GALOIS COVERINGS OF THE PLANE BY K3 SURFACES

A. MUHAMMED ULUDAĞ

Abstract. We give a complete list of branched Galois coverings of the projective plane by smooth K3 surfaces, with the branching curve of degree $d < 6$, and some examples of such coverings with $d = 6$. These coverings are not assumed to be abelian.

1. Introduction

The aim of the present article is to study the problem: "classify the branched Galois coverings of $\mathbf{P}^2$ by *smooth* K3 surfaces". A basic example of such a covering is the following: Let $C \subset \mathbf{P}^2$ be a smooth quartic given by the homogeneous polynomial $f(x, y, z)$. Then the surface $X_1 \subset \mathbf{P}^3$ with equation $w^4 = f(x, y, z)$ is a smooth K3 surface. The projection $\psi : [x : y : z : w] \in X_1 \to [x : y : z] \in \mathbf{P}^2$ is a covering with $\mathbb{Z}/(4)$ as the Galois group, branched along C with branching index 4. Viewing the pair $\mathcal{O} := (\mathbf{P}^2, 4C)$ as an *orbifold*, the projection $\psi : X \to \mathcal{O}$ is a *uniformization* of this orbifold. The curve C is called the *locus* of $\mathcal{O}$. Our approach to the problem consists in the study of orbifolds over $\mathbf{P}^2$ uniformized by smooth $K3$ surfaces. As a result, we obtain a complete list of orbifolds over $\mathbf{P}^2$ uniformized by smooth K3 surfaces and with an abelian Galois group (Theorem 1) and a complete list of orbifolds over $\mathbf{P}^2$ with a locus of degree ≤ 5, uniformized by smooth K3 surfaces and with non-abelian Galois groups (Theorem 2). We also present some K3 orbifolds over $\mathbf{P}^2$ with a sextic locus. There are no K3 orbifolds with a locus of degree > 6.

An alternative approach to this problem is to study the K3 surfaces X with the action of a group $G \subset Aut(X)$ such that $X/G \simeq \mathbf{P}^2$. A triple (G, X, ρ) is called a *K3 group* if X is a K3 surface and G is a group with a faithful action $\rho : G \times X \to X$. A K3 group (G, X, ρ) is called *symplectic* if G preserves a nowhere vanishing 2-form ω_X on X. Symplectic K3 groups were classified by Nikulin [19] for the abelian case and by Mukai [18] in the general case: they are 80 subgroups of a certain set of 11 subgroups of the Mathieu group M_{23}. Mukai also shows that if (G, X, ρ) is symplectic, then the minimal desingularization of

X/G is also a K3 surface, see also [29] and [13] . The case where X/G is a Kummer surface has been studied in detail, see [25],[17], [23].

Since the form ω_X is unique up to a constant multiple, any K3 group (G, X, ρ) has a 1-dimensional representation $g \to \alpha(g) \in \mathbb{C}$, where α is defined by $g^*\omega_X = \alpha(g)\omega_X$. The kernel of G_N of this representation preserves ω_X, and hence is a symplectic K3 group. Hence, the classification of K3 groups can be reduced to the study of "geometrically realizable" extensions of symplectic groups. In a recent preprint [21], Oguiso announced that this classifification will be carried out in [11]. In contrast with the extensive literature on K3 groups (X, G, ρ), there are not many results concerning the orbifold structure of X/G (see however [30]

A classical construction in the theory of algebraic surfaces is that of a double plane, which has been applied extensively to plane sextics to obtain concrete examples of K3 surfaces [26], [1] [24]. Arguing in the opposite direction, Yang [32] used the classification theory of K3 surfaces to classify all irreducible plane sextics with only simple singularities. Yang's results can be used to extend our results to orbifolds with sextic locus.

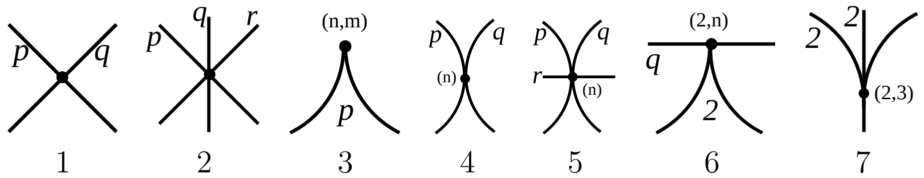
Note that the double plane construction yields a smooth surface X only when C is a smooth sextic. Excluding this case, we shall not discuss double covers in this article. We are exlusively concerned with those coverings $X \to \mathbf{P}^2$ where X is a "honestly" smooth K3 surface.

2. Orbifolds

2.1. **Definitions.** We shall mostly follow the terminology settled in [31]. One may also consult [6] and [33]. Let M be a connected smooth complex manifold, $G \subset \mathrm{Aut}(M)$ a properly discontinuous subgroup and put $X := M/G$. Then the projection $\phi : M \to X$ is a branched Galois covering endowing X with a map $\beta_\phi : X \to \mathbb{N}$ defined by $\beta_\phi(p) := |G_q|$ where q is a point in $\phi^{-1}(p)$ and G_q is the isotropy subgroup of G at q. In this setting, the pair (X, β_ϕ) is said to be uniformized by $\phi : M \to (X, \beta_\phi)$. An *orbifold* is a pair (X, β) of an irreducible normal analytic space X with a function $\beta : X \to \mathbb{N}$ such that the pair (X, β) is locally finitely uniformizable. In case $\gamma|\beta$, the orbifold (X, γ) is said to be a *suborbifold* of (X, β). Let $\phi : (Y, 1) \to (X, \gamma)$ be a uniformization of (X, γ), e.g. $\beta_\phi = \gamma$. Then $\phi : (Y, \beta') \to (X, \beta)$ is called an *orbifold covering*, where $\beta' := \beta \circ \phi / \gamma \circ \phi$. The orbifold (Y, β') is called the *lifting of (X, β) to the uniformization Y of (X, γ)* and will be denoted by $(X, \beta)/(X, \gamma)$.

Let (X,β) be an orbifold, $B_\beta := \mathrm{supp}(\beta-1)$ be its *locus* and let $B_1,\dots,B_n$ be the irreducible components of B_β. Then β is constant on $B_i \backslash \mathrm{sing}(B_\beta)$; so let b_i be this number. The *orbifold fundamental group* $\pi_1^{orb}(X,\beta)$ of (X,β) is the group defined by $\pi_1^{orb}(X,\beta) := \pi_1(X\backslash B_\beta)/\langle\langle \mu_1^{b_1},\dots,\mu_n^{b_n}\rangle\rangle$ where $\mu_i^{b_i}$ is a meridian of B_i and $\langle\langle\rangle\rangle$ denotes the normal closure. The *local orbifold fundamental group* $\pi_1^{orb}(X,\beta)_p$ *at a point* $p\in X$ is the orbifold fundamental group $\pi_1^{orb}(B,p)$ of the restriction of (X,β) to a sufficiently small neighbourhood of B of p.

2.2. **Orbifold singularities in dimension 2.** Let (X,β) be an orbifold with X being a smooth surface. Then the map β is determined by the numbers b_i; in fact $\beta(p)$ is the order of the local orbifold fundamental group at p. In case $\dim X = 2$ the orbifold condition (i.e. locally finitely uniformizability) is equivalent to the finiteness of the local fundamental groups. This implies that the locus B_β of (X,β) may have ADE singularities only, and the admissible values for b_i are very restricted. These singularities and the branching numbers can either be characterized as the branching loci of the quotients of $\mathbb{C}^2$ by finite reflection groups or as the germs with a finite orbifold fundamental group (see [33]). Below is a list of these singularities.



No	*Equation*	*Condition*	*Order*
1	xy	$--$	pq
2	$xy(x+y)$	$0<\rho:=\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1$	$4\rho^{-2}$
3	x^n-y^m $(\gcd(n,m)=1)$	$0<\rho:=\frac{1}{n}+\frac{1}{m}+\frac{1}{p}-1$	$\frac{4}{nm}\rho^{-2}$
4	x^2-y^{2n} $(n\geq 2)$	$0<\rho:=\frac{1}{p}+\frac{1}{q}+\frac{1}{n}-1$	$\frac{4}{n}\rho^{-2}$
5	$y(x^2-y^{2n})$	$0<\rho:=\frac{1}{p}+\frac{1}{q}+\frac{1}{nr}-1$	$\frac{4}{n}\rho^{-2}$
6	$y(x^2-y^n)$ $(n\ \ odd)$	$--$	$2nq^2$
7	$x(x^2-y^3)$	$--$	96

In the last column of the table above, order of the local orbifold fundamental group of the corresponding singularity is given. For the node (1), this order is evidently pq. For the case (2), i.e. triple point, the orders of the corresponding reflection groups are well known, see e.g. [33]. For the remaining cases, the orders can be computed by using

Yoshida's result that any of the remaining singular loci is an orbifold covering of one of the triple points in (2) ([33], pp 167-169).

2.3. **Finite uniformizations.** Let X be a smooth manifold, (X,β) an orbifold, and let $f:\pi_1^{orb}(X,\beta)\to G$ be a surjection onto a finite group G. Then there is a corresponding branched covering $\phi: M\to (X,\beta)$ with G as the Galois group. This is a finite smooth uniformization of X if and only if ϕ coincides locally with the local finite uniformizations of (X,β). In terms of G, this condition is expressed as follows:

Lemma 1. *Let $\iota_p:\pi_1^{orb}(X,\beta)_p\to\pi_1^{orb}(X,\beta)$ be the homomorphism induced by the inclusion. Then M is a finite smooth uniformization of (X,β) if and only if for any $p\in X$, the map $f\circ\iota_p:\pi_1^{orb}(X,\beta)_p\to G$ is an injection.*

Let (X,β) be an orbifold, (X,γ) a suborbifold, $(Y,1)\to(X,\gamma)$ a universal uniformization, and $(Y,\beta')\simeq(X,\beta)/(X,\gamma)$. Then one has the exact sequence of fundamental groups

$$0\to\pi_1^{orb}(Y,\beta')\to\pi_1^{orb}(X,\beta)\to\pi_1^{orb}(X,\gamma)\to 0$$

so that if $\pi_1^{orb}(X,\beta)$ is finite, then $|\pi_1^{orb}(X,\beta)|=|\pi_1^{orb}(X,\gamma)||\pi_1^{orb}(Y,\beta')|$.

The following lemma will be useful.

Lemma 2. *Let $\mathcal{O}=(\mathbf{P}^2,B)$ be an orbifold where $B=\sum_{i=1}^n b_iB_i$ with B_i being irreducible plane curve of degree d_i. If B is a nodal curve then*

$$\pi_1^{orb}(\mathcal{O})\simeq\langle\mu_1,\mu_2,\dots,\mu_n\,|\,\mu_1^{b_1}=\mu_2^{b_2}=\cdots=\mu_n^{b_n}=\mu_1^{d_1}\mu_2^{d_2}\dots\mu_n^{d_n}=1\rangle$$

Proof. By the Zariski conjecture proved by Deligne and Fulton (see [5]) the fundamental group of a nodal curve is abelian. Therefore

$$\pi_1(\mathbf{P}^2\backslash B)\simeq H_1(\mathbf{P}^2\backslash B,\mathbb{Z})\simeq\langle\mu_1,\mu_2,\dots,\mu_n\,|\,\mu_1^{d_1}\mu_2^{d_2}\dots\mu_n^{d_n}=1\rangle$$

the classes μ_i being representatives of meridians around B_i for $i\in[1,n]$. The lemma follows by adding the orbifold relations to the above presentation. $\square$.

2.4. **Orbifold Chern numbers.** Hereafter, we shall exclusively be concerned with orbifolds (X,β) with X being a smooth algebraic surface. Since in this case β is determined by its values b_i on $B_i\backslash\mathrm{sing}(B_\beta)$, an orbifold can alternatively be defined as a pair (X,B) where $B:=b_1B_1+\dots b_nB_n$ is a divisor on X with $b_i\geq 1$. For an orbifold (X,B) the corresponding map $X\to\mathbb{N}$ will be denoted by β_B.

Let $\mathcal{O}:=(\mathbf{P}^2,B)$ be an orbifold where $B=b_1B_1+\dots+b_nB_n$ with B_i being an irreducible curve of degree d_i.

Definition. The *orbifold Chern numbers of* $\mathcal{O}$ are defined as

$$c_1^2(\mathcal{O}) := \left[-3 + \sum_{1\le i\le n} d_i \left(1 - \frac{1}{b_i}\right)\right]^2$$

$$e(\mathcal{O}) := 3 - \sum_{1\le i\le n}\left(1-\frac{1}{b_i}\right) e(B_i\backslash \operatorname{sing}(B)) - \sum_{p\in \operatorname{sing}(B)}\left(1-\frac{1}{\beta_B(p)}\right)$$

Let $\mathcal{O} := (\mathbf{P}^2, B)$ be an orbifold where $B = b_1B_1 + \cdots + b_nB_n$ with B_i being an irreducible curve of degree d_i. If $(X, 1) \to (\mathbf{P}^2, \beta)$ is a finite uniformization of degree d then the Chern numbers of X are given by $e(X) = de(\mathcal{O})$ and $c_1^2(X) = dc_1^2(\mathcal{O})$ (see [12]).

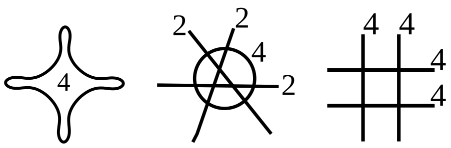
2.5. **K3 orbifolds.** We call $\mathcal{O} := (\mathbf{P}^2, B)$ a *K3 orbifold* if there is a finite uniformization $\phi : M \to \mathcal{O}$ by a smooth $K3$ surface M. Since M is simply connected, ϕ must be the universal uniformization of $\mathcal{O}$, which means that the corresponding Galois group is the full group $\pi_1^{orb}(\mathcal{O})$, and the conditions of Lemma 1 must hold for $G = \pi_1^{orb}(\mathcal{O})$. Moreover, since $c_1^2(M) = 0$ and $e(M) = 24$, one has the following lemma:

Lemma 3. *Let* $\mathcal{O} := (\mathbf{P}^2, B)$ *be a K3 orbifold where* $B = b_1B_1 + \cdots + b_nB_n$ *with* B_i *being an irreducible curve of degree* d_i*. Then*
(i) One has $c_1^2(\mathcal{O}) = 0$*. In particular,* $4 \le \deg(B) \le 6$*, and if* $\deg(B)$*=6, then* $b_1 = b_2 = \cdots = b_n = 2$.
(ii) $\pi_1^{orb}(\mathcal{O})$ *is of order* $24/e(\mathcal{O})$*. In particular,* $24/e(\mathcal{O})$ *is an integer.*

3. Abelian K3 orbifolds

In 3.1-3.5, we shall give a complete list of K3 orbifolds with abelian uniformizing Galois group. The completeness of the list will be proved in 3.6. The locus of any orbifold considered in this section is assumed to be a nodal curve.

3.1. **The K3 orbifolds $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$.** These orbifolds are defined as follows: $\mathcal{C}_1$ is the orbifold $(\mathbf{P}^2, 4K)$ where K is a smooth quartic; $\mathcal{C}_2$ is the orbifold $(\mathbf{P}^2, 2L_1 + 2L_1 + 2L_3 + 4Q)$ where L_1, L_2 and L_3 are three distinct lines and Q is a smooth quadric; $\mathcal{C}_3$ is the orbifold $(\mathbf{P}^2, 4L_1 + 4L_1 + 4L_3 + 4L_4)$ where L_1, L_2, L_3 and L_4 are four distinct lines.



$\mathcal{C}_1$ (4) $\mathcal{C}_2$ (16) $\mathcal{C}_3$ (64)

3.1.1. *The orbifold* $\mathcal{C}_1$. This is the orbifold mentioned in the beginning of the introduction. Obviously, one has $|\pi_1^{orb}(\mathcal{C}_1)| = 4$.

3.1.2. *The orbifold* $\mathcal{C}_2$. Suppose that L_1, L_2, L_3 and Q are given respectively by the equations $x = 0$, $y = 0$, $z = 0$ and $P[x : y : z] = 0$. The suborbifold $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3)$ of $\mathcal{C}_2$ is uniformized by $\mathbf{P}^2$ via the covering $\phi_2 : [x : y : z] \to [x^2 : y^2 : z^2]$. Lifting the orbifold $\mathcal{C}_2$ to this uniformization gives the orbifold $\mathcal{C}_1 = (\mathbf{P}^2, 4K)$, where K is the smooth quartic $P(x^2, y^2, z^2) = 0$. In other words, one has $\mathcal{C}_1 \simeq \mathcal{C}_2/\mathcal{A}_2$, and there is a covering $\phi_2 : \mathcal{C}_1 \to \mathcal{C}_2$ of degree 4. The orbifold $\mathcal{C}_1$ in turn is uniformized by the K3 surface X_2 defined by the equation $w^4 = P(x^2, y^2, z^2)$. One has

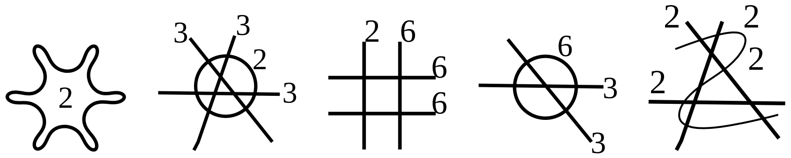
$$|\pi_1^{orb}(\mathcal{C}_2)| = |\pi_1^{orb}(\mathcal{A}_2)||\pi_1^{orb}(\mathcal{C}_1)| = 4 \times 4 = 16$$

3.1.3. *The orbifold* $\mathcal{C}_3$. Since any two arrangement of four lines in general position in $\mathbf{P}^2$ are projectively equivalent, one can assume that the lines L_1, L_2, L_3 and L_4 are given by the equations $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 0$ respectively. Consider the suborbifold $\mathcal{A}_4 := (\mathbf{P}^2, 4L_1 + 4L_2 + 4L_3)$ of $\mathcal{C}_3$, which is uniformized by $\mathbf{P}^2$ via the covering $\phi_4 : [x : y : z] \to [x^4 : y^4 : z^4]$. Lifting the orbifold $\mathcal{C}_3$ to this uniformization gives the orbifold $\mathcal{C}_1 = (\mathbf{P}^2, 4C)$, where C is the Fermat quartic given by the equation $x^4 + y^4 + z^4 = 0$. In other words, one has $\mathcal{C}_1 \simeq \mathcal{C}_3/\mathcal{A}_4$, and there is a covering $\phi_4 : \mathcal{C}_1 \to \mathcal{C}_3$ of degree 16. The orbifold $\mathcal{C}_1$ in turn is uniformized by the Fermat K3 surface X_3 defined by the equation $w^4 = x^4 + y^4 + z^4$. One has

$$|\pi_1^{orb}(\mathcal{C}_3)| = |\pi_1^{orb}(\mathcal{A}_4)||\pi_1^{orb}(\mathcal{C}_1)| = 16 \times 4 = 64$$

Note that $\mathcal{A}_2' := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3)$ is also a suborbifold of $\mathcal{C}_3$, which is uniformized by $\mathbf{P}^2$ via the covering $\phi_2' : [x : y : z] \to [x^2 : y^2 : z^2]$. Lifting the orbifold $\mathcal{C}_3$ to this uniformization gives the orbifold $\mathcal{C}_2 = (2L_1 + 2L_2 + 2L_3 + 4Q)$, where L_1, L_2 and L_3 are respectively given by the equations $x = 0$, $y = 0$, $z = 0$ and Q is the smooth quadric given by the equation $x^2 + y^2 + z^2 = 0$. In other words, one has $\mathcal{C}_2 \simeq \mathcal{C}_3/\mathcal{A}_2'$, and there is a covering $\phi_2' : \mathcal{C}_2 \to \mathcal{C}_3$ of degree 4.

3.2. **The K3 orbifolds $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_4$ and $\mathcal{D}_5$.** Let L_1, L_2, L_3, L_4 be four distinct lines. $\mathcal{D}_1$ is the orbifold $(\mathbf{P}^2, S)$ where S is a smooth sextic; $\mathcal{D}_2$ is the orbifold $(\mathbf{P}^2, 3L_1 + 3L_2 + 3L_3 + 2Q)$ where Q is a smooth quadric; $\mathcal{D}_3$ is the orbifold $(\mathbf{P}^2, 6L_1 + 6L_2 + 6L_3 + 2L_4)$; $\mathcal{D}_4$ is the orbifold $(\mathbf{P}^2, 3L_1 + 3L_2 + 6Q)$ where Q is a smooth quadric. $\mathcal{D}_5$ is the orbifold $(\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3 + 2C)$ where C is a smooth cubic.



$\mathcal{D}_1$ (2) $\mathcal{D}_2$ (18) $\mathcal{D}_3$ (72) $\mathcal{D}_4$ (18) $\mathcal{D}_5$ (8)

3.2.1. *The orbifold* $\mathcal{D}_1$. The double covering Y_1 of the plane along the sextic is the uniformization in this case. One has $|\pi_1^{orb}(\mathcal{D}_1)| = 2$.

3.2.2. *The orbifold* $\mathcal{D}_2$. Consider the suborbifold $\mathcal{A}_3 := (\mathbf{P}^2, 3L_1 + 3L_2 + 3L_3)$ of $\mathcal{D}_3$, which is uniformized by $\mathbf{P}^2$ via ϕ_3. Assume that L_1, L_2, L_3 and Q are defined respectively by $x = 0$, $y = 0$, $z = 0$ and $P[x : y : z] = 0$. Lifting $\mathcal{D}_2$ by ϕ_3 gives the orbifold $\mathcal{D}_1 = (\mathbf{P}^2, 2S)$, where S is the smooth sextic $P(x^3, y^3, z^3) = 0$. In other words, one has $\mathcal{D}_1 \simeq \mathcal{D}_2/\mathcal{A}_3$, and there is a covering $\phi_3 : \mathcal{D}_1 \to \mathcal{D}_2$ of degree 9. The orbifold $\mathcal{D}_1$ in turn is uniformized by the K3 surface Y_2 which is the double cover of the plane along S. One has

$$|\pi_1^{orb}(\mathcal{D}_2)| = |\pi_1^{orb}(\mathcal{A}_3)||\pi_1^{orb}(\mathcal{D}_1)| = 9 \times 2 = 18$$

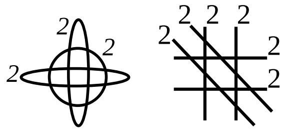
3.2.3. *The orbifold* $\mathcal{D}_3$. Assume that L_1, L_2, L_3 and L_4 are defined respectively by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 0$. Consider the suborbifold $\mathcal{A}_6 := (\mathbf{P}^2, 6L_1 + 6L_2 + 6L_3)$ of $\mathcal{D}_3$, which is uniformized by $\mathbf{P}^2$ via ϕ_6. Lifting $\mathcal{D}_3$ by ϕ_6 gives $\mathcal{D}_1 = (\mathbf{P}^2, 2S)$, where S is the Fermat sextic $x^6 + y^6 + z^6 = 0$. In other words, one has $\mathcal{D}_1 \simeq \mathcal{D}_3/\mathcal{A}_6$, and there is a covering $\phi_6 : \mathcal{D}_1 \to \mathcal{D}_3$ of degree 36. The orbifold $\mathcal{D}_1$ in turn is uniformized by the K3 surface Y_3 which is the double cover of the plane along S. One has

$$|\pi_1^{orb}(\mathcal{D}_3)| = |\pi_1^{orb}(\mathcal{A}_6)||\pi_1^{orb}(\mathcal{D}_1)| = 36 \times 2 = 72$$

3.2.4. *The orbifold* $\mathcal{D}_4$. One has $\mathcal{D}_4 \simeq \mathcal{D}_3/\mathcal{A}_2$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_2 + 2L_3 + 2L_4)$. Hence $|\pi_1^{orb}(\mathcal{D}_4)| = |\pi_1^{orb}(\mathcal{D}_3)|/|\pi_1^{orb}(\mathcal{A}_2)| = 72/4 = 18$.

3.2.5. *The orbifold* $\mathcal{D}_5$. One has $\mathcal{D}_5 \simeq \mathcal{D}_3/\mathcal{A}_3$, where $\mathcal{A}_3 := (\mathbf{P}^2, 3L_1 + 3L_2 + 3L_3)$. Hence $|\pi_1^{orb}(\mathcal{D}_5)| = |\pi_1^{orb}(\mathcal{D}_3)|/|\pi_1^{orb}(\mathcal{A}_3)| = 72/9 = 8$.

3.3. **The K3 orbifolds $\mathcal{E}_1$ and $\mathcal{E}_2$.** These are defined as follows: $\mathcal{E}_1$ is the orbifold $(\mathbf{P}^2, 2Q_1 + 2Q_2 + 2Q_3)$ where Q_1, Q_2, Q_3 are three smooth quadrics; $\mathcal{E}_2$ is the orbifold $(\mathbf{P}^2, \sum_{i=1}^{6} 2L_i)$ where L_i ($i \in [1, 6]$) are six distinct lines.



$\mathcal{E}_1$ (8) $\mathcal{E}_2$ (32)

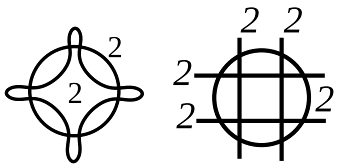
3.3.1. *The orbifold* $\mathcal{E}_2$. By Lemma 2 the orbifold fundamental group is the abelian group with the presentation

$$\pi_1^{orb}(\mathcal{E}_2) \simeq \langle \lambda_1, \dots, \lambda_6 \mid \lambda_1^2 = \dots \lambda_6^2 = \lambda_1 \dots \lambda_6 = 1 \rangle,$$

which is isomorphic to $(\mathbb{Z}/(2))^5$ and is of order 32. In order to show that $\mathcal{E}_1$ is uniformizable, one must verify that the subgroup $\langle \lambda_i \rangle$ is the group $\mathbb{Z}/(2)$ and the subgroup $\langle \lambda_i, \lambda_j \rangle$ is the group $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ for any $i \neq j \in [1, 6]$. This is easily verified. Moreover, the euler number of this orbifold is easily seen to be $3/4$, so that the euler number of the uniformization is $3|\pi_1^{orb}(\mathcal{E}_2)|/4 = 24$. Since the first Chern number of $\mathcal{E}_2$ is 0, it follows that this is a K3 orbifold.

3.3.2. *The orbifold* $\mathcal{E}_1$. One has $\mathcal{E}_1 \simeq \mathcal{E}_2/\mathcal{A}_2$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3)$. Hence $|\pi_1^{orb}(\mathcal{E}_1)| = |\pi_1^{orb}(\mathcal{E}_2)|/|\pi_1^{orb}(\mathcal{A}_2)| = 32/4 = 8$.

3.4. **The K3 orbifolds $\mathcal{F}_1$ and $\mathcal{F}_2$.** These are defined as follows: $\mathcal{F}_1$ is the orbifold $(\mathbf{P}^2, 2Q + 2K)$ where Q is a smooth quadric and K is a smooth quartic; $\mathcal{F}_2$ is the orbifold $(\mathbf{P}^2, 2Q + \sum_{i=1}^4 2L_i)$ where L_i ($i \in [1, 4]$) are four distinct lines and Q is a smooth quadric.



$\mathcal{F}_1$ (4) $\mathcal{F}_2$ (16)

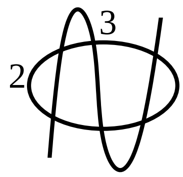
3.4.1. *The orbifold* $\mathcal{F}_2$. By Lemma 2 the orbifold fundamental group is the abelian group with the presentation

$$\pi_1^{orb}(\mathcal{F}_2) \simeq \langle \kappa, \lambda_1, \dots, \lambda_4 \mid \lambda_1^2 = \dots \lambda_4^2 = \lambda_2 \lambda_3 \lambda_4 \lambda_5 = 1 \rangle,$$

which is isomorphic to $(\mathbb{Z}/(2))^4$ and is of order 16. In order to show that $\mathcal{F}_1$ is uniformizable, one must verify that the subgroups $\langle \kappa \rangle$ and $\langle \lambda_i \rangle$ is isomorphic to $\mathbb{Z}/(2)$ and the subgroups $\langle \kappa, \lambda_j \rangle$ and $\langle \lambda_i, \lambda_j \rangle$ are isomorphic to $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ for any $i \neq j \in [1, 4]$. These are easily verified. Moreover, the euler number of this orbifold is easily seen to be $3/4$, so that the euler number of the uniformization is $3|\pi_1^{orb}(\mathcal{F}_2)|/4 = 24$. Since the first Chern number of $\mathcal{F}_2$ is 0, it follows that this is a K3 orbifold.

3.4.2. *The orbifold* $\mathcal{F}_1$. One has $\mathcal{F}_1 \simeq \mathcal{F}_2/\mathcal{A}_2$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3)$. Hence $|\pi_1^{orb}(\mathcal{F}_1)| = |\pi_1^{orb}(\mathcal{F}_2)|/|\pi_1^{orb}(\mathcal{A}_2)| = 16/4 = 4$.

3.5. **The K3 orbifold $\mathcal{G}$.** This is the orbifold $(\mathbf{P}^2, 2Q + 3C)$ where Q is a smooth quadric and C is a smooth cubic.



$\mathcal{G}$ (6)

By Lemma 2 one has

$$\pi_1^{orb}(\mathcal{O}) \simeq \langle \lambda, \kappa \mid \lambda^2 = \kappa^3 = 1 \rangle \simeq \mathbb{Z}/(6),$$

In order to show that $\mathcal{G}$ is uniformizable, one must verify that the subgroups $\langle \kappa \rangle$ and $\langle \lambda \rangle$ are isomorphic respectively to $\mathbb{Z}/(3)$ and $\mathbb{Z}/(2)$, and the subgroup $\langle \kappa, \lambda \rangle$ is isomorphic to $\mathbb{Z}/(2) \oplus \mathbb{Z}/(3)$ since the local orbifold fundamental group at any node of $Q \cup C$ is generated by λ and κ. These are easily verified. Moreover, the euler number of $\mathcal{G}$ is easily seen to be 4, so that the euler number of the uniformization is $4|\pi_1^{orb}(\mathcal{F}_2)| = 24$. Since the first Chern number of $\mathcal{G}$ is 0, it follows that this is a K3 orbifold.

3.6. **Completeness of the list.**

Theorem 1. *The list* $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{D}_5$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{G}$ *is the complete list of K3 orbifolds with an abelian fundamental group.*

Suppose that $\mathcal{O} = (\mathbf{P}^2, B)$ is a K3 orbifold over $\mathbf{P}^2$, the locus of $B = \sum_{i=1}^n b_i B_i$ being a plane curve of degree d. By Lemma 3 one has $4 \le d \le 6$ and $\sum_{i=1}^n \deg(B_i)/b_i = 1$ if $d = 4$. In case $d = 5$ one has $\sum_{i=1}^n \deg(B_i)/b_i = 2$. If $d = 6$, one has $b_1 = \cdots = b_n = 2$. Hereafter it will be assumed that these conditions holds.

On the other hand, the local fundamental group of an orbifold singularity is abelian if and only if the singularity is nodal. By the condition of smoothness on the uniformization, the local orbifold fundamental group of each singular point should inject into the global group $\pi_1^{orb}(\mathcal{O})$. Since this latter group is assumed to be abelian, and since the only orbifold singularities with abelian local fundamental group are nodes, B must be a nodal curve. We claim moreover that irreducible components of B must be smooth: Assume the contrary; e.g. suppose that an irreducible component B_i of B has a node p.

The local orbifold fundamental group of this node has the presentation $\langle \mu_1, \mu_2 \,|\, [\mu_1, \mu_2] = \mu_1^{b_i} = \mu_2^{b_i} \rangle \simeq \mathbb{Z}/(b_i) \oplus \mathbb{Z}/(b_i)$, where μ_1 and μ_2 are meridians of the branches of B_i meeting at p. However, since B_i is irreducible, the meridians μ_1 and μ_2 of B_i are conjugate elements of the group $\pi_1^{orb}(\mathcal{O})$. Since $\pi_1^{orb}(\mathcal{O})$ is abelian, one actually has $\mu_1 = \mu_2$. Hence, the subgroup of $\pi_1^{orb}(\mathcal{O})$ generated by μ_1 and μ_2 is at most $\mathbb{Z}/(b)$ and cannot be isomorphic to the local orbifold fundamental group $\mathbb{Z}/(b_i) \oplus \mathbb{Z}/(b_i)$ at p.

The proof of Theorem 1 will be given in 3.6.1-3.6.3.

3.6.1. *Quartics.* Suppose that $B = b_1L_1 + b_2L_2 + b_3L_3 + b_4L_4$ with L_1, L_2, L_3, L_4 being four lines in general position. There are 14 four-tuples (b_1, b_2, b_3, b_4) with $1/b_1 + 1/b_2 + 1/b_3 + 1/b_4 = 1$. The euler number of $\mathcal{O}$ reads

$$e(\mathcal{O}) = 1 - \left(\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4}\right) + \sum_{1 \le i \ne j \le 4} \frac{1}{b_ib_j}$$

By Lemma 3, the number $24/e(\mathcal{O})$ should be an integer. The only 4-tuples (b_1, b_2, b_3, b_4) satisfiying this are $(4, 4, 4, 4)$ and $(2, 6, 6, 6)$, corresponding respectively to the orbifolds $\mathcal{C}_3$ and $\mathcal{D}_3$.

Suppose that $B = b_1L_1 + b_2L_2 + bQ$ where L_1, L_2 are two lines and Q is a smooth quadric. The only triples (b_1, b_2, b) satisfying $1/b_1 + 1/b_2 + 2/b = 1$ are $(12, 4, 3)$, $(6, 6, 3)$, $(4, 4, 4)$ and $(3, 3, 6)$. One has

$$e(\mathcal{O}) = 2 - \left(\frac{1}{b_1} + \frac{1}{b_2} + \frac{2}{b}\right) + \frac{1}{b_1b_2} + \frac{2}{b_1b} + \frac{2}{b_2b}$$

One has $24/e(\mathcal{O}) \in \mathbb{N}$ only for the triple $(3, 3, 6)$, corresponding to the orbifold $\mathcal{D}_4$.

Suppose that $B = b_1Q_1 + b_2Q_2$ where Q_1, Q_2 are two smooth quadrics. The only pairs (b_1, b_2) satisfying $2/b_1 + 2/b_2 = 1$ are $(4, 4)$ and $(6, 3)$. One has $e(\mathcal{O}) = 3 - (2/b_1 + 2/b_2) + 4/(b_1b_2)$. The number $24/e(\mathcal{O})$ is never an integer.

Suppose that $B = b_1L + b_2C$ where L is a line and C is a smooth cubic. The only pairs (b_1, b_2) satisfying $1/b_1 + 3/b_2 = 1$ are $(4, 4)$ and $(2, 6)$. One has $e(\mathcal{O}) = 3 - (1/b_1 + 3/b_2) + 3/(b_1b_2)$. The number $24/e(\mathcal{O})$ is never an integer.

Suppose that $B = b_1K$, where K is a smooth quartic. Then $b_1 = 4$ and $e(\mathcal{O}) = 6$, corresponding to the orbifold $\mathcal{C}_1$.

3.6.2. *Quintics.* Suppose that $B = \sum_{i=1}^{5} b_iL_i$, where $L_1, \ldots, L_5$ are five lines in general position. There are only 3 five-tuples $(2, 2, 2, 3, 6)$,

$(2,2,2,4,4)$, $(2,2,3,3,3)$ with reciprocals summing up to 2. The euler number of $\mathcal{O}$ reads

$$e(\mathcal{O}) = 3 - \sum_{1\leq i\leq 5} \frac{2}{b_i} + \sum_{1\leq i\neq j\leq 5} \frac{1}{b_i b_j}$$

None of the above 5-tuples satisfies $24/e(\mathcal{O}) \in \mathbb{N}$.

Suppose that $B = b_1L_1 + b_2L_2 + b_3L_3 + bQ$ with L_1, L_2 and L_3 being three lines and Q a smooth quadric. The only four-tuples (b_1, b_2, b_3, b) satisfying $1/b_1+1/b_2+1/b_3+2/b = 2$ are $(2,3,6,2)$, $(2,2,2,4)$, $(2,4,4,2)$, $(3,3,3,2)$ and $(2,2,3,3)$. One has

$$e(\mathcal{O}) = 4 - \left(\frac{2}{b_1} + \frac{2}{b_2} + \frac{2}{b_3} + \frac{4}{b}\right) + \frac{1}{b_1b_2} + \frac{1}{b_2b_3} + \frac{1}{b_3b_1} + \frac{2}{b_1b} + \frac{2}{b_2b} + \frac{2}{b_3b}$$

The number $24/e(\mathcal{O})$ is integer only for the four-tuples $(2,2,2,4)$ and $(3,3,3,2)$ which corresponds to the orbifolds $\mathcal{C}_2$ and $\mathcal{D}_2$ respectively.

Suppose that $B = bL_1 + b_1Q_1 + b_2Q_2$ with Q_1, Q_2 being two smooth quadrics, and L a line. The only triples (b, b_1, b_2) satisfying $1/b_, +2/b_1 + 2/b_2 = 2$ are $(2,2,4)$, and $(3,3,2)$. One has

$$e(\mathcal{O}) = 5 - \left(\frac{4}{b_1} + \frac{4}{b_2} + \frac{2}{b}\right) + \frac{2}{bb_1} + \frac{2}{b_2b} + \frac{4}{b_1b_2}$$

The number $24/e(\mathcal{O})$ is never an integer.

Suppose that $B = b_1L_1 + b_2L_2 + bC$ where C is a smooth cubic. The only triples (b_1, b_2, b) satisfying $1/b_1 + 1/b_2 + 3/b = 2$ are $(4,4,2)$, $(3,6,2)$ and $(2,2,3)$. One has

$$e(\mathcal{O}) = 6 - \left(\frac{2}{b_1} + \frac{2}{b_2} + \frac{6}{b}\right) + \frac{1}{b_1b_2} + \frac{3}{b_3b_1} + \frac{3}{b_2b_3}$$

The number $24/e(\mathcal{O})$ is never an integer.

Suppose that $B = b_1Q_+b_2C$ where Q is a smooth quadric and C is a smooth cubic. The only pairs (b_1, b_2) satisfying $2/b_1 + 3/b_2 = 2$ are $(4,2)$ and $(2,3)$. One has $e(\mathcal{O}) = 7-(4/b_1+6/b_2)+6/(b_1b_2)$. One has $24/e(\mathcal{O}) \in \mathbb{N}$ only for the pair $(2,3)$, corresponding to the orbifold $\mathcal{G}$.

In case $B = b_1L_1 + b_2K$ with K being a smooth quartic, there are no pairs (b_1, b_2) satisfying $1/b_1 + 4/b_2 = 2$. Similarly, in case $B = bP$ with P being a smooth quintic, one never has $5/b = 2$.

3.6.3. *Sextics.* It suffices to show that if $\deg(B) = 6$ and $\mathcal{O}$ is not one of the orbifolds $\mathcal{D}_1$, $\mathcal{D}_5$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{F}_1$, $\mathcal{F}_2$ then $\mathcal{O}$ is not a K3 orbifold.

If $B = 2(L + P)$ where P is a smooth quintic, then by Lemma 2 one has $\pi_1^{orb}(\mathcal{O}) \simeq \langle \lambda, \kappa \,|\, \lambda^2 = \kappa^2 = \lambda\kappa^5 = 1\rangle \simeq \mathbb{Z}/(2)$, whereas the local orbifold fundamental group of a node of B is $\mathbb{Z}/(2)\oplus\mathbb{Z}/(2)$, which cannot be a subgroup of $\pi_1^{orb}(\mathcal{O})$. Hence $\mathcal{O}$ is not uniformizable.

If $B = 2(C_1 + C_2)$ where C_1 and C_2 are smooth cubics, then one has $\pi_1^{orb}(\mathcal{O}) \simeq \langle \lambda, \kappa \,|\, \lambda^2 = \kappa^2 = \lambda^3\kappa^3 = 1\rangle \simeq \mathbb{Z}/(2)$, so that $\mathcal{O}$ is not uniformizable.

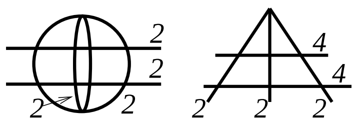
If $B = 2(L_1 + L_2 + K)$ with K being a smooth quartic then one has $\pi_1^{orb}(\mathcal{O}) \simeq \langle \lambda_1, \lambda_2, \kappa \,|\, \lambda_1^2 = \lambda_2^2 = \kappa^2 = \lambda_1\lambda_2 = 1\rangle$. Consider the intersection point p of L_1 and L_2. The local orbifold fundamental group at the node p is generated by λ_1 and λ_2 and is isomorphic to $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$, whereas the subgroup of $\pi_1^{orb}(\mathcal{O})$ generated by λ_1 and λ_2 is isomorphic to $\mathbb{Z}/(2)$. Hence $\mathcal{O}$ is not uniformizable.

If $B = 2(L + Q + C)$ with Q being a quadric and C a cubic, then one has $\pi_1^{orb}(\mathcal{O}) \simeq \langle \lambda, \kappa, \mu \,|\, \lambda^2 = \kappa^2 = \mu^2 = \lambda\mu = 1\rangle$. Consider the intersection point p of L and C. The local orbifold fundamental group at the node p is generated by λ and μ and is isomorphic to $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$, whereas the subgroup of $\pi_1^{orb}(\mathcal{O})$ generated by λ and μ is isomorphic to $\mathbb{Z}/(2)$. Hence $\mathcal{O}$ is not uniformizable.

If $B = 2(L_1 + L_2 + Q_1 + Q_2)$ with Q_1 and Q_2 being smooth quadrics then $\pi_1^{orb}(\mathcal{O}) \simeq \langle \lambda_1, \lambda_2, \kappa_1, \kappa_2 \,|\, \lambda_1^2 = \lambda_2^2 = \kappa_1^2 = \kappa_2^2 = \lambda_1\lambda_2 = 1\rangle$. Consider the intersection point p of L_1 and L_2. The local orbifold fundamental group at the node p is generated by λ_1 and λ_2 and is isomorphic to $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$, whereas the subgroup of $\pi_1^{orb}(\mathcal{O})$ generated by λ_1 and λ_2 is isomorphic to $\mathbb{Z}/(2)$. Hence $\mathcal{O}$ is not uniformizable.

4. Non-Abelian K3 orbifolds

4.1. **The K3 orbifolds $\mathcal{H}_1$ and $\mathcal{H}_2$.** These are defined as follows: $\mathcal{H}_1$ is the orbifold $(\mathbf{P}^2, 2L_1 + 2L_2 + 2Q_1 + 2Q_2)$, where Q_1, Q_2 are two smooth quadrics tangent to each other at two distinct points, and L_1, L_2 are two lines in general position with respect to $Q_1 \cup Q_2$; $\mathcal{H}_2$ is the orbifold $(\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3 + 4L_4 + 4L_5)$, where L_1, L_2 and L_3 are three lines with a common point and L_4, L_5 are two lines in general position with respect to $L_1 \cup L_2 \cup L_3$.



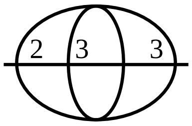
$\mathcal{H}_1$ (64) $\qquad$ $\mathcal{H}_2$ (16)

4.1.1. *The orbifold* $\mathcal{H}_2$. One has $\mathcal{H}_2/\mathcal{A}_2 \simeq \mathcal{C}_2$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_4)$. Since $\mathcal{C}_2$ is a K3 orbifold, so is $\mathcal{H}_2$. One has

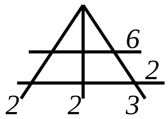
$$|\pi_1^{orb}(\mathcal{H}_2)| = |\pi_1^{orb}(\mathcal{A}_2)||\pi_1^{orb}(\mathcal{C}_2)| = 4 \times 16 = 64.$$

4.1.2. *The orbifold* $\mathcal{H}_1$. One has $\mathcal{H}_1 \simeq \mathcal{H}_2/\mathcal{A}_2'$, where $\mathcal{A}_2' := (\mathbf{P}^2, 2L_1 + 2L_4 + 2L_5)$. Hence $|\pi_1^{orb}(\mathcal{H}_1)| = |\pi_1^{orb}(\mathcal{H}_2)|/|\pi_1^{orb}(\mathcal{A}_2)| = 64/4 = 16.$

4.2. **The K3 orbifolds $\mathcal{I}_1$ and $\mathcal{I}_2$.** These are defined as follows: $\mathcal{I}_1$ is the orbifold $(\mathbf{P}^2, 2L_1 + 2Q_1 + 2Q_2)$, where Q_1, Q_2 are two smooth quadrics tangent to each other at two distinct points, and L_1 is a line in general position with respect to $Q_1 \cup Q_2$; $\mathcal{I}_2$ is the orbifold $(\mathbf{P}^2, 2L_1 + 2L_2 + 3L_3 + 2L_4 + 6L_5)$ where L_1, L_2, L_3 are three lines with a common point and $L_4 \cup L_5$ is in general position with respect to $L_1 \cup L_2 \cup L_3$.



$\mathcal{I}_1$ (18)

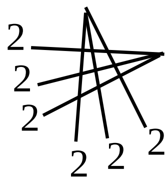


$\mathcal{I}_2$ (72)

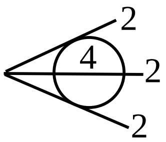
4.2.1. *The orbifold $\mathcal{I}_2$.* One has $\mathcal{I}_2/\mathcal{A}_2 \simeq \mathcal{D}_4$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_4)$. Hence $|\pi_1^{orb}(\mathcal{I}_2)| = |\pi_1^{orb}(\mathcal{A}_2)||\pi_1^{orb}(\mathcal{D}_4)| = 4 \times 18 = 72$. Moreover, one has $\mathcal{I}_2/\mathcal{A}_2' \simeq \mathcal{D}_2$, where $\mathcal{A}_2' := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_5)$.

4.2.2. *The orbifold $\mathcal{I}_1$.* One has $\mathcal{I}_1 \simeq \mathcal{I}_2/\mathcal{A}_2''$, where $\mathcal{A}_2'' := (\mathbf{P}^2, 2L_1 + 2L_4 + 2L_5)$. Hence $|\pi_1^{orb}(\mathcal{I}_1)| = |\pi_1^{orb}(\mathcal{I}_2)|/|\pi_1^{orb}(\mathcal{A}_2'')| = 72/4 = 18$.

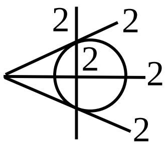
4.3. **The K3 orbifolds $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$, $\mathcal{J}_4$ and $\mathcal{J}_5$.** These are defined as follows: $\mathcal{J}_1$ is the orbifold $(\mathbf{P}^2, \sum_{i=1}^6 2L_i)$, where the lines L_1, L_2, L_3 meet at a point, L_3, L_4, L_5 meet at another point, and $L_1 \cup L_2 \cup L_2$ is in general position with respect to $L_4 \cup L_5 \cup L_6$; $\mathcal{J}_2$ is the orbifold $(\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3 + 4Q)$ where L_1, L_2, L_3 meet at a point not lying on the smooth quadric Q and L_1, L_3 are tangents to Q; $\mathcal{J}_3$ is the orbifold $(\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3 + 2L_4 + 2Q)$ where L_1, L_2, L_3 meet at a point not lying on Q, the lines L_1, L_3 are tangents to Q and L_4 pass through these tangence points; $\mathcal{J}_4$ is the orbifold $(\mathbf{P}^2, 4L_1 + 2L_2 + 2L_3 + 2L_4 + 4L_5)$ where L_3 pass through the points $L_1 \cap L_2$ and $L_4 \cap L_5$; $\mathcal{J}_5$ is the orbifold $(\mathbf{P}^2, 2L_1 + 2L_2 + 2K)$ where K is a quartic with a tacnode and L_1, L_2 are two flexes (with multiplicity 4) of K.



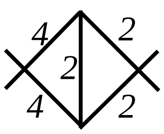
$\mathcal{J}_1$ (64)



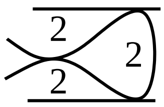
$\mathcal{J}_2$ (64)



$\mathcal{J}_3$ (64)



$\mathcal{J}_4$ (256)



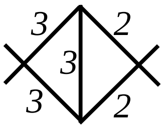
$\mathcal{J}_5$ (16)

One has $\mathcal{J}_4/\mathcal{A}_2 \simeq \mathcal{C}_3$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_2 + 2L_3 + 2L_4)$ is a suborbifold of $\mathcal{J}_4$. Hence $|\pi_1^{orb}(\mathcal{J}_4)| = |\pi_1^{orb}(\mathcal{A}_2)||\pi_1^{orb}(\mathcal{C}_3)| = 4 \times 64 = 256$. Moreover, let $\mathcal{A}_2' := (\mathbf{P}^2, 2L_1 + 2L_3 + 2L_5)$, $\mathcal{A}_2'' := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_4)$ and $\mathcal{A}_2''' := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_5)$ be suborbifolds of $\mathcal{J}_4$. Then one has

$\mathcal{J}_4/\mathcal{A}_2' \simeq \mathcal{J}_1$, $\mathcal{J}_4/\mathcal{A}_2'' \simeq \mathcal{J}_2$ and $\mathcal{J}_4/\mathcal{A}_2''' \simeq \mathcal{J}_3$, showing that $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ are K3 orbifolds.

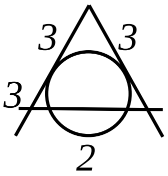
On the other hand, one has $\mathcal{J}_1/\mathcal{A}_2 \simeq \mathcal{H}_1$ where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_4)$ is a suborbifold of $\mathcal{J}_1$, $\mathcal{J}_3/\mathcal{A}_2 \simeq \mathcal{H}_1$ where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1 + 2L_3 + 2L_4)$ is a suborbifold of $\mathcal{J}_3$, and $\mathcal{J}_3/\mathcal{A}_2' \simeq \mathcal{J}_5$ where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_4)$ is a suborbifold of $\mathcal{J}_3$.

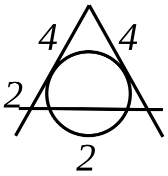
4.4. **The K3 orbifold $\mathcal{K}$.** This is the orbifold $(\mathbf{P}^2, 3L_1 + 3L_2 + 3L_3 + 2L_4 + 4L_5)$ where L_3 pass through the points $L_1 \cap L_2$ and $L_4 \cap L_5$. One has $\mathcal{K}/\mathcal{A}_3 \simeq \mathcal{J}_1$, showing that $\mathcal{K}$ is a K3 orbifold.

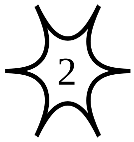


$\mathcal{K}$ (576)

4.5. **The K3 orbifolds $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$.** The orbifolds $\mathcal{L}_1$, $\mathcal{L}_2$ are defined respectively as $(\mathbf{P}^2, 3L_1 + 3L_2 + 3L_3 + 2Q)$ and $(\mathbf{P}^2, 4L_1 + 4L_2 + 2L_3 + 2Q)$ where L_1, L_2 are tangent to Q and L_3 is in general position with respect to $L_1 \cup L_2 \cup Q$. By definition, $\mathcal{L}_3$ is the orbifold $\mathcal{L}_1/\mathcal{A}_3$, where $\mathcal{A}_3 := (\mathbf{P}^2, 3L_1 + 3L_2 + 3L_3)$ is a suborbifold of $\mathcal{L}_1$. The locus of $\mathcal{L}_3$ is an irreducible sextic with 6 cusps and no other singularities. For other examples of non-abelian branched coverings along sextic see [27].







$\mathcal{L}_1$ (54) $\mathcal{L}_2$ (64) $\mathcal{L}_3$ (6)

4.5.1. *The orbifold $\mathcal{L}_2$.* One has $\mathcal{L}_2/\mathcal{A}_2 \simeq \mathcal{P}_2$, which is a K3 orbifold (see 4.8).

4.5.2. *The orbifold $\mathcal{L}_1$.* An application of Zariski-Van Kampen (see [7]) gives the (redundant) presentation

$$\pi_1(\mathbf{P}^2 \backslash (L_1 \cup L_2 \cup Q)) \simeq \langle \kappa, \tau_1, \tau_2 \,|\, (\kappa\tau_1)^2 = (\tau_1\kappa)^2, \quad \kappa^2\tau_1\tau_2 = 1 \rangle,$$

where κ is a meridian of Q and τ_i is a meridian of L_i. Since L_3 is in general position with respect to $L_1 \cup L_2 \cup Q$, the fundamental group $\pi_1(\mathbf{P}^2 \backslash (L_1 \cup L_2 \cup L_3 \cup Q))$ admits the presentation

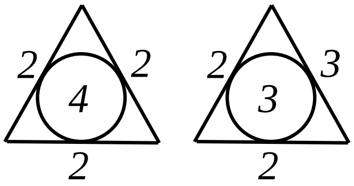
$$\langle \kappa, \tau_1, \tau_2, \tau_3 \,|\, (\kappa\tau_1)^2 = (\tau_1\kappa)^2, \quad [\tau_3, \kappa] = [\tau_3, \tau_1] = [\tau_3, \tau_2] = \kappa^2\tau_3\tau_2\tau_1 = 1 \rangle,$$

so that the group $\pi_1^{orb}(\mathcal{L}_1)$ admits the presentation (see [9])

$$(1) \qquad \pi_1^{orb}(\mathcal{L}_1) \simeq \left\langle \kappa, \tau_1, \tau_2, \tau_3 \;\middle|\; \begin{array}{l} (\kappa\tau_1)^2 = (\tau_1\kappa)^2 \\ [\tau_3, \kappa] = [\tau_3, \tau_1] = [\tau_3, \tau_2] = 1 \\ \tau_3\tau_2\tau_1 = 1 \\ \tau_1^3 = \tau_2^3 = \tau_3^3 = \kappa^2 = 1 \end{array} \right\rangle$$

The group theory software GAP [8] gives $|\pi_1^{orb}(\mathcal{L}_1)| = 54$. In order to show that $\mathcal{L}_1$ is a K3 orbifold, one must verify (i) $|\langle\tau_i\rangle| = 3$, (ii) $|\langle\tau_i, \tau_j\rangle| = 9$ $(i \neq j)$, (iii) $|\langle\kappa\rangle| = 2$, (iv) $|\langle\kappa, \tau_3\rangle| = 6$ and (v) $|\langle\kappa, \tau_1\rangle| = |\langle\kappa, \tau_2\rangle| = 18$ in the group $\pi_1^{orb}(\mathcal{L}_1)$. The conditions (i)-(iv) already holds in the abelianized orbifold fundamental group and are easily verified. By symmetry, it suffices to verify only $|\langle\kappa, \tau_1\rangle| = 18$ in (v). This latter is a routine computer check by GAP. Hence, $\mathcal{L}_1$ is uniformized by a smooth surface Y with $c_1^2(Y) = 0$. Since $e(\mathcal{L}_1) = 3/2$, one has $e(Y) = |\pi_1^{orb}(\mathcal{L}_1)|e(\mathcal{L}_1) = 24$, and Y is a K3 surface.

4.6. **The orbifolds $\mathcal{M}_1$ and $\mathcal{M}_2$.** The orbifolds $\mathcal{M}_1$, $\mathcal{M}_2$ are defined respectively as $(\mathbf{P}^2, 2L_1+2L_2+2L_3+4Q)$ and $(\mathbf{P}^2, 2L_1+2L_2+3L_3+3Q)$ where Q is a smooth quadric and L_1, L_2, L_3 are three distinct tangents. Note that any two such arrangements are projectively equivalent.



$\mathcal{M}_1$ (256) $\quad$ $\mathcal{M}_2$ (576)

4.6.1. *The orbifold $\mathcal{M}_1$.* One has $\mathcal{M}_1/\mathcal{A}_2 \simeq \mathcal{C}_3$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3)$. Note that $\mathcal{M}_1$ is invariant under the S_3 action on $\mathbf{P}^2$.

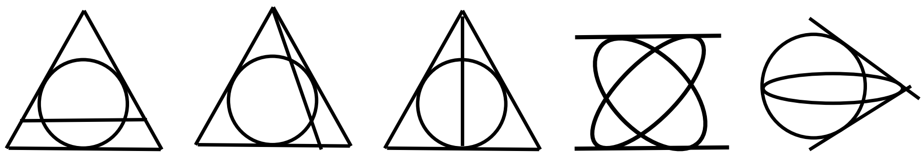
4.6.2. *The orbifold $\mathcal{M}_2$.* The group $\pi_1(\mathbf{P}^2 \backslash (Q \cup L_1 \cup L_2 \cup L_3))$ is the braid group on two strands of the thrice punctured sphere, and its presentation is known, see [2] and see [3], [15] for generalizations. Adding the orbifold relations to this presentation gives

$$(2) \quad \pi_1^{orb}(\mathcal{M}_2) \simeq \left\langle \begin{array}{l} \tau_1, \tau_2, \tau_3, \\ \kappa_1, \kappa_2, \kappa_3 \end{array} \;\middle|\; \begin{array}{l} \kappa_i = \tau_{i-1}\kappa_{i-1}\tau_{i-1}^{-1},\ 2 \leq i \leq 3 \\ (\kappa_i\tau_i)^2 = (\tau_i\kappa_i)^2,\ 1 \leq i \leq 3 \\ \left[\kappa_i^{-1}\tau_i\kappa_i, \tau_j\right] = 1,\ 1 \leq i < j \leq 3 \\ \tau_3\tau_2\tau_1\kappa_1^2 = \tau_1^2 = \tau_2^2 = \tau_3^3 = \kappa^3 = 1 \end{array} \right\rangle$$

where κ_i are meridians of Q and τ_i is a meridian of T_i for $1 \leq i \leq 3$. GAP gives $|\pi_1^{orb}(\mathcal{M}_2)| = 576$.

As a by-product of the method used in the computation of the group $\pi_1(\mathbf{P}^2 \backslash (Q \cup L_1 \cup L_2 \cup L_3))$ in [2], the generators of the local fundamental groups around the singular points of $Q \cup L_1 \cup L_2 \cup L_3$ can be read easily from the above presentation. For the nodes the local groups are $\langle \kappa_i^{-1}\tau_i\kappa_i, \tau_j\rangle$, and for the tangency points they are $\langle \kappa_i, \tau_i\rangle$. In order to show that $\mathcal{M}_2$ is uniformizable, one needs to verify the followings: (i) $|\langle \kappa_1\rangle| = |\langle \tau_3\rangle| = 3$ and $|\langle \tau_1\rangle| = |\langle \tau_2\rangle| = 2$, (ii) $|\langle \kappa_1^{-1}\tau_1\kappa_1, \tau_2 j\rangle| = 4$ and $|\langle \kappa_1^{-1}\tau_1\kappa_1, \tau_3\rangle| = |\langle \kappa_2^{-1}\tau_2\kappa_2, \tau_3\rangle| = 6$, (iii) $|\langle \kappa_1, \tau_1\rangle| = |\langle \kappa_2, \tau_2\rangle| = 18$ and $|\langle \kappa_3, \tau_3\rangle| = 72$. This is a routine check by GAP. Hence, the orbifold $\mathcal{M}_2$ is uniformizable by a smooth surface Z. Since $e(\mathcal{M}_2) = 1/24$, one has $e(Z) = e(\mathcal{M}_2)|\pi_1^{orb}(\mathcal{M}_2)| = 24$, so that Z is a K3 surface, showing that $\mathcal{M}_2$ is a K3 orbifold.

4.7. **The sextic orbifolds $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_4$ and $\mathcal{N}_5$.** The orbifolds $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$ are defined as follows: Let $B = 2L_1 + 2L_2 + 2L_3 + 2L_4 + 2Q$, where L_1, L_2, L_3 are three tangents of Q. The case where L_4 is in general position with respect to $Q \cup L_1 \cup L_2 \cup L_2$ gives $\mathcal{N}_1$. The case where L_4 passes through the point $L_1 \cap L_2$ and not through the point $L_3 \cap Q$ gives $\mathcal{N}_2$. Finally, the case where L_4 is the line through the points $L_1 \cap L_2$ and $L_3 \cap Q$ gives $\mathcal{N}_3$. The orbifolds $\mathcal{N}_4$ and $\mathcal{N}_5$ are defined as follows: Let $B = 2L_1 + 2L_2 + 2Q_1 + 2Q_2$, where L_1, L_2 are two common tangents of the quadrics Q_1 and Q_2. Assume that the lines do not pass through any point in $Q_1 \cap Q_2$. The case where $Q_1 \cap Q_2$ consists of four distinct points gives $\mathcal{N}_4$, and the case where $Q_1 \cap Q_2$ consists of three distinct points gives $\mathcal{N}_5$.

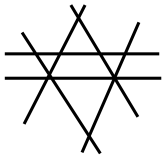


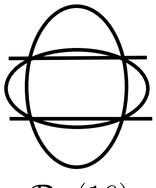
$\mathcal{N}_1$ (64) $\mathcal{N}_2$ (128) $\mathcal{N}_3$ (256) $\mathcal{N}_4$ (32) $\mathcal{N}_5$ (64)

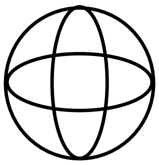
Let $\mathcal{A}_2 := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_3)$ and $\mathcal{A}_2' := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_4)$ be suborbifolds of $\mathcal{N}_1$. One has $\mathcal{N}_1/\mathcal{A}_2 \simeq \mathcal{F}_2$. Since $\mathcal{F}_2$ is a K3 orbifold, so is $\mathcal{N}_1$ and $|\pi_1^{orb}(\mathcal{N}_1)| = |\pi_1^{orb}(\mathcal{A}_2)||\pi_1^{orb}(\mathcal{F}_2)| = 4 \times 16 = 64$. On the other hand, one has $\mathcal{N}_1/\mathcal{A}_2' = \mathcal{N}_3$, $\mathcal{N}_3/\mathcal{A}_2 = \mathcal{J}_2$ and $\mathcal{N}_3/\mathcal{A}_2' = \mathcal{N}_5$.

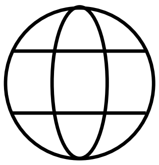
One has $\mathcal{N}_2/\mathcal{A}_2 \simeq \mathcal{E}_2$. Since $\mathcal{E}_2$ is a K3 orbifold, so is $\mathcal{N}_2$ and $|\pi_1^{orb}(\mathcal{N}_1)| = |\pi_1^{orb}(\mathcal{A}_2)||\pi_1^{orb}(\mathcal{F}_2)| = 4 \times 16 = 64$. On the other hand, one has $\mathcal{N}_2/\mathcal{A}_2' \simeq \mathcal{N}_4$.

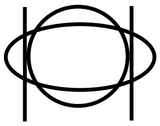
4.8. **The sextic K3 orbifolds $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$ and $\mathcal{P}_5$.** Let $\mathcal{P}_1$ be the orbifold $(\mathbf{P}^2, B)$, where $B = \sum_{i=1}^{\infty} 2L_i$ such that L_1, L_2 and L_3 meet at a point, L_3, L_4 and L_5 meet at another point, and L_6 is in general position with respect to $\cup_{i=1}^{5} L_i$. One has $\mathcal{P}_1/\mathcal{A}_2 \simeq \mathcal{F}_2$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_2 + 2L_3 + 2L_4)$. Since $\mathcal{F}_2$ is a K3 orbifold, so is $\mathcal{P}_1$. On the other hand, one has
one has $\mathcal{P}_1/\mathcal{A}_2' \simeq \mathcal{P}_2$ where $\mathcal{A}_2' := (\mathbf{P}^2, 2L_1 + 2L_2 + 2L_6)$;
$\mathcal{P}_1/\mathcal{A}_2'' \simeq \mathcal{P}_3$ where $\mathcal{A}_2'' := (\mathbf{P}^2, 2L_2 + 2L_4 + 2L_6)$;
$\mathcal{P}_1/\mathcal{A}_2''' \simeq \mathcal{P}_4$ where $\mathcal{A}_2''' := (\mathbf{P}^2, 2L_2 + 2L_3 + 2L_6)$;
$\mathcal{P}_1/\mathcal{A}_2'''' \simeq \mathcal{P}_5$ where $\mathcal{A}_2'''' := (\mathbf{P}^2, 2L_2 + 2L_3 + 2L_4)$.


$\mathcal{P}_1$ (64)

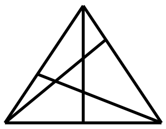

$\mathcal{P}_2$ (16)

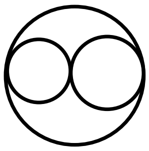

$\mathcal{P}_3$ (16)


$\mathcal{P}_4$ (16)


$\mathcal{P}_5$ (16)

4.9. **The sextic K3 orbifolds $\mathcal{Q}_1$ and $\mathcal{Q}_2$.** Let $\mathcal{Q}_1$ be the orbifold $(\mathbf{P}^2, B)$, where $B = \sum_{i=1}^{\infty} 2L_i$ such that L_1, L_2, L_3 meet at a point, L_3, L_4, L_5 meet at another point, and L_5, L_6, L_1 meet at another point. The orbifold $\mathcal{Q}_2$ is defined to be the lifting $\mathcal{Q}_1/\mathcal{A}_2$, where $\mathcal{A}_2 := (\mathbf{P}^2, 2L_2 + 2L_4 + 2L_6)$; its locus consists of three quadrics Q_1, Q_2, Q_3 with six tacnodes. On the other hand, one has $\mathcal{Q}_1/\mathcal{A}_2' \simeq \mathcal{E}_2$, where $\mathcal{A}_2' := (\mathbf{P}^2, 2L_1 + 2L_3 + 2L_5)$, showing that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are both K3 orbifolds.


$\mathcal{Q}_1$ (128)


$\mathcal{Q}_2$ (32)

4.10. **Completeness of the list.**

Theorem 2. *The list $\mathcal{H}_2$, $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{J}_2$, $\mathcal{J}_4$, $\mathcal{K}$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{M}_1$, $\mathcal{M}_2$ is the complete list of K3 orbifolds over $\mathbf{P}^2$, with a locus of degree ≤ 5 and with a non-abelian fundamental group.*

It is easily shown that there are no quartic K3 orbifolds with a non-abelian fundamental group, see 4.10.1. Suppose that $\mathcal{O} = (\mathbf{P}^2, B)$ is a quintic K3 orbifold with a non-abelian orbifold fundamental group. In particular, the group $\pi_1(\mathbf{P}^2 \backslash B)$ should be non-abelian. All quintic curves with a non-abelian fundamental group has been listed and computed by Degtyarev [4], so it suffices to consider the orbifolds with loci in Degtyarev's list. Evidently, those quintics with non-simple singularities are immediately eliminated. The proof of the theorem for quintic orbifolds will be given in 4.10.2- 4.10.6.

4.10.1. *Quartics.* Suppose that $\mathcal{O} = (\mathbf{P}^2, B)$ is a quartic K3 orbifold with a non-abelian orbifold fundamental group, where $B = b_1B_1 + \cdots + b_nB_n$. The condition $c_1^2(\mathcal{O}) = 0$ forces $b_1 = \cdots = b_n = 4$. Since $\pi_1^{orb}(\mathcal{O})$ is non-abelian, B must have a non-nodal singular point. This point cannot be a reducible singularity of B, since two branches of a reducible non-nodal orbifold point cannot have branching 4 simultaneously. Hence, B may have irreducible singularities only, in particular B itself is irreducible. The only irreducible quartic with a non-abelian fundamental group is the 3-cuspidal quartic K (see [34], [7]). The group $\pi_1(\mathbf{P}^2\backslash K)$ is non-abelian of order 12. The euler number of the orbifold $\mathcal{O} = (\mathbf{P}^2, 4K)$ is easily computed: Since K is rational, one has $e(\mathcal{E}) = 2$ and $e(\mathbf{P}^2\backslash K) = 1$. Since $e(C\backslash \text{sing}(K) = 2 - 3 = -1$, one has $e(\mathcal{O}) = 1 - 1/4 + 3/96 = 25/32$, so that $24/e(\mathcal{O}) \notin \mathbb{N}$.

4.10.2. *Linear quintics.* For the Sections 4.10.2- 4.10.6 set $\beta_i := 1/b_i$. Let $\mathcal{O} = (\mathbf{P}^2, B)$, where $B = \sum_{i=1}^5 b_iL_i$. One has $c_1^2(\mathcal{O}) = 0$ if and only if $\sum_{i=1}^5 \beta_i = 2$. There are only 3 five-tuples $(2, 2, 2, 3, 6)$, $(2, 2, 2, 4, 4)$, $(2, 2, 3, 3, 3)$ satisfying this condition.

Suppose that L_1, L_2, L_3 meet at a point, and that $L_4 \cup L_5$ is in general position with respect to $L_1 \cup L_2 \cup L_3$. Then $e(\mathcal{O})$ is

$$2-(\beta_1+\beta_2+\beta_3+2\beta_4+2\beta_5)+(\beta_4+\beta_5)(\beta_1+\beta_2+\beta_3)+\frac{1}{4}[\beta_1+\beta_2+\beta_3-1]^2$$

For the triple point to be an orbifold point one must have: $\beta_1+\beta_2+\beta_3 > 1$. The only five-tuples satisfiying this are $(2, 2, 2, 4, 4)$ and $(2, 2, 3, 6, 2)$, corresponding to the orbifolds $\mathcal{H}_2$ and $\mathcal{I}_2$ respectively.

Suppose that L_1, L_2, L_3 meet at a triple point, and L_1, L_4, L_5 meet at another point. Then $e(\mathcal{O})$ is

$$1 - (\beta_2 + \beta_3 + \beta_4 + \beta_5) + (\beta_4 + \beta_5)(\beta_2 + \beta_3)+ \\ \frac{1}{4}[\beta_1 + \beta_2 + \beta_3 - 1]^2+\frac{1}{4}[\beta_1 + \beta_5 + \beta_4 - 1]^2$$

For the triple points to be orbifold points one must have: $\beta_1+\beta_2+\beta_3 > 1$ and $\beta_1 + \beta_4 + \beta_5 > 1$. The only five-tuples satisfiying these conditions are $(2, 4, 2, 2, 4)$ and $(3, 3, 2, 3, 2)$, corresponding to the orbifolds $\mathcal{J}_4$ and $\mathcal{K}$ respectively.

4.10.3. *A quadric with three lines.* Let $B = b_1L_1 + b_2L_2 + b_3L_3 + bQ$. One has $c_1^2(\mathcal{O}) = 0$ if and only if $\beta_1 + \beta_2 + \beta_3 + 2\beta = 2$. The possible values of (b_1, b_2, b_3, b) are (i) $(2, 3, 6, 2)$, (ii) $(2, 4, 4, 2)$, (iii) $(2, 2, 2, 4)$, (iv) $(3, 3, 3, 2)$ and (v) $(2, 2, 3, 3)$.

Suppose that L_1, L_2, L_3 have a common point and $L_1 \cup L_2 \cup L_3$ is in general position with respect to Q. Then

$$e(\mathcal{O}) = 3-(\beta_1+\beta_2+\beta_3+4\beta)+2\beta_4(\beta_1+\beta_2+\beta_3)+\frac{1}{4}\left[\beta_1 + \beta_2 + \beta_3 - 1\right]^2$$

For the triple point to be an orbifold point one must have: $\beta_1+\beta_2+\beta_3 > 1$. This occurs only in (iii), which forces $b_1 = b_2 = b_3 = 2$ and $b = 4$. But then $e(\mathcal{O}) = 21/16$, so that $24/e(\mathcal{O}) \notin \mathbb{N}$.

Suppose that L_1, L_2, L_3 have a common point, L_1 is tangent to Q, and $L_2 \cup L_4$ is in general position with respect to Q. Then $e(\mathcal{O})$ is

$$2-(\beta_2+\beta_3+3\beta)+2\beta(\beta_2+\beta_3)+\frac{1}{4}\left[\beta_1 + \beta_2 + \beta_3 - 1\right]^2+\frac{1}{2}\left[\beta_1 + \beta - \frac{1}{2}\right]^2$$

For the triple point to be an orbifold point one must have: $\beta_1+\beta_2+\beta_3 > 1$. This occurs only in (iii), which forces $b_1 = b_2 = b_3 = 2$ and $b = 4$. But then $e(\mathcal{O}) = 31/16$, so that $24/e(\mathcal{O}) \notin \mathbb{N}$.

Suppose that L_1, L_2, L_3 have a common point and L_1, L_2 are tangent to Q. For the triple point to be orbifold point one must have: $\beta_1 + \beta_2 + \beta_3 > 1$. This occurs only in (iii), which forces $b_1 = b_2 = b_3 = 2$ and $b = 4$. But then $\mathcal{O}$ is the orbifold $\mathcal{J}_2$.

Suppose that L_1, L_2, L_3 have no point in common, L_1, L_2 are tangent to Q and L_3 is in general position with respect to Q. Then

$$e(\mathcal{O}) = 2 - (\beta_1 + \beta_2 + 2\beta_3 + 2\beta) + \beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1 + 2\beta\beta_3+ \frac{1}{2}\left[\beta + \beta_1 - \frac{1}{2}\right]^2+\frac{1}{2}\left[\beta + \beta_2 - \frac{1}{2}\right]^2$$

For the tangence points to be orbifold singularities one must have: $\beta+\beta_1 > 1/2$ and $\beta+\beta_2 > 1/2$. The number $24/e(\mathcal{O})$ is an integer only for 4 four-tuples (b_1, b_2, b_3, b): The four-tuples $(3, 3, 3, 2)$ and $(4, 4, 2, 2)$ yield the K3 orbifolds $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. The remaining two orbifolds are not uniformizable: Indeed, for $(3, 6, 2, 2)$, one should have $|\pi_1^{orb}(\mathcal{O})| = 24/e(\mathcal{O}) = 64$, whereas a meridian of L_1 in this group should be of order 3. For $(2, 6, 3, 2)$ one has the presentation

$$\pi_1^{orb}(\mathcal{L}_1) \simeq \left\langle \kappa, \tau_1, \tau_2, \tau_3 \;\middle|\; \begin{array}{l} (\kappa\tau_1)^2 = (\tau_1\kappa)^2 \\ [\tau_3, \kappa] = [\tau_3, \tau_1] = [\tau_3, \tau_2] = 1 \\ \tau_3\tau_2\tau_1\kappa^2 = 1 \\ \tau_1^2 = \tau_2^6 = \tau_3^3 = \kappa^2 = 1 \end{array} \right\rangle \tag{3}$$

This group is of order 24. But then the corresponding orbifold is not uniformizable since $24/e(\mathcal{O}) = 54 \neq |\pi_1^{orb}(\mathcal{O})|$.

Suppose that L_1, L_2, L_3 have no point in common, L_1, L_2 are tangent to Q, the line L_3 pass through $L_1 \cap Q$ and in general position with respect to $L_2 \cup Q$. Then $e(\mathcal{O})$ is

$$1-(\beta_2+\beta_3+\beta)+\beta_1\beta_2+\beta_2\beta_3+\beta\beta_3+\frac{1}{2}\left[\beta+\beta_1+\frac{\beta_3}{2}-1\right]^2+\frac{1}{2}\left[\beta+\beta_2-\frac{1}{2}\right]^2$$

For the tangence points to be orbifold points one must have: $\beta+\beta_1+\beta_3/2 > 1$ and $\beta+\beta_2 > 1/2$. The number $24/e(\mathcal{O})$ is never an integer.

Let $B = b_1L_1 + b_2L_2 + b_3L_3 + bQ$, where L_1, L_2, L_3 have no point in common, L_1, L_2 are tangent to Q and L_3 pass through the tangence points. Then

$$e(\mathcal{O}) = \beta_1\beta_2 + \frac{1}{2}\left[\beta+\beta_1+\frac{\beta_3}{2}-1\right]^2 + \frac{1}{2}\left[\beta+\beta_2+\frac{\beta_3}{2}-1\right]^2$$

For the tangence points to be orbifold singularities one must have: $\beta+\beta_1+\beta_3/2 > 1$ and $\beta+\beta_2+\beta_3/2 > 1$. The number $24/e(\mathcal{O})$ is never an integer.

Let $B = b_1L_1 + b_2L_2 + b_3L_3 + bQ$, where L_1, L_2, L_3 are tangent to Q. This case was settled in [28]. It yields the orbifolds $\mathcal{M}_1$ and $\mathcal{M}_2$.

4.10.4. *Two quadrics with a line.* Let $B = bL + b_1Q_1 + b_2Q_2$. Since $\beta + 2\beta_1 + 2\beta_2 = 2$, one has $(b, b_1, b_2) \in \{(2, 2, 4), (3, 2, 3)\}$.

Suppose that Q_1, Q_2 have an intersection point of multiplicity 4. The point $Q_1 \cap Q_2$ is an orbifold point only when $Q_1 \cap Q_2 \notin L$ and (b_1, b_2) is one of $(2, 2)$, $(3, 2)$ or $(2, 3)$. Since $\beta+2\beta_1+2\beta_2 = 2$, the case $(b_1, b_2) = (2, 2)$ cannot occur. This leaves the possibilities $(3, 2, 3)$ and $(3, 3, 2)$ for (b, b_1, b_2). If L is in general position with respect to $Q_1 \cup Q_2$, then for $(b, b_1, b_2) = (3, 3, 2)$ one has $24/e(\mathcal{O}) = 384/17 \notin \mathbb{N}$. On the other hand, if L is tangent to Q_1, then for the triple $(3, 2, 3)$ one has $e(\mathcal{O}) = -7/144$, whereas for the triple $(3, 3, 2)$ one has $e(\mathcal{O}) = 25/48$.

Suppose that Q_1 and Q_2 are tangent at two distinct points. If L is in general position with respect to $Q_1 \cup Q_2$, then

$$e(\mathcal{O}) = 3 - 2(\beta + \beta_1 + \beta_2) + 2\beta(\beta_1 + \beta_2) + \left[\beta_1 + \beta_2 - \frac{1}{2}\right]^2$$

Since $\beta + 2(\beta_1 + \beta_2) = 2$, this becomes $e(\mathcal{O}) = (5 + 2\beta - 3\beta^2)/4$. Recall that b can only take the values 2 and 3. For $b = 2$, one has $24/e(\mathcal{O}) = 384/21 \notin \mathbb{N}$. For $b = 3$ one has $24/e(\mathcal{O}) = 18$. The corresponding triple is $(b, b_1, b_2) = (3, 3, 2)$, which gives the orbifold $\mathcal{I}_1$.

If L is tangent to Q_1 and in general position with respect to Q_2, then

$$e(\mathcal{O}) = 2-(\beta+\beta_1+2\beta_2)+2\beta\beta_2+\frac{1}{2}\left[\beta+\beta_1-\frac{1}{2}\right]^2+\left[\beta_1+\beta_2-\frac{1}{2}\right]^2,$$

and $24/e(\mathcal{O})$ is never an integer. If L intersects $Q_1\cup Q_2$ transversally at one or both of the points of $Q_1\cap Q_2$, then this point is an orbifold point only if $\beta/2+\beta_1+\beta_2>1$, which contradicts the condition $\beta+2\beta_1+2\beta_2=2$. If L is tangent to $Q_1\cup Q_2$ at one of the points $Q_1\cap Q_2$, then this point is an orbifold point only if $b=b_1=b_2=2$. But then one can not have $\beta+2\beta_1+2\beta_2=2$.

4.10.5. *A cubic with two lines.* Let $B:=bC+b_1L_1+b_2L_2$, where C is a cubic curve. The condition $3\beta+\beta_1+\beta_2=2$ implies that (b,b_1,b_2) is one of $(4,4,2)$, $(2,2,3)$ or $(3,6,2)$. In this subsection, we shall adopt Degtyarev's notation for quintic curves [4] and give only the euler numbers of the orbifolds. Non of these orbifolds are uniformizable.

The quintic $C_3(A_2)\sqcup\{\times 3\}\sqcup\{\times 2,\times 1\}$: The euler number $e(\mathcal{O})$ is

$$1-(2\beta+\beta_2)+\beta_2(\beta+\beta_1)+\frac{3}{2}\left[\beta-\frac{1}{6}\right]^2+\frac{1}{2}\left[\beta_1+\beta-\frac{1}{2}\right]^2+\frac{3}{4}\left[\beta_2+\beta-\frac{2}{3}\right]^2$$

The quintic $C_3(A_2)\sqcup\{A_2^*\}\sqcup\{\times 3\}$: The line L_1 is tangent to $C_3(A_2)$ at the cusp A_2. This is an orbifold point only if $b=b_1=2$. But then one cannot have $3\beta+\beta_1+\beta_2=2$.

The quintics $C_3(A_2)\sqcup\{\underline{\times 3}\}\sqcup\{A_2,\underline{\times 1}\}$ and $C_3(A_2)\sqcup\{\underline{\times 3}\}\sqcup\{\underline{\times 1},\times 1,\times 1\}$: The flex point of $C_3(A_2)$ is an orbifold point only if $\beta+\beta_1+\beta_2/3>1$. But then one cannot have $3\beta+\beta_1+\beta_2=2$.

The quintic $C_3(A_2)\sqcup\{\times 3\}\sqcup\{A_2,\times 1\}$: Euler number $e(\mathcal{O})$ is

$$e(\mathcal{O})=1-(\beta_2+\beta)+\beta_2(\beta_1+\beta)+\frac{3}{4}\left[\beta+\beta_1-\frac{2}{3}\right]^2+\frac{3}{2}\left[\beta+\beta_2/3-\frac{1}{2}\right]^2$$

The quintic $C_3(A_2)\sqcup\{\times 3\}\sqcup\{\times 1,\times 1,\times 1\}$: Euler number $e(\mathcal{O})$ is

$$e(\mathcal{O})=2-(2\beta_2+3\beta)+\beta_2(3\beta+\beta_1)+\frac{3}{2}\left[\beta-\frac{1}{6}\right]^2+\frac{3}{4}\left[\beta+\beta_1-\frac{2}{3}\right]^2$$

The quintic $C_3(A_2)\sqcup\{\underline{\times 2},\times 1\}\sqcup\{\times 2,\underline{\times 1}\}$: Euler number $e(\mathcal{O})$ is

$$e(\mathcal{O})=1-2\beta+\beta_1\beta+\frac{3}{2}\left[\beta-\frac{1}{6}\right]^2+\frac{1}{2}\left[\beta+\beta_2-\frac{1}{2}\right]^2+\frac{1}{2}\left[\beta+\beta_2/2+\beta_1-1\right]^2$$

The quintic $C_3(A_2)\sqcup\{A_2,\underline{\times 1}\}\sqcup\{\times 2,\underline{\times 1}\}$: Euler number $e(\mathcal{O})$ is

$$e(\mathcal{O})=1-\beta+\frac{1}{2}\left[\beta+\beta_2-\frac{1}{2}\right]^2+\frac{1}{4}\left[\beta+\beta_2+\beta_1-1\right]^2+\frac{3}{2}\left[\beta+\beta_1/3-\frac{1}{2}\right]^2$$

The cusp of C is an orbifold point only if $\beta + \beta_1/3 > \frac{1}{2}$, which forces $b = 2$. But then $\beta_1 + \beta_2 + 3\beta = 2$ implies that $\beta + \beta_1 + \beta_2 = 1$. On the other hand, the triple point is an orbifold point only if $\beta + \beta_1 + \beta_2 > 1$.

The quintic $C_3(A_1) \sqcup \{\times 3\} \sqcup \{\times 3\}$: Euler number $e(\mathcal{O})$ is

$$e(\mathcal{O}) = 1 - 2\beta + \beta^2 + \beta_1\beta_2 + \frac{3}{4}\left[\beta + \beta_1 - \frac{2}{3}\right]^2 + \frac{3}{4}\left[\beta + \beta_2 - \frac{2}{3}\right]^2$$

The quintic $C_3(A_1) \sqcup \{\underline{\times 3}\} \sqcup \{\times 2, \underline{\times 1}\}$: The point of intersection of the lines is an orbifold point only if $\beta = \beta_1 = 2$. But then $\beta_1 + 3\beta = 2$, so that one cannot have $\beta_1 + \beta_2 + 3\beta = 2$.

The quintic $C_3(A_1) \sqcup \{\times 2, \underline{\times 1}\} \sqcup \{\times 2, \underline{\times 1}\}$: Euler number $e(\mathcal{O})$ is

$$2 - 3\beta + \beta^2 + \frac{1}{2}\left[\beta + \beta_2 - \frac{1}{2}\right]^2 + \frac{1}{2}\left[\beta + \beta_1 - \frac{1}{2}\right]^2 + \frac{1}{4}\left[\beta + \beta_2 + \beta_1 - 1\right]^2$$

4.10.6. *A cubic with a quadric.* Let $B = b_1 Q + b_2 C$. Then (b_1, b_2) is one of $(4, 2)$ or $(2, 3)$. Suppose that Q intersects C at two points of multiplicity 3 at each. Then

$$e(\mathcal{O}) = 1 - \beta_2 + \frac{3}{2}\left[\beta_2 - \frac{1}{6}\right]^2 + \frac{3}{2}\left[\beta_1 + \beta_2 - \frac{2}{3}\right]^2$$

For $(\beta_1, \beta_2) = (4, 2)$ one has $e(\mathcal{O}) = 65/96$. For $(b_1, b_2) = (2, 3)$ one has $24/e(\mathcal{O}) = 32$. But if $|\pi_1^{orb}(\mathcal{O})| = 32$ then this group cannot contain elements of order 3. Hence, this orbifold is not uniformizable neither.

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Galatasaray University, Department of Mathematics, Ortaköy/İstanbul, Turkey
e-mail adress: muludag@gsu.edu.tr