j(x,t)$ are defined as follows:

$L_p(x,t):=\cosh qa_p(x-x_0-4q^2a_p^2t)$, $a_p=p$, if p is odd
$L_p(x,t):=\sinh qa_p(x-x_0-4q^2a_p^2t)$, $a_p=p$, if p is even,
$1\leq p\leq m-n$;
$a_{m-n+j}=m-n+2j, \quad j=1,\dots,n.$
$L_{m-n+j}(x,t):=\cosh qa_{m-n+j}(x-x_0-4q^2a_{m-n+j}^2t)$, if $m-n$ is odd
$L_{m-n+j}(x):=\sinh qa_{m-n+j}(x-x_0-4q^2a_{m-n+j}^2t)$, if $m-n$ is even.

3 Differential identities

Formulas (2) and (3) lead to some differential identities. Let $\widehat{D}$ be a differential operator defined for any integer $m>2$ by the formula

$$(\widehat{D}_mf)(x):=\left(\prod_{k=2}^{m}\partial_{x_k}^{k-1}\right)f(x_1,x_2,\dots,x_m)\Big|_{x_1=x_2=\dots=x_m=x} \tag{37}$$

Let Q_m be the following determinant: $Q_m=\det a_{j,k}$, $a_{j,k}=\sin jx_k$, $1\leq j, \quad k\leq m$.
Clearly, we have

$$\widehat{D}_mQ_m=\det\frac{\partial^{j-1}\sin kx}{\partial x^{j-1}}=W(\sin x,\sin 2x,\dots,\sin mx)=W_{m0}. \tag{38}$$

The determinant Q_m can be easily computed by use of (13) and further obvious reduction to a Vandermond determinant:[4]

$$Q_m = 2^{\frac{m(m-1)}{2}} \prod_{k=1}^{m} \sin x_k \prod_{k>j} (\cos x_k - \cos x_j) \tag{39}$$

Now, substituting this expression for Q_m in (44) and taking into account the formula (3) for W_{m0}, we get the following identity:

$$\widehat{D}_m \left[\left(\prod_{k=1}^{m} \sin x_k \right) \cdot \left(\prod_{j>k} (\cos x_k - \cos x_j) \right) \right] \\ = \prod_{k=1}^{m-1} k! \, \sin^{\frac{m(m+1)}{2}} x \qquad m \geq 2 \tag{40}$$

Similar identities might be obtained along the same lines , comparing (3) with $\widehat{D}_m \det B$, where $B_{jk} = \varphi_p(x_j)$, $j, p = 1, \ldots, m$, and the functions $\varphi_p(x)$ were defined in (1). For instance for $n = 1, q = 1, x_0 = 0$ this leads to the following identity:

$$\widehat{D}_m \left[\left(\prod_{k=1}^{m} \sin x_k \right) \cdot \left(\sum_{j=1}^{m} \cos x_j \right) \cdot \left(\prod_{k>j} (\cos x_j - \cos x_k) \right) \right] \\ = m! \left(\prod_{k=1}^{m-2} k! \right) \cos x \cdot \sin^{\frac{m(m+1)}{2}} x, \tag{41}$$

where the operator $\widehat{D}_m$ is defined by (37).

It is possible to generalize and unify the identities above in a single formula (47) using a slightly different approach.

It is obvious that the Wronskian

$$\tilde{W}_{mn} := W(1, \cos x, \cos 2x, \cos dx, \cos(d+2)x, \cos(d+4)x, \ldots, \cos(d+2n)x)$$

is proportional to W_{mn}:

$$\tilde{W}_{mn} = (-1)^m d! \prod_{k=1}^{n} (d+2k) \cdot W_{mn}, \\ \tilde{W}_{m0} = (-1)^m m! \cdot W_{m0}, \tag{42}$$

[4]The same expression, (up to a different constant normalization factor and the sign factor), as the RHS of (39) recently appeared [24] (Th. A, Th. 5.1) as a density of probability measure describing the asymptotic distribution of Frobenius roots on the m -dimensional Abelian varieties over finite fields F_q when $q \to \infty$.

This follows immediately from the expansion of $\tilde{W}_{mn}$ by the elements of its first column. Assuming that $n = 0$, and introducing the determinant

$$G_m := \det B, B_{jk} = \cos j\, x_k,\ j = 0, 1, \ldots, m;\ k = 1, 2, \ldots m+1, \tag{43}$$

we obviously have the equality

$$\tilde{W}_{m0} = \widehat{D}_{m+1} G_m, \tag{44}$$

where $\widehat{D}_m$ is defined by (37). Now G_m can be easily computed using the expansion

$$\cos jx_k = 2^{j-1} \cos^j x_k + \text{sum of the lower powers of } \cos x_k.$$

Replacing $\cos jx_k$ by the first term of this expansion in G_m we get the Vandermond determinant equal to G_m. Therefore, for G_m the following expression holds

$$G_m = 2^{\frac{m(m-1)}{2}} \prod_{m+1 \geq j > k \geq 1} (\cos x_j - \cos x_k). \tag{45}$$

Comparing (45) and (44) with explicit expression for $\tilde{W}_{m0}$, following immediately from (42), and replacing m by $m-1$ we obtain the following identity :

$$\widehat{D}_m V_m = \left(\prod_{k=1}^{m-1} k! \right) \cdot \sin^{\frac{m(m-1)}{2}} x, \qquad V_m := \prod_{m \geq k > j \geq 1} (\cos x_j - \cos x_k). \tag{46}$$

Replacing $\widehat{D}_m$ by any operator $\tilde{D}$ obtained by removing from $\widehat{D}_m$ some partial differentiations we have the obvious relation $\tilde{D} \cdot G_{m-1} = 0$. Combining this relation with the fact that V_m is proportional to G_{m-1}, we get the following identity:

$$\widehat{D}_m (V_m \cdot f(x_1, \ldots, x_m)) = f(x_1, \ldots, x_m)|_{x_1 = x_2 = \ldots = x_m = x}\ \widehat{D}_m(V_m), \tag{47}$$

where explicit expression for $\widehat{D}_m(V_m)$ is given by (46). It is also obvious that

$$\tilde{D}(V_m \cdot f(x_1, \ldots, x_m)) = 0, \tag{48}$$

where again $\tilde{D}$ means the differential operator obtained from $\widehat{D}_m$ by removing some partial differentiations.

The identities (40,41) now can be considered as immediate consequence of (47) and (46) corresponding to particular choices of the function $f(x_1, x_2, \ldots x_m)$ in (47).

4 Additional comments

- Initial motivation of this article was the following question. How to select the solutions f_j of the free Schrödinger equation $-f_j'' = E_j f_j$ in order to get the DPT potential from Crum formula, (20) with $v = 0$. The choice of E_j was clear from [5] but at every point E_j we have the generic solution $A_j \sin(\sqrt{E_j}x + x_j)$, where A_j play no role in computing $\partial_x^2 \log W(f_1, \ldots, f_m)$. In general, there is no compact expression for the Wronskian provided that the phases are chosen arbitrary. It was also clear that some selection of phases x_j leading to DPT potential via (20) should exist but the fact that trivial phases provide the right answer apriory was not obvious and it was necessary to prove (3), (at least modulo constant normalization factor c_{mn}), in order to confirm it.

- There is some similarity of (3), with $n = 0$ with addition formulas for elliptic functions obtained by Kiepert, Brioschi, Hermite, Frobenius and Stickelberger [6, 23] although we were not able to find a direct connection with their results. In particular, we do not know the compact expression for $W(\sigma(x), \sigma(2x), \ldots, \sigma(nx))$, where $\sigma(x)$ is a Weierstrass sigma function.

- Taking $f = e^{ikx}, k^2 = E$ in the formula (21) for the solutions of (26) we see that the related function $\psi(x, E) = P(x, k) \cdot e^{ikx}$ where $P(x, k)$ is a polynomial of k. In general the potentials allowing the solutions of this form are called Bargmann potentials. All Bargmann potentials might be considered as a degenerate case of the hyperelliptic finite gap potentials [3]. In particular, all the DPT potentials (27),(29), (28),(30) might be obtained from the general m-gap hyperelliptic potentials by taking appropriate limit in Its-Matveev formula ([3]), which is somehow technically less convinient with respect to direct algebraic approach used here.

- After Darboux, integrability of the hyperbolic version of the same potentials was rediscovered [18] by Pöschl and Teller (father of the US hydrogenous bombe). These authors certainly did not observed that the same (and more general) equation was already solved by Darboux but they found important applications of the DPT potentials in quantum theory of two atomic molecules. Later the Schrödinger equation with the same potentials, (mainly called Pöschl-Teller potentials in quantum-mechanical literature,) were discussed by many authors for different boundary conditions in connection with different physical applications, see for instance [1, 2, 14, 10, 12, 11].

- Using the change of variable reducing the Schrödinger equation with DFT potential to the Fuchs equation with 3 singular points Darboux found another form of the solution of (26):

$$y = \sin^m x \cos^n x F(\alpha, \beta, \gamma, \sin^2 x), \tag{49}$$
$$\gamma = \frac{m+1}{2}, \quad \alpha = \frac{m+n}{2} + \frac{1}{2}\sqrt{\lambda},$$
$$\beta = \frac{m+n}{2} - \frac{1}{2}\sqrt{\lambda},$$

  where F is any solution of the Gauss hypergeometrical equation. This form of the solution was also rediscovered several times by some of the aforementioned authors.

- Darboux [8] also found the integrable elliptic generalization of the DPT potentials also representing the 4-parametric generalization of the Lame equation. This elliptic Darboux potential has the following form:

$$u(x) = \frac{m(m-1)k^2 cn^2 x}{dn^2 x} + \frac{q(q-1)dn^2 x}{cn^2 x} + \frac{p(p-1)}{sn^2 x} + n(n-1)k^2 sn^2 x \tag{50}$$

  In (50) sn, cn and dn are usual Jacobi elliptic functions of modulus k and m, q, p, n are arbitrary non negative integers. In the case m=q=p=0, this equation reduces to Lamé equation first integrated by Hermite. The potentials of the form (50) were rediscovered relatively recently by Trebich and Verdier [22] as a special reduction of the general formula [9, 13, 3] describing all finite-gap periodic or quasiperiodic potentials . More general connections between the finite gap potentials and Fuchsian equations were recently discussed by Smirnov [20].

  In the limit when one of the periods of the elliptic functions tends to infinity the Darboux-Vedrier-Trebich potential reduces to the Darboux-Pöschl-Teller potential

- The DPT potentials and the Darboux-Verdier-Trebich potentials recently found some applications to the algebraic Bethe Anzatz construction for trigonometric case of the quantum Calogero-Moser multi particle system. [21]

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