ON CONTRACTIONS OF SMOOTH VARIETIES

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On contractions of smooth varieties.*

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Abstract. Let $\varphi : X \to Z$ be a proper surjective map from a smooth complex manifold X onto a normal variety Z. If φ has connected fibers and $-K_X$ is φ-ample then φ is called a good contraction. In the present paper we study good contractions, fibers of which have dimension less or equal than two: after describing possible two dimensional isolated fibers we discuss their scheme theoretic structure and the geometry of $\varphi : X \to Z$ nearby such a fiber. If $dimX = 4$ and φ is birational with an isolated 2 dimensional fiber then we obtain a complete description of φ. We provide also a description of a 4 dimensional conic fibration with an isolated fiber which is either a plane or a quadric. We construct pertinent examples.

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Introduction.

The present paper is about maps of complex algebraic varieties. A *contraction* $\varphi : X \to Z$ is a proper surjective map of normal varieties with connected fibers. The contraction φ is called *good* if the anticanonical divisor $-K_X$ of X is φ-ample. Good contractions occur naturally in classification theories of algebraic varieties. This is most apparent in the Minimal Model Program where the point is to use good contractions and birational transformation related to them to produce a minimal model of a given variety. In what follows we study the local structure of a good contraction of a smooth variety.

As they appear naturally in classification of algebraic varieties, good contractions have been studied on many occasions. First, let us note that a Fano manifolds X admits a good contraction to a point and from this position the theory of good contractions is just a very broad generalization of the theory of Fano manifolds. Next let us notice that the classical and the contemporary adjunction theory is about the adjunction morphism which is nothing else but a special good contraction. In particular, the classical works of Enriques and Castelnuovo provide a complete description of good contractions of smooth surfaces. However, it had to come to the case of threefolds to establish the fundamental role of good contractions in classification of algebraic varieties. A complete description of (elementary) good contractions of smooth 3-folds was given by S. Mori [Mo1] as the base of the Minimal Model Program in dimension 3. Also, the objective of the Program in dimension 3, that is the existence of minimal models, was achieved by studying good contractions of varieties admitting good (terminal) singularities, see [Mo2]. Subsequently, the theory of good contractions of terminal threefolds was extended in [Ko-Mo].

In this paper we look at good contractions of smooth varieties of dimension $n \geq 3$ with low dimensional fibers (i.e. of dimension ≤ 2). In particular we provide a complete list of isolated two dimensional fibers. Then, after reviewing the 3 dimensional case we focus on the case $n = 4$ and φ birational. In this range some results were already proved by T. Ando [An], M. Beltrametti [Be] and Y.Kawamata [Ka1].

In order to understand the local structure of a good contraction $\varphi : X \to Z$ we will assume that the target Z is affine and $z \in Z$ is a fixed (geometric) point. First we will describe the geometric structure of the fiber $F = \varphi^{-1}(z)$. Subsequently we will discuss its normal sheaf of F and the fiber scheme structure on F. Finally we will provide a description of the singularity of Z at z and the description of φ around F.

The paper is divided into six sections. The first two sections are preparatory. In Section 1 we recall first the definition of good and crepant contractions and subsequently we present our main technical tools. This includes theorems about vanishing, relative base point freeness and deformation of rational curves. In Section 2 we collect pertinent results about blow-ups and about rank two vector bundles on Fano varieties. In particular we discuss in detail the blow-up of a smooth variety along a codimension two locally complete intersection subvariety and we prove a generalized Castelnuovo's contraction theorem (2.4).

In Section 3 we present several examples of good contractions. They are constructed with various algebraic geometry techniques: as complete intersections in projective bundles, or as blow-ups and blow-downs of special varieties, or as double coverings of special varieties, or as toric varieties. Although the constructions are not particularly difficult we found the results of some of them rather surprising. Actually, the list of examples grew together with our understanding of the classification of good contractions. In the subsequent section we show that the list of examples covers (almost) all possible two dimensional isolated fiber of a good contraction. In Section 4 we recall also the case of one dimensional fiber and at the end of the section we reprove the theorems for good contractions of a 3-fold (see (4.13)). The classification of 2 dimensional fibers is achieved in a two step argument. First we prove that a two dimensional fiber is normal and it has Fujita's Δ-genus equal to zero (see (4.2.1)) Then, to get the final list of possible fibers, we compare deformations of rational curves inside the fiber with the deformations of such curves inside the ambient variety. The main results are summarized in Table II and (4.5), (4.7) and (4.11).

In Section 5 we discuss the scheme theoretic structure of a fiber of a good contractions. We prove that if the fiber F is a locally complete intersection and the blow-up along this fiber has good singularities then the conormal bundle of the fiber provides a lot of information about the contraction φ around F. In particular, if the conormal bundle is spanned by global sections then the fiber structure on F is trivial and the contraction can be factored through the blow-up of Z along the maximal ideal of z (see (5.5)). Subsequently, we focus on the case $n = 4$ and we prove that the good situation mentioned above occurs for any two dimensional isolated fiber of a birational good contraction. The strategy in this section is as follows: we take a general smooth divisor $X' \in |-K_X|$ and then we consider the map $\varphi' := \varphi_{|X'} : X' \to Z'$ which is a *crepant contraction* of a smooth 3-fold. The structure of such a map is rather well understood by results of the Minimal Model

Program in dimension 3 (we discuss it in detail in (5.6)). Then we apply an *ascending lemma* (5.7.2) which gives the spannedness of the conormal of the fiber.

Section 6 concludes the paper with a geometric description of a good birational contraction φ of a smooth 4 fold. In particular, we show that φ can be *resolved* in terms of some special blow-ups and blow-downs. Next we describe the singularities of the target Z and of the image of the exceptional locus $\varphi(E) = S$. Here we apply the classification of spanned rank two vector bundles on Fano manifolds presented in Section 2 as well as arguments involving Hilbert scheme.

The following is a summary of the 4 dimensional birational result:

Theorem. *Let $\varphi : X \to Z$ be a birational good contraction from a smooth variety X of dimension 4 onto a normal variety Z (possibly affine). Let $F = \varphi^{-1}(z)$ be a (geometric) fiber of φ such that $dimF = 2$. Assume that all other fibers of φ have dimension < 2 and all components of the exceptional locus E of φ meet F (this may be achieved by shrinking Z to an affine neighbourhood of z and restricting φ to its inverse image, if necessary).*

If φ is not divisorial then $E = F \simeq \mathbf{P}^2$ and its normal is $N_{F/X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. In this situation the flip of φ exists. (This was the situation studied in [Ka1]).

If E is a divisor then Z as well as $S := \varphi(E)$ are smooth outside of z. Moreover, outside of F the map φ is a simple blow-down of the divisor E to the surface $S \subset Z$. The scheme theoretic fiber structure over F is trivial, that is the ideal $\mathcal{I}_F$ of F is equal to the inverse image of the maximal ideal of z, that is $\mathcal{I}_F = \varphi^{-1}(m_z) \cdot \mathcal{O}_X$.

The fiber F and its conormal $\mathcal{I}_F/\mathcal{I}_F^2$ as well as the singularity of Z and S at z can be described as follows

F	$N^*_{F/X}$	$SingZ$	$SingS$
$\mathbf{P}^2$	$T(-1) \oplus \mathcal{O}(1)/\mathcal{O}$	*cone over* $\mathbf{Q}^3$	*smooth*
$\mathbf{P}^2$	$\mathcal{O}^{\oplus 4}/\mathcal{O}(-1)^{\oplus 2}$	*smooth*	*cone over rational twisted cubic*
quadric	*spinor bundle from* $\mathbf{Q}^4$	*smooth*	*non-normal*

The quadric fiber can be singular, even reducible, and in the subsequent table we present a refined description of its conormal bundle. The last entry in the table provides information about the ideal of a suitable surface S which is computed via degeneracy locus technique in Example 3.2.

quadric	*conormal bundle*	*the ideal of S in* $\mathbf{C}[[x,y,z,t]]$
$\mathbf{P}^1 \times \mathbf{P}^1$	$\mathcal{O}(1,0) \oplus \mathcal{O}(0,1)$	(xz, xt, yz, yt)
quadric cone	$0 \to \mathcal{O} \to N^* \to \mathcal{J}_{line} \to 0$	*generated by 5 cubics*
$\mathbf{P}^2 \cup \mathbf{P}^2$	$T_{\mathbf{P}^2}(-1) \cup (\mathcal{O} \oplus \mathcal{O}(1))$	*generated by 6 quartics*

A similar classification is expected for contractions of fiber type from a smooth 4 dimensional projective variety with an isolated two dimensional fiber. The list of possible fibers is given in Section 4 (see (4.11)) and many examples are discussed in Section 3. In this case similar results were proved also by Kachi [Kac]. The understanding of the normal of such a fiber requires a new approach, i.e. the *ascending lemma* (5.7.2) has to be replaced by a *trace argument* (see 5.9). As the result we obtain a structure theorem for 4 dimensional conic fibrations with an isolated 2 dimensional fiber which is either $\mathbf{P}^2$ or a quadric.

Due to the relative base point freeness proved in [A-W], the 4-dimensional results can be extended for adjoint contractions of varieties of higher dimension (see the corollary (5.8.1))

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0. Notation and assumptions.

We work with schemes defined over complex numbers. In particular, a variety is a puredimensional reduced separated scheme of finite type over $\mathbf{C}$, a curve (a surface) is a variety of pure dimension 1 (2, respectively) (thus it does not have to be irreducible).

On a variety X by K_X we will denote its canonical divisor. If K_X is Cartier the associated line bundle will be denoted by the same name. More generally: we will confuse Cartier divisors and line bundles whenever it makes sense. We will also identify vector bundles and locally free sheaves. Whenever possible, the Chern or Segre classes will be identified with integers. If $\mathcal{E}$ is a vector bundle over a variety X then

$$\mathbf{P}(\mathcal{E}) = Proj_X(\bigoplus_{m>0} S^m\mathcal{E})$$

is a *projective bundle* with a relatively ample line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. If $\mathcal{I}$ is a coherent sheaf of ideals on X then

$$\hat{X} = Proj_X(\bigoplus_{m\geq 0} \mathcal{I}^m)$$

is the *blowing up of X with respect to the coherent sheaf of ideal $\mathcal{I}$*. If F is the closed subscheme of X corresponding to $\mathcal{I}$ we also call $\hat{X}$ the *blowing up of X along F*.

A Hirzebruch surface $\mathbf{F}_r$ is a $\mathbf{P}^1$-bundle $\mathbf{P}(\mathcal{O}(r) \oplus \mathcal{O})$ over projective line $\mathbf{P}^1$ with a unique section $C_0 \subset \mathbf{F}_r$ (isomorphic to $\mathbf{P}^1$) such that $C_0^2 = -r \leq 0$. A fiber of the projection $\mathbf{F}_r \to \mathbf{P}^1$ will be denoted by f. A (normal) cone $\mathbf{S}_r$ is defined by contracting $C_0 \subset \mathbf{F}_r$ to a normal point; in terms of projective geometry $\mathbf{S}_r$ is a cone over $\mathbf{P}^1 \hookrightarrow \mathbf{P}^r$ embedded via Veronese map (r-uple embedding). The restriction of the hyperplane section line bundle from $\mathbf{P}^{r+1}$ to $\mathbf{S}_r$ will be denoted by $\mathcal{O}_{\mathbf{S}_r}(1)$; the pull-back of this bundle to $\mathbf{F}_r$ is $\mathcal{O}(C_0 + rf)$.

Let $\mathbf{Q}_4 \simeq Gr(1,3)$ be the smooth 4-dimensional quadric, identified with the Grassmaniann of lines in $\mathbf{P}^3$, and let $\mathcal{S}$ be the universal bundle which we call also the spinor bundle on $\mathbf{Q}_4$. Consider a codimension 2 linear section $i : V \hookrightarrow Gr(1,3)$. The surface V is again a quadric, either $\mathbf{Q}_2 = \mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$, or the quadric cone $\mathbf{S}_2$, or a reducible quadric, i.e.

$V \simeq \mathbf{P}^2 \cup \mathbf{P}^2$ where the two $\mathbf{P}^2$ intersect along a line l. We will denote by $\mathcal{O}_V(1)$ (or just $\mathcal{O}(1)$) the restriction of $\mathcal{O}(1)$ from $\mathbf{Q}_4$. The *spinor bundle* over V, which we denote again by $\mathcal{S}$, is the rank two vector bundle defined as $i^*(\mathcal{S})$. We note that if $V \simeq \mathbf{P}^1 \times \mathbf{P}^1$ is a smooth quadric then $\mathcal{S} = \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)$; if $\pi : \mathbf{F}_2 \to \mathbf{S}_2 = V$ is the resolution of the vertex of the singular quadric, then the pull-back of $\mathcal{S}$ is in the non-splitting extension $0 \to \mathcal{O}(-f) \to f^*\mathcal{S} \to \mathcal{O}(-f - C_0) \to 0$. If $V \simeq \mathbf{P}^2 \cup \mathbf{P}^2 = P_1 \cup P_2$, then $\mathcal{S}$ restricted to P_1 is $T\mathbf{P}^2(-2)$ while restricted to P_2 is $\mathcal{O} \oplus \mathcal{O}(-1)$.

Let $\mathcal{E}$ be a rank r vector bundle on a projective variety X and let $C \cong \mathbf{P}^1 \subset X$ be a rational curve in X. The splitting type of E on C is a sequence of r numbers, $(a_1, \ldots, a_r)$ such that $E_{|C} = \mathcal{O}(a_1) \oplus ... \oplus \mathcal{O}(a_r)$; we assume that $a_1 \leq a_2 ... \leq a_r$.

For other definitions and notations connected with the theory of minimal models that we will use in the paper we refer in general the reader to [K-M-M]; for the main ones see also the next section.

1. Good contractions: definitions and fundamentals tools.

(1.0) A *contraction* is a proper map $\varphi : X \to Z$ of normal irreducible varieties with connected fibers. We assume that a contraction is not an isomorphism. The map φ is birational or otherwise $dimZ < dimX$, in the latter case we say that φ is of fiber type. The exceptional locus $E(\varphi)$ of a birational contraction φ is equal to the smallest subset of X such that φ is an isomorphism on $X \setminus E(\varphi)$.

Throughout the paper we will assume that X is smooth. In this situation the contraction φ is called *good* if the anti-canonical divisor $-K_X$ is φ-ample. If the map φ is birational and if K_X is a pull-back of a line bundle from Y then we say that φ is crepant. We say that φ is *elementary* if $PicX/\varphi^*(PicZ) \simeq \mathbf{Z}$. An elementary contraction is called *small* if its exceptional locus is of codimension ≥ 2.

In the present paper we are interested in the local description of a contraction: we would like to know a structure of the target Z and of the fiber of the map φ. Thus we choose a point $z \in Z$, we assume that the target Z is affine, and we consider the topological, (or set theoretical) fiber of φ over z, that is $\varphi^{-1}(z)$. The set $\varphi^{-1}(z)$ may be reducible, however we will usually assume that (unless otherwise specified) $\varphi^{-1}(z)$ is equidimensional (this is because we deal with low dimensional fibers, see (4.1)). We have two natural scheme structures on $\varphi^{-1}(z)$. One is the scheme theoretic fiber structure, which we denote by $\tilde{F}$, which is the closed subscheme of X defined by the ideal $\mathcal{I}_{\tilde{F}} := \varphi^{-1}(m_z) \cdot \mathcal{O}_X$. Since Z is affine and normal, and φ is proper with connected fiber, the ideal $\varphi^{-1}(m_z) \cdot \mathcal{O}_X$ is generated by global functions on X vanishing along $\varphi^{-1}(z)$. The other is what we can call the geometric structure, which we denote by F; this is the smallest scheme structure on $\varphi^{-1}(z)$. With this structure F is a variety, that is it is reduced and it has no embedded component.

For a good contraction φ we will also consider a φ-ample line bundle L such that $K_X + L$ is a pullback of a line bundle from Z, if Z is affine then $K_X + L = \varphi^*(\mathcal{O}_Z)$.

(1.1) Let us begin with a classical example. The following is the list of all possible good contractions $\varphi : X \to Z$ of smooth surfaces:

(a) Z is a point and X is a del Pezzo surface;
(b) Z is a smooth curve and $\varphi : X \to Z$ is a conic or $\mathbf{P}^1$-bundle, in particular every fiber $\tilde{F}$ is reduced and isomorphic to $\mathbf{P}^1$ or to union of two $\mathbf{P}^1$'s meeting transversally;
(c) Z is a smooth surface (thus φ is birational) and the exceptional locus consists of disjoint smooth rational curves with normal bundle $\mathcal{O}(-1)$, thus φ is a composition of blow-downs of disjoint rational curves to smooth points on Z.

Similarly one can describe crepant birational contractions of smooth surfaces (see (1.5.2).

The description of 2-dimensional contractions was known classically. To understand them it is enough to apply adjunction formula, Grauert criterion and the theory of divisors on surfaces.

To understand higher dimensional contractions one has to use some other properties of good contractions. The fundaments of the theory were set in the 80's by S. Mori, Y. Kawamata, J. Kollár, M. Reid. In particular, in the famous paper [Mo1], S. Mori produced the list of all possible good (elementary) contractions for smooth three dimensional projective varieties (which will be reproved here in (4.1) and (4.14)).

The aim of this section is to recall to the reader some properties of good contractions and to state them in the form which is convenient for our purposes.

The chief tool is the vanishing theorem due to Y.Kawamata, E.Viehweg and J. Kollár (see [K-M-M], section (1-2), or [E-W], corollary 6.11):

Theorem (1.2). (Vanishing theorem) *Let $\varphi : X \to Z$ be a good or crepant contraction with target Z being affine. Assume that L is a φ-ample line bundle. Then for any non-negative integer t we have*

$$H^i(X, tL) = 0 \text{ for } i > 0.$$

If φ is a good contraction and $K_X + L = \varphi^*(\mathcal{O}_Z)$ then also

$$H^i(X, -L) = 0 \text{ for } i > dimX - dimZ.$$

Let us note that although in the present paper we discuss only the case of smooth X, the vanishing theorem and many of its consequences remain true if we allow that X has log terminal singularities.

In the case of good contractions the fiber structure scheme $\tilde{F}$ has nice properties. Namely, its structural sheaf admits all the vanishings which hold for the ambient space. The following lemma is used very often; for the proof we refer the reader to the following papers: ([Mo1];(3.20) and (3.25.1)), ([Fu];(11.3)), ([An]; (2.2)), ([Y-Z], lemma 4).

Lemma (1.2.1). *Let $\varphi : X \to Z$ be a good contraction and let L, $z \in Z$, F and $\tilde{F}$ be as in (1.0). Moreover, let $\hat{F}$ be a scheme structure on $\varphi^{-1}(z)$ defined by an ideal $\varphi^{-1}(\mathcal{I}) \cdot \mathcal{O}_X$, where $\mathcal{I}$ is an ideal of a zero dimensional subscheme of Z supported at z (in particular $\hat{F} = \tilde{F}$ if $\mathcal{I} = m_z$). If either $t \geq 0$ and $i \geq 1$ or $t = -1$ and $i > dimX - dimZ$ then*

(a) $$H^i(\hat{F}, tL_{|\hat{F}}) = 0.$$

Let F' be any subscheme of X whose support is contained in F so that $\varphi(F') = z$. If either $t \geq 0$ and $r = dimF$ or $t = -1$ and $r \geq max\{dimF, dimX - dimZ + 1\}$ then

$$(b) \qquad H^r(F', tL_{|F'}) = 0.$$

Proof. Suppose that $\mathcal{I}$ is generated by functions $\{f_1, \ldots, f_r\}$. Then let us consider a sequence $X = S_0 \supset S_1 \supset \ldots \supset S_r = \hat{F}$ of subschemes of X, each S_k defined in X by functions $\{f_1 \circ \varphi, \ldots, f_k \circ \varphi\}$. Now vanishing (a) is proved for all schemes S_k by induction on k if we consider sequences

$$0 \longrightarrow tL_{|S_k} \xrightarrow{\cdot f_{k+1}} tL_{|S_k} \longrightarrow tL_{|S_{k+1}} \longrightarrow 0$$

and we start with the vanishing (1.2) — see also [An] or [Fu]. To prove (b) let us note that any F' supported on $\varphi^{-1}(z)$ is contained in a subscheme $\hat{F}$ which is supported on $\varphi^{-1}(z)$ and defined by global functions. Since $F' \subset \hat{F}$ the restriction map $tL_{|\hat{F}} \to tL_{|F'}$ is surjective and thus also $H^r(\hat{F}, tL_{|\hat{F}}) \to H^r(F', tL_{|F'})$ is surjective for $r \geq dimF$. This proves the part (b).

Lemma (1.2.2). *Let $\varphi : X \to Z$ be a good contraction and let L, $\hat{F}$ and F' be as in (1.2.1). Let also $X' \in |L|$ be the zero locus of a non-trivial section of L. Then we have*

$$(a) \qquad H^i(\hat{F} \cap X', tL_{|\hat{F} \cap X'}) = 0$$

if either $t \geq 1$ and $i \geq 1$ or $t = 0$ and $i \geq max\{dimX - dimZ, 1\}$; we also have

$$(b) \qquad H^r(F' \cap X', tL_{|F' \cap X'}) = 0$$

if either $t \geq 1$ and $r = dimF \cap X'$, or $t = 0$ and $r \geq max\{dimX - dimZ, dimF \cap X'\}$.

Proof. The first part of the lemma follows from the previous one if we consider the cohomology sequence associated to the exact sequence

$$0 \to -L_{|\hat{F}} \to \mathcal{O}_{\hat{F}} \to \mathcal{O}_{\hat{F} \cap X'} \to 0$$

tensored by tL. The part (b) is proved similarly as in the previous lemma.

Lemma (1.2.3). *Let $\varphi : X \to Z$ be a crepant contraction and let $\hat{F}$, F and F' be as in (1.2.1). Then*

$$(a) \qquad H^i(\hat{F}, \mathcal{O}_{\hat{F}}) = 0 \ \text{ for } \ i \geq 1 \text{ and}$$

$$(b) \qquad H^r(F', \mathcal{O}_{F'}) = 0 \ \text{ for } \ r \geq dimF.$$

Proof. The same as of the above.

We also note the following important consequence of the vanishing:

Theorem (1.2.4). *Let $\varphi : X \to Z$ be a good contraction of a smooth variety X of dimension n onto a normal variety Z of dimension m. Then Z has rational singularities and $R^{n-m}\varphi_*(K_X) = \omega_Z$, where ω_Z is the dualizing sheaf of Z.*

Proof. The rationality of singularities follows immediately from the vanishing (see, [Kol, Cor. 7.4] for φ of fiber type. The descending property of the canonical sheaf was noted first by Kempf [Ke, pp. 49–51] in case $m = n$ and for $m \le n$ it was proved by Kollár [Kol, Prop. 7.6] in case Z is smooth. The general case is obtained by compilation of these two result and application of Grothendieck spectral sequence. Namely, let $\alpha : \tilde{Z} \to Z$ be a desingularisation of Z and let $\tilde{X}$ be a desingularisation of the fiber product $X \times_Z \tilde{Z}$ with the induced morphisms $\beta : \tilde{X} \to X$ and $\tilde{\varphi} : \tilde{X} \to \tilde{Z}$, such that $\varphi \circ \beta = \alpha \circ \tilde{\varphi}$. Thus, by Grothendieck spectral sequence, the sequences $R^i\varphi_*(R^j\beta_*(K_{\tilde{X}}))$ and $R^i\alpha_*(R^j\tilde{\varphi}_*(K_{\tilde{X}}))$ have the same limit. But, because of Grauert-Riemenschneider vanishing $R^j\tilde{\varphi}_*(K_{\tilde{X}}) = 0$ for $j > 0$ and $\tilde{\varphi}_*(K_{\tilde{X}}) = K_{\tilde{Z}}$ and by Kempf's result $R^{n-m}(\alpha \circ \tilde{\varphi})_*(K_{\tilde{X}}) = \omega_Z$. On the other hand $R^i\varphi_*(R^j\beta_*(K_{\tilde{X}})) = 0$ for $i > 0$, [Kol, Thm. (3.8.i)], and $\beta_*(K_{\tilde{X}}) = K_X$ so that the other sequence degenerates to $R^i(\varphi \circ \beta)_*(K_{\tilde{X}}) = R^i\varphi_*(K_X)$, and we are done.

Another feature of good contractions is the special behaviour of some divisors, we will use it to apply some inductive arguments. Namely, in the set up of (1.0) we will choose a good section of $(K_X + L)$ or of L and we restrict to this section. We call this procedure *vertical*, respectively *horizontal*, *slicing*; in order to do this we need the following (for a proof see [A-W], (2.5) and (2.6)). (We note that vertical slicing was already used in the proof of (1.2.1).)

Lemma (1.3). *Let $\varphi : X \to Z$ be a good contraction of a smooth variety, assume moreover that Z is affine and $K_X + L = \mathcal{O}_X$.*

(1.3.1) **(Vertical slicing)** *Assume that $X'' \subset X$ is a non-trivial divisor defined by a global function $h \in H^0(X, K_X + L) = H^0(X, \mathcal{O}_X)$. Then for a general choice of h, X'' is smooth and any section of L on X'' extends to X.*

(1.3.2) **(Horizontal slicing)** *Let X' be a general divisor from the linear system $|L|$. Then , outside of the base point locus of $|L|$, X' is smooth and any section of L on X' extends to X.*

(i) If $\varphi' := \varphi_{|X'}$, then $K_{X'}$ is φ'-trivial

(ii) Let $Z' := Spec(X', \mathcal{O}_{X'})$. If φ is birational then the induced map $Z' \to Z$ is a closed immersion. Therefore the map φ restricted to X' has connected fibers.

The above lemma on horizontal slicing is particularly effective if we can prove a relative base point free theorem for the line bundle L, which means that the evaluation $\varphi^*\varphi_*L \to L$ is surjective. The next result is in this direction and it is a special case of the main theorem of [A-W] (i.e. Theorem (5.1) in [A-W]).

Proposition (1.3.3). Relative spannedness. *Let $\varphi : X \to Z$ be a good birational contraction of a smooth n-fold. Assume that a fiber F of φ is of dimension ≤ 2. Then the evaluation morphism $\varphi^*\varphi_*L \to L$ is surjective at every point of F (we say that L is φ-spanned).*

Proposition (1.3.4). *In the same hypothesis of the proposition (1.3.3) (or, more generally, of theorem 5.1 from [A-W]) the bundle L is φ-very ample which means that there exists an embedding $X \to Z \times \mathbf{P}^N$ over Z such that L is the pull-back of $\mathcal{O}(1)$.*

Proof. The proof is the same as the one of the theorem (5.1) in [A-W]: in the hypothesis one can slice horizontally until fibers of φ are zero dimensional. In this case the φ-very ampleness is clear. By (1.3.2) the sections of L extend up from a divisor from $|L|$ so does the map to $Z \times \mathbf{P}^N$.

A useful consequence of the relative very ampleness is an estimate on the normal of a linear subspace of a fiber of a contraction:

Lemma (1.3.5). (c.f. [Ei]) *Let $\varphi : X \to Z$ be a good contraction with a φ-very ample line bundle L. Assume that $S \simeq \mathbf{P}^r$ is contained in a fiber of φ and $L_{|S} \simeq \mathcal{O}(1)$. Then the twisted conormal bundle $N^*_{S/X}(1)$ is spanned by global sections.*

Proof. The inclusions $S \subset X \subset \mathbf{P}^N$ yield a surjection $N^*_{S/\mathbf{P}^N} \to N^*_{S/X}$. Since $N^*_{S/\mathbf{P}^N} \simeq \mathcal{O}(-1)^{N-r}$ the lemma follows.

Another fundamental property of good contractions we will use is the existence of rational curves in fibers: through any point of the contracted locus there passes a rational curve. A proof of this fine property requires deformation theory. We only note that the vanishing (1.2.2) implies that any 1-dimensional component of a fiber must be $\mathbf{P}^1$ (see (2.1)).

We will study the deformations of rational curves as well as their chains and for this we need the following estimate on the dimension of a component of the Hilbert scheme containing the class of a curve $C \subset X$.

Definition (1.4). A proper curve C is called smoothable if there is an irreducible pointed variety $0 \in T$ and a proper flat family of curves $g : W \to T$ such that $C = g^{-1}(0)$ and the generic fiber of g is smooth.

There are two natural examples of reducible smoothable curves of genus 0 a tree and a bunch of rational curves. A curve $C = \cup_i R_i$ is a (connected) tree of rational curves if:

(i) any R_i is a smooth rational curve

(ii) R_i intersects $\sum_{i-1}^{j=1} R_J$ in a single point which is an ordinary node of C.

The smoothing of a tree of rational curves is obtained by modifying (blowing-up) the special fiber in the trivial family $\mathbf{P}^1 \times T$.

A bunch of m rational curves is a projective cone over m generic points in $\mathbf{P}^{m-1}$. In other words such a bunch is a section of $\mathbf{S}_m$ by a hyperplane which passes through the vertex of $\mathbf{S}_m$. The smoothing of the bunch of rational curves is obtained by considering a generic pencil of sections of $\mathbf{S}_m$ which contains the section in question. Let us also note that if we attach to a one of the stems of a bunch of rational curves a tree (so that the meeting point is an ordinary node) then the resulting curve is again smoothable.

Proposition (1.4.1). *Let C be a proper curve without embedded points. Suppose that $f : C \to X$ is an immersion of C into a smooth variety X and that C is smoothable.*

Then any component of the Hilbert scheme containing $f(C)$ has dimension $-K_X.C + (n-3)\chi(\mathcal{O}_C)$ at least.

We will apply the proposition in the case C is a tree or a bunch of rational curves; in particular we will have $\chi(\mathcal{O}_C) = 1$ (see the section 4, in particular (4.5) and the next).

The proposition is proved in the book of J. Kollár (see [Ko],ch. II, theorem (1.14)). Another version of this result for irreducible rational curves was used by Mori to prove the existence of rational curves in fibers of good contractions (see [Mo1]). It was also used to make relation between the dimension of a fiber and the dimension of the exceptional locus of φ (see [Wi], Theorem (1.1) and [Io], Theorem 0.4).

(1.5). Our inductive proofs will frequently lead from good contractions to crepant contractions. Therefore we will use some results on crepant contraction. As an immediate application of the Lemma (1.2.1) we obtain

Corollary (1.5.1). *Any one dimensional component F' of a fiber of a good or crepant contraction is a smooth rational curve and $H^1(F', \mathcal{J}/\mathcal{J}^2) = 0$, where $\mathcal{J}$ is the sheaf of ideals of F'. If F is one dimensional fiber of a good or crepant contraction then $H^1(F, \mathcal{O}_F) = 0$ and thus the graph of F, with edges representing its components and vertexes representing their incidence points, is simply connected.*

Remark (1.5.2). A complete description of the one dimensional fibers of a good contraction of a smooth variety X will be given in (4.1). If φ is creapant contraction of a smooth variety X then a more detailed and refined description of the configuartion of the curves in F can be given if $n = 2$ or $n = 3$; some more subtle arguments are needed. For $n = 2$ the incidence of the curves is described by a dual graph which is isomorphic to one of the following Dynkin diagrams: A_n, D_n, E_6, E_7, E_8 (see for instance [B-P-V]). If $n = 3$ the incidence of the curves is described by a dual graph which is isomorphic to one obtained by contracting any subset (possibly the empty subset) of the (-2)-curves of a Dynkin diagram $(A_n, D_n, E_6, E_7, E_8)$ (this was proved in [Re]).

We restrict our attention now to the case of a crepant contraction of a smooth 3-fold; this was studied first by M. Reid (see [Re]) and subsequently by J. Kollár (see [C-K-M]), D.Morrison-S.Katz (see [Ka-M] and Y.Kawamata (see [Ka2]).

In particular M.Reid (see [Re], section 1, Theorem (1.4 iii) and its proof) proved, among other results, the following:

Proposition (1.5.3). *Let $\varphi : X \to Z$ be a small crepant contraction of a smooth 3-fold X. Then Z has only a terminal-Gorenstein singularity or, equivalently, Z has only cDV singularity. If p is a singular point of Z and H is a generic divisor through p (therefore p is a rational double point on H by the definition of cDV singularity) then $\varphi^{-1}H$ is non singular at the general point of any curve of $\varphi^{-1}(p)$. In particular $\varphi^{-1}H$ is normal and $\varphi^{-1}H \to H$ is a partial resolution of H.*

2. Generalities on blow-ups and vector bundles.

In the present section we collect some general results concerning blow-ups and vector bundles which we will use in the sequel. Our notation is consistent with that of Hartshorne's book [Ha, Sect.II.7].

The first results are about blow-ups of locally complete intersections.

Lemma (2.1). *Let $S \subset X$ be a locally complete intersection subvariety in a smooth variety X defined by a sheaf of ideals $\mathcal{I}_S$. Suppose that $\beta : \hat{X} \to X$ is the blow-up of X along the subvariety S. Then the exceptional set $E = E(\beta)$ is a Cartier divisor on $\hat{X}$ and $\beta_E : E \to S$ a projective bundle over S isomorphic to $\mathbf{P}(\mathcal{I}_S/\mathcal{I}_S^2)$. Moreover $\beta_* \mathcal{O}_{\hat{X}}(-E) = \mathcal{I}_S$. If S is connected then E generates the Picard group of $\hat{X}$ over $PicX$. Moreover $\hat{X}$ is Gorenstein and $K_{\hat{X}} = \beta^* K_X + (dimX - dimS - 1)E$.*

Proof. The variety $\hat{X}$ can be locally embedded into $X \times \mathbf{P}^{k-1}$, where k is the codimension of S. Indeed, if $f_1, \dots f_k$ are the functions defining locally the ideal $\mathcal{I}_S$ then $\hat{X}$ is defined in $X \times \mathbf{P}^{k-1}$ by equations $f_i t_j = f_j t_i$, where t_i are coordinates in $\mathbf{P}^{k-1}$. Using this observation one may verify all the assertions. We refer the reader to [EGA] and [Ha] for this and further properties of blow-ups.

Let X be a smooth variety of dimension $n \geq 3$. Assume that S_1 and S_2 are two codimension 2 smooth subvarieties of X which meet transversaly along a set Δ of dimension $n-3$. That is, in local coordinates $(z_0, z_1, \dots, z_{n-1})$ in a neighbourhood of a given point $x \in \Delta$, the subvarieties S_1 and S_2 are defined by, respectively, $z_0 = z_1 = 0$ and $z_0 = z_2 = 0$. Therefore the reducible subvariety $S := S_1 \cup S_2$ is defined locally by two functions and in these coordinate system its equations are $z_0 = z_1 z_2 = 0$. In particular, the sheaf $N_S^* := \mathcal{I}_{S_1 \cup S_2}/\mathcal{I}_{S_1 \cup S_2}^2$ is locally free rank 2 over $S_1 \cup S_2$.

Lemma (2.2). *There is an exact sequence of $\mathcal{O}_{S_1}$-modules:*

$$0 \longrightarrow (N_{S/X}^*)_{|S_1} \longrightarrow N_{S_1/X}^* \longrightarrow N_{\Delta/S_2}^* \longrightarrow 0.$$

Proof. We have an embedding of sheaves of ideals $\mathcal{I}_S \hookrightarrow \mathcal{I}_{S_1}$ which is the identity outside of S_2. For $x \in \Delta$ and local coordinates as above, this inclusion can be expressed as $(z_0, z_1 z_2) \subset (z_0, z_1)$. Therefore over Δ the quotient $\mathcal{I}_{S_1}/(\mathcal{I}_S + \mathcal{I}_{S_1}^2)$ is generated by the function z_1. On the other hand, the inclusion $\Delta \subset S_2$ is related to $\mathcal{I}_{S_2} \hookrightarrow \mathcal{I}_\Delta$ and thus to the inclusion $(z_0, z_2) \subset (z_0, z_1, z_2)$. Therefore $\mathcal{I}_\Delta/(\mathcal{I}_\Delta^2 + \mathcal{I}_{S_2})$ is generated by z_1 as well.

The geometric meaning of the above sequence can be described as follows. Let $\pi_1 : \hat{X}_1 \to X$ be the blow-up of X along S_1 with the exceptional divisor $\hat{E}_1$. Then $\hat{E}_1 = \mathbf{P}(N_{S_1}^*)$ and $N_{S_1}^* = (\pi_1)_* \mathcal{O}_{\hat{E}_1}(-\hat{E}_1)$. Let $\hat{S}_2 \subset \hat{X}_1$ be the strict transform of S_2. We note that $\hat{S}_2 \simeq S_2$ and $\hat{S}_2$ meets E_1 along a section of the $\mathbf{P}^1$-bundle $\pi_1 : \hat{\Delta} = \pi_1^{-1}(\Delta) \to \Delta$. Let us call the section Δ_1. We note that the section Δ_1 is associated to the surjective morphism $(N_{S_1}^*)_{|\Delta} \to N_{\Delta/S_2}^* \to 0$ of $\mathcal{O}_\Delta$-modules (dually: Δ_1 parametrizes normal vectors along which S_2 enters into Δ). Now we blow-up $\hat{X}_1$ along $\hat{S}_2$ and we call the result $\tilde{X}_2$ (and the exceptional divisor $\tilde{E}_2$). The strict transform of $\hat{\Delta}$, call it $\tilde{\Delta}$, has now the normal whose restriction to any fiber of the ruling $\tilde{\Delta} \to \Delta$ is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Thus, we see that $\tilde{\Delta}$ can be contracted now in $\tilde{X}_2$. That is, we have a contraction $\tilde{X}_2 \to \bar{X}$ over X which is isomorphism outside of $\tilde{\Delta}$ and which contracts $\tilde{\Delta}$ to the set isomorphic to Δ and any point of this set is a quadric cone singularity on $\bar{X}$. The map $\bar{\pi} : \bar{X} \to X$ is the blow-up of the ideal of $S = S_1 \cup S_2$. The strict transform of $\hat{E}_1$, call it $\bar{E}_1$, is again a $\mathbf{P}^1$-bundle over S_1 and $N^*_{S|S_1}$ is the pushforward of $\mathcal{O}_{\bar{E}_1}(-\bar{E}_1 - \bar{E}_2)$.

Over S_1, however, the whole operation can be described as the blowing-up of the section Δ_1 in E_1 and then contracting of the strict transform of $\hat{\Delta}$. The birational map $\bar{X} \to \hat{X}_1$ is associated to the inclusion $\mathcal{I}_S \subset \mathcal{I}_{S_1}$ and also to the injection $0 \to N^*_{S|S_1} \to N^*_{S_1}$. The cokernel of the injection is $\mathcal{O}_{\Delta_1}(-E_1) = N^*_{\Delta/S_2}$. The details of this geometric interpretation of "vector bundle surgery" are explained in general by Maruyama in [Ma, pp.116–117].

The above blow-up argument is useful to understand the subsequent situation.

Lemma (2.3). *With the above notation*

$$N^*_{\hat{S}_2/\hat{X}_1} \simeq (N^*_S)_{|S_2} \otimes \mathcal{O}_{S_2}(\Delta).$$

Proof. Indeed, the exceptional set $\tilde{E}_2$ of the blow-up $\tilde{X}_2 \to \hat{X}_1$ is not influenced by the contraction of $\tilde{\Delta}$, that is its strict tranform $\bar{E}_2$ is the same as $\tilde{E}_2$. But $\tilde{E}_2$ and $\bar{E}_2$ are projectivisations of, respectively, left-hand and right-hand side of the above equality; thus the equality is proved up to the twisting. The twist follows by comparing the 1st Chern classes of $N^*_{\hat{S}_2/\hat{X}_1}$ and $N^*_{S|S_2}$; e.g. using the above lemma. Namely, in the above argument we noticed that (changing the indices appropriately) $(N^*_{S/X})_{|S_2} = (\bar{\pi})_* \mathcal{O}_{\bar{E}_2}(-\bar{E}_1 - \bar{E}_2)$ while $N^*_{\hat{S}_2/\hat{X}} = (\tilde{\pi}_2)_* \mathcal{O}_{\tilde{E}_2}(-\tilde{E}_2)$, where $\bar{\pi}$ and $\tilde{\pi}_2$ are the appropriate blow-downs. Another proof of the above equality can be obtained by calculation in local coordinates of the blow-up $\hat{X}_1$. That is, we can write the equations of all the needed sets in the local (mixed: affine-homogenous) coordinates $((z_0, z_1, \ldots, z_{n-1}), [\hat{z}_0, \hat{z}_1])$ around each point of Δ and we can compute the equality directly.

We conclude the part of the section devoted to blow-ups with a version of Castelnuovo theorem, see [Ha, V.5.7].

Proposition (2.4). *Let $\varphi : X \to Z$ be a projective morphism from a smooth variety X onto a normal variety Z with connected fibers (a contraction). Suppose that $z \in Z$ is a point of Z and $F = \varphi^{-1}(z)$ is the geometric fiber over z (see (1.0)). which is locally complete intersection in X. Let $\mathcal{I}$ denote the ideal of F and $r := dim H^0(F, \mathcal{I}/\mathcal{I}^2)$. Assume that for any positive integer k*

$$H^1(F, S^k(\mathcal{I}/\mathcal{I}^2)) = 0 \quad \text{and} \quad H^0(F, S^k(\mathcal{I}/\mathcal{I}^2)) = S^k H^0(F, (\mathcal{I}/\mathcal{I}^2))$$

then z is a smooth point of Z and $dim Z = r$.

Proof. Since $\varphi_* \mathcal{O}_X = \mathcal{O}_Z$ we can apply the theorem on formal function (see [Ha, III.11.1]) to describe the completion $\hat{\mathcal{O}}_z$ of the local ring of $z \in Z$

$$\hat{\mathcal{O}}_z = \varprojlim H^0(\hat{F}_k, \mathcal{O}_{F_k}),$$

where $\hat{F}_k$ is the closed subscheme of X defined by $\varphi^{-1}(m_z^k) \cdot \mathcal{O}_X$. Since $F = \varphi^{-1}(z)$ it follows that the sequence of ideals $\varphi^{-1}(m_z^k) \cdot \mathcal{O}_X$ is cofinal (in the sense of [Ha, p. 194]) with the sequence of ideals $\mathcal{I}^k$, so we may use schemes F_k defined by $\mathcal{I}^k$ instead of $\hat{F}_k$.

We will prove for each k that $H^0(F_k, \mathcal{O}_{F_k})$ is isomorphic to a truncated power series ring $A_k = \mathbf{C}[[x_1, ..., x_r]]/(x_1, ..., x_r)^k$. This will imply that $\hat{\mathcal{O}}_z \cong \mathbf{C}[[x_1, ..., x_r]]$ and therefore that z is a smooth point of Z.

For $k = 1$ we have that $H^0(F, \mathcal{O}_F) = \mathbf{C}$. For $k > 1$ we note that $\mathcal{I}^k/\mathcal{I}^{k+1} = S^k(\mathcal{I}/\mathcal{I}^2)$, because F is a local complete intersection. Thus the claim can be proved by induction, using the cohomology sequence associated to the exact sequence

$$0 \to \mathcal{I}^k/\mathcal{I}^{k+1} \to \mathcal{O}_{F_{k+1}} \to \mathcal{O}_{F_k} \to 0$$

and the cohomology hypothesis (see [Ha], proof of (V.5.7)).

Now we pass to the discussion of vector bundles. We start with a very simple result for vector bundles on a tree of rational curves.

Lemma (2.5). *Let $C = \bigcup C_i$ be a connected (possibly reducible) curve which is a tree of rational curves, $C_i \simeq \mathbf{P}^1$. Suppose that $\mathcal{E}$ is a vector bundle over C. Then the following conditions are equivalent:*

(a) The bundle $\mathcal{E}$ is nef; that is, $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is nef on $\mathbf{P}(\mathcal{E})$ (in particular, for any i and any surjective morphism over the i-th component $\mathcal{E}_{|C_i} \to \mathcal{E}_i'$ we have $deg(\mathcal{E}_i') \geq 0$).
(b) The bundle $\mathcal{E}$ is spanned by global sections at a generic point of any component of C; that is, for any i and a generic $x \in C_i$ the evaluation $\Gamma(C, \mathcal{E}) \to \mathcal{E}_x$ is surjective.
(c) The bundle $\mathcal{E}$ is spanned by global sections at every point of C; that is, for every $x \in C$ the evaluation $\Gamma(C, \mathcal{E}) \to \mathcal{E}_x$ is surjective.

Proof. The implications (c)⇒(b)⇒(a) are clear so we shall prove (a)⇒(c). We will proceed by induction with respect to the number of irreducible components of the curve C. The lemma is clearly true if $C \simeq \mathbf{P}^1$. Now the curve C which contains $n \geq 2$ irreducible components can be presented as the union $C' \cup C_n$ where C' has $n-1$ components, $C_n \simeq \mathbf{P}^1$ and C' and C_n meet at one point, say p. By the inductive assumption $\mathcal{E}_{C'}$ is globally generated by $\Gamma(C', \mathcal{E}_{C'})$ and $\mathcal{E}_{C_n}$ is globally generated by $\Gamma(C_n, \mathcal{E}_{C_n})$. The bundle $\mathcal{E}$ is obtained from $\mathcal{E}_{C_n}$ and $\mathcal{E}_{C'}$ by the identification of their fibers over p. Moreover $\Gamma(C, \mathcal{E})$ is obtained from $\Gamma(C', \mathcal{E}_{C'})$ and $\Gamma(C_n, \mathcal{E}_{C_n})$ by identification of the values of sections over p. Now it is clear that the spannedness of the restriction of $\mathcal{E}$ to both C' and C_n implies the spannedness of $\mathcal{E}$. For example: to get a section with a prescribed value v_x over $x \in C'$ we choose first a section $s' \in \Gamma(C', \mathcal{E}_{C'})$ such that $s'(x) = v_x$ and then $s_n \in \Gamma(C_n, \mathcal{E}_{C_n})$ such that $s_n(p) = s'(p)$. Now s' and s_n glue to a global section $s \in \Gamma(C, \mathcal{E}_C)$ such that $s(x) = v_x$.

The next result is from [S-W1]:

Table I

c_1 c_2	*description of bundle* $\mathcal{E}$	*description of* $\mathbf{P}(\mathcal{E}) \to \mathbf{P}(H^0(\mathbf{P}^2, \mathcal{E}))$
0 0	$\mathcal{E} = \mathcal{O} \oplus \mathcal{O}$	$\mathbf{P}^2 \times \mathbf{P}^1 \to \mathbf{P}^1$
1 0	$\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(1)$	*blow-up of a point at* $\mathbf{P}^3$
1 1	$\mathcal{E} = T\mathbf{P}^2(-1)$	$\mathbf{P}^1$*-bundle over* $\mathbf{P}^2$
2 0	$\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(2)$	*blow-up of vertex of a cone over* $\mathbf{P}^2 \subset \mathbf{P}^5$
2 2	$0 \to \mathcal{O} \to T\mathbf{P}^2(-1) \oplus \mathcal{O}(1) \to \mathcal{E} \to 0$	*blow-up of a line in a smooth quadric*
2 3	$0 \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus 4} \to \mathcal{E} \to 0$	*blow-up of a twisted rational curve in* $\mathbf{P}^3$
2 4	$0 \to \mathcal{O}(-2) \to \mathcal{O}^{\oplus 3} \to \mathcal{E} \to 0$	*conic bundle over* $\mathbf{P}^2$

Lemma (2.6). *Let $\mathcal{E}$ be a rank-2 vector bundle over $\mathbf{P}^2$. Suppose that $\mathcal{E}$ is spanned by global sections and $0 \leq c_1(\mathcal{E}) \leq 2$. Then $\mathcal{E}$ is isomorphic to one of the bundles listed in Table I in which we provide also its Chern classes together with a description of the map $\mathbf{P}(\mathcal{E}) \to \mathbf{P}(H^0(\mathbf{P}^2, \mathcal{E})^*)$ associated to the evaluation of sections.*

Lemma (2.7). *Let $\mathcal{E}$ be a rank-2 vector bundle over $\mathbf{P}^2$. If the splitting type of $\mathcal{E}$ on every line is the same, then $\mathcal{E}$ is either a twist of the tangent bundle or it is decomposable. If the splitting type of $\mathcal{E}$ on each line is either $(-1, 1)$ or $(-2, 2)$ then either $\mathcal{E}$ is decomposable or it is not semi-stable (which means that $\mathcal{E}(-1)$ has a section) with $c_2(\mathcal{E}) = 0$. In the latter case the section of $\mathcal{E}(-1)$ vanishes at one point only and thus $\mathcal{E}$ is in the following sequence*

$$0 \longrightarrow \mathcal{O}(1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_x(-1) \longrightarrow 0. \tag{2.7.0}$$

Proof. The first part of the lemma is just a theorem of Van de Ven. The second part follows from a theorem of Grauert—Muellich. Namely, because of its general splitting type the bundle $\mathcal{E}(-1)$ has a section which (again by the splitting type) vanishes at one point at most (see [O-S-S]).

Corollary (2.7.1). *Suppose that $\mathcal{E}$ is a rank-2 vector bundle over $\mathbf{P}^2$ such that $c_1(\mathcal{E}) = 2$ and for any line $l \subset \mathbf{P}^2$ we have $H^1(l, \mathcal{E}_l) = 0$. Then either the general splitting of $\mathcal{E}$ is $\mathcal{O}(1) \oplus \mathcal{O}(1)$ or $\mathcal{E}$ is decomposable, or it is not semistable as in (2.7.0).*

Now we pass to the discussion of vector bundles over quadrics.

Lemma (2.8). *Let $\mathcal{E}$ be a rank-2 vector bundle over a 2-dimensional quadric V which is either a smooth quadric $\mathbf{Q}_2 = \mathbf{F}_0$, or a quadric cone $\mathbf{S}_2$, or $\mathbf{P}^2 \cup \mathbf{P}^2$. Suppose that $det(\mathcal{E})$ is the restriction of $\mathcal{O}(1)$ from $\mathbf{Q}^4$ and that $\mathcal{E}$ is spanned by global sections. Then, up to an automorphism of the quadric, we have either $\mathcal{O} \oplus \mathcal{O}(1)$, or $\mathcal{E} \simeq \mathcal{S}(1)$ or $\mathcal{E}$ is a pull-back of $T\mathbf{P}^2(-1)$ via a double covering $V \to \mathbf{P}^2$.*

Proof. The case of $\mathbf{F}_0$ is discussed in [S-W2]. The case of $\mathbf{S}_2$ can be done similarly. We note only that since $\mathcal{E}$ is spanned then its second Chern class c_2 as well as its Segre class

$s_2 = c_1^2 - c_2 = 2 - c_2$ have to be non-negative and thus $c_2 = 0,\ 1,\ 2$. Because $\mathbf{P}(\mathcal{E})$ is a Fano variety one can apply vanishing and get $h^0(\mathcal{E}) = 5 - c_2$.

The case of $\mathbf{P}^2 \cup \mathbf{P}^2$ can be easily done with the help of Van de Ven theorem. Namely, the restriction of $\mathcal{E}$ to each of the components is either $\mathcal{O} \oplus \mathcal{O}(1)$ or $T\mathbf{P}^2(-1)$ and the glueing of these two bundles along $\mathbf{P}^2 \cap \mathbf{P}^2$ is unique as it follows from an easy argument concerning extensions of automorphisms of these bundles from a line to $\mathbf{P}^2$.

Remark (2.9). Let F be a 2-dimensional quadric which is either a smooth quadric $\mathbf{Q}_2 = \mathbf{F}_0$, or a quadric cone $\mathbf{S}_2$, or a reducible quadric $\mathbf{P}^2 \cup \mathbf{P}^2$, and let $\mathcal{E} = \mathcal{S}(1)$. Then the map $\varphi : \mathbf{P}(\mathcal{E}) \to \mathbf{P}(H^0(\mathbf{P}^2, \mathcal{E})^*)$, associated to the evaluation of sections, is a birational map to $\mathbf{P}^3$. If $F = \mathbf{F}_0$ then φ is the blow-up of two disjoint lines in $\mathbf{P}^3$: up to a change of coordinates $[x_0, x_1, x_2, x_3]$ in $\mathbf{P}^3$ the morphism φ is the blow-up of the ideal $(x_0x_2, x_0x_3, x_1x_2, x_1x_3)$. If $F = \mathbf{S}_2$ then φ is the blow-up of a double line associated to the ideal $(x_0^2, x_1^2, x_0x_1, x_0x_2 + x_1x_3)$. If F is the reducible quadric then $\mathbf{P}(\mathcal{E})$ is reducible and it consists of two components: $\mathbf{P}(\mathcal{E}) = \mathbf{P}(\mathcal{E}_{|F_1}) \cup \mathbf{P}(\mathcal{E}_{|F_2}) = \mathbf{P}(T_{\mathbf{P}^2}(-1)) \cup \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(1))$ and $\varphi_{|\mathbf{P}(T_{\mathbf{P}^2}(-1))}$ is a $\mathbf{P}^1$-bundle over $\mathbf{P}^2 \subset \mathbf{P}^3$ while $\varphi_{|\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(1))}$ is the blow up of a point $x \in \mathbf{P}^2 \subset \mathbf{P}^3$. We note also that, since each of the quadrics is a linear section of the Grassmann variety of lines on $\mathbf{P}^3$, each of the projective bundles has a natural interpretation in terms of incidence of lines in $\mathbf{P}^3$ (c.f. 3.4.0).

Suppose now that V is an irreducible quadric, that is either $\mathbf{F}_0$ or the quadric cone $\mathbf{S}_2$ with the resolution $\pi : \mathbf{F}_2 \to \mathbf{S}_2$ of its vertex. Line bundles over $\mathbf{F}_0$ are of the form $\mathcal{O}(a_1, a_2) = \mathcal{O}(a_1f_1 + a_2f_2)$, where (f_1, f_2) are fibers of two different ruling over $\mathbf{P}^1$ and a_1, a_2 are integers. A conic on a quadric is an element of the $|\mathcal{O}(1)|$, or a linear section of the surface embeded into $\mathbf{P}^3$. If $V = \mathbf{F}_0$ then $\mathcal{O}(1) = \mathcal{O}(1,1)$ and if $V = \mathbf{S}_2$ then $\pi^*\mathcal{O}(1) = \mathcal{O}(C_0 + 2f)$ where C_0 is the exceptional divisor of π and f is a fiber of the ruling $\mathbf{F}_2 \to \mathbf{P}^1$.

Lemma (2.10.1). *Let $\mathcal{E}$ be a rank 2-vector bundle over $\mathbf{F}_0$. Suppose that the restriction of $\mathcal{E}$ to a generic conic splits into a direct sum $\mathcal{O}(a) \oplus \mathcal{O}(b)$, where $a \geq b$. If $a - b \geq 2$ then there exists a zero-dimensional subscheme $Z \subset \mathbf{F}_0$ such that the bundle $\mathcal{E}$ is in the following exact sequence*

$$0 \longrightarrow \mathcal{O}(a_1, a_2) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Z \otimes \mathcal{O}(b_1, b_2) \longrightarrow 0,$$

where $a_1 + a_2 = a$ and $b_1 + b_2 = b$.

Lemma (2.10.2). *Let $\mathcal{E}$ be a rank 2-vector bundle over $\mathbf{S}_2$. Suppose that the restriction of $\mathcal{E}$ to a generic conic splits into a direct sum $\mathcal{O}(a) \oplus \mathcal{O}(b)$, where $a \geq b$. If $a - b \geq 2$ then there exists a zero-dimensional subscheme $Z \subset \mathbf{F}_2$ such that the bundle $\pi^*\mathcal{E}$ is in the following exact sequence*

$$0 \longrightarrow \mathcal{O}(a_0C_0 + af) \longrightarrow \pi^*\mathcal{E} \longrightarrow \mathcal{I}_Z \otimes \mathcal{O}(b_0C_0 + bf) \longrightarrow 0,$$

where $2(a_0 + b_0) = a + b$.

Proof. The proof of both lemmata depends on a criterion from [F-H-S], Theorem (4.2). Although the quoted theorem is formulated for smooth varieties, it is not hard to see that the construction of a rank 1 subsheaf of $\mathcal{E}$ holds also in the case of the resolution of the singular irreducible quadric. We note that the condition of Lemma 4.1 of [ibid] is satisfied. This is because the incidence variety $Z \to V$ of conics on the quadric V is obtained from the flag manifold of planes in $\mathbf{P}^3 \supset V$ and one computes easily that the restriction of $T_{Z/V}$ to the unique section $\hat{C}$ of $Z \to V$ over a smooth conic C is equal to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Lemma (2.11). *Let $\mathcal{E}$ be a rank 2 vector bundle over an irreducible quadric V such that $det\mathcal{E} = \mathcal{O}(1)$. Assume that the first cohomology of the restriction of the bundle $\mathcal{E}$ to any conic is zero. Then either the restriction of $\mathcal{E}$ to a generic smooth conic is $\mathcal{O}(1) \oplus \mathcal{O}(1)$ or one of the following is true:*

(a) $V = \mathbf{F}_0$ and either

(i) $\mathcal{E}$ is decomposable and (up to the change of the order in the product $\mathbf{P}^1 \times \mathbf{P}^1$) isomorphic to $\mathcal{O}(1,1) \oplus \mathcal{O}$ or $\mathcal{O}(2,0) \oplus \mathcal{O}(-1,1)$, or $\mathcal{O}(2,1) \oplus \mathcal{O}(-1,0)$, or

(ii) there exists a point $x \in V$ such that $\mathcal{E}$ fits in the exact sequence

$$0 \longrightarrow \mathcal{O}(1,1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_x \to 0.$$

(b) $V = \mathbf{S}_2$ and if $\mathcal{E}' = \pi^*\mathcal{E}$ then either

(i) $\mathcal{E}'$ is decomposable and isomorphic to $\mathcal{O}(C_0 + 2f) \oplus \mathcal{O}$ or

(i') $\mathcal{E}'$ fits in one of the sequences

$$0 \longrightarrow \mathcal{O}(2C_0 + 2f) \longrightarrow \mathcal{E}' \longrightarrow \mathcal{O}(-C_0) \longrightarrow 0 \qquad (2.11.0)$$

$$0 \longrightarrow \mathcal{O}(2C_0 + 3f) \longrightarrow \mathcal{E}' \longrightarrow \mathcal{O}(-C_0 - f) \longrightarrow 0$$

(ii) there exists a point $x \in \mathbf{F}_2 \setminus C_0$ such that $\mathcal{E}'$ fits in the exact sequence

$$0 \longrightarrow \mathcal{O}(C_0 + 2f) \longrightarrow \mathcal{E}' \longrightarrow \mathcal{I}_x \to 0.$$

Proof. The assumption on vanishing of the cohomology implies that the splitting type of $\mathcal{E}$ ony any smooth conic is either $(1,1)$ or $(2,1)$, or $(3,-1)$ while the splitting type on any line is either $(1,0)$ or $(2,-1)$. If the generic spliting type is not $(1,1)$ then we can use the previous two lemmata to place $\mathcal{E}$ (or $\mathcal{E}'$) as a middle term of an exact sequence. We moreover know that the intersection of the invertible subsheaf $\mathcal{L}_1$ of $\mathcal{E}$ (resp. $\mathcal{E}'$), provided by this sequence, with a conic is 2 or 3, while the intersection with any line is ≤ 2. In case of $\mathbf{S}_2$ we also know that $\mathcal{L}_1.C_0 \leq 0$. This gives a list of possible $\mathcal{L}_1$. Similarly we find the condition on the degeneracy of the map $\mathcal{L}_1 \to \mathcal{E}$, that is on the 0-cycle Z from the previous lemmata. If $V = \mathbf{F}_0$ then we note that $length(Z) \leq 1$ because otherwise we would find a smooth conic containing at least two of the points from Z and the spliting type of $\mathcal{E}$ on such a conic would not be admissible. Considering the restriction of $\mathcal{E}$ to lines passing through Z we find out that $Z \neq 0$ only if $\mathcal{L}_1 = \mathcal{O}(1,1)$. The argument for $V = \mathbf{S}_2$ is similar: considering lines passing through Z we find out that $Z \neq 0$ only if $\mathcal{L}_1 = \mathcal{O}(C_0 + 2f)$, in such a case however $Z \cap C_0 = \emptyset$ because $\mathcal{L}_1.C_0 = 0$. Also, Z consits of one (reduced) point because we can consider the splitting type of $\mathcal{E}'$ on a conic passing through two points from Z.

Corollary (2.11.1). *Suppose that $\mathcal{E}$ is as in the above lemma and its generic splitting type on conics is neither $(1,1)$ nor $(2,0)$, then $h^0(V, \mathcal{E} \otimes \mathcal{O}(-1)) - h^1(V, \mathcal{E} \otimes \mathcal{O}(-1)) > 0$.*

Proof. It follows immediately from the previous lemma.

3. Good contractions: examples.

In this section we will present several examples of good contractions of a smooth projective variety X; we look especially at the case in which all fibers have dimension ≤ 2.

(3.1) Projective bundles.

Let $(F, \mathcal{E})$ be a pair consisting of a smooth variety F and a numerically effective (for example spanned) vector bundle $\mathcal{E}$ such that $-K_F - det\mathcal{E}$ is ample. The bundle $\mathcal{E}$ is called Fano since its projectivisation is a Fano manifold, see [S-W1, S-W2] and Table I. Let then $X := \mathbf{P}(\mathcal{E} \oplus \mathcal{O})$ and let ξ denote the tautological line bundle on X, that is $\mathcal{O}_{\mathbf{P}(\mathcal{E}\oplus\mathcal{O})}(1)$. Let us consider the section of the projective bundle $X \to F$ determined by the surjection $\mathcal{E} \oplus \mathcal{O} \to \mathcal{O} \to 0$, we will call it again F. It is easy to check that $N^*_{F/X} = \mathcal{E}$. Since ξ is nef and $\xi - K_{\mathbf{P}(\mathcal{E}\oplus\mathcal{O})}$ is ample, thus by Kawamata-Shokurov contraction theorem it follows that $m\xi$ is base point free for $m \gg 0$ and it defines a good contraction $\varphi : X \to Z$. In particular $Z = Proj(\bigoplus_{m\geq 0} H^0(F, S^m(\mathcal{E} \oplus \mathcal{O})))$ and if $\mathcal{E}$ is spanned then φ is the connected part of the Stein factorsation of the map given by the linear system $|\xi|$. The map φ contracts F to a point.

Example (3.1.1). To get 4-dimensional birational contractions we consider pairs

$$(\mathbf{P}^2,\ (T\mathbf{P}^2(-1) \oplus \mathcal{O}(1))/\mathcal{O}),\quad (\mathbf{P}^2,\ \mathcal{O}^{\oplus 4}/\mathcal{O}(-1)^{\oplus 2})$$
$$(\mathbf{P}^1 \times \mathbf{P}^1,\ \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)),\quad (\mathbf{P}^2,\ \mathcal{O}(1) \oplus \mathcal{O}(1)).$$

By the theorem of Leray and Hirsch one gets that $\xi^4 = c_1^2(\mathcal{E} \oplus \mathcal{O}) - c_2(\mathcal{E} \oplus \mathcal{O}) > 0$. This implies that the divisor ξ is big so that the map φ is birational. In each of the above cases the map φ will have exactly one 2-dimensional fiber, namely F. If $(F, \mathcal{E}) = (\mathbf{P}^2,\ \mathcal{O}(1) \oplus \mathcal{O}(1))$ then the contraction is small while in each of the remaining cases it will contract a divisor (see also (6.1) and (6.2)). For the pair $(F, \mathcal{E})$ different from $(\mathbf{P}^1 \times \mathbf{P}^1,\ \mathcal{O}(1,0) \oplus \mathcal{O}(0,1))$ the map φ is elementary. Let us also note that X can be described in terms of the rational map $\mathbf{P}(H^0(\mathcal{E})) \supset \varphi(X) - \to F$. For example, if $\mathcal{E} = \mathcal{O}^{\oplus 4}/\mathcal{O}(-1)^{\oplus 2}$ then the map is defined by a net of quadrics which contain the rational twisted cubic in $\mathbf{P}^3$.

Similarly, considering other bundles from (2.6) and (2.8) we get a series of good contractions of fiber type with the special fiber F equal to $\mathbf{P}^2$ or $\mathbf{F}_0$. More precisely, to get an isolated 2-dimensional fiber one has to consider $T\mathbf{P}^2(-1)$ and $\mathcal{O}^{\oplus 3}/\mathcal{O}(-2)$ over $\mathbf{P}^2$ and the pullback of $T\mathbf{P}^2(-1)$ to $\mathbf{F}_0$.

Example (3.1.2). Using the results of [S-W2] one can produce examples of contractions of a manifold $X = \mathbf{P}(\mathcal{E} \oplus \mathcal{O})$, $dimX = n \geq 5$, with an isolated 2-dimensional fiber $F \simeq \mathbf{P}^2$. Birational contractions can be obtained for $n = 5$, 6 if we take $\mathcal{E} = \mathcal{O}^{\oplus n}/\mathcal{O}(-1)^{\oplus 2}$ (which means that $\mathcal{E}(1)$ is the quotient of $\mathcal{O}(1)^{\oplus n}$ divided by two general sections) or $\mathcal{E} = T\mathbf{P}^2(-1) \oplus \mathcal{O}(1)$ for $n = 5$. Fiber type contractions (conic fibrations with an isolated $\mathbf{P}^2$ fiber) will be produced this way if we take $\mathcal{E} = \mathcal{O}^{\oplus n-1}/\mathcal{O}(-2)$ for $n \leq 7$. Similarly, if we take $\mathcal{E} = \mathcal{O}^{\oplus 4}/\mathcal{O}(-1,-1)$ over $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$ then $\mathbf{P}(\mathcal{E} \oplus \mathcal{O})$ has a structure of a conic fibration over $\mathbf{P}^4$ with an isolated 2 dimensional fiber equal to $\mathbf{F}_0$.

(3.2) Complete intersections in projective bundles.

Example (3.2.1). Let $\mathbf{Q}_3$ be a smooth 3-dimensional quadric and let $\mathcal{S}$ be the spinor bundle over $\mathbf{Q}_3$. (As described in section 0 the bundle $\mathcal{S}$ is a rank two vector bundle which is the pull back of the universal bundle under the inclusion $i : \mathbf{Q}_3 \to Gr(1,3)$.) Let us consider the projectivization $M := \mathbf{P}(\mathcal{O} \oplus \mathcal{S}(1))$ with the projection $p : M \to \mathbf{Q}_3$. By ξ let us denote the line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{O}\oplus\mathcal{S}(1))}(1)$. The complete linear system $|\xi|$ gives a contraction of M onto $\mathbf{P}^4$ such that the section corresponding to the trivial factor of $\mathcal{O} \oplus \mathcal{S}(1)$ is contracted to a point by φ and all the other fibers of φ are $\mathbf{P}^1$'s.

Let then X be a smooth divisor in the linear system $|\xi + p^*\mathcal{O}_{\mathbf{Q}^3}(1)|$. Since this linear system is ample $Pic(X)$ is the same as $Pic(M)$ and thus φ restricted to X — call it again φ — is an elementary contraction. Moreover, it is easy to see that $\varphi : X \to \mathbf{P}^4$ is birational and divisorial. Also, it will have a 2-dimensional fiber F which comes from a 3-dimensional fiber of $M \to \mathbf{P}^4$, thus F is a quadric. For a general choice of X the fiber F is a smooth quadric $\mathbf{P}^1 \times \mathbf{P}^1$. Thus we can realise the pair $(F, N_{F/X}) = (\mathbf{P}^1 \times \mathbf{P}^1,\ \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1))$ in an elementary contraction (c.f. the previous series of examples).

Moreover, we claim that there exists a smooth X such that F is the quadric cone $\mathbf{S}^2$. Indeed, let us choose a section $\alpha \in H^0(\mathbf{Q}^3, \mathcal{O}(1))$ which vanishes along a quadric cone with a vertex at w and a section $\beta \in H^0(\mathbf{Q}^3, \mathcal{S}(2))$ which does not vanish at w. Then the section of $\xi + p^*\mathcal{O}(1)$ associated to the section (α, β) of $\mathcal{O}(1) \oplus \mathcal{S}(2)$ is smooth along the 2-dimensional fiber of the contraction. (This can be verified locally — as in the subsequent example.) Thus we can produce an example of the a fiber and its normal being $(\mathbf{S}_2, \mathcal{S}_{|\mathbf{S}_2})$.

Example (3.2.2). Now we extend the above example to the codimension 2 complete intersection in a projective bundle over the smooth 4 dimensional quadric $\mathbf{Q}_4$. This time we consider a spinor bundle $\mathcal{S}$ over the quadric $\mathbf{Q}_4$ and the projective bundle $p : M = \mathbf{P}(\mathcal{O} \oplus \mathcal{S}(1)) \to \mathbf{Q}_4$. Again, let ξ denote the relative hyperplane bundle — the evaluation map associated to the system $|\xi|$ is onto $\mathbf{P}^4$ and contract a section Q_0 to a point $v \in \mathbf{P}^4$. The quadric $\mathbf{Q}^4$ parametrizes planes in $\mathbf{P}^4$ containing v and M is the incidence variety of points on the planes.

Again, we consider the linear system $\Lambda = |\xi + p^*\mathcal{O}(1)|$ over M and X will be a complete intersection of two divisors from Λ. The intersection $F := X \cap Q_0$ is a linear section of the 4-dimesional quadric. The codimension 2 linear section of $Q_0 \simeq \mathbf{Q}^4$ can be either a smooth quadric $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$ or a quadric cone $\mathbf{S}_2$, or a union of two planes meeting along a line. We will show that each of these cases can be realised so that the complete intersection X is smooth. The first two cases were dealt with in the preceeding examples so we focus on the case of the two $\mathbf{P}^2$s. We distinguish the two planes calling them P_1 and P_2, so that $\mathcal{S}(1)_{|P_1} = T\mathbf{P}^2(-1)$ and $\mathcal{S}(1)_{|P_2} = \mathcal{O} \oplus \mathcal{O}(1)$. In particular, over the line $l := P_1 \cap P_2$ the bundle $\mathbf{S}(1)$ decomposes into $\mathcal{O} \oplus \mathcal{O}(1)$.

Sections of $\xi + p^*\mathcal{O}(1)$ are associated to sections of $\mathcal{O}(1) \oplus \mathcal{S}(2)$; the decomposition of the later bundle enables to write the sections in the form (α, β), where $\alpha \in H^0(\mathbf{Q}^4, \mathcal{O}(1))$ and $\beta \in H^0(\mathbf{Q}^4, \mathcal{S}(2))$. For a suitable trivialisation of $\mathcal{S}(2)$ (and thus of $\mathbf{P}(\mathcal{S}(1) \oplus \mathcal{O})$) the zero locus of the section of $\xi + p^*\mathcal{O}(1)$ associated to (α, β) is given by the equation

$$\alpha(x)z_0 + \beta^1(x)z_1 + \beta^2(x)z_2 = 0$$

where $[z_0, z_1, z_2]$ are the associated uniform coordinates in the fiber of p and x is a coordinate in $\mathbf{Q}^4$. Moreover, we may assume that the trivialisation of $\mathcal{S}(2)$ over l coincides with the splitting $\mathcal{S}(2)_l = \mathcal{O}(1) \oplus \mathcal{O}(2)$ and therefore ${\beta^k}_{|l} \in H^0(l, \mathcal{O}(k))$ for $k = 1,\ 2$. In these coordinates the section Q_0 is where $z_1 = z_2 = 0$ and thus in affine coordinates around Q_0 the zero locus is described by the equation

$$\alpha(x) + \beta^1(x)z_1 + \beta^2(x)z_2 = 0.$$

Let α_1 and α_2 be the two sections of $\mathcal{O}_{\mathbf{Q}^4}(1)$ such that their common zero is the union of two $\mathbf{P}^2$'s, i.e. $P_1 \cup P_2$. Then each of α_i's is a quadric cone with the vertex on the line $P_1 \cap P_2$. The partial derivative $\partial\alpha_i/\partial x_j$ vanishes only at the vertex of the cone defined by the section α_i. Therefore, the Jacobi matrix $(\partial\alpha_i/\partial x_j)$ (where (x_j) are local coordinates on $\mathbf{Q}^4$) is of rank one along $l = P_1 \cap P_2$. Now we choose two sections, β_i with $i = 1,\ 2$, of $\mathcal{S}(2)$ such that over l they coincide with the splitting, that is $\beta_{1|l} = (\beta_1^1, 0)$ and $\beta_{2|l} = (0, \beta_2^2)$, where $\beta_i^i \in H^0(l, \mathcal{O}(i))$. (We can make such a choice because the restriction map $H^0(\mathbf{Q}^4, \mathcal{S}(2)) \to H^0(l, \mathcal{S}(2)_{|l})$ is surjective — which is easy to verify e.g. by considering cohomology.) Moreover, we may assume that the zero set of the section β_i^i does not include the vertex of the cone defined by α_i. And we consider two sections $f_i \in H^0(M, \xi + p^*\mathcal{O}(1)) \simeq H^0(Q^4, \mathcal{O}(1) \oplus \mathcal{S}(2))$ which locally are of the form

$$f_1 := \alpha_1(x)z_0 + \beta_1 z_1 \qquad f_2 := \alpha_2(x)z_0 + \beta_2 z_2$$

Now we can compute the matrix of derivatives $(\partial f_i/\partial x_j, \partial f_i/\partial z_k)$ in local coordinates (x_j, z_1, z_2) around $l \subset Q_0$. The result evaluated on l is the matrix:

$$\begin{pmatrix} \frac{\partial\alpha_1}{\partial x_j} & \beta_1^1 & 0 \\ \frac{\partial\alpha_2}{\partial x_j} & 0 & \beta_2^2 \end{pmatrix}$$

Because of our assumption on the zero locus of β_i^i it follows that the above matrix is of rank 2 everywhere on l. Therefore, the complete intersection of divisors defined by f_i's is smooth.

Remark 1. *Discussion of normal bundles of P_i's in X.*

The conormal of the total fiber $\mathcal{I}_{P_1\cup P_2}/\mathcal{I}^2_{P_1\cup P_2}$ is the restriction of the spinor bundle; in particular, its restriction to P_1 is $T\mathbf{P}^2(-1)$ and to P_2 is $\mathcal{O} \oplus \mathcal{O}(1)$. Using Lemma 2.2 one proves that the conormal of P_1 and P_2 is, respectively, a stable bundle with $c_1 = 2$ $c_2 = 4$ which is spanned outside of l (see [S-W1]) and, respectively, an unstable, semistable bundle with $c_1 = 2$, $c_2 = 3$. In both cases l is the unique jumping line with the splitting type $\mathcal{O}(-1) \oplus \mathcal{O}(3)$. (For further discussion see Section 6.)

Remark 2. *Digression on the geometry of surfaces in $\mathbf{P}^4$.*

The above series of examples with special fiber a two dimensional quadric can be described from the point of view of the geometry of surfaces in $\mathbf{P}^4$. Namely, in each of the cases X is a blow up of $\mathbf{P}^4$ along a surface S with an isolated singularity at the point v. Equivalently, X is a graph of a rational map $\mathbf{P}^4 - \to \mathbf{Q}^s \subset \mathbf{P}^{s+1}$ for $s = 2,\ 3,\ 4$ (respectively for (3.1.1), (3.2.1), (3.2.2)). The resolution of the sheaf of ideals $\mathcal{I}_S$ is as follows:

$$0 \longrightarrow \mathcal{O}(-s-2) \longrightarrow \mathcal{O}(-s-1)^{\oplus(s+2)} \longrightarrow \mathcal{O}(-s)^{\oplus(s+2)} \longrightarrow \mathcal{I}_S \longrightarrow 0.$$

If $s = 2$ then S consists of two planes meeting transversally at v and the above sequence can be computed directly. In the other two cases this can be computed by looking at S as degeneracy locus of sections of sheaves. Namely, the projective bundle M (over $\mathbf{Q}^3$ or $\mathbf{Q}^4$, respectively) in which we consider the complete intersection has a projective bundle structure over $\mathbf{P}^4$ outside of the point v. If we pull the sheaf $\mathcal{O}(1)$ from the quadric to M and then we push it forward to $\mathbf{P}^4$ then the resulting sheaf, call it $\mathcal{F}$, will have an isolated singularity at v and outside of v it will be locally free. The sheaf $\mathcal{F}$ is of rank 2 or 3 and S is cut out by its 1 or 2 sections of $\mathcal{F} \otimes \mathcal{O}(1)$, respectively. Moreover, since M comes from a flag variety, one finds out easily that the restriction of $\mathcal{F}$ to any linear $\mathbf{P}^3 \subset \mathbf{P}^4$ which does not contain v is equal to $\mathcal{N}(1)$ ($\mathcal{N}$ denoting the well known null-correlation bundle on $\mathbf{P}^3$) or $\Omega_{\mathbf{P}^3}(2)$, respectively. This provides enough information to get the above resolution and more. In particular one finds that the degree of S is 5 and 9, and the ideal I_S is defined by five cubics and six quartics, respectively. Using *CoCoa* program [CC] for symbolic computations we found the following example of the ideal $I_S = (txy - x^2y - txz + x^2z - t^2u, tx^2 - xyz + xz^2 - xyu + tzu, t^3 - ty^2xy^2 - xyz + tz^2, t^2x - xyz + xz^2 - tyu, txy + t^2z - y^2z + z^3 - y^2u)$, where $[x, y, z, t, u]$ are homogeneus coordinates and $v = [0, 0, 0, 0, 1]$. If we blow up the ideal I_S in $\mathbf{P}^4$ we obtain a special fiber of dimension two which is the quadric cone $\mathbf{S}_2$. The reader may easily verify that the tangent cone of S at v gives a double line scheme.

(3.3) **A conic bundle with a special fiber $\mathbf{S}_3$.**
Let $\beta : V \to \mathbf{P}^3$ be the blow-up of $\mathbf{P}^3$ at a point x_0 with the exceptional divisor V_0 and another projection $\alpha : V \to \mathbf{P}^2$ which makes V the $\mathbf{P}^1$ bundle $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(1))$. Over V we consider the pull-back bundle $\mathcal{E} := \alpha^*(\mathcal{O}^4/\mathcal{O}(-1)^2)$ and its projectivization $\pi : W = \mathbf{P}(\mathcal{E}) \to V$. Over W we have the relative hyperplane section bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$, which we denote by ξ. The bundle ξ generates *PicW* together with $H := (\alpha\pi)^*(\mathcal{O}(1))$ and $D := (\beta\pi)^*(\mathcal{O}(1))$. Let us note that all the above three line bundles are spanned and they define maps onto $\mathbf{P}^4$, $\mathbf{P}^2$ and $\mathbf{P}^3$, respectively. Moreover, we can compute that $-K_W = 3\xi + 2D$.

Let $W_0 := (\beta\pi)^{-1}(x_0)$, W_0 is the unique effective divisor in $|D - H|$. We note that W_0 is a 4-fold whose case was discussed in a previous series of examples: it has a structure of a $\mathbf{P}^2$ bundle over $\mathbf{P}^2$ and another good contraction supported by the divisor ξ which contracts an irreducible divisor $Y_0 \subset W_0$ to a cone $\mathbf{S}_3 \subset \mathbf{P}^4$. The divisor Y_0 is the unique effective divisor in W_0 equivalent to the restriction of $2\xi - H$. A section of $W_0 \to \mathbf{P}^2$ is contracted to the vertex of $\mathbf{S}_3$, call it Π_0. Let us note that Y_0 is smooth outside of Π_0. The set $Y_0 \subset W_0$ is contracted by the birational map supported by $a\xi + bD$ where a, $b > 0$. Moreover, let us note that $N^*_{\Pi_0/W} = (\mathcal{O}^4/\mathcal{O}(-1)^{\oplus 2}) \oplus \mathcal{O}(1)$.

Claim . Y_0 extends to a smooth divisor $Y \subset W$ such that $Y \in |2\xi - H + mD|$ where $m \gg 0$.

Proof. Let us note that the linear system $|2\xi - H + mD|$ is base point free outside of W_0 for $m \gg 0$. On the other hand, from the sequence

$$0 \to (2\xi + (m-1)D) \to (2\xi - H + mD) \to (2\xi - h + mD)_{|W_0} \to 0$$

and from Kawamata-Viehweg vanishing, it follows that Y_0 extends. If Y is an extension of Y_0 then since $Y_0 = Y \cap W_0$ we have the smoothness of Y at the smooth points of Y_0, that is outside of Π_0. Thus, since smoothnes is an open property, we will be done if we show a divisor in $|2\xi - H + mD|$ which is smooth along Π_0. To this end, note that it is enough to take a reducible $Y' = W_0 + Y''$ such that the divisor $Y'' \in |2\xi + (m-1)D|$ does not meet Π_0. The existence of such Y'' is clear since $2\xi + (m-1)D$ is trivial on Π_0.

Thus we can produce a smooth conic fibration $\psi = (\beta \circ \pi)_{|Y} : Y \to \mathbf{P}^3$ with a non-normal divisorial fiber Y_0 over x_0 such that $-K_Y = \xi + H + m'D$. Now a good supporting divisor $\xi + D$ defines a divisorial contraction of $\rho : Y \to X \subset \mathbf{P}^4 \times \mathbf{P}^3$ of Y over $\mathbf{P}^3$. The contraction ρ contracts the divisor Y_0 to the cone $\mathbf{S}_3 = X \cap (\mathbf{P}^4 \times \{x_0\})$. Thus X is smooth and the resulting contraction $\varphi : X \to \mathbf{P}^3$ is a conic fibration with an exceptional fiber $\mathbf{S}_3$.

Remark. A different, simpler and very nice construction of a conic bundle with the special fiber $\mathbf{S}_3$ was given by N. Shepherd-Barron. The example is reported in a paper of Y. Kachi (see [Kac], example (11.6)).

(3.4) **Blow-ups, blow-downs.**
A very convenient way to produce a conic fibration is to alter another fibration. We will use this method to produce non-elementary fibrations also with reducible fibers. The fibration which will be the base of the construction is either a simple $\mathbf{P}^1$-bundle or a non-equidimensional 4-dimensional scroll $\psi : Y \to Z$ with an exceptional fiber $V \simeq \mathbf{P}^2$, as dealt with in [A-W, Remark (4.12)], and [B-W].
Example (3.4.0) A 4 dimensional scroll with an exceptional fiber $\mathbf{P}^2$.
Our favorite example of such a scroll comes from incidence construction (c.f. 3.2.2)): we set $Z := \mathbf{P}^3$ and we fix a point $v \in \mathbf{P}^3$. Then we consider the incidence variety

$$Y := \{(z, \Pi) \in \mathbf{P}^3 \times Grass(\mathbf{P}^2, \mathbf{P}^3) \colon z \in \Pi \text{ and } v \in \Pi\}$$

with the projections $\psi \colon Y \to Z = \mathbf{P}^3$ and $\pi \colon Y \to \mathbf{P}^2$. The π makes Y a projective bundle $\mathbf{P}(\mathcal{O} \oplus T\mathbf{P}^2(-1))$ (c.f.[A-W],(4.12)) while ψ is the scroll with a unique 2-dimensional fiber $V \simeq \mathbf{P}^2$ over v. If (z_0, z_1, z_2) are coordinates in the affine neighbourhood of $v = (0,0,0)$ and $[t_0, t_1, t_2]$ are homogenous coordinates in the $\mathbf{P}^2$ then the equation of Y in $\mathbf{C}^3 \times \mathbf{P}^2$ is $t_0z_0 + t_1z_1 + t_2z_2 = 0$ (this is just a duality pairing between the plane at infinity of $\mathbf{C}^3$ with its dual).

Example (3.4.1) A conic fibration with exceptional fiber $\mathbf{F}_1$.
Let $S \subset Y$ be a smooth surface meeting V transversaly at one point and the other fibers of ψ at one point (transversaly!) at most. Then the blow-up of Y along S: $\alpha : X \to Y$ with the morphism

$$\varphi := \psi \circ \alpha \colon X \longrightarrow Z$$

is a conic bundle with discriminant divisor $\Delta = \psi(S)$ and a special fiber over v isomorphic to $\mathbf{F}_1$. Let us note that X admits another good contraction β which is a simple blow-down map of X to a $\mathbf{P}^1$-bundle over Z. In local coordinates $S = \{([t_0, t_1, t_2], (z_0, z_1, z_2)) : z_0 = t_1 = t_2 = 0\}$, $\beta \circ \alpha^{-1}$ is the projection from S and its inverse $\alpha \circ \beta^{-1}$ can be described as follows:

$$\mathbf{P}^1 \times \mathbf{C}^3 \ni ([\tau_1, \tau_2], (z_0, z_1, z_2)) \mapsto ([-\tau_1 z_1 - \tau_2 z_2, \tau_1 z_0, \tau_2 z_0], (z_0, z_1, z_2)) \in V.$$

The exceptional set of $\alpha \circ \beta^{-1}$ is the simple blow-up of the surface Δ at v, that is $\{([\tau_1, \tau_2], (z_0, z_1, z_2) : z_0 = \tau_1 z_1 + \tau_2 z_2 = 0\}$.

Example (3.4.2) A conic fibration with exceptional fiber $\mathbf{F}_0$.
Let us consider a rational map $\gamma : \mathbf{P}^1 \times \mathbf{C}^3 - \rightarrow \mathbf{P}^1 \times \mathbf{C}^3$ given by the formula

$$\gamma([\tau_1, \tau_2], (z_0, z_1, z_2)) = ([\tau_1 z_0 + \tau_2 z_2,\ \tau_1 z_1 + \tau_2 z_0], (z_0, z_1, z_2)).$$

The inverse of γ is $([\tau_1, \tau_2], (z_0, z_1, z_2)) \mapsto ([\tau_1 z_0 - \tau_2 z_2,\ -\tau_1 z_1 + \tau_2 z_0], (z_0, z_1, z_2))$ and the exceptional set of each of these two maps is a resolution of the quadric singularity $\{z_0^2 = z_1 z_2\}$. If X is the resolution of γ then it is smooth and the map $X \rightarrow \mathbf{C}^3$ is a conic fibration with the discriminant $\Delta = \{(z_0, z_1, z_2) : z_0^2 = z_1 z_2\}$ and an exceptional 2-dimensional fiber equal to $\mathbf{F}_0$.

Example (3.4.3) A conic fibration with exceptional fiber $\mathbf{F}_1 \cup \mathbf{P}^2$.
Let $S \subset Y$ be a smooth surface meeting V along a line and such that $\psi_{|S}$ is a blow-down of a (-1) curve. For example, in the local coordinates $([t_0, t_1, t_2], (z_0, z_1, z_2))$, which we introduced above, we can take S given by equations $t_0 = z_0 = 0$. Then the blow-up of Y along S, $\alpha : X \rightarrow Y$, with morphism $\varphi := \psi \circ \alpha$ is a conic fibration with discriminant divisor $\Delta = \psi(S)$ and special fiber over v isomorphic to $\mathbf{F}_1 \cup \mathbf{P}^2$.

Alternatively, X is the closure of the graph of a rational map of Y:

$$Y \ni ([t_0, t_1, t_2], (z_0, z_1, z_2)) \mapsto ([t_0, t_1 z_0, t_2 z_0], (z_0, z_1, z_2)) \in W$$

where $W \subset \mathbf{P}^2 \times \mathbf{C}^3$ is given by an equation $t_0 z_0^2 + t_1 z_1 + t_2 z_2 = 0$ and thus it has an isolated quadric cone singularity at $([1, 0, 0], (0, 0, 0))$. The contraction $X \rightarrow W$ has an isolated two dimensional fiber $\simeq \mathbf{P}^2$ with normal $(T\mathbf{P}^2(-1) \oplus \mathcal{O}(1))/\mathcal{O}$. The exceptional set of the inversed rational map is the smooth surface $\{z_0 = t_1 = t_2 = 0\} \subset W \subset \mathbf{P}^2 \times \mathbf{C}^3$.

Example (3.4.4) A conic fibration with exceptional fiber $\mathbf{F}_0 \cup \mathbf{P}^2$.
A similar argument as in the previous example leads to a conic fibration with an exceptional fiber $\mathbf{F}_0 \cup \mathbf{P}^2$. In this case, however, we choose Δ having a quadric cone singularity at v so that the map $\psi_{|S} : S \rightarrow \Delta$ is a contraction of a (-2) curve. Namely, let us consider S given in $Y \subset \mathbf{P}^2 \times \mathbf{C}^3$ by equations $t_1 = t_0 z_0 + t_2 z_2 = t_2 z_0 + t_0 z_1 = 0$ (c.f. (3.4.2)). Then S is a resolution of $\Delta = \{(z_0, z_1, z_2) : z_0^2 = z_1 z_2\}$ and the blow-up of Y along S is a conic fibration over Z with an exceptional fiber $\simeq \mathbf{F}_0 \cup \mathbf{P}^2$.

Similarly as before one can describe X as a graph of a rational map. Let us consider $\gamma : Y - \rightarrow \mathbf{P}^1 \times \mathbf{C}^3$ such that $\gamma([t_0, t_1, t_2], (z_0, z_1, z_2)) = ([t_0 z_1 + t_2 z_0, t_1], (z_0, z_1, z_2))$. Then the inverse of γ is as follows

$$([\tau_0, \tau_1], (z_0, z_1, z_2)) \mapsto ([\tau_0 z_2 + \tau_1 z_0 z_1, \tau_1(z_1 z_2 - z_0^2), -\tau_0 z_0 - \tau_1 z_1^2], (z_0, z_1, z_2)).$$

The exceptional set of γ is the smooth surface S which we have just described above, while the exceptional set of γ^{-1} is a surface with an isolated singularity at $([0, 1], (0, 0, 0))$ which is of the cubic cone type. Thus X has two elementary contractions: a simple blow-down to Y and the birational contraction to $\mathbf{P}^1 \times \mathbf{C}^3$ with an exceptional $\mathbf{P}^2$ whose conormal is $\mathcal{O}^{\oplus 4}/\mathcal{O}(-1)^{\oplus 2}$.

Example (3.4.5) Conic fibrations with (non-isolated) 2-dim fibers equal to $\mathbf{P}^2 \cup \mathbf{F}_2$ and $\mathbf{P}^2 \cup \mathbf{F}_1 \cup \mathbf{P}^2$.
In Y or W, which we have discussed above, we consider a smooth surface $S = \{z_0 = t_1 = z_2 = 0\}$. Then the blow-up of either Y, or respectively, W along S is smooth and the induced map $\varphi_Y : X_Y \to \mathbf{C}^3$ (resp. $\varphi_W : X_W \to \mathbf{C}^3$) is a good contraction. The fibers of these contractions are of the following type: $\varphi_Y^{-1}((0,0,0)) = \mathbf{P}^2 \cup \mathbf{F}_2$, $\varphi_W^{-1}((0,0,0)) = \mathbf{P}^2 \cup \mathbf{F}_1 \cup \mathbf{P}^2$, $\varphi_Y^{-1}(0,z_1,0)) = \varphi_W^{-1}(0,z_1,0) = \mathbf{F}_0$ for $z_1 \neq 0$ and all the other fibers are $\mathbf{P}^1$. We note that the exceptional fiber is a limit of two dimensional fibers and apart of the strict tranform of the special fiber of the initial scroll, it contains the specialization of the pair $(\mathbf{F}_0, C_0 + 2f)$. We also note that in both cases the contraction factors through a small contraction which contracts the strict tranform of the special fiber of the initial scroll.

(3.5) **Double coverings.** Let $\psi : Y \to Z'$ be a good contraction of a smooth variety Y. Assume that L is a ψ-ample line bundle and $-K_Y - 2L = \psi^*(L')$ for some line bundle L' over Z'. If $B \in |2L|$ is a smooth divisor then we can construct a double covering $\pi : X \to Y$ which is branched along B (see for instance [B-P-V], pp.42–43); the variety X is then smooth and $-K_X = \pi^*(L + \psi^*(L'))$ so that $-K_X$ is $\psi \circ \pi$-ample.

If ψ is of fiber type then fibers of $\varphi := \psi \circ \pi$ are connected and thus $\varphi : X \to Z'$ is a good contraction. If ψ is birational then $\psi \circ \pi$ is generically $2:1$. In this case however, the connected part of the Stein factorisation of $\psi \circ \pi$, which we denote by $\varphi : X \to Z$, is a good contraction. The finite part of the Stein factorisation $\pi' : Z \to Z'$ is a double covering branched along $\psi(B)$.

Let us note that a similar construction with L such that $K_X + L = \psi^*(L')$ leads from a good birational contraction ψ to a crepant contraction $\varphi : X \to Z$. The first example concerns this case.

Example (3.5.1) A crepant elementary divisorial contraction of a smooth 3-fold.

Let us consider a product $Y = \mathbf{F}_1 \times \mathbf{C}$ with projections $p_1 : Y \to \mathbf{F}_1$, $p_2 : Y \to \mathbf{C}$. Y admits a good contraction $\psi : Y \to \mathbf{P}^2 \times \mathbf{C}$ supported by $p_1^*(C_0 + f)$ which is a simple blow-down of the exceptional divisor $C_0 \times \mathbf{C}$. Let $L := p_1^*(C_0 + 2f)$. We claim that there exists a smooth divisor $B \in |2L|$ such that $B_t := B \cap \mathbf{F}_1 \times \{t\}$ is a smooth curve of genus 2 for general t and $B_0 = C_0 \cup C_1$ where $C_1 \in |C_0 + 4f|$ is a smooth rational curve meeting C_0 transversally at 3 points. This follows from a general

Lemma (3.5.2). *Let Λ be a base-point-free linear system on a smooth variety X. Let $D_0 \in \Lambda$ be a divisor which is smooth except finite number of points. Then there exists a linear pencil of divisors $\{D_\lambda : \lambda \in \mathbf{P}^1 \subset \Lambda\}$ which contains D_0 and such that the divisor $D_{\mathbf{P}^1} := \bigcup_\lambda D_\lambda$ in $X \times \mathbf{P}^1$ is smooth.*

Proof. Let $D_\Lambda \subset X \times \Lambda$ be the universal divisor (incidence variety); locally D_Λ is defined by a function $f(x, \lambda)$ with variables x and λ being coordinates in X and Λ, respectively. Since Λ is base-point-free it follows that D_λ is non-singular (it has projective bundle structure over X) and thus the vector of partial derivatives $(\partial f/\partial x, \partial f/\partial \lambda)$ is nowhere zero on D_Λ. If D_{λ_0} is singular at x_0 then $\partial f/\partial x$ vanishes at (x_0, λ_0). However in such a situation $\partial f/\partial \lambda$ does not vanish so thus if we choose a linear pencil D_λ which contains

λ_0 but is not contained in the kernel of $\partial f/\partial\lambda$ then the resulting divisor $\bigcup_\lambda D_\lambda$ will be smooth at (x_0, λ_0). Now the lemma follows easily.

Example (3.5.1), continued. Let $\varphi : X \to Z$ be a crepant contraction obtained from $\psi : Y \to \mathbf{P}^2 \times \mathbf{C}$ with a double covering $\pi : X \to Y$ branched along B. Note that Z admits a morphism onto $\mathbf{C}$ so that fiberwise we have a family of maps $\varphi_t : X_t \to Z_t$ parametrized by $t \in \mathbf{C}$. For a general t the surface Z_t is a double cover of $\mathbf{P}^2$ branched along a quartic which has a simple double point and $\varphi_t : X_t \to Z_t$ is a resolution of the resulting A_1 singularity. Let E denotes the exceptional divisor of the map φ and by E_t let us denote the reduced fiber $E \cap X_t$. We claim that

(a) for any t, $E_t \simeq \mathbf{P}^1$, $E.E_t = -2$ for a general t and $E.E_0 = -1$,
(b) E is non-normal along E_0 and smooth elsewhere,
(c) the normal bundle of E_t in X is $\mathcal{O} \oplus \mathcal{O}(-2)$ for a general t and $\mathcal{O}(1) \oplus \mathcal{O}(-3)$ for $t = 0$.

The first two properties follow easily from the construction while the third one comes from

Lemma (3.5.3). *Let $\varphi : X \to Z$ be a crepant divisorial contraction of a smooth 3-fold with the exceptional irreducible divisor E which is contracted to a curve $C \subset Z$. Let 0 be a fixed point on C. For any point $t \in C$ let E_t denote the fiber $\varphi^{-1}(t)$ with the reduced structure. Then the property (a) above yields property (c).*

Proof. We are to prove that the normal of E_0 in X is $\mathcal{O}(1) \oplus \mathcal{O}(-3)$. By (5.6.2) it is enough to exclude possibility that the normal is either $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{O} \oplus \mathcal{O}(-2)$. Let us blow-up X along E_0 and call the resulting variety $\hat{X}$, the exceptional divisor by A and the strict transform of E by $\hat{E}$. In $\hat{X}$ we have a family of effective 1-cycles E_t parametrized by $C \setminus \{0\}$. By $\hat{E}_0$ let us call the limit cycle of E_t as $t \to 0$. Then $\hat{E}_0$ is an effective cycle, it is supported on the set $\hat{E} \cap A$ and $A.\hat{E}_0 = 0$. This last equality alone excludes the possibility of the normal $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ — indeed, in this case the divisor $-A$ is ample on A, so it has positive intersection with any effective cycle on A. If the normal of E_0 were $\mathcal{O} \oplus \mathcal{O}(-2)$ then $-A$ would be nef on $A = F_2$ and $\hat{E}_0$ would be supported on the unique curve $C_0 \subset F_2$ whose intersection with $-A$ is zero. But then $\hat{E}_{|A}$ would be supported on C_0 so that

$$\gamma C_0 = \hat{E}_{|A} = -\alpha A + (E.E_0)f = \alpha C_0 + (2\alpha + E.E_0) \cdot f$$

where α and γ would be positive integers. Since $E.E_0 = -1$ this is impossible.

Example (3.5.4). Divisorial elementary contraction of a 4-fold with quadric fibers.

This is a 4-dimensional version of the previous example: let $V_1 \to \mathbf{P}^3$ be the blow-up of $\mathbf{P}^3$ at one point, by S_0 let us denote the exceptional divisor of the blow-up. V_1 has a $\mathbf{P}^1$-bundle structure, $V_1 = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbf{P}^2$, by H let us denote the pull-back of the line. Let us consider a product $Y := V_1 \times \mathbf{C}$ with projections p_1, p_2 onto factors. Over Y we have a line bundle $L := p_1^*(S_0 + 2H)$. The data consisting of the contraction morphism $\psi : Y \to \mathbf{P}^3 \times \mathbf{C}$, and ψ-ample line bundle L can be plugged to the construction described above as soon as we provide a smooth divisor B in $|2L|$. The resulting elementary contraction is divisorial and the exceptional divisor E is contracted to $\mathbf{C}$. From the theory of 3-dimensional good contration [Mo1] we know that a general fiber E_t of $E \to \mathbf{C}$ is either a smooth quadric or a quadric cone. The singularity of a special fiber E_0 will depend on the singularity of $B_0 := B \cap (S_0 \times \{0\})$. In particular, if we find a smooth B such that this intersection is a double line then E_0 will be a reducible quadric, that is, a union of two planes meeting along a line. The construction of such a B can be done as follows: first, using the notation introduced in [Wi, p.154] an arguing as in [ibid, pp.156–157] we can prove that there exists a divisor B_0 on V_1 such that: $B_0 \in 2S_0 + 4H$, $B_0 \cap S_0$ is a double line and B_0 has only isolated singularities. Then using lemma (3.5.2) we can extend B_0 to a smooth B on $V_1 \times \mathbf{C}$.

Example (3.5.5). Conic fibrations.
A similar argument will enable us to construct different conic fibration: also non-equidimensional and also with reducible 2-dimensional fibers. If we use a $\mathbf{P}^1$ bundle as the base of our covering then the resulting contraction will be equidimensional i.e. a conic bundle. If however, we begin with the scroll from (3.4) then we get a contraction with an isolated 2 dimensional fiber. Moreover, according to the choice of the branching divisor, the special fiber will be one of the following:

(i) a smooth quadric if the branching divisor intersects the 2-dimensional fiber (i.e. $\mathbf{P}^2$) along a smooth conic,
(ii) a quadric cone if the intersection is a reducible conic,
(iii) two planes if the intersection is a double line.

It can be shown that there exists no smooth branch divisor which contains the 2-dimensional fiber (so that the case of a "double $\mathbf{P}^2$" can not be produced this way).

(3.6) **Toric examples.** An especially nice class of examples comes with the geometry of toric varieties. We refer the reader to Oda [Od], Danilov [Da] or Fulton [Fl] for the language and notation of toric geometry.
Example (3.6.1). Blow-up of two transversal planes meeting at a point (see also (3.1) and the Remark 2 in (3.2)).

The blow-up of two transveral planes in $\mathbf{C}^4$ can be realised as follows. Let $N_{\mathbf{R}}$ be a 4-dimensional real vector space with a basis $\{e_1,\ e_2,\ e_3,\ e_4\}$ and let $N = \mathbf{Z}e_1+\mathbf{Z}e_2+\mathbf{Z}e_3+\mathbf{Z}e_4$ be a lattice in $N_{\mathbf{R}}$. Then we can subdivide the cone spanned by the basis by adding two vertices $u := e_1 + e_2$, $v := e_3 + e_4$ and considering the following 4 cones

$$\langle u, v, e_1, e_3\rangle, \quad \langle u, v, e_1, e_4\rangle, \quad \langle u, v, e_2, e_3\rangle, \quad \langle u, v, e_2, e_4\rangle$$

where $\langle \ldots \rangle$ denotes the cone spanned by appropriate vectors. The resulting fan describes a toric variety X together with a contraction onto $\mathbf{C}^4$ which is the blow-up along two transversal planes defined (in suitable coordinate system) by equations $z_1 = z_2 = 0$ and $z_3 = z_4 = 0$.

The projection of the space N_R along the linear subspace spanned on u and v given by a matrix

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

maps the above defined fan onto a fan giving $\mathbf{P}^1 \times \mathbf{P}^1$. Actually, the reader may want to verify the resulting construction describes the closure of the the graph of the rational map $\mathbf{C}^4 \to \mathbf{P}^1 \times \mathbf{P}^1$ defined as follows

$$(z_1, z_2, z_3, z_4) \to ([z_1, z_2],\ [z_3, z_4])$$

The map onto $\mathbf{P}^1 \times \mathbf{P}^1$ makes X the total space of the bundle $\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)$ as described in (3.1).

Example (3.6.2). Conic fibration which a 2-dimensional fiber which consists of two planes meeting in one point. (A non-toric version of this example was presented in [Kac]; recently J. Włodarczyk exdended this example to higher dimensions.)

Again, we consider a 4-dimensional vector space $N_{\mathbf{R}}$ with a fixed basis $\{e_1,\ e_2,\ e_3,\ e_4\}$ and the lattice N as above. We take the following vectors in $N_{\mathbf{R}}$:

$$v_1 = -e_1 - e_3, \quad v_2 = -e_2 - e_3 - e_4$$

and consider a fan in $N_{\mathbf{R}}$ with vertices in $e_1,\ v_1,\ e_2,\ e_4,\ -e_4$ containing the following 5 simpleses:

$$\langle e_1, v_1, e_2, e_4 \rangle, \quad \langle e_1, v_1, v_2, e_4 \rangle, \quad \langle e_1, v_1, e_2, v_2 \rangle, \quad \langle e_1, e_2, v_2, -e_4 \rangle, \quad \langle v_1, e_2, v_2, -e_4 \rangle.$$

The toric variety associated to this fan is smooth. Now we consider the projection of $N_{\mathbf{R}}$ along the last coordinate to a 3-dimensional vector space $N'_{\mathbf{R}}$ with a basis $\{e'_1,\ e'_2,\ e'_3\}$. The projection maps the above fan into a cone spanned by $e'_1,\ e'_2,\ -e'_1 - e'_3,\ -e'_2 - e'_3$, which is a fan of the affine quadric cone (i.e. the affine cone over $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$). We consider the induced map of toric varieties $X \to X'$. All fibers of this map except the one over the vertex of the cone are $\mathbf{P}^1$'s. The fiber over the vertex of the quadric cone consits of two 2-dimensional orbits associated to simpleses $\langle e_,\ v_1 \rangle$ and $\langle e_2,\ v_2 \rangle$. Each of these two components is isomorphic to $\mathbf{P}^2$ and they meet at a point which is the orbit associated to $\langle e_1,\ v_1,\ e_2,\ v_2 \rangle$. Moreover, let us note that the divisors D_1 and D_2 associated to cones $\langle e_4 \rangle$ and $\langle -e_4 \rangle$ map to X' so that we get the usual toric flop $D_1 \to X' \leftarrow D_2$.

4. Geometric fiber.

(4.0) Let $\varphi : X \to Z$ be a contraction of a smooth n-fold X. We assume that φ is either a good contraction or a crepant contraction. If φ is a good contraction we fix a relatively ample line bundle $L := -K_X$. In the present section we want to study the geometric structure of a positive dimensional fiber F of φ. The fiber F will be considered with a reduced structure. We will say that F is an *isolated* fiber of dimension k if $\dim F = k$ and all fibers of φ in some neighbourhood of F have dimension $< k$.

Let us start with the following well known description of 1-dimensional fibers of good contractions (see [Mo1] and [An]).

Proposition (4.1) *Let $\varphi : X \to Z$ be a good contraction of a smooth variety. Suppose that a fiber F of φ contains an irreducible component of dimension 1; then F is of pure dimension 1 and all components of F are smooth rational curves.*

(1) If φ is birational then F is irreducible and it is a line relatively to L, moreover the scheme theoretic fiber structure on F is reduced.

(2) If φ is of fiber type then F is a conic relatively to L, that is: either

(i) F is a smooth $\mathbf{P}^1$ and $L \cdot F = 2$, or

(ii) F is a union of two smooth rational curves meeting at one point and each of these curves is a line with respect to L, or

(iii) F is a smooth $\mathbf{P}^1$, $L \cdot F = 1$ and the fiber structure on F is of multiplicity 2 (a reduced conic).

In the cases (i) and (ii) the fiber structure is reduced.

In each one of the above cases the variety Z is smooth at $\varphi(Z)$.

Proof. Let F be the fiber in question. We are to prove first that F is of pure dimension 1. Let us consider an irreducible 1-dimensional component C of F. Lemma (1.2.1) implies that $h^1(\mathcal{O}_C) = 0$ and thus $C \simeq \mathbf{P}^1$. By deformation arguments (see (1.4.1)) we have that $dim_{[C]}Hilb(X) \geq n - 3 - K_X.C = n - 3 + L.C$ and therefore the deformations of C sweep at least a divisor. Moreover if $L.C \geq 2$ then C must move in a $n-1$ dimensional family and therefore φ cannot be birational. On the other hand the deformation locus meets the other components of F along a subset of C and thus the components of F which meet C have to intersect the deformation locus at a finite number of points hence they are of dimension 1. Because the fiber is connected this applies to any component of F and thus we have that F is of pure dimension 1. The configuration of curves in the fiber was described in the proposition (1.5.1). From deformation theory, as above using (1.4.1), we know that F moves in a family of dimension $L \cdot F + n - 3$ at least. Thus we see that $L \cdot F = 1$ in the birational case and $L \cdot F \leq 2$ in the fiber type case. The description of the morphism (smoothness of Z fiber, structure of non-reduced fibers) requires either studying Hilbert scheme of fibers (we do not consider it now, see [An]) or one can use the results in (5.6) and in (2.4).

(4.2) We now pass to the case where F has dimension two. That is, $\varphi : X \to Z$ is a good contraction of a smooth n-fold with a fiber F of (pure) dimension 2. Since the target Z may be assumed affine and it can be shrunk, if necessary, we may assume that all fibers of φ are of dimension ≤ 2. As in (4.0) $L := -K_X$ and we assume it is φ-spanned by global sections; this is true in the birational case by the proposition (1.3.3) and in the conic fibration case if $n \leq 4$ by a result of Kachi-Kawamata (see [Kac]). Let us also note that some of the subsequent results which depend on the vanishing (1.2) remain true if we allow X to have log terminal singularities. This concerns e.g. (4.2.1) and (4.3.3).

Proposition (4.2.1). *In the above hypothesis any component F' of F is normal. The pair $(F', L_{|F'})$ has Fujita Δ-genus 0 and it is among the following:*
(1) $(\mathbf{P}^2, \mathcal{O}(e))$, *with* $e = 1, 2$,
(2) $(\mathbf{F}_r, C_0 + kf)$ *with* $k \geq r+1$, $r \geq 0$,
(3) $(\mathbf{S}_r, \mathcal{O}_{\mathbf{S}_r}(1))$ *with* $r \geq 2$.

Proof. The line bundle $L' = L_{|F'}$ is base point free and for any section C of $L_{|F'}$ we have $g(C) := h^1(C, \mathcal{O}_C) = 0$. This is a consequence of lemma (1.2.2) since $C = F' \cap D$, where D is is a section of L (see (1.3.2)). Thus the proposition follows from the following non-normal version of a very well known characterization of projective normal surfaces with sectional genus 0 (see [Fu]).

Proposition (4.3). *Let F' be an irreducible (reduced) variety of dimension 2 and let L' be an ample and spanned line bundle such that for* **any** *$C \in |L'|$ we have $g(C) =: h^1(\mathcal{O}_{|C}) = 0$. Then F' is normal, L' is very ample and the pair (F', L') has delta genus zero, $\Delta(F', L') = 0$ thus the pair (F', L') is one of the pairs in the proposition (2.3).*

Remark (4.3.1). The proposition is not true if we only assume that $g(C) = 0$ for a general C. For instance take the variety obtained by identifying two points on $\mathbf{P}^n$ (remark at p. 30 of [Fu]).

A similar result holds for varieties of higher dimension where C is any curve obtained as intersection of $(n-1)$ elements of $|L'|$.

We need the following lemma.

Lemma (4.3.2). *Let F' be as in the proposition and let L' be a line bundle on F' such that the zero locus of a general section of L' is reduced and connected. Assume that through a point $x \in F'$ there is a section $C \in |L'|$ which is generically reduced, connected and such that $h^1(\mathcal{O}_C) = h^1(\mathcal{O}_{C'})$, where C' is a general section of $|L'|$. Then x is a Cohen-Macaulay point of F'.*

Proof. Notice that $\chi(\mathcal{O}_C) = \chi(\mathcal{O}'_C)$; this follows from the exact sequence

$$0 \to L'^{-1} \to \mathcal{O}_{F'} \to \mathcal{O}_C \to 0,$$

true for every $C \in |L'|$, and the fact that $\chi(\mathcal{O}_{F'})$ and $\chi(L'^{-1})$ does not depend on C. This and the hypothesis imply that $h^0(\mathcal{O}_C) = h^0(\mathcal{O}_{C'}) = 1$. We then consider the exact sequence

$$0 \to S \to \mathcal{O}_C \to \mathcal{O}_{C_{red}} \to 0,$$

where S is a skycraper sheaf supported on the non Cohen-Macaulay points of C; since $h^0(\mathcal{O}_C) = h^0(\mathcal{O}_{C_{red}}) = 1$, we have that $S = 0$ and therefore C is Cohen-Macaulay. Since C is a Cartier divisor every point of C is a Cohen-Macaulay point of F'.

Proof of proposition (4.2.1). We are to prove that F' is normal then the rest will follow from Fujita's result. Since L' is ample and spanned by the Bertini theorem we have through every point $x \in F'$ a section $C \in |L'|$ which is generically reduced and connected. By hypothesis all $C \in |L'|$ have genus 0 and therefore we can apply the lemma and say that F' has Cohen-Macaulay singularities.

On the other hand a general element of $|L'|$, being irreducible and reduced and of sectional genus 0, is a smooth rational curve. Therefore F' itself is smooth along the point of a general C; since L' is ample this implies that F' is smooth in codimension 2.

By Serre criterion F' is therefore normal.

To conclude the proof we apply results of T. Fujita; in the subsequnet lines we refer to his book [Fu]. First by the proposition (3.4) it follows that $\Delta(F', L') = 0$; secondly by the corollary (4.12) we have that L' is very ample. Finally applying the theorem (5.15) and the remark (5.16) we have the complete proposition.

We note that the argument applied in the course of the proof of (4.3.2) to a component F' can be actually used for the whole fiber F and it yields the following

Lemma (4.3.3). *Let F be a two dimensional fiber of a good contration as in (4.2). Then F is Cohen-Macauley unless the zero locus of a general section $\in |L_F|$ is disconnected.*

The example of non-Cohen-Macaulay fiber is obtained in (3.6.2): the meeting point of the two components of the fiber is its unique non-C-M point. If φ is birational then the hyperplane section of F is connected which follows from the subsequent result.

Lemma (4.4) (Horizontal slicing). *Assume that φ is a good contraction as in (4.2). Let X' be a general section of $|L|$ not containing any component of the special fiber F. Consider the restriction of φ to X', $\varphi_{|X'} : X' \to \varphi(X')$, and let*

$$X' \xrightarrow{\varphi'} Z' \longrightarrow Z$$

be the Stein factorization of $\varphi_{|X'}$. Then X' is a smooth $(n-1)$-dimensional variety and φ' is a crepant (birational) contraction with at most 1-dimensional fibers onto a normal variety Z'. If φ is birational, then $\varphi' = \varphi_{|X'}$. If φ is of fiber type than $Z' \to Z$ is a double covering. Moreover for $n \geq 4$ non trivial fibers of $\varphi' : X' \to Z'$ are parametrized on Z' by a subvariety of dimension $\geq (n-4)$. If $n = 4$ and F is an isolated 2-dimensional fiber of φ then X' may be chosen so that non trivial fibers of φ' are isolated.

Proof. The first part of the lemma follows from (1.3.2) and adjunction formula. Let E' be the exceptional locus of φ'. We claim that $dim\varphi'(E') \geq (n-4)$. Indeed, take any rational curve in the fiber of φ'. Since $K_{X'}.C = 0$ then from deformation theory (1.4.1) we have that C moves at least in a $(n-1)-3$ family. The last part is obtained by a standard dimension counting.

The deformation argument which we have just applied implies that the hyperplane section of the fiber F (or its part, if it is reducible) moves in a nontrivial family, if only $n \geq 5$. Thus, knowing the classification of good contractions for $n \leq 4$ we can estimate the degree of F with respect to L. The idea of moving the rational curves and estimating the exceptional locus of a good or crepant contraction was used in the proof of the length-fiber-locus inequality in [Wi, (1.1)], (see also [Io, 0.4]) which related a general non-trivial fiber of a good contracion with the dimension of its exceptional locus.

A similar idea can be used to distinguish the locus of fibers of different dimension. Namely, let us set consider the locus of fibers of dimension k. More precisely, for $k \geq 1$, let $E_k(\varphi)$ denote the closure of the set $\{x \in X : \dim\varphi^{-1}(\varphi(x)) = k\}$. Let E_k be an irreducible component of $E_k(\varphi)$. Then either E_k is an irreducible component of the exceptional locus $E(\varphi)$ of φ, or it is contained in $E_m(\varphi)$ for some $m < k$.

Proposition (4.5). *Let E_2 be an irreducible component of $E_2(\varphi)$. Then $dimE_2 \geq n-2$ if E_2 is also a component of $E(\varphi)$. Otherwise*
(i) $dimE_2 \geq n-4$ if φ is birational,
(ii) $dimE_2 \geq n-5$ if $dimX - dimZ = 1$.
(All the above estimates are best possible, see (3.1).)

Proof. The first part of the lemma is just [Wi, (1.1)]. Thus we may assume that $E_2 \subset E_1(\varphi)$. Moreover, we may asume that E_2 is just a component of a 2-dimensional isolated fiber of φ. Indeed, we can consider X'', the intersection of pull-back of general $\dim\varphi(E_2)$ very ample divisors on Z, and the restriction of φ to X'' (vertical slicing in [A-W]). Then the formula proved on X'' remains valid also on X.

Thus E_2 is equal to one of the surfaces listed in (4.2.1). Suppose first that $d := L^2 \cdot E_2 > 2$. Let C be a general curve from the linear system $|L_{E_2}|$. Then C is a rational curve and since $H^0(E_2, L_{E_2}) = d+2$ it follows that it moves inside E_2 in a family of dimension $d+1$. On the other hand, because of (1.4.1) $\dim_{[C]}Hilb(X) \geq d+n-3$. Since the degree of neighbouring fibers of φ is 2 at most and, being chosen generally, C can not move to another component of the fiber of φ, it follows that $\dim_{[C]}Hilb(E_2) \geq \dim_{[C]}Hilb(X)$ which implies $n \leq 4$.

If $L^2 \cdot E_2 \leq 2$ then E_2 is either $\mathbf{P}^2$ or $\mathbf{F}_0$, or $\mathbf{S}_2$. Let $C \subset E_2$ be a general conic. If φ is birational then all neighbouring non-trivial fibers of φ have degree 1 with respect to L. Thus again, deformations of C should remain inside E_2. This, however, implies the inequalities:

$$\dim_{[C]}Hilb(E_2) \geq \dim_{[C]}Hilb(X) \geq L \cdot C + n - 3 \qquad (4.5.1)$$

where the right-hand-side inequality comes from (1.4.1). Therefore, using the description of E_2 we have the following bound:

$$n \leq \dim_{[C]}Hilb(E_2) + 3 - L \cdot C \leq 5 + 3 - 2 = 6. \qquad (4.5.2)$$

This proves the birational case.

In the fiber type case the degree of a general fiber of f is 2, so C can move out of E_2 and therefore the argument has to be adjusted accordingly. (We note that we may repeat the above argument for curves of degree > 2 but the result is not satisfactory.) Let $\mathcal{H}$ be an irreducible component of the Hilbert scheme $Hilb(X)$ which contains a general fiber of the conic fibration $\varphi : X \to Z$. Over $\mathcal{H}$ we have the incidence variety $\mathcal{C}$. That is, there exist a conic bundle $p : \mathcal{C} \to \mathcal{H}$ and a dominant morphism $q : \mathcal{C} \to X$ which maps fibers of p to conics contracted by φ. We note that the map q is birational with its exceptional locus in $q^{-1}(E_2)$ and the composition $\varphi\circ q$ is proper. By possible normalization of $\mathcal{H}$ and base change of the conic bundle we may assume that $\mathcal{C}$ is normal. Over E_2 the component $\mathcal{H}$ meets other components which are associated to conics inside E_2. In particular, the components of $q^{-1}(E_2)$ are contained is some incidence varieties of conics inside E_2. Suppose that the dimension of a component $\mathcal{H}_{E_2}$ of $Hilb(E_2)$ is smaller than $n-1$, i.e. $\dim_{[C]} Hilb(E_2) < n-1$ for some conic $C \subset E_2$ which is the case if $n \geq 7$. Then, because of the deformation argument, $\mathcal{H}_{E_2} \subset \mathcal{H}$ and $A := p^{-1}(\mathcal{H}_{E_2})$ is a component of $q^{-1}(E_2)$ However, in view of [Ko, VI.1.5], A is a divisor in $\mathcal{C}$ and therefore

$$n - 1 = \dim A = \dim \mathcal{H}_{E_2} + 1 \tag{4.5.3}$$

which proves the fiber type case.

The argument which we presented above can be extended to deal with possibly reducible fibers if we note the following two easy observations

Lemma (4.6.1). *Let $C = \bigcup C_k$ be a (reduced) connected curve (with irreducible components C_k) contained in a variety $F = \bigcup F_k$, where F_k are irreducible components of F. Suppose that the generic point of any irreducible component C_k is contained in $F_k - (\bigcup_{j\neq k} F_j)$. Then a small deformation of C in F, call it C', has a decomposition into irreducible components C'_k and the generic point of C'_k is contained in $F_k - (\bigcup_{j\neq k} F_j)$.*

Lemma (4.6.2). *Let F be a (possibly reducible) surface and L an ample and spanned line bundle on it. Suppose that a general curve $C \in |L|$ is a connected curve of genus 0. Then $H^0(S, L) \leq L^2 \cdot F + 2$.*

The subsequent result provides a description of isolated (c.f. (4.0)) 2-dimensional fibers of contractions of varieties of dimension ≥ 5.

Proposition (4.7). *Let F be an isolated 2 dimensional fiber of a good contraction, as in (4.0) and (4.2). If φ is birational then $n = \dim X \leq 6$ and moreover $F = \mathbf{P}^2$ if $n = 5,\ 6$. If φ is of fiber type (and L is φ-spanned) then $n \leq 7$ and for $n \geq 5$ the pair (F, L_F) is either $(\mathbf{P}^2, \mathcal{O}(1))$ or, for $n = 5$, it is a smooth or reducible quadric (i.e. $(\mathbf{F}_0, C_0 + f)$, or a union of two planes meeting along a line and L restricted to each of planes is $\mathcal{O}(1)$).*

Proof. The preceeding lemma provides the upper bound on n. Also the birational case is an immediate consequence of the preceeding argument. Namely, if $L^2 \cdot F \geq 2$ then we take a general curve $C \in |L_F|$ and consider its deformations to get $n \leq 4$ (we note that C is connected, possibly reducible and we can apply (1.4.1) directly to C or possibly to a connected smoothable sub-curve of degree ≥ 2). This concludes the birational case. For the fiber type we need the following

Lemma (4.7.1). *In the situation of Proposition (4.7) assume that F is not irreducible. Let F_1, F_2 be two intersecting irreducible components of F. If they meet along a curve K then it is a line relatively to L. If they have an isolated common point then $n = 4$, φ is of fiber type and $F_1 = F_2 = \mathbf{P}^2$, $L_{F_i} \simeq \mathcal{O}(1)$.*

Proof. If the curve K is not a line then, taking a general smooth section $X' \in |L|$, the map $\varphi' := \varphi_{|X'} : X' \to Z'$ would be a crepant contraction with a fiber that contains two curves meeting in more then one point. This is in contradiction with the proposition (1.5.1). Suppose now that x is an isolated point of the intersection $F_1 \cap F_2$. Then obviously $n \geq 4$. Let us take a general section of L whose zero locus contains x. Then we get two curves $C_i \subset F_i$, with $d_i := L \cdot C_i = L^2 \cdot F_i$, meeting at the point x. We may assume that the curve $C = C_1 \cup C_2$ is smoothable. This is clear if both C_i are smooth (then they have to meet tranversaly because $H^1(C_1 \cup C_2, \mathcal{O}) = 0$) otherwise one of F_i is a cone and then we can take a proper subcurve of C_i so that C is connected and has degree ≥ 3. We claim that $\dim_{[C]} Hilb(F_1 \cup F_2) = d_1 + d_2$. Indeed, any deformation of C is obtained by deforming each of C_is with x fixed and therefore $\dim_{[C]} Hilb(F) = \dim|L_{F_1} \otimes J_x| + \dim|L_{F_2} \otimes J_x|$. If $L \cdot C = d_1 + d_2 > 2$ then argueing like in the first part of the proof of (4.5) we get $n \leq 3$, a contradiction. If $L \cdot C = 2$ then the argument which led to (4.5.3) provides us with the following inequality:

$$n \leq \dim_{[C]} Hilb(F_1 \cup F_2) + 2 = 4$$

which concludes the proof of (4.7.1).

Conclusion of the proof of (4.7). Now, in view of the above Lemma we can repeat the computations from the proof of (4.5). Indeed, a general $C \in |L_F|$ is now a connected, possibly reducible, curve of genus 0 and therefore the computations are in fact the same. Thus, for $n \geq 5$ we get $L^2 \cdot F \leq 2$. Suppose that $L^2 \cdot F = 2$. Then as in (4.5.3) we get $(n-1) \leq \dim_{[C]} Hilb(F) + 1 = 4$ and thus $n \leq 5$. To conclude the proof of the proposition we are only to exclude the case $F = \mathbf{S}_2$ for $n = 5$. In this case we consider $C \subset \mathbf{S}_2$ which is a union of a conic and a line. Then it is not hard to see that $\dim_{[C]} Hilb(\mathbf{S}_2) = 4 = L \cdot C + 1$ and thus this case can occur for $n \leq 4$ only.

Remark In (3.1.2) we provided examples of contractions of manifolds of dimension ≥ 5 which show that the results of Proposition (4.7) are best possible; the only exception is the case of $F = \mathbf{P}^2 \cup \mathbf{P}^2$ for $n = 5$.

In the remaining part of this section we will deal with the case $n = 4$ and 3. First we prove

Lemma (4.8). *If $\varphi : X \to Z$ is a good contraction of a 4-fold with an isolated 2-dimensional fiber F and a general section $X' \in |L|$ is disconnected then F is a union of two copies of $\mathbf{P}^2$ meeting at one point (we denote it by $\mathbf{P}^2 \bullet \mathbf{P}^2$).*

Proof. We argue as in the proof of (4.7.1). Namely, we can find a decomposition of $F = F_1 \cup F_2$ so that F_1 and F_2 have an isolated common point x. Now we take a general $C \in |L_F|$ which contains x and arrive to a contradiction if $C \cdot L \geq 3$.

With this argument we exhausted the technique of using deformations of $C \in |L_F|$ to get informations about F. (We note that e.g. the inequality (4.4.2), whose different versions we considered above, becomes useless if $C \in |L_F|$ and $n \leq 4$.) From now on we will choose the curve C for each case separately. First we discuss when a surface S among these listed in (4.2.1) can be a component of the fiber F. This will be done with the help

Table II.

N^o	*pair* (S, L_S)	*curve* C	$L \cdot C$	$\dim_{[C]} Hilb(S)$
(1)	$(\mathbf{P}^2, \mathcal{O}(2))$	*line*	2	*2*
(2)	$(\mathbf{S}_3, \mathcal{O}(1))$	*two lines*	2	*2*
(3a)	$(\mathbf{F}_r, C_0 + kf)$, $r \geq 1$, $k > r$	$C_0 + f$	$k - r + 1$	≤ 2
(3b)	$(\mathbf{F}_r, C_0 + (r+1)f)$, $r \geq 3$	$C_0 + 2f$	3	*2*
(4)	$(\mathbf{F}_2, C_0 + 3f)$	$C \in \lvert C_0 + 2f \rvert$	3	*3*
(5a)	$(\mathbf{F}_1, C_0 + 2f)$	$C \in \lvert C_0 + f \rvert$	2	*2*
(5b)	$(\mathbf{F}_1, C_0 + 2f)$	$C \in \lvert C_0 + 2f \rvert$	3	*4*
(6a)	$(\mathbf{F}_0, C_0 + 2f)$	C_0	2	*1*
(6b)	$(\mathbf{F}_0, C_0 + 2f)$	$C \in \lvert C_0 + f \rvert$	3	*3*
(7a)	$(\mathbf{F}_0, C_0 + f)$	$C \in \lvert C_0 + f \rvert$	2	*3*
(7b)	$(\mathbf{F}_0, C_0 + f)$	$C \in \lvert C_0 + 2f \rvert$	3	*5*

of the Table II in which we indicate the cases which we have to discuss together with our choice of the curve $C \subset S$ and the numerical data which comes up from the choice.

(4.9). Using Table II we can proceed with the argument that we explained in the course of the proof of (4.5). We use the inequality (4.5.1) which gives a fundamental constrain on a possible component S of the fiber F. That is, if $L \cdot C > 1$ in the birational case and, respectively, $L \cdot C > 2$ in the fiber type case, then $\dim_{[C]} Hilb(S) + 3 - L \cdot C \geq n$.

Thus using the respective entries from Table II we get:

(1) $(\mathbf{P}^2, \mathcal{O}(2))$ may be a component of an isolated 2-dimensional fiber of a birational (respectively fiber type) contraction only if $n \leq 3$ (resp. $n \leq 4$). Moreover $(\mathbf{P}^2, \mathcal{O}(2))$ is not a component of a reducible fiber by (4.7.1) and (4.8).

(2) $(\mathbf{S}_3, \mathcal{O}(1))$ is not a component of an isolated 2-dimensional fiber of birational type contraction. Moreover we note that the vertex of the cone $(\mathbf{S}_r, \mathcal{O}(1))$ has embedding dimension equal to $r + 1$ so for $n \leq 4$ only $\mathbf{S}_3$ and $\mathbf{S}_2$ have to be discussed.

(3) if $(\mathbf{F}_r, C_0 + kf)$ is a component of an isolated 2-dimensional fiber and $r > 1$ then $L_{|S} = C_0 + (r+1)f$ and therefore $L \cdot C_0 = 1$. Indeed, either C_0 is not contained in another component of the fiber and we may use the estimate from the table or it is contained in another component of the fiber and thus in view of () $L \cdot C_0 = 1$. Moreover $\mathbf{F}_r$ can not be a component of an isolated 2-dimensional fiber if $r \geq 3$. To see this note a small deformation C' of $C = C_0 + 2f$ has to remain inside S because it will contain a component of degree 2 through a generic point of S. Since a deformation C' of C will be connected, the components of C' passing through a generic point of S (union of two disjoint lines) will define uniquely the remaining component of degree 1, which must be the curve C_0.

(4) $\mathbf{F}_2$ may be a component of an isolated 2-dimensional fiber only if $n \leq 3$.
(5) $\mathbf{F}_1$ may be a component of an isolated 2-dimensional fiber of a good birational (resp. fiber type) contraction only if $n \leq 3$ (resp. $n \leq 4$).
(6) $(\mathbf{F}_0, C_0+2f)$ can not be a component of an isolated 2-dimensional fiber of a birational contraction and it may be a component of an isolated 2-dimensional fiber of a fiber type contraction only if $n = 3$.
(7) $(\mathbf{F}_0, C_0 + f)$ may be a component of an isolated 2-dimensional fiber of a birational (respectively fiber type) contraction only if $n \leq 4$ (resp. $n \leq 5$)

Therefore we obtained a finite list of possible components of an isolated 2-dimensional fiber. The results of the preceeding discussion are listed in Table III. For clarity we also included the case when $n \geq 5$. The last column of the table indicates the number of an example which admits a fiber having S as an irreducible component in the case $n > 3$.

Table III

N^o	*component* (S, L_S)	φ *birational*	φ *of fiber type*	*example (for* $n > 3$*)*
(0)	$(\mathbf{P}^2, \mathcal{O}(1))$	$n \leq 6$	$n \leq 7$	*(3.1) and other*
(1)	$(\mathbf{P}^2, \mathcal{O}(2))$	$n \leq 3$	$n \leq 4$	*(3.4.0)*
(2)	$(\mathbf{S}_3, \mathcal{O}(1))$	——	$n = 4$	*(3.3)*
(3)	$(\mathbf{S}_2, \mathcal{O}(1))$	$n \leq 4$	$n \leq 4$	*(3.2.1) and (3.5.5)*
(4)	$(\mathbf{F}_2, C_0 + 3f)$	$n = 3$	$n = 3$	
(5)	$(\mathbf{F}_1, C_0 + 2f)$	$n = 3$	$n \leq 4$	*(3.4.1), (3.4.3)*
(6)	$(\mathbf{F}_0, C_0 + 2f)$	——	$n = 3$	
(7)	$(\mathbf{F}_0, C_0 + f)$	$n \leq 4$	$n \leq 5$	*(3.1) and other*

(4.10). Now we discuss the question of reducible fibers.

Lemma (4.10.1). *The cone* $\mathbf{S}_3$ *can not be a component of a reducible 2-dimensional fiber if* $n = 4$. *Also* $\mathbf{S}_2$ *can not be a component of a reducible fiber if* $n = 3$ *and in dimension 4 it meets one component at most. No three, respectively, four components of* F *can meet in a single point if* $n = 3$ *or 4, respectively.*

Proof. We can take a general $X' \in |L|$ which passes through the point x which denotes the vertex of the cone or the meeting point of components. Such X' is then smooth because there exist lines passing through x which are not contained in X'. Therefore, in view of (1.5.2) at most 2, or respectively, 3 components of $X' \cup F$ should meet in x if $n = 3$ or 4, respectively. So the last statement of the lemma is clear. Since the cones meet other components along lines in the ruling, which contain x, this implies the other claims too.

Lemma (4.10.2). *Let F be an isolated 2 dimensional fiber of a good contraction φ of a 4 fold. Let S_1 and S_2 be two (different) irreducible components of F which meet along a line l. Then at least one of them is $\mathbf{P}^2$ and if φ is birational then both of them are $\mathbf{P}^2$. Moreover, if one of the components is $\mathbf{F}_1$ then the meeting line is C_0.*

Proof. In the union $S = S_1 \cup S_2$ we construct a connected curve $C = C_1 \cup C_2$, with irreducible components $C_i \subset S_i$ passing through the generic point of each S_i and having a common point at the line $l = S_1 \cap S_2$. We assume that $S_1 \neq \mathbf{P}^2$. In Table IV we indicate the choice of $C_1 \subset S_1$ which will depend on the position of l in S_1. The curve $C_2 \subset S_2$ will be always a general member of $|L_{S_2}|$ passing through $C_1 \cap l$. We note that if $d_2 = L \cdot C_2 = L^2 \cdot S_2$ then $\dim|L_{S_2}| = d_2 + 1$. Then we compute the degree of C and the space of its deformations inside S and the lemma follows by comparing these two numbers. Namely, it follows that the curve C moves out of the fiber F, so that its degree has to be bounded by 1 or 2, if the contraction is birational or of fiber type, respectively.

Table IV.

N^o	*S_1 and l*	*curve $C_1 \subset S_1$*	$L \cdot C$	*$\dim_{[C]}Hilb(S)$*
(A)	$\mathbf{S}_2$, $l =$ *line*	*line*	$1 + d_2$	$1 + d_2$
(B)	$\mathbf{F}_r$, $r \geq 0$, $l = C_0$	f	$1 + d_2$	$1 + d_2$
(C)	$\mathbf{F}_r$, $r \geq 0$, $l = f$	$C_1 \in \lvert rf + C_0\rvert$	$r + 1 + d_2$	$r + 1 + d_2$

A similar argument is made in the proof of

Lemma (4.10.3). *In the above situation no three components can meet along a line.*

Proof. Assume the contrary, i.e. let S_1, S_2 and S_3 meet along a line l. From what we have just proved we know that two of them are $\mathbf{P}^2$. Then, argueing exactly like in the previous lemma we conclude that the third must be also $\mathbf{P}^2$. Namely, we consider $S_1 = \mathbf{F}_r$ and C_1 as in Table IV, while $C_2 \subset \mathbf{P}^2 \cup \mathbf{P}^2$ consists of two lines, one in each component. Now we claim that the conormal of l in X is (at least generically) spanned by the conormals of l in each of S_i. Indeed, if we consider a general linear section $X' \in |L|$ then the tangent vectors of lines $S_i \cap X'$ will spann the tangent space at $X' \cap l$. This however implies that we have an embedding

$$0 \longrightarrow N_{l/S_1} \oplus N_{l/S_2} \oplus N_{l/S_3} = \mathcal{O}(1)^{\oplus 3} \longrightarrow N_{l/X} \longrightarrow T$$

where T is a torsion sheaf on l. This is absurd, since, because of adjunction, $c_1(N_{l/X}) = -1$.

Now we are ready to prove the classification of isolated 2 dimensional fibers in dimension 4:

Proposition (4.11). *Let $\varphi : X \to Z$ be a good contraction of a smooth 4-fold and F is an isolated 2-dimensional fiber. If φ is birational then the pair (F, L_F) is isomorphic to one of the following:*

$$(\mathbf{P}^2, \mathcal{O}(1)), \quad (\mathbf{F}_0, C_0 + f), \quad (\mathbf{S}_2, \mathcal{O}(1)), \quad (\mathbf{P}^2 \cup \mathbf{P}^2, \mathcal{O}(1)).$$

If φ is of fiber type and F is irreducible then (F, L_F) is one of the following:

$$(\mathbf{P}^2, \mathcal{O}(1)), \quad (\mathbf{P}^2, \mathcal{O}(2)), \quad (\mathbf{S}_3, \mathcal{O}(1)), \quad (\mathbf{S}_2, \mathcal{O}(1)), \quad (\mathbf{F}_1, C_0 + 2f), \quad (\mathbf{F}_0, C_0 + f).$$

If φ is of fiber type and F is reducible then it has at most three components and it is one of the following (we supress the description of L which is obvious):

$$\mathbf{P}^2 \cup \mathbf{P}^2, \quad \mathbf{P}^2 \cup \mathbf{F}_0, \quad \mathbf{P}^2 \cup_{C_0} \mathbf{F}_1, \quad \mathbf{P}^2 \cup \mathbf{S}_2, \quad \mathbf{P}^2 \cup \mathbf{P}^2 \cup \mathbf{P}^2, \quad \mathbf{P}^2 \cup_f (\mathbf{F}_0)_{C_0} \cup \mathbf{P}^2,$$

where any two components intersect along a line (indicated by a subscript, when needed), and the exceptional case of $\mathbf{P}^2 \bullet \mathbf{P}^2$ when the two components intersect at an isolated point.

Proof. The birational case is an immediate consequence of Table III as well as (4.10.2), (4.10.3) and (4.7.1). Indeed, if a fiber is reducible then all its components are $\mathbf{P}^2$. If it had more than two components then, since three of them do not meet along a common line, two of the components would meet along a common isolated point, contradictory to (4.7.1).

Now we pass to the fiber type case. The description of irreducible fibers and fibers of two components is already known due to Table III and (4.10.2). Thus let us pass to fiber which contain at least three components, call them S_i. From (4.10.2) we know that from a pair of two components meeting along a line, at least one is $\mathbf{P}^2$. Moreover, no three components make a cycle. That is, it is not possible that S_1 meets S_2 along $l_{1,2}$, S_2 meets S_3 along $l_{2,3}$ and S_3 meets S_1 along $l_{3,1}$ and all three lines are different, because then the linear section of $S_1 \cup S_2 \cup S_3$ would contain a cycle of rational curves.

We recall that according to (4.7.1) if two components meet at a single point then both are $\mathbf{P}^2$. Thus, if S_2 meets S_1 and S_3 along two different lines which intersect then $S_1 = S_3 = \mathbf{P}^2$. According to Table III and lemmata (4.10.1) and (4.10.2) the central component S_2 is either $\mathbf{P}^2$ or $\mathbf{F}_0$. If the central component S_2 meets the other two components (equal to $\mathbf{P}^2$) along two non-intersecting lines then it has to be $\mathbf{F}_0$. In this case, however, we can take a conneted curve C of degree 3 consisting of three lines, each in one component, and we get contradiction to the deformation principle because $\dim_{[C]} Hilb(F) = 3$.

The case of more than 3 components is ruled out similarly. First we note that all component would be $\mathbf{P}^2$ and then we can apply argument similar to the one in (4.7.1) to get contradiction.

Remark . In Section 3 we have examples of appropriate 2 dimensional fibers except cases $\mathbf{P}^2 \cup \mathbf{S}_2$, $\mathbf{P}^2 \cup \mathbf{P}^2 \cup \mathbf{P}^2$ and $\mathbf{P}^2 \cup_f (\mathbf{F}_0)_{C_0} \cup \mathbf{P}^2$. A list of possible exceptional fibers of a fiber type contraction of a 4-fold was obtained independently by T. Kachi [Kac], whom we owe our thanks for pointing out missing cases in our preliminary list.

(4.12) The last part of this section is dedicated to the case $n = 3$; this case is known (and proved by S. Mori in the case of elementary contractions) but the subsequent discussion may be a good test for our approach. Our goal is to describe all possible two dimensional isolated fibers of a good contraction of threefolds. A particular feature of this dimension is the fact that these fibers are actually divisors.

The list of possible components of a fiber is set up in Table III. Also, because of (4.7.1) and (4.10.1) neither $(\mathbf{P}^2, \mathcal{O}(2))$ nor $\mathbf{S}_2$ is a component of a reducible fiber and — the fact which will be used constantly in our arguments — no three components meet at one point.

Let us discuss first the question of reducible fibers: assume therefore that F contains at least two intersecting components, F_1 and F_2. Then $F_1 \cap F_2 := R$ is a line relative to L and the two components intersect in R tranversally (see Lemma (4.4) and (1.5))

We use now the intersection theory of divisors to prove a useful formula. Let $N_{R/X}$ denotes the normal bundle of R in X; we have $N_{R/X} = N_1 \oplus N_2$, where N_i denotes the normal bundle of R in F_i, and $deg(det(N_{R/X})) = -1$, by adjunction formula. It follows that

$$-1 = deg(det(N_{R/X})) = degN_1 + degN_2 = F_1 \cdot R + F_2 \cdot R.$$

Since $F_i \cdot R = (K_{F_i} - K_X) \cdot R = K_{F_i} \cdot R + 1$, we obtain $K_{F_1} \cdot R + K_{F_2} \cdot R = -3$.

The curve R is a line relatively to L; therefore, modulo numerical equivalence, it is unique in $\mathbf{P}^2$ while it can be either f or C_0 in $\mathbf{F}_r$, the last case only if $L_{\mathbf{F}_r} = C_0 + (r+1)f$. This implies that $K_{F_i} \cdot R = -3$, if $F_i \simeq \mathbf{P}^2$, and $K_{F_i} \cdot R = (r-2)$ or -2, if $F_i \simeq \mathbf{F}_r$. This observation added to the above formula gives only two possibilities (up to possible renumeration of components), namely the following:

(i) $F_1 = \mathbf{P}^2$ and $F_2 = \mathbf{F}_2$, R is a line in $\mathbf{P}^2$ and the section C_0 in $\mathbf{F}_2$;
(ii) $F_1 = \mathbf{F}_1$ and $F_2 = \mathbf{F}_r$, R is the section C_0 in $\mathbf{F}_1$ and a fiber f in $\mathbf{F}_r$.

If we are in the case (ii) we can apply the argument of deforming of a rational curve of degree 3 which is the union of $f \subset \mathbf{F}_1$ and of $C_0 + f \subset \mathbf{F}_r$ (none of these curves is contained in another component because our "no three meet" rule). Since $\dim_{[C]} Hilb(F) \leq 2$ for $r > 0$, we obtain a contradiction unless $r = 0$. Then $F = \mathbf{F}_0 \cup \mathbf{F}_1$. If the fiber F has at least three components, by the above, we must have that two of them are $\mathbf{F}_1$ and one is $\mathbf{F}_0$, the $\mathbf{F}_1$ are disjoint and they intersect with the $\mathbf{F}_0$ along two disjoint fibers of its ruling. Deforming a degree 3 rational curve in F which is the union of the section C_0 in $\mathbf{F}_0$ and of two fibers, one in each $\mathbf{F}_1$, we obtain a contradiction, as above.

Now we know all possible fibers and it remains to distinguish the fibers of birational contractions from these of fiber type contractions. To this end we note that if the fiber in question has ample conormal bundle (i.e. $\mathcal{O}_F(-F)$ is ample) then it is an isolated positive dimensional fiber of the contraction (see also (6.1)). Indeed, the limit of 1-dimensional fibers approaching the fiber F would produce a curve in F whose intersection with F would be zero. Since $\mathcal{O}_F(-F)$ can be computed easily by adjunction, we find out all the pairs (F, L_F) which may be non-isolated positive dimensional fibers; these are the following:

$$(\mathbf{F}_0, C_0 + 2f), \quad (\mathbf{F}_1, C_0 + 2f), \quad (\mathbf{F}_0 \cup_{C_0} \mathbf{F}_1,\ L_{\mathbf{F}_0} = C_0 + f,\ L_{\mathbf{F}_1} = C_0 + 2f).$$

Moreover, only in the case $(\mathbf{F}_1, C_0, +2f)$ the normal $F_F = C_0 + f$ has zero intersection with the unique curve C_0 while in the remaining two cases the locus of curves which have intersection 0 with F coincides with F. This distinguishes the birational and fiber case. Thus we have proved

Proposition (4.13). *Let φ be a good contraction of a smooth 3-fold with an isolated two dimensional fiber F; as usually $L = -K_X$. If φ is birational then (F, L_F) is one of the following pairs*

$$(\mathbf{P}^2, \mathcal{O}(1)), \quad (\mathbf{P}^2, \mathcal{O}(2)), \quad (\mathbf{S}_2, \mathcal{O}_{\mathbf{S}_2}(1)), \quad (\mathbf{F}_0, C_0 + f), \quad (\mathbf{F}_1, C_0 + 2f),$$

or $\mathbf{P}^2 \cup_{C_0} \mathbf{F}_2$ and $L_{|\mathbf{P}^2} = \mathcal{O}(1)$, $L_{|F_2} = C_0 + 3f$. In case when $F = \mathbf{F}_1$ the contraction φ contracts also a smooth divisor $E \subset X$ to a smooth curve $E \cap F_1 = C_0 \subset F_1$. In the remaining birational cases the fiber F is an isolated poistive dimensional fiber of φ. If φ is of fiber type with generic fiber of dimension 1 (a conic fibration) and L is φ-spanned then the pair (F, L_F) is either $(\mathbf{F}_0, C_0 + 2f)$ or $F = \mathbf{F}_0 \cup_{C_0} \mathbf{F}_1$ and $L_{|\mathbf{F}_0} = C_0 + f$, $L_{|\mathbf{F}_1} = C_0 + 2f$.

Let us note that all the above cases exist. Indeed, the construction of elementary contractions with appropriate fibers is done by Mori in [Mo1]. All cases which are not in the Mori's list are obtained from Mori's contractions. Namely, the cases $(\mathbf{F}_1, C_0 + 2f)$ and $\mathbf{P}^2 \cup \mathbf{F}_2$ are obtained from $(\mathbf{P}^2, \mathcal{O}(1))$ by a blow-up of a curve. The case $(\mathbf{F}_0, C_0 + 2f)$ is just a product of $\mathbf{P}^1$ and a simple blow-up of a surface while the case of $\mathbf{F}_0 \cup \mathbf{F}_1$ is obtained by blowing a line C_0 in the previous one. Alternatively, the last case is the blow-up of the quadric cone singularity of the scroll

$$\mathbf{P}^2 \times \mathbf{C}^2 \supset \{([t_0, t_1, t_2], (z_1, z_2) : t_1 z_1 + t_2 z_2 = 0\} \longrightarrow \mathbf{C}^2$$

c.f. Example (3.4.3) and (3.2) in [B-W].

5. Scheme theoretic fiber.

(5.0) Let $\varphi : X \to Z$ be a good contraction of a smooth n-fold X. As in (4.0) we fix a relative ample line bundle $L := -K_X$ and we choose $F = \varphi^{-1}(z)$ to be a positive dimensional fiber of φ.

Suppose that we already know the geometric (reduced) structure of the fiber F of the map φ. Then the next step in understanding of the map φ is the conormal sheaf (bundle) of F in X, which is defined as the quotient $N^*_{F/X} =: \mathcal{I}_F/\mathcal{I}_F^2$, where $\mathcal{I}_F$ is the ideal of F with the reduced structure.

If F is a locally complete intersection then $N^*_{F/X}$ is locally free over F (a vector bundle) and its dual is the normal bundle $N_{F/X}$. We note that almost all among possible fibers which we have described in Section 4 are locally complete intersections (exceptions are $\mathbf{S}_3$ and most of the reducible fibers) and moreover the blow-up of X along F has always terminal singularities.

If F is locally complete intersection then we have the adjunction formula: $K_F = (K_X)_{|F} + det N_{F/X}$. In particular, if F and $L_{|F} = -K_{X|F}$ are fixed then the first Chern class of the normal bundle is given. If F has codimension 1 the normal bundle is thus fixed.

It turns out that in many instances the knowledge of the fiber and its conormal allows to describe the contraction map. A typical example of such reasoning is the Castelnouvo theorem which we recalled in (2.4). The formal function theorem which was used in the proof of the Castelnouvo theorem allowed us to compare "asymptotic" behaviour of the ideals $\mathcal{I}_F$ and m_z. This leads to a natural question of comparison of ideals $\mathcal{I}_F$ and $\varphi^{-1}m_z \cdot \mathcal{O}_X$, that is of understanding the fiber structure on F.

Let us begin with the following general observation.

Lemma (5.1). *Let $\varphi : X \to Z$ be a projective morphism of normal varieties such that $\varphi_*\mathcal{O}_X = \mathcal{O}_Z$. For a point $z \in Z$ let $F = \varphi^{-1}(z)$ be the geometric fiber of φ with the ideal (sheaf of ideals) $\mathcal{I}_F$. Then $\varphi_*\mathcal{I}_F \subset \mathcal{O}_Z$ is the maximal ideal m_z of the point z and the scheme theoretic structure on F is defined by the ideal $\mathcal{I}_{\tilde{F}}$ which is the image of the evaluation $\varphi^{-1}\varphi_*(\mathcal{I}_F) \to \mathcal{I}_F$.*

Therefore, in order to understand the structure of the map φ we will analyse the behaviour of pull-backs and push-forwards.

A very useful example is the contraction to the vertex of a cone.

Example (5.1.1) Let Y be a smooth variety and let $\mathcal{L}$ be an ample line bundle over Y. Let $X := Spec_Y(\bigoplus_{k\geq 0} k\mathcal{L})$ be the total space of the dual bundle $\mathcal{L}^*$ with a zero section Y_0. Consider the collapsing $\varphi : X \to Z$ of Y_0 to the vertex z of a cone Z. That is, $Z = Spec(\bigoplus_{k\geq 0} H^0(Y, k\mathcal{L}))$ and the map φ is associated to the evaluation of $m\mathcal{L}$. If Y is Fano and $-K_Y - \mathcal{L}$ is ample then the contraction φ is good. The maximal ideal of the vertex is $m_z = \bigoplus_{m>0} H^0(Y, k\mathcal{L})$. Let us note that $m_z = \varphi_*\mathcal{O}_X(-Y_0)$.

The fiber structure of the fiber Y_0 is defined by sections of bundles $k\mathcal{L}$. More precisely, at a point $y \in Y_0$ the fiber structure ideal is generated by functions $s_0^k \cdot s_k$ where s_0 is a local generator of the reduced ideal of Y_0 in X (zero section of $\mathcal{L}^*$) and $s_k \in H^0(Y, k\mathcal{L})$. Therefore the scheme theoretic structure of the fiber coincides with the geometric structure at y if and only if the exists $s_1 \in H^0(Y, \mathcal{L})$ which does not vanish at y (or $\mathcal{L}$ is generated at

y). In particular, the multiplicity of the fiber is 1, or the fiber structure coincides with the geometric structure at a general point if and only if the bundle $\mathcal{L}$ has a non-zero section. In such a case the fiber structure has embeded components at base-points of the line bundle $\mathcal{L}$. In this situation, the natural property $H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}) = \mathbf{C}$ is not true.

Let us note that the gradation in the ring $\bigoplus_{k\geq 0} H^0(Y, k\mathcal{L})$ may not coincide with the gradation of the maximal ideal m_z of the vertex. In fact these two coincide if and only if $\mathcal{L}$ is projectively normal i.e. the map $S^k(H^0(Y,\mathcal{L})) \to H^0(Y, k\mathcal{L})$ is surjective for all $k \geq 0$.

Then Z is an affine cone over the Y embedded in a projective space by $|\mathcal{L}|$. In such a case X is the blow-up of Z at z and $m/m^2 \simeq H^0(Y,\mathcal{L})$. In fact, we have the following criterion relating the contraction to the vertex and the blow-up of the vertex.

Lemma (5.2). *(c.f. [EGA], (8.8.3)) Suppose that $H^0(Y,\mathcal{L}) \neq 0$. Then X is the blow-up of Z at z i.e. $X = Proj_Z(\bigoplus_k m_z^k)$ if and only if $\mathcal{L}$ is spanned over Y.*

Proof. If $X \to Z$ is the blow-up of m_z then $\varphi^{-1}m_z$ is invertible on X and defines the fiber structure over Y_0. Therefore $\varphi^{-1}m_z = \mathcal{O}_X(-kY_0)$ for some $k \geq 1$. Since $H^0(Y,\mathcal{L}) \neq 0$, the ideal is generically reduced and hence $\varphi^{-1}m_z = \mathcal{O}_X(-Y_0)$. Thus $\mathcal{L} = \mathcal{O}_{Y_0}(-Y_0)$ is spanned. Conversely, suppose that $\mathcal{L}$ is spanned. Then, according to what we have said above, $\varphi^{-1}m_z = \mathcal{O}(-Y_0)$ and thus by the universal property of the blow-up (see [EGA] or [Ha] (7.14)) we have a map $X \to Proj_Z(\bigoplus_k m_z^k)$ which is an isomorphism.

A similar situation occurs for semiample vector bundles.

Example (5.1.2) Let $\mathcal{E}$ be a rank r semiample bundle on a smooth variety Y. That is $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is semiample, or equivalently, the symmetric power $S^k(\mathcal{E})$ is generated by global sections for $m \gg 0$ (see for instance Example (3.1)). Then similarly as above we consider $X = Spec_Y(\bigoplus_{k\geq 0} S^k\mathcal{E})$, the total space of the dual bundle $\mathcal{E}^*$ with a zero section Y_0, and the collapsing $\varphi : X \to Z = Spec(\bigoplus_{k\geq 0} H^0(Y, S^k\mathcal{E}))$ of Y_0 to the special point $z \in Z$. If Y is Fano and $-K_Y - det\mathcal{E}$ is ample then the contraction φ is good. Moreover φ is birational if and only if its top Segre class is positive. If $\mathcal{E}$ is spanned then the scheme theoretic fiber $\tilde{Y}_0$ is reduced and φ factors through the blow up of Z at z.

The good properties of the above examples can be extended to arbitrary good contractions; the assumption which is needed is nefness of the conormal of the fiber. Then the map behaves similarly as the contraction to the cone because the sections of the conormal of the fiber extend.

Lemma (5.3). *Let $\varphi : X \to Z$ be a good or crepant contraction of a smooth variety with a fiber $F = \varphi^{-1}(z)$. Assume that F is locally complete intersection and the blow up $\beta : \hat{X} \to X$ of X along F has log terminal singularities. By $\hat{F}$ we denote the exceptional divisor of the blow-up. Let $\mathcal{L}$ be an a line bundle on X such that $-K_X + \mathcal{L}$ is φ-big and nef. If the conormal bundle $N^*_{F/X}$ is nef then:*

(a) The line bundle $\mathcal{O}_{\hat{X}/X}(1) = -\hat{F}$ is $\varphi \circ \beta$-nef.

(b) Any section of N^*_F extends to a function in $\Gamma(X, \mathcal{O}_X)$ vanishing along F, that is the natural map

$$\begin{array}{rcl} \varphi^! : m_z & \longrightarrow & H^0(F, N^*_F) = H^0(\hat{F}, \mathcal{O}_{\hat{F}}(-\hat{F})) \\ m_z \ni f & \mapsto & (x \mapsto [\varphi \circ f] \in (\mathcal{I}_F/\mathcal{I}_F^2)_x) \end{array}$$

is surjective.

(c) *For $i > 0$ and $t \geq 0$ we have vanishing $H^i(F, N^*_{F/X} \otimes \mathcal{L}) = 0$.*

(d) *Some positive multiple $\mathcal{O}_{\hat{X}/X}(k) = -k\hat{F}$ is $\varphi \circ \beta$-spanned and it defines a good contraction over Z:*

$$\hat{\varphi} : \hat{X} \longrightarrow \hat{Z} = Proj_Z(\bigoplus_k (\varphi \circ \beta)_* \mathcal{O}_{\hat{X}}(-k\hat{F});$$

the scheme $\hat{Z}$ is a blow-up of Z along some ideal of a scheme supported at z.

Proof. The nefness of $-\hat{F}$ is clear. To prove (b) we consider a sequence

$$0 \longrightarrow \mathcal{O}_{\hat{X}}(-2\hat{F}) \longrightarrow \mathcal{O}_{\hat{X}}(-\hat{F}) \longrightarrow \mathcal{O}_{\hat{F}}(-\hat{F}) \longrightarrow 0.$$

Since, by assumption, $-2\hat{F} - K_{\hat{X}} = -(dimX - dimF + 1)\hat{F} - \beta^* K_X$ is $\varphi \circ \beta$-big and nef, and moreover $\hat{X}$ has good singularities it follows that $H^1(\hat{X}, \mathcal{O}_{\hat{X}}(-2\hat{F})) = 0$ and sections of $\mathcal{O}_{\hat{F}}(-\hat{F})$ extends to $\hat{X}$. This implies (b).

To prove (c) we note that $H^i(F, N^*_{F/X} \otimes \mathcal{L}) = H^i(\hat{F}, (-\hat{F} + \beta^*\mathcal{L})_{|\hat{F}})$. Moreover, as above, we note that the line bundle $-s\hat{F} + \beta^*\mathcal{L} - K_{\hat{X}}$ is $\varphi \circ \beta$-nef and big for $s \geq 0$. Now we can apply the Kawamata-Viehweg vanishing to the cohomology of the sequence

$$0 \longrightarrow (-2\hat{F} + \beta^*\mathcal{L}) \longrightarrow (-\hat{F} + \beta^*\mathcal{L}) \longrightarrow (-\hat{F} + \beta^*\mathcal{L})_{|\hat{F}} \longrightarrow 0$$

to get (c).

The claim (d) follows from (a) by Kawamata-Shokurov base-point-free theorem. Indeed, since $K_{\hat{X}} = (dimX - dimF - 1)\hat{F} + \beta^* K_X$ then $-\hat{F} - K_{\hat{X}} = -(dimX - dimF)\hat{F} - \beta^* K_X$ is $\varphi \circ \beta$-nef and big and some multiplicity of $-\hat{F}$ is spanned by global sections on $\hat{X}$. The statement on the blow-up of the ideal is general — see [Ha], (7.14).

Corollary (5.3.1). *In the situation of the previous Lemma the exceptional set $\hat{G}$ of the blow-up $\hat{Z} \to Z$ is equal to $Proj(\bigoplus_k H^0(F, S^k(N^*_F)))$ and the map $\hat{\varphi}_{\hat{F}} : \hat{F} \to \hat{G}$ is defined by the evaluation $\bigoplus_k H^0(F, S^k(N^*_F)) \to \bigoplus_k S^k(N^*_F)$. If $\hat{G}$ is irreducible and $H^0(F, N^*_F) \neq 0$ then $\hat{G}$ is a Cartier divisor in $\hat{Z}$ and $\mathcal{O}_{\hat{G}}(\hat{G}) \simeq \mathcal{O}(-1)$, where the latter bundle is defined naturally on the Proj of the graded ring. If moreover $\hat{G}$ is smooth then $\hat{Z}$ is smooth along $\hat{G}$.*

Proof. The connected part of the Stein factorization of the map $\hat{\varphi}_{|\hat{F}}$ is the evaluation map $\hat{F} \to Proj(\bigoplus_k H^0(F, S^k(N^*_F)))$ defined above. Then the map $Proj(\bigoplus_k H^0(F, S^k(N^*_F)))$ to $\hat{Z}$ is associated to the restriction

$$(\varphi \circ \beta)_* \mathcal{O}_{\hat{X}}(-k\hat{F}) \longrightarrow H^*(\hat{F}, \mathcal{O}_{\hat{F}}(-\hat{F})) \simeq H^0(F, N^*_F)$$

which is surjective because of Kawamata-Viehweg vanishing theorem.

Therefore the scheme $Proj(\bigoplus_k H^0(F, S^k(N_F^*)))$ embeds in $\hat{Z}$ as the exceptional set $\hat{G}$ of $\alpha : \hat{Z} \to Z$.

By our construction $\hat{\varphi}^*(\mathcal{O}_{\hat{Z}}(1) = \mathcal{O}_{\hat{X}}(-\hat{F})$. If $s \in H^0(F, N_F^*)$ is a non-zero section then it extends to a section of $\mathcal{O}_{\hat{X}}(-\hat{F})$ over $\hat{X}$ and it descends to a divisor $D \in \mathcal{O}_{\hat{Z}}(1)$ which does not contain $\hat{G}$. Now, using the embedding $\mathcal{O}_{\hat{X}}(-\hat{F}) \subset \mathcal{O}_{\hat{X}} \simeq \mathcal{O}_Z$ we find a global function f on Z such that $\alpha^*(f) = D + a\hat{G}$. Since the multiplicity of $\beta^*(f)$ along (at least one of the components of) $\hat{F}$ is 1 it follows that f vanishes with multiplicity 1 along $\hat{G}$ and thus $\hat{G}$ is Cartier and $\mathcal{O}_{\hat{G}}(-\hat{G}) \simeq \mathcal{O}_{\hat{G}}(1)$.

Finally to get the last claim of the Corollary we note that if a Cartier divisor is smooth then the ambient variety is smooth along the divisor in question — this is a general fact.

Proposition (5.4). *Let $\varphi : X \to Z$ be a good or crepant contraction of a smooth variety. Assume that $F = \varphi^{-1}(z)$, a geometric fiber of φ, is locally complete intersection with the conormal bundle $N_{F/X}^* = \mathcal{I}_F/\mathcal{I}_F^2$. Suppose moreover that the blow up $\beta : \hat{X} \to X$ of X along F has log terminal singularities. Then the following conditions are equivalent:*

(a) the bundle $N_{F/X}^*$ is generated by global sections on F,

(b) the invertible sheaf $\mathcal{O}_{\hat{X}}(-\hat{F})$ is generated by global sections at any point of $\hat{F}$.

(c) $\varphi^{-1}m_z \cdot \mathcal{O}_X = \mathcal{I}_F$ or, equivalently, the scheme-theoretic fiber structure of F is reduced and contains no embedded components, i.e. $\tilde{F} = F$.

(d) there exists a good contraction $\hat{\varphi} : \hat{X} \to \hat{Z} = Proj_Z(\bigoplus_k m_z^k)$ onto a blow-up of Z at the maximal ideal of z, and $\varphi^*(\mathcal{O}_{\hat{Z}}(1)) = \mathcal{O}_{\hat{X}}(1)$.

Proof. (a) implies (b) because of the previous Lemma, part (b). Claims (b) and (c) are equivalent because $\beta_*\mathcal{O}_{\hat{X}}(-\hat{F}) = \mathcal{I}_F$ and $\beta^{-1}(\mathcal{I}_F) = \mathcal{O}_{\hat{X}}(-\hat{F})$. The implication (b)⇒(d) follows by the universal property of the blow-up, since by (b) $(\varphi \circ \beta)^{-1}m_z \cdot \mathcal{O}_X = \mathcal{I}_{\hat{F}}$. The implication (d)⇒(a) is clear since $\mathcal{O}_{\hat{Z}/Z}(1)$ is spanned over Z.

Before stating the last result of this subsection let us recall that for a local ring $\mathcal{O}_{Z,z}$ with the maximal ideal m_z one defines the graded **C**-algebra $gr(\mathcal{O}_{Z,z}) := \bigoplus_k m_z^k/m_z^{k+1}$. The knowledge of the ring $gr(\mathcal{O}_{Z,z})$ allows sometimes to describe the completion ring $\hat{\mathcal{O}}_{Z,z}$, like in the Castelnuovo theorem (2.4). Also, we will say that a spanned vector bundle $\mathcal{E}$ on a projective variety Y is p.n.-spanned (p.n. stands for projectively normal) if for any $k > 0$ the natural morphims $S^kH^0(Y, \mathcal{E}) \to H^0(Y, S^k\mathcal{E})$ is surjective. As we noted while discussing the contraction to the vertex, projective normality allows us to compare gradings of rings "upstairs" and "downstairs".

Proposition (5.5). *(c.f. [Mol], 3.32]) Let $\varphi : X \to Z$ be a contraction as in the previous Proposition. Suppose moreover that N_F^* is p.n.-spanned. Then $\varphi_*(\mathcal{I}_F^k) = m_z^k$, $\varphi^{-1}(m_z^k) \cdot \mathcal{O}_X = \mathcal{I}_F^k$ and there is a natural isomorphism of graded **C**-algebras:*

$$gr(\mathcal{O}_{Z,z}) \simeq \bigoplus_k H^0(F, S^k(N_F^*)).$$

Proof. See [Mol], p.164.

(5.6). The normal bundle of a 1-dimensional fiber.
We first recall the case in which F is a fiber of dimension 1. This is well known and we will give it as a warm up before discussing the two dimensional case. Let C be an irreducible component of F. As we have seen in (4.1) C is a rational curve and it can be either a line or a conic with the respect to L; in the last case φ is of fiber type.

Let $\mathcal{I}$ be the ideal of $C \subset X$ (with the reduced structure) and consider the exact sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}_X/\mathcal{I}^2 \longrightarrow \mathcal{O}_X/\mathcal{I} \longrightarrow 0.$$

In the long cohomology sequence associated the map of global sections $H^0(\mathcal{O}_X/\mathcal{I}^2) \to H^0(\mathcal{O}_X/\mathcal{I})$ is surjective; moreover, by (1.2.1), $H^1(\mathcal{O}_X/\mathcal{I}^2) = 0$. Therefore $H^1(\mathcal{I}/\mathcal{I}^2) = 0$ which gives a bound on $N^*_{C/X} = \mathcal{I}/\mathcal{I}^2$. Namely if $N_{C/X} = \oplus\mathcal{O}(a_i)$ then $a_i < 2$. On the other hand, by adjunction, $det(N_{C/X}) = \Sigma a_i = \mathcal{O}(-2 - K_X.C)$ and thus the list of possible values of $(a_1, ..., a_{n-1})$ are finite.

If φ is a good birational contraction then we have even a better bound because, similarly as above and using (1.2.1), we actually get $H^1(N^*_{C/X} \otimes \mathcal{O}(K_X.C)) = 0$. Therefore, since $K_X.C = 1$, there is only one possibility, namely $N_{C/X} = \mathcal{O}(-1) \oplus \mathcal{O}^{(n-2)}$.

If φ is of fibre type then the estimate coming from this technique is not sufficient and one has to use other arguments. More precisely, one has to use a scheme associated to a double structure on C — see [An] — or one may use arguments coming from the deformation theory as it follows. Namely the possibilities which can occur from the above vanishing, if $n = 3$, are the following:

$$\mathcal{O} \oplus \mathcal{O}, \quad \mathcal{O} \oplus \mathcal{O}(-1), \quad \mathcal{O}(1) \oplus \mathcal{O}(-2), \quad \mathcal{O}(1) \oplus \mathcal{O}(-1).$$

We will show that the last possibility does not occur. The argument which we apply is related to the deformation technique (1.4.1) and it will be used later to deal with 2 dimensional fibers too.

Lemma (5.6.1). *The normal bundle $N_{C/X}$ cannot be $\mathcal{O}(1) \oplus \mathcal{O}(-1)$.*

Proof. Assume the contrary and let $\psi : \hat{X} \to X$ be the blow-up of X along C; let $E := \mathbf{P}(\mathcal{O}(1) \oplus \mathcal{O}(-1))$ be the exceptional divisor. Let C_0 be the curve contained in E which is the section of the ruled surface $E \to C$ corresponding to the line bundle $\mathcal{O}(-1)$. We have immediately that $E \cdot C_0 = 1$ and that ψ_{C_0} is a $1-1$ map from C_0 to C; therefore $K_{\hat{X}} \cdot C_0 = K_X \cdot C + E \cdot C_0 = -1$. In particular this implies that C_0 moves at least in a 1-dimensional family on $\hat{X}$ (see (1.4.1)); since it does not move on E it means that it goes out of E. Since C_0 is contracted by $\varphi \circ \psi$ it implies that $E.C_0 = 0$, but this is a contradiction since $E.C_0 = 1$.

For crepant contractions we have the following useful results.

Proposition (5.6.2). *(see [C-K-M, (16.6)]) Let $\varphi : X \to Z$ be a crepant contraction of a smooth 3- fold X with an irreducible (reduced) 1-dimensional fiber $f := \varphi^{-1}(z)_{\mathrm{red}}$. Then $f \simeq \mathbf{P}^1$ and the conormal bundle $N^*_{f/X} = I_f/I_f^2$ is isomorphic to either $\mathcal{O}(1) \oplus \mathcal{O}(1)$ or* *to $\mathcal{O} \oplus \mathcal{O}(2)$ or to $\mathcal{O}(-1) \oplus \mathcal{O}(3)$. In the first two cases the ideal $\mathcal{I}_f$ is spanned by global functions on X. More precisely $\mathcal{I}_f = \varphi^{-1}m_z \cdot \mathcal{O}_X$.*

Proof. The description of $N^*_{f/X}$ follows from the above arguments using the vanishing in (1.2.1) and the fact that $K_X \cdot f = 0$. The last part of the proposition follows from (5.4) (see also the proof of the next proposition).

Proposition (5.6.3). *Let $\varphi : X \to Z$ be a crepant contraction of a smooth 3- fold X with a (reduced) 1-dimensional fiber f consisting of two components, $f = f_1 \cup f_2$. Then each of the components is a smooth $\mathbf{P}^1$, they meet transversally at one point and the conormal $N^*_{f/X} = I_f/I_f^2$ is locally free and restricted to the component f_i it is isomorphic to either $\mathcal{O} \oplus \mathcal{O}(1)$ or $\mathcal{O}(-1) \oplus \mathcal{O}(2)$. If the restriction to both components is $\mathcal{O} \oplus \mathcal{O}(1)$ then $\mathcal{I}_f = \varphi^{-1}m_z \cdot \mathcal{O}_X$.*

Proof. The description of the structure of f is clear and follows from (1.5.1). The fiber then is locally complete intersection in X and the conormal is locally free. Then, as above we get $H^1(f, \mathcal{I}_f/\mathcal{I}_f^2) = 0$. This, because of the restriction to $f_i \subset f$, yields the vanishing of $H^1(f_i, (N^*_{f/X})_{|f_i})$ and thus it implies the splitting type of N^*_f on each f_i.

The global structure of N^*_f over f is determined by its restriction to each of f_i and also by the glueing above the common point $x_0 := f_1 \cap f_2$. Geometrical meaning of this is as follows: if $F_i = \mathbf{P}((N^*_{f/X})_{|f_i})$ is the appropriate ruled surface over f_i then $F := \mathbf{P}(N^*_f)$ is obtained by glueing of these two surfaces along the fiber over x_0. There are two possibilities: either the two negative sections of each of them meet together over x_0 or they do not meet. For example, if the restriction to each of the component is $\mathcal{O} \oplus \mathcal{O}(1)$ then either the bundle is globally decomposable or not. The two cases can be cohomologically distinguished when we twist the bundle by a line bundle which on each of the components is $\mathcal{O}(-1)$. Then the decomposable bundle has both cohomology of dimension 1 while both cohomology vanish if the glueing yields a non-decomposable bundle. We note that in both cases the bundle N^*_f is spanned by global sections.

The rest of the proposition follows then from (5.4) noting that the variety $\hat{X}$ has a unique singular point $\hat{x}_0$ over x_0 and that the singularity of $\hat{X}$ at $\hat{x}_0$ is of the quadric cone, in particular it is terminal.

Remark (5.6.4). In the previous proof we have pointed out 6 types of the normal bundle N_f — depending on the splitting type on each of the components and on the glueing over the common point. Among these, two types with a "double" splitting $\mathcal{O} \oplus \mathcal{O}(-1)$ are similar to the types $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and $\mathcal{O} \oplus \mathcal{O}(-2)$ in the irreducible case. Let us also note that the cohomology yields non-existence of reducible fiber and decomposable normal bundle with double splitting type $\mathcal{O}(1) \oplus \mathcal{O}(-2)$ (because then the conormal has nonzero 1-st cohomology). To the authors' knowledge the relation between the splitting type and the singularities of the general plane section is not yet completely understood (see [Re], [Ka-Mo], [Ka2]).

From (2.5) we get the following more general

Corollary (5.6.5). *Let C be a reduced 1 dimensional fiber of a good or crepant contraction of a smooth manifold X. Then the following properties are equivalent:*
(a) $N^*_{C/X}$ is nef,
(b) $N^*_{C/X}$ is spanned at a generic point of any component of C,
(c) $N^*_{C/X}$ is spanned everywhere on C.

(5.7) The normal bundle of a two dimensional fiber.

In order to understand higher dimensional fibers of good contractions we will slice them down. Thus we will need some kind of "ascending property".

Suppose that $\varphi : X \to Z$ is a good contraction of a smooth variety, $\mathcal{L}$ is an ample line bundle on X such that $-K_X - \mathcal{L}$ is φ-big and nef. Let $F = \varphi^{-1}(z)$ be a (geometric) fiber of φ. Suppose that F is locally complete intersection. Let $X' \in |\mathcal{L}|$ be a normal divisor which does not contain any component of F. Then the restriction of φ to X', call it φ', is a contraction, either good or crepant (see (1.3.2) and (4.4)). The intersection $F' = X' \cap F$ is then a fiber of φ'. The regular sequence of local generators $(g_1, \ldots, g_r)$ of the ideal of the fiber F in X descends to a regular sequence in the local ring of X' which defines a subscheme $F \cdot X'$ supported on $F' = F \cap X'$, call it $\bar{F}'$. Let us note that if the divisor X' has multipicity 1 along each of the components of F then, since a locally complete intersection has no embedded components, we get $\bar{F}' = F'$.

Lemma (5.7.1). *The scheme $\bar{F}'$ is locally complete intersection in X' and*

$$N^*_{\bar{F}'/X'} \otimes_{\mathcal{O}_{\bar{F}'}} \mathcal{O}_{F'} \simeq (N^*_{F/X})_{|F'}.$$

If moreover X' is smooth, $\mathcal{L}$ is spanned and $dim F' = 1$ then $H^1(F', (N^*_{F/X})_{|F'}) = 0$.

Proof. The first part of the lemma follows from the preceding discussion so it is enough to prove the vanishing. Let $\mathcal{J}$ be the ideal of $\bar{F}'$ in X'. From (1.2.1) we know that $H^1(\bar{F}', \mathcal{O}_{X'}/\mathcal{J}^2) = 0$ and since we have an exact sequence

$$0 \longrightarrow \mathcal{J}/\mathcal{J}^2 = N^*_{\bar{F}'/X'} \longrightarrow \mathcal{O}_{X'}/J^2 \longrightarrow \mathcal{O}_{X'}/\mathcal{J} = \mathcal{O}_{\bar{F}'} \longrightarrow 0$$

then we will be done if we show $H^0(\bar{F}', \mathcal{O}_{\bar{F}'}) = \mathbf{C}$. Since $H^1(\bar{F}', \mathcal{O}_{\bar{F}'}) = 0$ then this is equivalent to $\chi(\mathcal{O}_{\bar{F}'}) = 1$. The equality $H^0(\bar{F}', \mathcal{O}_{\bar{F}'}) = \mathbf{C}$ is clear if $\bar{F}'$ is reduced. But since $\mathcal{L}$ is spanned and F is locally complete intersection then there exists a flat deformation of $\bar{F}'$ to another intersection $F \cdot X''$ which is reduced. This is what we need, because flat deformation preserves Euler characteristic.

Now let us consider the following ascending property. Let us consider a point $x \in F'$. Suppose that the ideal of F', or equivalently $N^*_{F'/X'}$, is generated by global functions from X'. That is, there exist global functions $g'_1, \ldots g'_r \in \Gamma(X', \mathcal{O}_{X'})$ which define F' at x. Then, since $H^1(X, -\mathcal{L}) = 0$ these functions extend to $g_1, \ldots g_r \in \Gamma(X, \mathcal{O}_X)$ which define F. Thus passing from the ideal $\mathcal{I}$ to its quotient $\mathcal{I}/\mathcal{I}^2$ we get the first part of

Lemma (5.7.2). *If $N^*_{F'/X'}$ is spanned by global functions from $\Gamma(X', \mathcal{O}_{X'})$ at a point $x \in F'$ then $N^*_{F/X}$ is spanned at x by functions from $\Gamma(X, \mathcal{O}_X)$. If $N^*_{F'/X'}$ is spanned by global functions from $\Gamma(X', \mathcal{O}_{X'})$ everywhere on F' then $N^*_{F/X}$ is nef.*

Proof. We are only to proof the second claim of the Lemma. Since $F' \subset F$ is an ample section then the set where $N^*_{F/X}$ is not generate by global sections is finite in F. Therefore the restriction $(N^*_{F/X})_{|C}$ is spanned generically for any curve $C \subset F$ and consequenly it is nef.

If the fiber is of dimension 2 then we have a better extension property.

Lemma (5.7.3). *Let $\varphi : X \to Z$ be a good birational contraction of a smooth variety with an isolated 2-dimensional fiber F which is a locally complete intersection. As usually $L = -K_X$ is a φ-ample line bundle which can be assumed φ-very ample (see (1.3.4)). Then the following conditions are equivalent:*
(a) $N^*_{F/X}$ is generated by global sections at any point of F
(b) for a generic (smooth) divisor $X' \in |L|$ the bundle $N^*_{F'/X'}$ is generated by global sections at a generic point of any component of F'

Proof. The implication (a)⇒(b) is clear. To prove the converse we assume the contrary. Let S denote the set of points on F where $N^*_{F/X}$ is not spanned. Because of the extension property (5.7.2) and Corollary (5.6.5) the set does not contain F' and thus it is finite. Now we choose another smooth section $X'_1 \in |L|$ which meets F along a (reduced) curve F'_1 containing a point of S. (We can do it because L is φ-very ample.) The bundle $N^*_{F'_1/X'_1}$ is generated on a generic point of F'_1 so it is generated everywhere but this, because of the extension property, implies that $N^*_{F/X}$ is generated at some point of S, a contradiction.

Lemma (5.7.4). *Let $\varphi : X \to Z$ be a good birational contraction of a smooth 4-fold with an isolated 2-dimensional fiber $F = \varphi^{-1}(z)$. As usually $L = -K_X$ is a φ-ample line bundle which may be assumed to be φ-very ample. Then the fiber structure $\tilde{F}$ coincides with the geometric structure F unless one of the following occurs:*

(a) the fiber F is irreducible and the restriction of N_F to any smooth curve $C \in |L_{|F}|$ is isomorphic to $\mathcal{O}(-3) \oplus \mathcal{O}(1)$,
(b) $F = \mathbf{P}^2 \cup \mathbf{P}^2$ and the restriction of N_F to any line in one of the components is isomorphic to $\mathcal{O}(-2) \oplus \mathcal{O}(1)$.

Proof. Let us consider an arbitrary curve $C \in |L_{|F}|$. Since L is φ-very ample we can take a smooth $X' \in |L|$ such that $\varphi' = \varphi_{|X}$ is a crepant contraction and $C = F \cap X'$. Then considering the embeddings $C = F \cap X' \subset F \subset X$ and $C = F \cap X' \subset X' \subset X$

$$N_{C/X} = N_{C/X'} \oplus L_C = (N_{F/X})_{|C} \oplus L_C$$

and therefore $N_{C/X'} = (N_{F/X})_{|C}$. Now we apply the propositions (5.6.2) and (5.6.3) to describe $N_{C/X'}$. In particular it follows that if neither (a) nor (b) occurs then the fiber structure of the contraction φ' is reduced. Thus, using our ascending proposition (5.7.3) and the equivalence in (5.4), we conclude that $\tilde{F} = F$.

Now, let us discuss the possible exceptions described in the above lemma. If $F = \mathbf{P}^2$ then, because of the theorem of Van de Ven (see (2.7)), $N_{F/X}$ would be decomposable and in particular $h^0(N_{F/X}) - h^1(N_{F/X}) > 0$. Because of Lemma (2.11) the possible exception over an irreducible quadric would satisfy the same inequality (we use the vanishing in (5.7.1)). This, however, because of the theory of deformation see, e.g. [Ko] would imply that F moves in X which contradicts our assumption that F is an isolated 2-dimensional fiber. A similar argument works for a reducible quadric. Namely, because of (5.7.4.(b)) we can apply the theorem of Van de Ven to claim that for a component F' of F we have $(N_{F/X})_{|F'} = \mathcal{O}(-2) \oplus \mathcal{O}(1)$. Thus, in view of (2.3) if we blow-up the other component and consider the strict transform of F' then its normal would be $\mathcal{O}(-3) \oplus \mathcal{O}$. Now we see that the strict transform would move which is clearly impossible. Therefore we have proved

Theorem (5.7.5). *Let $\varphi : X \to Z$ be a good birational contraction of a smooth 4-fold with an isolated 2-dimensional fiber $F = \varphi^{-1}(z)$. Then the fiber structure $\tilde{F}$ coincides with the geometric structure F and the conormal $N^*_{F/X}$ is spanned by global sections.*

Now we can verify which one among spanned vector bundles with the appropriate c_1 is actually the conormal of an isolated 2-dimensional fiber. The result is the following

Theorem (5.7.6). *Let $\varphi : X \to Z$ be a good birational contraction of a smooth 4-fold with an isolated 2-dimensional fiber $F = \varphi^{-1}(z)$. If $F = \mathbf{P}^2$ then $N^*_{F/X}$ is either $\mathcal{O}(1) \oplus \mathcal{O}(1)$ or $T(-1) \oplus \mathcal{O}(1)/\mathcal{O})$, or $\mathcal{O}^{\oplus 4}/\mathcal{O}(-1)^{\oplus 2})$. If is a quadric (possibly singular or even reducible) then $N^*_{F/X}$ is the spinor bundle $\mathcal{S}(1)$.*

Proof. We use the classification results in (2.6) and (2.8) together with the following observations. Because of the deformation argument we know that $h^0(N_{F/X}) - h^1(N_{F/X}) \leq 0$ and thus $N^*_{F/X}$ can not be decomposable with a trivial factor. On the other hand none of the bundles with $s_2 = c_1^2 - c_2 = 0$ can occur as $N^*_{F/X}$ because of the following

Lemma (5.7.7). *Let φ be a good birational contraction from a smooth 4-fold and let F be an isolated two dimensional fiber. Then $s_2(N^*_{F/X}) > 0$.*

Proof. Assume by contradiction that $s_2 = 0$. Then from the classification of such bundles, (2.6) and (2.8), we see that $H^0(S^n(N^*_{F/X})) = S^n(H^0(N^*_{F/X}))$ is of dimension $\binom{n+3}{2}$. Thus, by (2.4) the contraction should be to a a 3-dimensional smooth point contrary to the fact that $dim Z = 4$.

Remark The cases with $s_2 = 0$ do occur in the fiber type contractions; see examples (3.5.5).

In some respect the above results about the fiber structure of a 2-dimensional fiber are nicer than one may expect. Namely, there is no multiple fiber structure, the conormal is nef and the normal of the geometric isolated fiber has no section. Thus the situation is better than for 1-dimensional isolated fibers in dimension 2 and 3: the fundamental cycle of a Du Val $D - E$ surface singularity is non-reduced and in dimension 3 one may contract an isolated $\mathbf{P}^1$ with the normal $\mathcal{O}(1) \oplus \mathcal{O}(-3)$. On the other hand, using the double covering construction (see the Examples section, (3.5)) in dimension 5 one may contract a quadric fibration over a smooth 3-dimensional base with an isolated fiber equal to $\mathbf{P}^2$, scheme theoretically the fiber is a double $\mathbf{P}^2$. Using the sequence of normal bundles and the deformation of lines argument, one may verify that in this case $N_F \simeq \mathcal{O}(1) \oplus \mathcal{E}^*$ where $\mathcal{E}$ is a rank 2 spanned vector bundle with $c_1 = 2$, $c_2 = 4$, so that $dim H^1(\mathcal{E}^*) = -\chi(\mathcal{E}^*) = 3$.

Let us also note that for a divisorial fibers we have the following:

If $F = \bigcup F_i$ is a divisorial fiber of a surjective map $X \to Y$, where X is smooth and $dim Y \geq 2$ then for some $k > 0$ the line bundle $\mathcal{O}_{F_i}(-kF_i)$ has non-trivial section and thus no multiple of $\mathcal{O}_{F_i}(F_i)$ has a section. In particular, if $rank(Pic(X/Y)) = 1$ then $\mathcal{O}_F(-F)$ is ample.

One can then try to conjecture that if F is an isolated fiber of a (good) contraction which is locally complete intersection and with "small" codimension then $H^0(F, N_F) = 0$.

(5.8) The above result on contractions of 4-folds can be generalized for the adjunction mappings of an n-fold. Namely, suppose that $\varphi : X \to Y$ is a good contraction of a smooth n-fold X supported by a divisor $K_X + (n-3)H$, where H is a φ-ample divisor on X. Since we are interested in the local desription of φ around a non-trivial fiber $F = \varphi^{-1}(z)$, we may assume that the variety Z is affine.

Corollary (5.8.1). *Let us assume that φ is birational and that F is an isolated fiber of dimension $n-2$. If $n \geq 5$ then the contraction φ is small and F is an isolated non-trivial fiber. More precisely $F \simeq \mathbf{P}^{n-2}$ and $N_{F/X} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and there exists a flip of φ (see [A-B-W]).*

Proof. The proof follows by easy slicing method: by [A-W], (5.1), the line bundle H is φ-spanned, so there exists a smooth hypeplane section $\in |H|$ which has the same properties as X. Indeed, the restriction of φ to the hyperplane section is a good contraction (see [A-W], (2.6)) supported by $K + (n-4)H$ and, because of Bertini theorem, the hyperplane section of F is an isolated fiber of dimension $n-3$. Thus we can arrive to the 4-dimensional linear section X' of X which we know by (5.7.6). In particular we know the surface section F' of the fiber F and its normal $N_{F'/X'}$. As in (5.7.1) we note that $N_{F'/X'} \simeq (N_{F/X})_{F'}$. Therefore it is enough to verify which of the pairs $(F', N_{F'/X})$ is a plane section of a higher-dimensional pair. Also, we note that because of Lemma (5.7.2) $N^*_{F'/X'}$ extends to a nef vector bundle.

If $F' \simeq \mathbf{P}^2$ then it must be a section of the projective space. But then, among the bundles occuring in (5.7.6), only the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and $T\mathbf{P}^2(-1) \oplus \mathcal{O}/\mathcal{O}(-1)$ extend to nef vector bundles on $\mathbf{P}^3$ (see [Sz-W2]). The bundle $T\mathbf{P}^2(-1) \oplus \mathcal{O}/\mathcal{O}(-1)$ extends to the null-correlation bundle on $\mathbf{P}^3$ and this one can occur as the normal of an isolated 3-dimensional fiber of a fiber type contraction. Thus, by a similar argument as in the proof of (5.7.7) this case can not occur in the case of a birational contraction. Indeed, because of the nefness we have the vanishing of $H^1(F, S^k(N^*_{F/X}))$ and since F with such a normal is a fiber of a good contraction to a smooth 4-dimensional point it follows that we can apply (2.4): The existence of flip and other results are proved in the Lemma (6.1) of the next section.

A similar argument works if F' is a quadric. First we note that F can not be reducible quadric since $T\mathbf{P}^2$ does not extend. A quadric $\mathbf{Q}^3$ with the normal $\mathcal{S}$ can be a fiber of a fiber type contraction to a smooth point and again, by cohomological argument, which actually depends only on the nefness and Chern classes of the bundles in question, it can not be a fiber of a birational contraction.

(5.9) **The case of a conic fibration.**
Note that the preceding arguments, which led to the classification of the birational 4-dimensional case, depend on the isomorphism $\varphi'_*\mathcal{O}_{X'} \simeq \mathcal{O}_Z \simeq \varphi_*\mathcal{O}_X$. This fails to be true if φ is of fiber type. Namely, let $\varphi : X \to Z$ be a conic fibration, i.e. a good contraction such that $dimZ = dimX - 1$. As usually, we will assume that F is an isolated 2 dimensional fiber of φ and $L = -K_X$ is φ-spanned. Then the restriction of φ to a general section $X' \in |L|$ is generically $2:1$ covering of Z. Let us assume that X' is connected. The push-forward of the divisorial sequence for X' yields the exact sequence

$$0 \longrightarrow \varphi_*\mathcal{O}_X = \mathcal{O}_Z \longrightarrow \varphi_*\mathcal{O}_{X'} \longrightarrow R^1\varphi_*\mathcal{O}_X(K_X) \longrightarrow 0 \qquad (5.9.1)$$

of $\mathcal{O}_Z$-modules. Moreover, by (1.2.4) $R^1\varphi_*\mathcal{O}_X(K_X) = \omega_Z$. If φ is an elementary contraction then Z is algebraically factorial hence $\omega_Z = K_Z$ is Cartier and Z is Gorenstein. Thus $\varphi_*\mathcal{O}_{X'}$ is locally free $\mathcal{O}_Z$-module.

Let $\varphi = \pi\circ\varphi'$ be the Stein factorization of φ, where the map $\pi : Z' \to Z$ is a $2:1$ cover and φ' a crepant contraction (see (4.4)) so that Z' is Gorenstein. Since $\pi_*\mathcal{O}_{Z'} = \varphi_*\mathcal{O}_{X'}$ is locally free, it follows that π is flat. We consider the *trace map*, $tr : \pi_*\mathcal{O}_{Z'} \to \mathcal{O}_Z$, which over an open $U \subset Z$ is defined as it follows (for details see [A-K], p. 123). The sheaf $\pi_*\mathcal{O}_{Z'}$ is free over $\mathcal{O}_Z$ and every element f in $\pi_*\mathcal{O}_{Z'}(U)$ defines a $\mathcal{O}_Z(U)$-homomorphism of the free $\mathcal{O}_Z(U)$-module $\pi_*\mathcal{O}_{Z'}(U)$; we define $tr(f)$ to be the trace of this homomorphism. The map tr splits (5.9.1).

Another proof of the splitting of (5.9.1), independent on the assumption that φ is elementary, is provided by [Ko2, Cor. 2.25]. We note that also in this case, since $\varphi_*\mathcal{O}_{X'}$ is reflexive (because Z is normal and $\varphi_{|X'}$ contracts no divisor) thus, by the splitting, $R^1\varphi_*\mathcal{O}_X(K_X)$ is reflexive, hence invertible. Therefore, as before, Z is Gorenstein and π is flat.

So π is a double cyclic cover. The kernel of its trace map is equal to $\pi_*\mathcal{I}_R = \mathcal{I}_B$, where R and B are, respectively, ramification and branching divisors of π, and both are Cartier.

Now suppose that F is as in (5.0), i.e. it is locally complete intersection and the blow-up of X at F has log terminal singularities. Let, as in (5.7), $F' = F \cap X'$ and set $N^* = N^*_{F/X}$. We note that Lemma (5.7.1) remains true also in this case and in particular we have the restriction $res : H^0(F, N^*) \to H^0(F', N^*_{|F'}) = H^0(F', N^*_{F'/X'})$. Suppose that $N^*_{F'/X'}$ is nef. Then, acccording to (5.3.b) any section of $N^*_{F'/X'}$ extends to a section of X' and we can apply tr to this extension. As the result we obtain a well defined map $tr : H^0(F', N^*_{F'/X'}) \to H^0(F, N^*_{F/X})$ such that $res \circ tr = id$. Since R is Cartier and it gives the kernel of $tr : \pi_*\mathcal{O}_{Z'} = \pi_*(\varphi'_*\mathcal{O}_{X'}) \to \mathcal{O}_Z$ it follows

Lemma (5.9.2). *The kernel of* $tr : H^0(F', N^*_{F'/X'}) \to H^0(F, N^*_{F/X})$ *is of dimension one at most and it is generated by the class of* $f_R \circ \varphi'$ *where the function* f_R *generates the ideal of* R *at* $z' = \varphi'(F')$.

Now, using Lemma (5.9.2) we investigate the structure of the conic type contraction in dimension 4. We will concentrate on the case when F is a projective space or an irreducible quadric.

Lemma (5.9.3). *Let* $\varphi : X \to Z$ *be a conic fibration of a smooth 4-fold. Suppose that* F *is an irreducible isolated 2-dimensional fiber of* φ *such that* (F, L_F) *is either* $(\mathbf{P}^2, \mathcal{O}(1))$, *or* $(\mathbf{F}_0, C_0 + f)$, *or* $(\mathbf{S}_2, \mathcal{O}(1))$. *Then either* $N^*_{F/X}$ *is nef or it is one of the exceptional bundles: either (2.7.0) or* $\mathcal{O}(2,0) \oplus \mathcal{O}(-1,1)$, *or (2.11.0), respectively.*

Proof. We use the notation introduced above, i.e. $X' \subset X$ is a linear section of X, $F' = X' \cap F$ and so on. In view of the vanishing (1.2.2) and Lemma (4.7.1), the first cohomology of the restriction of $N^* := N^*_{F/X}$ to $F' \subset F$ vanishes. In view of the results of Section 2 (2.7–2.11), either the general splitting type of N^* is $(1,1)$ or N^* is among some very special exceptions.

We claim that if for a general smooth $F' \subset F$ there is $N^*_{|F'} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ then the bundle N^* is nef. Let consider the image of trace $\Lambda := im(tr) \subset H^0(F, N^*)$ which gives rise to (at least) 2 dimensional linear system $|\Lambda| \subset |H^0(\mathbf{P}(N^*, \mathcal{O}_{\mathbf{P}(N^*)}(1)))|$. Since $N^*_{|F'} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ thus over $F' \subset F$ the system $|\Lambda|$ has at most one base point. Therefore, if a curve $l_0 \subset \mathbf{P}(N^*)$ has negative intersection with $\mathcal{O}_{\mathbf{P}(N^*)}(1)$ then it descends to a line $l \subset F$. Because of the vanishing (1.2.2) we have $N^*_{|l} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(3)$ or $\mathcal{O}(-1) \oplus \mathcal{O}(2)$, depending on whether F is $\mathbf{P}^2$ or a quadric. Therefore $l_0 \cdot \mathcal{O}_{\mathbf{P}(N^*)}(1) = -1$. Now the argument is similar to the one in the proof of (5.6.1). Namely, let us consider a blow-up $\hat{X}$ of X along F. We compute that $l_0 \cdot K_X = 0$ and thus, because of (1.4.1), l_0 moves in one dimensional family in $\hat{X}$ (if $F = \mathbf{S}^2$ then $\hat{X}$ is singular and one has to use the full strength of [Ko, II.1.14]). This implies that deformations of l_0 sweep out a divisor in $\mathbf{P}(N^*)$ which dominates F and contradicts the fact that the base locus of $|\Lambda|$ over F' has one point only.

Now it remains to verify which of the bundles listed in (2.7.1) and (2.11) are good candidates for N^* in case when it is not nef. The decomposable bundles on $\mathbf{P}^2$ as well as some on the quadrics do not verify $h^0(N) - h^1(N) \leq 0$ condition. Moreover, the bundles presented in (2.11) (a-ii) and (b-ii) contain only a finite number of lines with splitting type $(-1, 2)$ and thus are excluded by the above deformation argument. Thus we are left with the exceptions which are listed in the present lemma.

A similar argument proves the following

Lemma (5.9.4). *In the situation of (5.9.3) if* $(F, L_F) = (\mathbf{P}^2, \mathcal{O}(2))$ *then* $N^* \simeq T\mathbf{P}^2(-1)$.

Proof. In view of (2.7) it is enough to prove that the splitting type of N^* on any line is $(0,1)$. (It is easy to check that if $N^* \simeq \mathcal{O}(1) \oplus \mathcal{O}$ then F is a non-isolated 2-dimensional fiber.) If it is not the case, then as above we consider the blow-up $\hat{X}$ and in $\hat{X}$ over a line $l \subset X$ we have a rational curve l_0 such that $l_0 \cdot K_{\hat{X}} = -1$. Now l_0 moves in a 2 dimensional family which would imply that all lines in F have splitting type $(-1,2)$ and thus $N^* \simeq \mathcal{O}(-1) \oplus \mathcal{O}(2)$ which contradicts $h^0(N) - h^1(N) \leq 0$ condition.

Now we deal with the exceptional bundles which are singled out in (5.9.3). First, however, let us make an observation concerning its proof. We use the notation from the proof of (5.9.3), i.e. we consider a smooth section $F' \subset F$, and we assume that $N^*_{|F'} \simeq \mathcal{O} \oplus \mathcal{O}(2)$. Let us consider a 2-dimensional linear system $|\Lambda| \subset |H^0(\mathbf{P}(N^*), \mathcal{O}(1))|$ which is the image of tr. Then the argument, which shows the nefness of N^*, fails only if the base point locus of $|\Lambda|$ over F' is the exceptional (-2)-curve $C_0 \subset \mathbf{P}(N^*_{|F'}) \simeq \mathbf{F}_2$. This is possible only if the kernel of $tr : H^0(F', N^*_{|F'}) = H^0(\mathbf{P}(N^*_{|F'}), \mathcal{O}(1)) \to H^0(F, N^*)$ contains a section, zero locus of which does not meet C_0. In other words, suppose that we take $\beta' : \hat{X}' \to X'$, the blow-up of X' along F' with the exceptional divisor $\hat{F}'$, and we take the strict transform $\hat{R}$ of the ramification divisor $R = \{f_R = 0\}$ then $\hat{R} \equiv (\beta' \circ \varphi')^*(R) - \hat{F}'$ and $\hat{R} \cap \hat{F}'$ does not meet C_0 — see Lemma (5.9.2).

Now, f_R^2 is in $\mathcal{O}_Z$ and it defines the branch divisor B. Let us note that $\varphi^* B \cdot X' = 2\varphi'^* R$. We take the blow-up $\beta : \hat{X} \to X$ of X along F with the exceptional divisor $\hat{F}$. Then, the strict transform $\hat{B}$ is equivalent to $-2\hat{F}$ and over F' it is equal to $2(R \cdot \hat{F}')$. But — as we noticed in the proof of (5.9.3) — the base point locus of $-2\hat{F}_{|\hat{F}} = \mathcal{O}_{\mathbf{P}(N^*)}(2)$ contains a divisor sweapt out by curves which have negative intersection with $-\hat{F}_{|\hat{F}}$ and, as it follows from the description of the bundles in question, over F' the base point locus contains the curve C_0. This is in contradiction with our observation that $\hat{R} \cap \hat{F}' \cap C_0 = \emptyset$. Thus we have proved

Proposition (5.9.5). *Let $\varphi : X \to Z$ be a conic fibration of a smooth 4-fold. Suppose that F is an irreducible isolated 2-dimensional fiber of φ which is either a projective plane or a quadric. Then the conormal bundle $N^*_{F/X}$ is nef.*

Since $N^*_{F/X}$ is nef then one can use the results presented in the beginning of the section to describe the contraction φ around the fiber F. Indeed, by the results of [S-W1] (c.f. (5.7.3)), nefness of such bundles implies their spannedness and one can use their explicit classification (see (2.6) and (2.8)). In particular one gets the following result, the proof of which is similar to that of (5.7.6) (see also Section 6):

Theorem (5.9.6). *Let $\varphi : X \to Z$ be a conic fibration of a smooth 4-fold. Suppose that $F = \varphi^{-1}(z)$ is an irreducible isolated 2-dimensional fiber. If $F \simeq \mathbf{P}^2$ then $N^*_{F/X} \simeq \mathcal{O}^3/\mathcal{O}(-2)$. If F is an irreducible quadric then $N^*_{F/X}$ is the pullback of $T\mathbf{P}^2(-1)$ via some double covering of $\mathbf{P}^2$. In both cases the fiber structure $\tilde{F}$ coincides with the geometric structure on F and Z is smooth at z.*

The remaining cases of 4 dimensional conic fibrations with isolated 2 dimensional fibers, including these which are non-reducible or non-locally complete intersection, will be treated separately in our forthcoming paper.

6. Geometry of a contraction of a 4-fold.

(6.0) In this section $\varphi : X \to Z$ will be a birational contraction of a smooth 4-fold with an isolated two dimensional fiber $F = \varphi^{-1}(z)$. In the previous section we described the normal of F. We proved that $N^*_{F/X}$ is spanned by global sections and thus we have a contraction map over Z of the blow-up $\hat{X}$ of X at $\mathbf{F}$ to $\hat{Z}$, the blow-up of Z at z (see (5.4)). Over the exceptional divisor $\hat{F}$ the map $\hat{\varphi}$ is associated to the evaluation of sections of $N^*_{F/X}$. Therefore, knowing the list of possible normal bundles, we can deduce the description of $\hat{\varphi}$. In particular, we note that $\hat{\varphi}$ is an isomorphism if and only if $F = \mathbf{P}^2$ and $N^*_F = \mathcal{O}(1) \oplus \mathcal{O}(1)$ which is exactly the case when φ is a small contraction. Indeed, we have a more general

Lemma (6.1). *Let $\varphi : X \to Z$ be a good or crepant contraction of a smooth variety with a fiber $F = \varphi^{-1}(z)$. Assume that F is locally complete intersection and the blow up $\beta : \hat{X} \to X$ of X along F has log terminal singularities. If $N^*_{F/X}$ is ample then the exceptional locus of φ is equal to F. If F is not a divisor and $\mathcal{L}$ is a φ-ample line bundle then a flip of φ is defined by a contraction of $\hat{X}$ supported by $-\hat{F} - \tau\mathcal{L}$, where τ is a rational number such that $-\hat{F} - \tau\mathcal{L}$ is nef but not ample.*

Proof. Outside of F and, respectively, $\hat{F}$ the exceptional loci of φ and $\hat{\varphi}$ coincide. On the other hand, because of the local nature of our set-up, $\hat{\varphi}$ is an isomorphism on $\hat{X}$ if and only if N^*_F is ample. This proves the first statement. The last part of the lemma is clear.

The above lemma can not be inversed since in dimension 3 we have a contraction of an isolated $\mathbf{P}^1$ with normal $\mathcal{O} \oplus \mathcal{O}(-2)$ or $\mathcal{O}(1) \oplus \mathcal{O}(-3)$. In our case, however, we can by using either the result of Kawamata [Ka1] or applying a direct argument based on the deformation theory, especially the inequality in [Wi] (Theorem (1.1)). More precisely, using this inequality one proves:

Lemma (6.2). *Let $\varphi : X \to Z$ be a good contraction of a smooth variety with an irreducible fiber F which is locally complete intersection in X. Suppose that $\hat{X}$ is smooth, N^*_F is nef (therefore semiample) and that the map of $\mathbf{P}(N^*_F)$ associated to high multiple of $\mathcal{O}_{N^*_F}(1)$ is birational with general non trivial fiber of dimension 1. Then the exceptional set of φ contains a divisor.*

Proof. By our assumptions the map $\hat{\varphi}$ supported by $-\hat{F}$ is birational with some 1-dimensional fibers and therefore, by dimension estimate (see [Wi]) the exceptional set of $\hat{\varphi}$ contains a divisor which is not $\hat{F}$ (because the map is birational on $\hat{F}$).

Corollary (6.2.1). *Let $\varphi : X \to Z$ be a birational good contraction of a 4-fold with an isolated two dimensional fiber F. Then φ is small if and only if $F = \mathbf{P}^2$ and $N_{F/X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.*

Now we assume that $n = 4$ and that the contraction $\varphi : X \to Z$ is divisorial with an exceptional set E mapped to a surface S and the isolated 2-dimensional fiber $F = \varphi^{-1}(z)$. Using the results from the previous section we can consider $\beta : \hat{X} \to X$, the blow-up of X along F with the exceptional divisor $\hat{F}$, $\alpha : \hat{Z} \to Z$, the blow-up of Z at the maximal ideal of z with the exceptional set $\hat{G}$, and the map $\hat{\varphi} : \hat{X} \to \hat{Z}$, which is a good contraction supported by $-\hat{F}$. Let $\hat{E}$ and $\hat{S}$ denote, respectively, the strict transforms of E and S. All the objects are presented on the following diagram.

$$
(*) \qquad
\begin{array}{ccccc}
 & & \hat{X} \supset (\hat{E}, \hat{F}) & & \\
 & \beta \swarrow & & \searrow \hat{\varphi} & \\
X \supset (E \supset F) & & & & \hat{Z} \supset (\hat{S}, \hat{G}) \\
 & \varphi \searrow & & \swarrow \alpha & \\
 & & Z \supset (S \ni z) & &
\end{array}
$$

We will provide a description of all these objects and, in particular, the singularities of Z at z. Actually, we can get the description of Z at z by using (2.4) and (5.5), however, we find it interesting to provide a complete description of all the objects which occur in this blow-up-blow-down construction.

Proposition (6.3). *Let $\varphi : X \to Z$ be a divisorial contraction of a 4-fold X with an isolated two dimensional fiber $F \simeq \mathbf{P}^2$. Then $\hat{Z}$ and $\hat{S}$ are smooth varieties and $\hat{\varphi}$ is the blow-up of $\hat{Z}$ along $\hat{S}$.*

(a) If $N^*_{F/X} = T(-1) \oplus \mathcal{O}(1)/\mathcal{O}$ then $\hat{G}$ is a smooth quadric $\mathbf{Q}^3$ and $N_{\hat{G}/\hat{Z}} = \mathcal{O}_{\mathbf{Q}^3}(-1)$. The map α is the contraction of $\mathbf{Q}^3$ to an isolated singular point $z \in Z$ which is analytically isomorphic to the quadric cone singularity. The surface $S \subset Z$ is smooth.

(b) If $N^*_{F/X} = \mathcal{O}^{\oplus 4}/\mathcal{O}(-1)^{\oplus 2}$ then $\hat{G}$ is equal to $\mathbf{P}^3$ and $N_{\hat{G}/\hat{Z}} = \mathcal{O}_{\mathbf{P}^3}(-1)$. The map α is a blow-down of $\hat{G}$ to a smooth point $z \in Z$. The surface S has a singularity in z of the type of the vertex of a cone over a twisted cubic.

Proof. The variety $\hat{X}$ is smooth and by Lemma (2.6) the map $\hat{\varphi}_{|\hat{F}}$ is either the blow-up of a smooth three dimensional quadric along a line or the blow-up of $\mathbf{P}^3$ along a twisted cubic curve. Consequently, the map $\hat{\varphi}$ is a divisorial contraction with all non trivial fiber of dimension 1. Therefore, by (6.1), $\hat{E}$ is smooth and $\hat{\varphi}$ blows it down to a smooth $\hat{S}$ in smooth $\hat{Z}$.

If m is the multiplicity of E along F, then we have the numerical equivalence

$$K_{\hat{X}} = \beta^* K_X + \hat{F} = \beta^* E + \hat{F} = \hat{E} + (m+1)\hat{F}$$

and also $K_{\hat{X}} = \hat{\varphi}^* K_{\hat{Z}} + \hat{E}$. Therefore $\hat{\varphi}^* K_{\hat{Z}} = (m+1)\hat{F}$ and, using the adjunction formula $(K_{\hat{Z}} + \hat{G})_{|\hat{G}} = K_{\hat{G}}$, we obtain that $(m+2)\hat{G}_{|\hat{G}} = \mathcal{O}(-3)$ or $\mathcal{O}(-4)$ if $\hat{G}$ is a quadric or $\mathbf{P}^3$, respectively. This implies that the normal of $\hat{G}$ in $\hat{Z}$ is in both cases $\mathcal{O}(-1)$ (which we know also from Corollary (5.3.1)) and also that $m = 1$, if $\hat{G}$ is a quadric, while $m = 2$ if $\hat{G}$ is $\mathbf{P}^3$ (this will be reproved later in (6.7)). Since the above multiplicity coincides with the degree of the projection $\hat{E} \cap \hat{F} \to F$ (which we know because of the description of N^*_F) it follows that $\hat{E}$ and $\hat{F}$ intersect transversaly. The description of the singularity of Z follows thus immediately (see also (5.5)).

Now we discuss the singularity of S at z. For this purpose we consider the intersection curve $f := \hat{G} \cap \hat{S}$. In both cases $f \simeq \mathbf{P}^1$ and the curve is either a line or a twisted cubic curve, if $\hat{G} \simeq \mathbf{Q}^3$ or $\mathbf{P}^3$, $N_{f/\hat{S}} = \mathcal{O}(-1)$ or $\mathcal{O}(-3)$ if $G \simeq \mathbf{Q}^3$ or $\mathbf{P}^3$, respectively. This provides the description of the singularity of S.

Proposition (6.4). *Let $\varphi : X \to Z$ be a divisorial contraction of a 4-fold X with an isolated two dimensional fiber F which is a, possibly singular (even reducible), quadric with conormal bundle equal to $\mathcal{S}(1)$. Then $\hat{Z}$ is smooth, $\hat{G} = \mathbf{P}^3$ and $N_{\hat{G}/\hat{Z}} = \mathcal{O}(-1)$. Thus $z \in Z$ is a smooth point and α is the blow-up of $z \in Z$. The surface S is smooth outside of z and it has an isolated non-normal point at z.*

Proof. Although we can make an argument similar to the one used in the previous Proposition, it is definitely much more convenient to use Corollary (5.3.1). First, however, we note that the blow-up of X along F has good singularities so that we can apply that result. Now, we note that $(Proj(\bigoplus_k H^0(F, S^k(N_F^*))), \mathcal{O}(1)) \simeq (\mathbf{P}^3, \mathcal{O}(1))$ — this follows e.g. from our examples (3.2) or can be verified directly. Thus, because of (5.3.1), $\hat{G} \simeq \mathbf{P}^3$, $\mathcal{O}_{\hat{G}}(-\hat{G}) \simeq \mathcal{O}(1)$, then $\hat{Z}$ is smooth along $\hat{G}$ and consequently, Z is smooth at z. The multiplicity of E along F can be computed similarly as in the case of $F \simeq \mathbf{P}^2$ — provided that the fiber is ireducible. Indeed, in such a case the singularities of $\hat{X}$ are $\mathbf{Q}$-factorial and one can make a computation involving Cartier divisors to prove that the multiplicity is 2. Also, in this case, one can describe the singularities of S in terms of $\hat{S}$: if $F \simeq \mathbf{P}^1 \times \mathbf{P}^1$, then $\hat{S}$ is smooth (by the same argument as in the case of $F = \mathbf{P}^2$) and the intersection $\hat{S} \cap \hat{G}$ is transversal and consits of two disjoint lines. Therefore the singularity of S is of type of two tranversal planes in a 4-space.

If F is a quadric cone, then the intersection $\hat{S} \cap \hat{G}$ consits of a line along which $\hat{S}$ is not normal. Indeed, we can take a hyperplane section Y of the manifold $\hat{Z}$ and then its inverse image $\hat{Y} \subset \hat{X}$. The variety $\hat{Y}$ is Gorenstein, it has an isolated singular points (quadric cone singularities) above the vertex of F and the contraction $\hat{Y} \to Y$ is a divisorial good contraction. Therefore by Cutkosky [Cu] the contraction $\hat{Y} \to Y$ is the blow-up of Y along a curve $Y \cap \hat{S}$ which is locally intersection and has (planar) singularities on $\hat{G}$. In other words, the singularities of $\hat{S}$ are of the double point type and the whole situation is a degeneration of the two non-meeting (-1)-lines which occur in the case of $F = \mathbf{F}_0$.

If $F \simeq \mathbf{P}^2 \cup \mathbf{P}^2$ then $\hat{X}$ is not $\mathbf{Q}$ factorial and in fact the exceptional divisor of the elementary contraction $\hat{f} : \hat{X} \to \hat{Z}$ is reducible and consists of the strict transform $\hat{E}$ and one of the components of $\hat{F}$ which is the projectivisation of $T\mathbf{P}^2(-1)$. The situation is better understood with an alternative approach based on the study of a component of a Hilbert scheme of X. We will do it in the remainder of this section, postponing the conclusion of the proof of the Proposition until the end of the section.

(6.5) An alternative way of describing the geometry of φ around the special fiber is based on the Hilbert scheme of lines in fibers of the good contraction. In the situation of a birational contraction as in (6.0) a *line* is a proper rational curve over Z whose intersection with L is 1. Suppose that the contraction φ is divisorial and the exceptional locus is an irreducible divisor E. The general fiber of $\varphi_{|E}$ is then a line with the normal $\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1)$. Let $\mathcal{H}$ be a component of the Hilbert scheme of X over Z which parametrizes general fibers of $\varphi_{|E}$. By general properties of Hilbert scheme the component $\mathcal{H}$ admits a morphism $\varphi_{\mathcal{H}} : \mathcal{H} \to Z$ such that $\varphi_{\mathcal{H}}([l]) = \varphi(l)$ where $[l]$ represents a line $l \subset X$. We note that $\varphi_{\mathcal{H}}$ maps $\mathcal{H}$ birationally to the surface $\varphi(E)$. The exceptional set of $\varphi_{\mathcal{H}}$ is over $z \in \varphi(E)$ and it parametrizes lines in $F = \varphi^{-1}(z)$ which are limits of *general lines*. We will say that such lines can *move out* of F. Since the incidence variety of lines over $\mathcal{H}$ maps onto E it follows that such lines cover F and thus $\varphi_{\mathcal{H}}^{-1}(z)$ is 1-dimensional.

Suppose that $F_1 \simeq \mathbf{P}^2$ is a component of the fiber F. Lines in F_1 are parametrized by another component of the Hilbert scheme of X, we call it $F_1^\vee$. The component $F_1^\vee$ is just a plane dual to F_1, i.e. $F_1^\vee \simeq \mathbf{P}^2$. The intersection $F_1^\vee \cap \mathcal{H}$ contains those lines which move out of F_1.

Lemma (6.6). *Suppose that the conormal bundle $N^*_{F_1/X}$ is as in (5.7.6). In particular, let us assume that $N^*_{F_1/X}$ is generically spanned and its restriction to a generic line is $\mathcal{O}(1) \oplus \mathcal{O}(1)$. A line $l \subset F_1$ moves out of F_1 if and only if it is a jumping line of the bundle $N_{F_1/X}$, which means that the restriction of the latter bundle to l is either $\mathcal{O} \oplus \mathcal{O}(-2)$ or $\mathcal{O}(1) \oplus \mathcal{O}(-3)$.*

Proof. We have an exact sequence of vector bundles on a line $l \subset F_1$

$$0 \longrightarrow N_{l/F_1} \simeq \mathcal{O}(1) \longrightarrow N_{l/X} \longrightarrow (N_{F_1/X})_l \longrightarrow 0$$

and therefore $H^1(l, N_{l/X}) \neq 0$ if and only if l is a jumping of $N_{F_1/X}$. If $l \in F_1^\vee \cap \mathcal{H}$ then the Hilbert scheme is singular at l and thus, by a general property of the Hilbert scheme, the above cohomology does not vanish. Conversely, suppose that l is a jumping line of $N_{F_1/X}$. We may assume that the restriction of $N_{F_1/X}$ to l is $\mathcal{O} \oplus \mathcal{O}(-2)$ (because the jumping lines of the other type are limits of these, if the conormal is spanned). Let $\pi : \hat{X} \to X$ be the blow-up of X along F_1 with the exceptional divisor $\hat{F}_1$. Then over l we have a unique curve l_0 such that $l_0.\hat{F}_1 = 0$. Therefore $K_{\hat{X}}.l_0 = -1$ and computing the deformation of l_0 on $\hat{X}$ we find out that it moves in in a 2-dimensional family. So l_0 moves out of $\hat{F}_1$ and thus l moves out of F_1.

Let us note that the above argument describes the intersection $\hat{E} \cap \hat{F}_1$ in the manifold $\hat{X}$, where $\hat{E}$ is the strict tranform of the divisor E. Namely, the intersection is equal to the locus of all sections l_0 which are described above. Knowing the list of possible bundles (see (5.7.6)) we find out that $\hat{E}_1 := \hat{F}_1 \cap \hat{E} \subset \hat{F}_1$ is the unique irreducible divisor in $\hat{F}_1$ such that $\hat{E}_1.l_0 = -1$. On the other hand, since l_0 moves out of $\hat{F}_1$ and becomes a fiber of contraction of $\hat{E}$ we seee that $\hat{E}.l_0 = -1$ as well. This implies however that the intersection of $\hat{F}_1$ and $\hat{E}$ is transversal, or equivalently, $\mathcal{O}_{\hat{F}_1}(\hat{E}) = \mathcal{O}_{\hat{F}_1}(\hat{E}_1)$. Therefore we can count the multiplicity of E along F_1 using the jumping lines of N_{F_1}. The result is as it follows.

Lemma (6.7).

$$mult_{F_1} E = c_2(N_{F_1/X}) - 1$$

Proof. The left-hand-side of the above formula is equal to the intersection number of the divisor $\hat{E}$ with a fiber of the blow-down $\hat{X} \to X$. The right-hand-side is equal to the degree of the curve of jumping lines of $N_{F_1/X}$ in the dual plane $F_1^\vee$ which is the same as the number of jumping lines passing through a generic point of F_1 or, equivalently, the degree of the map $\hat{E}_1 \to F_1$. Since $\mathcal{O}_{\hat{F}_1}(\hat{E}) = \mathcal{O}_{\hat{F}_1}(\hat{E}_1)$ then the above equality follows.

If F is a quadric $\mathbf{F}_0$ or $\mathbf{S}_2$ then a similar argument can be applied and the result is the following

Lemma (6.8). *If the fiber F is an irreducible quadric then $mult_F E = 2$.*

Proof. We use the notation from the proof of the previous lemma. If $F = F_1 = \mathbf{F}_0$ then $\hat{E}_1$ consits of two components which are sections of $\hat{F}_1 \to F$; consequently the argument can be applied to each of them. If $F = F_1 = \mathbf{S}_2$ then $\hat{E}_1$ is not Cartier but $2\hat{E}_1$ is Cartier and we check that, as above, $2\hat{E}_1 = \hat{E}_{F_1}$.

Let $\hat{\mathcal{H}}$ be a component of the Hilbert scheme of $\hat{X}$ which contains a (lift-up of) general line in E. Then there is a map $\hat{\mathcal{H}} \to \mathcal{H}$ which is identity outside of the set parametrizing lines lying in F. Moreover, if F is irreducible then, by the proof of Lemma (6.6) the map $\hat{\mathcal{H}} \to \mathcal{H}$ is bijective over F as well. If F is irreducible then all fiber of the contraction $\hat{X} \to \hat{Z}$ are of dimension 1 (we know that all the fibers of the map $\hat{E} \to \hat{S}$ are 1-dimensional) and therefore the natural map $\hat{\mathcal{H}} \to \hat{S}$ is bijective.

(6.9) Now let us discuss the case of $F = \mathbf{P}^2 \cup \mathbf{P}^2$. If F is reducible then the Hilbert scheme parametrizing lines consists of 3 irreducible components: $\mathcal{H}$ and $F_1^\vee$, $F_2^\vee$, the latter two are just dual $\mathbf{P}^2$s. Because of Lemma (6.7), after possible renumeration of them we have $mult_{F_1}(E) = 2$ and $mult_{F_2}(E) = 3$. According to the lemma (6.6) the components meet along subsets parametrizing jumping lines. In particular, the curve $F_1^\vee \cap \mathcal{H}$ is a reducible conic (two lines) in $F_1^\vee$ and $F_2^\vee \cap \mathcal{H}$ is a rational cubic in $F_2^\vee$. The components $F_1^\vee$ and $F_2^\vee$ meet at one point which is the singularity of each of the two preceeding curves. The point at which all the three components meet is associated to the unique line $F_1 \cap F_2$ with the normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-3)$.

The exceptional of divisor $\hat{F}$ consits of two components: $\hat{F}_1 = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(1))$ and $\hat{F}_2 = \mathbf{P}(T\mathbf{P}^2(-1))$. The map $\hat{\varphi}$ maps F_1 to $\hat{G} = \mathbf{P}^3$ contracting a section $\tilde{F}_1$ of $\hat{F}_1 \to F_1$ to a point $y_0 \in \hat{G}$. The component F_2 is contracted by $\hat{\varphi}$ to a plane in $\hat{G}$ containing y_0. The exceptional set of $\hat{\varphi}$ is equal to $\hat{F}_2 \cup \hat{E}$.

Using the results on blow-ups (Lemma (2.2) and the subsequent discussion which describes how to pass from $\mathbf{P}(N_{F_i/X})$ to $\mathbf{P}(N_{F_1 \cup F_2/X})$ using vector bundle surgery) and the above discussion on Hilbert we can describe the intersection of $\hat{E}$ with $\hat{F}_1$ and $\hat{F}_2$. The intersection $\hat{F}_1 \cap \hat{F}_2$ is a ruled surface $\mathbf{F}_1$ (ruled over $F_1 \cap F_2$) with a (-1) curve l_0. Then the locus of singular points of $\hat{X}$ is a section C_s of $\hat{F}_1 \cap \hat{F}_2 \to F_1 \cap F_2$ such that $C_s \cdot l_0 = 2$. In other words, if we contract l_0 in $\mathbf{F}_1$ then C_s becomes a rational cubic in $\mathbf{P}^2$. If we transform jumping lines of $N_{F_i/X}$ to $\hat{F}_i$ and use Lemma (2.2) then we can describe the locus of lines from $\hat{\mathcal{H}}$. Namely, in $\hat{F}_1$ these are the lines in $\tilde{F}_1$ which pass through one of the points in $l_0 \cap C_s$; in $\hat{F}_2$ these are the fibers in $\hat{\varphi}^{-1}(\hat{\varphi}(C_s))$. Thus $\hat{F}_1 \cdot \hat{E} = 2\tilde{F}_1$ and $\hat{F}_2 \cdot \hat{E} = \hat{\varphi}^{-1}\varphi(C_s)$ (the multiplicity of the intersection can be verified with the result of Lemma (6.7)).

From the above discussion it follows that the intersection $\hat{S} \cap \hat{G}$ is the rational cubic $\hat{\varphi}(C_s) \subset \hat{\varphi}(F_2) \subset \hat{G}$ which is singular at y_0. Also we find out that the map $\hat{\mathcal{H}} \to \mathcal{H}$ resolves the "triple component" singularity at the point which represents the curve with normal $\mathcal{O}(1)^2 \oplus \mathcal{O}(-3)$. That is, the lines in $\hat{\mathcal{H}}$ over the point z are parametrized by three rational curves forming a simple chain. The map $\hat{\mathcal{H}} \to \hat{S}$ contracts the two exterior lines and loops the central one to the rational cubic by glueing its meeting points. Therefore $\hat{S}$ is not normal at y_0.

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