POLARIZED PERIOD MAP FOR GENERALIZED K3 SURFACES AND THE MODULI OF EINSTEIN METRICS

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0. Introduction

The moduli space for marked polarized K3 surfaces or equivalently the moduli space for marked K3 surfaces with a Ricci-flat Einstein-Kähler metric is constructed in [T1] and [L]. This moduli space is isomorphic to an open dense subset $K\Omega^0$ of

$$K\Omega \overset{\text{def.}}{=} SO_0(3,19)/SO(2)\times SO(19) \ .$$

So, it is natural to ask what geometric objects correspond to the "hole" $K\Omega\backslash K\Omega^0$ of the moduli space. The purpose of the present paper is to make some contribution to this question from differential geometric point of view. Namely we consider the polarized period map for K3 surfaces with simple singular points. The flavor of our main result is most typical in the following:

Theorem 7. The moduli space of all Einstein metrics on a K3 surface, including Einstein-orbifold-metrics along simple singular points, is isomorphic to

$$\Gamma\backslash(SO_0(3,19)/SO(3)\times SO(19))\ ,$$

where Γ is the full group of isometries of the K3 lattice

$$2(-E_8)\oplus 3\begin{pmatrix}0&1\\1&0\end{pmatrix}\ .$$

For the proof of this theorem we need two main ingredients, one from algebraic geometry and the other from differential geometry. The algebro-geometric ingredient is the contribution due to mainly by Todorov [T1], Looijenga [L], and the generalization of their arguments by Morrison [Mr] which is very important in the present paper. The differential geometric ingredient is the solution of Calabi's conjecture due to Yau [Ya1] and the "equivariant version" of it which asserts the existence of a Ricci-flat Einstein-Kähler orbifold-metric on certain complex orbifolds. The existence of a Ricci-flat Einstein-Kähler orbifold-metric makes it possible to use the "<u>isometric</u> deformation" of Kähler structures on generalized K3 surfaces.

Einstein-Kähler orbifold-metrics were also used to characterize the <u>ball quotients</u> of finite volume in terms of numerical invariants of orbifolds [CY] and [Kb]. In differential geometric words, that is the criterion for the vanishing of the anti-self-dual Weyl tensor of the Einstein-Kähler orbifold-metric under consideration. In exactly the same spirit, we will obtain the criterion for the vanishing of the full curvature tensor of a Ricci-flat Einstein-Kähler orbifold-metric in terms of the numerical invariants of the orbifold:

<u>Theorem 9.</u> Let X be a compact complex surface with at worst simple singularities whose minimal resolution is a K3 surface. Then,

$$24 - \sum_{p \in \operatorname{Sing} X} \left(e(E_p) - \frac{1}{|G_p|}\right) \geq 0 ,$$

where E_p and $|G_p|$ is the exceptional divisor for the minimal resolution for $p \in \operatorname{Sing} X$ and the order of the corresponding local fundamental group G_p. The equality holds if and only if X is obtained by taking the quotient of a complex two torus with respect to the discrete group of Euclidean motions.

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1. Review on the moduli of K3 surfaces and the formulation of the problem

A K3 surface is a compact complex surface X which is connected and simply connected and has trivial canonical bundle K_X, i.e., X has a unique (up to constant) nowhere vanishing holomorphic 2-form ω_X. The notion of a K3 surface is invariant under deformation, i.e., any deformation of a K3 surface is a K3 surface [Kd]. Moreover any two K3 surfaces are deformations of each other [Kd]. So, there exists a unique underlying differentiable manifold of K3 surfaces which turns out to be a smooth quartic suface in $P_3(\mathbb{C})$. Hence the lattice $H^2(X;\mathbb{Z})$ with the cup bilinear form is the same for all K3 surfaces X and can be called the K3 lattice. K3 lattice L is the unique even unimodular lattice of rank 22 and index -16, i.e.,

$$L = 2(-E_8) \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

where E_8 is the even unimodular positive definite lattice associated with the Dynkin diagram of type E_8.

Definition. A choice of an isometry

$$\alpha : H^2(X;\mathbb{Z}) \xrightarrow{\sim} L$$

is called a marking of X. A pair (X,α) of a K3 surface X and a marking α is called a marked K3 surface.

Let (X,α) be a marked K3 surface. $H^{2,0}(X)$ is a $\mathbb{C}$-vector subspace of $H^2(X;\mathbb{C}) \cong L_{\mathbb{C}}$ of dimension 1 generated by ω_X, which satisfies

$$\langle\omega_X,\omega_X\rangle = 0 \quad \text{and} \quad \langle\omega_X,\overline{\omega_X}\rangle > 0 \ .$$

We can thus associate to (X,α) a point $[\alpha_{\mathbb{C}}(\omega_X)]$ in the <u>classical period domain</u>

$$\Omega = \{\omega \in L_{\mathbb{C}}; \langle\omega,\omega\rangle = 0,\ \langle\omega,\overline{\omega}\rangle > 0\}/\mathbb{C}^*$$

which is an open subset of a hyperquadric in $P_{21}(\mathbb{C})$.

The <u>classical period map</u> for marked K3 surfaces is the mapping sending a marked K3 surface (X,α) to the point $[\alpha_{\mathbb{C}}(\omega_X)]$ in Ω.

Another description for Ω which we will use later is the following: $\Omega = \{$all oriented two-planes $E \subset L_{\mathbb{R}}$ such that $\langle\,,\rangle|_E$ is positive definite$\}$. In this description for Ω the classical period map for marked K3 surfaces (X,α) is expressed as

$(X,\alpha) \mapsto$ [$\alpha_{\mathbb{R}}$-image of the two-plane E_X in $H^2(X;\mathbb{R})$ spanned by $\mathrm{Re}\,\omega_X$ and $\mathrm{Im}\,\omega_X$, where ω_X is a generator of $H^{2,0}(X)$ and the orientation of E_X is given by $(\mathrm{Re}\,\omega_X, \mathrm{Im}\,\omega_X)$.]

The classical period domain parametrizes effectively the local universal deformation (Kuranishi family) for any K3 surface X. If $p:(\mathfrak{X},X) \to (S,0)$ is the Kuranishi family for X, we then have a diffeomorphism $t: X\times S \to \mathfrak{X}$ such that $p\circ t = pr_2$. Once we choose a marking $\alpha: H^2(X;\mathbb{Z}) \longrightarrow L$, we get a <u>marking for the family</u> by setting

$$\begin{tikzcd} \alpha_s \overset{\text{def.}}{=} \alpha \circ t^*: H^2(X_s;\mathbb{Z}) \arrow[r] \arrow[dr, "t^*"'] & L \\ & H^2(X;\mathbb{Z}) \arrow[u, "\alpha"'] \end{tikzcd}$$

We thus get a <u>marked</u> Kuranishi family $(\mathfrak{X} \longrightarrow S,\alpha)$, which has a <u>period map</u> $\tau_S: S \longrightarrow \Omega$, defined by

$$S \ni s \longmapsto \tau_S(s) = [\alpha_{s_{\mathbb{C}}}(\omega_{X_s})] \ .$$

Now the local Torelli theorem due to Andreotti-Weil and Kodaira [Kd] states that the map τ_S is holomorphic and a local isomorphism at 0. Every point $x \in \Omega$ determines a Hodge structure of weight 2 on L in the following way: If $\omega \in L_{\mathbb{C}}$ is a representative for x, we define

$$\begin{aligned} H^{2,0}(x) &= \mathbb{C}\omega \subset L_{\mathbb{C}} \\ H^{0,2}(x) &= \mathbb{C}\bar{\omega} \subset L_{\mathbb{C}} \\ H^{1,1}(x) &= (H^{2,0}(x) + H^{0,2}(x))^{\perp} \subset L_{\mathbb{C}} \ . \end{aligned}$$

The classical period domain Ω classifies in a bijective way all K3 complex structures on the underlying differentiable manifold of a K3 surface <u>with a fixed marking.</u> See [T1],[BuR] and [LP]. But although the local Torelli theorem is true, one <u>cannot</u> construct a universal family of K3 surfaces on Ω. In fact, Atiyah [At] constructed two non-isomorphic families of K3 surfaces with the same period map. The reason is that the classical period domain sees only Hodge structures, although the rational curves on a K3 surface play an essential role in the construction of the fine moduli space. The following construction due to Burns-Rapoport [BuR] clarifies the importance of the rational curves on a K3 surface. For $x \in \Omega$, let $V^+(x)$ be one of the connected components of

$V(x) = \{\kappa \in H^{1,1}(x) \cap L_{\mathbb{R}} ; <\kappa,\kappa> = 1\}$, and let

$\Delta(x) = \{\delta \in H^{1,1}(x) \cap L ; <\delta,\delta> = -2\}$ and

$V^+_\Delta(x) = \{k \in V^+(x) ; <k,\delta> \neq 0$ for all $\delta \in \Delta(x)\}$. Since Ω is simply connected, it is possible to make a <u>continuous</u> choice of $V^+(x)$ with respect to $x \in \Omega$. For a K3 surface X , we define:

$\Delta(X) = \{\delta \in H^{1,1}(X) \cap H^2(X;\mathbb{Z}) ; <\delta,\delta> = -2\}$ and

$\Delta^+(X) = \{$all effective $\delta \in \Delta(X)$, i.e., δ corresponds to an effective divisor on $X\}$. By Riemann-Roch, δ or $-\delta$ is effective for all $\delta \in \Delta(X)$. So, $\Delta(X) = \Delta^+(X) \cup -\Delta^+(X)$ and if $\delta_1,\dots,\delta_k \in \Delta^+(X)$ and $\delta = \Sigma n_i\delta_i$ with $\mathbb{Z} \ni n_i \geq 0$

then $\delta \in \Delta^+(X)$. Let $V^+(X)$ be the connected component of $V(X) = \{\kappa \in H^{1,1}(X) \cap H^2(X;\mathbb{R})\ ;\ <\kappa,\kappa> = 1\}$ which contains a Kähler metric on X. We define
$V_P^+(X) = \{k \in V^+(X); <\kappa,\kappa> = 1,\ <\kappa,\delta>>0 \text{ for all } \delta \in \Delta^+(X)\}$
for a K3 surface X. The half cone

$$\begin{aligned} C_P^+(X) &= \mathbb{R}^+ \times V_P^+(X) \\ &= \{\kappa \in H^{1,1}(X) \cap H^2(X;\mathbb{R})\ ;<\kappa,\kappa>>0, <\kappa,\delta>>0 \\ &\quad \text{for all } \delta \in \Delta^+(X)\} \end{aligned}$$

over $V_P^+(X)$ is the <u>Kähler cone</u> for X. Note that every K3 surface admits a Kähler metric [Si], i.e., $V_P^+(X) \neq \phi$. The Kähler cone $C_P^+(X)$ for a Kähler surface X is originally defined by

$$(*) \qquad C_P^+(X) = \left\{ \begin{array}{l} \kappa \in H^{1,1}(X) \cap H^2(X;\mathbb{R})\ ;<\kappa,\kappa>>0 \text{ and } <\kappa,\delta>>0 \\ \text{for all effective classes } \delta \in H^{1,1}(X) \cap H^2(X;\mathbb{Z}) \end{array} \right\}.$$

But in the case of K3 surfaces, it is sufficient to check the property (*) for (-2)-effective classes $d \in \Delta^+(X)$. See, for example, [LP]. Burns-Rapoport [BuR] defined the <u>Burns-Rapoport period domain</u> $\tilde{\Omega}$ in the following way. Define two fiber spaces $K\Omega^0$, $K\Omega$ over Ω by

$K\Omega = \{(\kappa,[\omega]) \in L_{\mathbb{R}} \times \Omega; \kappa \in V^+(x)\}$, and
$K\Omega^0 = \{(\kappa,[\omega]) \in K\Omega;\ \kappa \in V_\Delta^+(x)\}$, where $x = [\omega]$.

Let $\pi: K\Omega^0 \longrightarrow \Omega = (K\Omega)^0/\sim$ be the quotient map defined by $(k,\omega) \sim (k',\omega')$ if and only if $\omega = \omega'$ and k and k' are in the same connected component of the fiber $pr_2^{-1}([\omega])$ in $K\Omega^0$. For each $x \in \Omega$, define the subgroup $W(x)$ of $\mathrm{Aut}(L) \cap SO(H^{1,1}(x) \cap L_{\mathbb{R}})$ generated by the reflections

$$s(\delta): x \longmapsto x + \langle x,\delta\rangle\delta \ ,$$

where δ runs over the whole $\Delta(x)$. Since $H^{1,1}(x) \cap L_{\mathbb{R}}$ has a signature $(1,19)$, $W(x)$ acts <u>properly discontinuously</u> on the hyperbolic 19-space $V^+(x)$, [Vn]. The set of fundamental domains in $V^+(x)$ is in one-to-one correspondence with the set of the partition of $\Delta(x)$ into $\Delta^+(x)$ and $-\Delta^+(x)$ with the property that if $\delta_1,\ldots,\delta_k \in \Delta^+(x)$ and $\delta = \Sigma n_i \delta_i$ with $\mathbb{Z} \ni n_i \geq 0$, then $\delta \in \Delta^+(x)$. For a partition $P: \Delta(x) = \Delta^+(x) \cup -\Delta^+(x)$, the corresponding fundamental domain $V_P^+(X)$ is $\{x \in V^+(x); \langle x,\delta\rangle > 0 \text{ for all } \delta \in \Delta^+(x)\}$ which turns out to be a <u>locally finite</u> "polyhedron" whose sides are given by hyperplanes $H_\delta = \{\delta\}^\perp$ for $\delta \in \Delta(x)$. The <u>Burns-Rapoport period map</u> associates to each marked K3 surface (X,α) the point in $\tilde{\Omega}$ determined by $\pi(\alpha_{\mathbb{R}}(\kappa), [\alpha_{\mathbb{R}}(\omega_X)])$, where κ is a Kähler class on X. For this period map, the following is known [BuR]:

<u>The Global Torelli Theorem.</u> Let X and X' be two K3 surfaces. If there is an isometry $\phi: H^2(X';\mathbb{Z}) \longrightarrow H^2(X;\mathbb{Z})$

satisfying $\phi_{\mathbb{C}}([\omega_X]) = c[\omega_{X'}]$ for some $c \in \mathbb{C}^*$ and $\phi_{\mathbb{R}}(V_P^+(X')) = V_P^+(X)$, then there is a unique isomorphism $\Phi: X \longrightarrow X'$ with $\Phi^* = \phi$.

This was first proved by Prateckii-Shapiro and Shafarevich [ShP] in the algebraic case and refined in the Kählerian case by Burns-Rapoport [BaR], simplified by Looijenga-Peters [LP]. This theorem means that any two marked K3 surfaces having the same Burns-Rapoport periods are isomoprhic in the unique way. For the surjectivity of this period map, Todorov [T1] proved

<u>Surjectivity Theorem.</u> For every $\tilde{x} \in \tilde{\Omega}$, there is a marked K3 surface whose Burns-Rapoport period is $\tilde{x}$.

For the proof, he used Yau's solution to Calabi's conjecture, i.e., the isometric deformation of Kähler structures with respect to a Calabi-Yau metric. The same technique is used in this paper, but now for a Ricci-flat <u>orbifold</u>-metric. We can thus use the Local Torelli Theorem, the Global Torelli Theorem, and the Surjectivity Theorem to glue up marked Kuranishi families (which should be small enough to be embedded in $\tilde{\Omega}$) via the Burns-Rapoport period map to identify $\tilde{\Omega}$ with the fine moduli space for marked K3 surfaces. As was shown by Atiyah [At] (see also [LP]), the space $\tilde{\Omega}$ is <u>not</u> Hausdorff. Moreover, Aut(L) <u>cannot</u> act on $\tilde{\Omega}$ in a properly discontinuous fashion. Morrison [Mr] made a great progress to avoid such unsatisfactory properties of

$\widetilde{\Omega}$ by introducing the polarized period map for generalized K3 surfaces instead of Burns-Rapoport period map for smooth K3 surfaces. The following definition are due to Morrison [Mr].

Definition. A compact complex surface X is called a generalized K3 surface if X has at worst simple singular points and its minimal resolution Y is a K3 surface.

Definition. Let X be a generalized K3 surface and $\rho: Y \longrightarrow X$ its minimal resolution. Let $\delta_1, \ldots, \delta_k \in H^{1,1}(Y) \cap H^2(Y;\mathbb{Z})$ be the classes of all (-2)-curves contracted by ρ. The root system $R(X)$ and the Weyl group $W(X)$ of X are defined by

$$R(X) \overset{\text{def.}}{=} \{\delta = \sum_{i=1}^{k} a_i\delta_i \in H^2(Y;\mathbb{Z}); a_i \in \mathbb{Z}, \langle\delta,\delta\rangle = -2\}$$

and $W(X) \overset{\text{def.}}{=}$ the group generated by $\{s(\delta); \delta \in R(X)\}$
$\subset \text{Isometry }(H^2(Y;\mathbb{Z}))$

Definition. We let

$I(X) \overset{\text{def.}}{=} H^2(Y;\mathbb{Z})^{W(X)}$, i.e., the set of all classes orthogonal to $R(X)$. Note that $I(X)_{\mathbb{C}}$ contains $H^{2,0}(Y)$ and so determines the Hodge structure of $H^2(Y;\mathbb{Z})$

Definition. A metric injection

$$\alpha: I(X) \longrightarrow L$$

is a <u>marking</u> of X if α is extendable to an isometry $\overline{\alpha}$ of $H^2(Y;\mathbb{Z})$ to L. A pair (X,α) is a <u>marked generalized K3 surface.</u>

<u>Definition.</u> For a generalized K3 surface X, we let

$$I(X) \supset V_P^+(X) \overset{\text{def.}}{=} \left\{ \kappa \in \overline{V_P^+(Y)};\ \begin{array}{l} \text{for all } \delta \in H^{1,1}(Y) \cap H^2(Y;\mathbb{Z}) \\ \text{with } \langle\delta,\delta\rangle = -2, \langle\kappa,\delta\rangle=0 \\ \text{if and only if } \delta \in R(X) \end{array} \right\},$$

where $V_P^+(Y) = \{\kappa \in H^{1,1}(Y) \cap H^2(Y;\mathbb{R})\ ; \langle\kappa,\kappa\rangle = 1$ and $\langle\kappa,\delta\rangle > 0$ for all effective (-2)-classes δ on $Y\}$

as before. The <u>Kähler cone</u> $C_P^+(X)$ is defined by

$$C_P^+(X) \overset{\text{def.}}{=} \mathbb{R}^+ \times V_P^+(X).$$

<u>Definition.</u> An element $\phi \in V_P^+(X)$ is called a <u>polarization</u> on X. A triple (X,ϕ,α) is a marked <u>polarized generalized K3 surface</u>. The <u>polarized period map</u> p for marked polarized generalized K3 surfaces sends (X,ϕ,α) to $p(X,\phi,\alpha) = (\alpha_{\mathbb{R}}(\phi),[\alpha_{\mathbb{C}}(\omega_Y)]) \in K\Omega$. For this map, Morrison [Mr] proved the Polarized Global Torelli Theorem for marked polarized generalized K3 surfaces:

<u>Theorem A [Mr]</u>. Let (X,ϕ) and (X',ϕ') be two polarized generalized K3 surfaces and let $\rho: Y \longrightarrow X$ and

$\rho':Y' \longrightarrow X'$ be their minimal resolutions. Suppose $\gamma: I^2(X') \longrightarrow I^2(X)$ is an isometry such that $\gamma_{\mathbb{C}}(H^{2,0}(Y')) = H^{2,0}(Y)$, $\gamma_{\mathbb{R}}(\phi') = \phi$, and extends to an isometry $\bar{\gamma}: H^2(Y',\mathbb{Z}) \longrightarrow H^2(Y;\mathbb{Z})$. Then there is a unique isomorphism $\phi: X \longrightarrow X'$ such that $\Phi^* = \gamma$. Φ comes from a unique isomorphism $\bar{\Phi}: Y \longrightarrow Y'$ which induces isomorphisms of exceptional sets for ρ and ρ'.

For the surjectivity of this period map, there is a strong result due to Looijenga [L] (see also [T1] and [Na]):

<u>Theorem B [L].</u> For every $(\kappa,x) \in K\Omega^0$, there is a marked polarized K3 surface (X,ϕ,α) such that $p(X,\phi,\alpha) = (\kappa,x)$ and the polarization ϕ contains a Kähler metric.

So it may be natural to ask what geometric objects correspond to holes $K\Omega \setminus K\Omega^0$ of the moduli space of marked polarized generalized K3 surfaces. Morrison proved the following weak version of Surjectivity Theorem:

<u>Theorem C [Mr].</u> For very $(\kappa,x) \in K\Omega$, there is a marked polarized generalized K3 surface (X,ϕ,α) such that $p(X,\phi,\alpha) = (\kappa,x)$.

Yau's solution to Calabi's conjecture tells us that $K\Omega^0$ is the moduli space for marked Einstein-Kähler K3 surfaces. On the other hand the point in the hole $K\Omega \setminus K\Omega^0$ corresponds to a K3 surface Y and a class $\phi \in H^{1,1}(Y)$

with $\langle\phi,\phi\rangle = 1$ such that the area of some effective curves are zero. So, the problem is to find the singular Ricci-flat Einstein-Kähler metric corresponding to ϕ. This question is asked by several authors [Be], [Mr]. We shall solve this problem in the following sections.

2. Ricci-flat orbifold-metrics on generalized K3 surfaces

In [Ya2], Yau presented some results for the existence of a singular Ricci-flat Kähler metric on certain complex manifolds. Since [Ya2] is not published as fas as the authors know, we include the proof of the equivariant version of the Calabi-Yau theorem in this section. There may be many ways arranging the material involved in Yau's proof of Calabi's conjecture. Here, we shall prove the simplest version sufficient for our purposes, namely filling the "holes" of the moduli space of Einstein metrics on a K3 surface.

Theorem 1. Let X be a compact complex surface with at worst isolated quotient singularities. Let $Y \longrightarrow X$ be the minimal resolution and $D = \sum_i D_i$ its exceptional sets decomposed into irreducible components. Choose non-negative rational numbers μ_i less than 1 such that $K_Y + \sum_i \mu_i D_i \sim 0$ near D (such μ_i's are uniquely determined). Assume that some tensor power of $K_Y + \sum_i \mu_i D_i$ is a trivial line bundle over Y. Then for any Kähler form ϕ in the sense of Fujiki-Moišezon (if exists) on X, we can find a unique real-valued orbifold-smooth function U on X

up to additive constants such that $\phi + \sqrt{-1}\partial\bar{\partial}U$ is a Ricci-flat Einstein-Kähler orbifold-metric form on X. Moreover the current $\phi + \sqrt{-1}\partial\bar{\partial}U$ on Y defines the same cohomology class as ω in $H^2(Y;\mathbb{R})$.

For the proof, we follow Yau's proof [Ya] of Calabi's conjecture partially simplified by Bourguignon [Bo 2], namely the simple proof for the C^0-estimate. It is easy to see that there exists a real valued orbifold-smooth function U_0 such that $\phi + \sqrt{-1}\partial\bar{\partial}U_0 =: \phi_0$ is an orbifold-Kähler form. Since the resolution of quotient singularities involve only polynomial functions and there exist nonnegative rational numbers μ_i such that $K_Y + \sum_i \mu_i D_i$ is trivial near D, the following estimates hold:

$$dz \approx O(\gamma^{\varepsilon-1}),\ dw \approx O(\gamma^{\varepsilon-1}) \ ,$$

where ε is a small positive number, (z,w) are holomorphic local coordinates near D and $\gamma = (|\lambda|^2 + |\mu|^2)^{1/2}$ is the distance function on the local uniformization $\mathbb{B}^2 : (\lambda,\mu)$ of the quotient singularity corresponding to D. It follows that for any orbifold-smooth function U on X the current $\phi + \sqrt{-1}\partial\bar{\partial}U$ defines the same cohomology class as $\phi \in H^2(Y;\mathbb{R})$. Indeed, we have only to show that

$$\int_Y dd^c u \wedge \Psi = 0,$$

for any smooth closed 2-form Ψ on Y. This is equivalent to showing that

$$\lim_{r\to 0} \int_{S^3(r)} d^c\tilde{u} \wedge \tilde{\Psi} = 0,$$

where $\sim$ means that the lifting to the local uniformization and $S^3(r)$ is the sphere of radius r centered at the origin. But this is clear from the above estimates.
Since some tensor power of $K_Y + \sum_i \mu_i D_i$ is trivial on Y, there exists a Ricci-flat volume from V on X which is orbifold-smooth. Thus we have

$$\phi_0^2 = e^{-f} V$$

for some orbifold-smooth function f on X. To find U in Theorem 1, we solve the following Monge-Ampère equation:

(1) $$(\phi_0 + \sqrt{-1}\partial\bar{\partial}U)^2 = e^f \phi_0^2 \quad \text{on} \quad X.$$

The proof of the uniqueness of orbifold-smooth U is exactly the same as in [Ya] and [Bo 2]. To solve the equation (1) we use the continuity method.

Let $C^{k,\alpha}(X)$ be the Banach space of all C^k functions on X whose k-th derivative are Hölder continuous of exponent α . This means that any element f in $C^{k,\alpha}(X)$ is of class $C^{k,\alpha}$ on local uniformations. The norm on $C^{k,\alpha}(X)$ is defined in completely analogous way as in the usual Hölder space. Consider the 1-parameter family of equations:

$$(1)_t \qquad (\phi_0 + \sqrt{-1}\partial\bar{\partial}U)^2 = e^{tf}\left(\frac{\int_X \phi_0^2}{\int_X e^{tf}\phi_0^2}\right) \phi_0^2 \, .$$

$(1)_0$ has a solution 0 and $(1)_1$ is what we want to solve. We show that the non-empty set
$A = \{t \mid t \in [0.1]$ and $(1)_t$ has a solution in $C^{k,\alpha}(X)\}$
is open and closed. Let E be defined by
$E = \{u \mid u \in C^{k,\alpha}(X)$ and $\int_X u\phi_0^2 = 0\}$, a closed subspace of $C^{k,\alpha}(X)$. Suppose $u \in C^{k,\alpha}(X)$ is a solution of (1) and let $\tilde{\phi} := \phi_0 + \sqrt{-1}\partial\bar{\partial}U$. Define

$$H := \{h \mid h \in C^{k-2,\alpha}(X) \text{ and } \int_X h\tilde{\phi}^2 = 0\} \, .$$

The openess of A follows from:

<u>Lemma.</u> $\Delta_{\tilde{\phi}}: E \longrightarrow H$ is an isomorphism.

<u>Proof.</u> Since $\int_X v\Delta_{\tilde{\phi}} v = \int_X |dv|^2_{\tilde{\phi}}$ is true in the orbifold category the map $\Delta_{\tilde{\phi}}: E \longrightarrow H$ is injective. To show the surjectivity it suffices to construct the Green's function on the Riemannian orbifold $(X,\tilde{\phi})$. The standard technique to construct Green's function works in our case. See, for example, [Au].

Q.E.D.

The closedness follows from the a-priori C^0-estimate for U. For the complex Monge-Ampère equation $(\phi + \sqrt{-1}\partial\bar{\partial}U)^n = e^F\phi^n$ on the compact Kähler manifold, where ω and F is given, the C^0-bound for U is obtained in terms of C^0-bound for F. The main ingredients are: (a) the construction of Green's function, and (b) the Sobolev inequality for L_1^2-functions f with $\int f=0$
$c(\int |f|^{\frac{2n}{n-2}})^{\frac{n-2}{2n}} \le (\int |df|^2)^{\frac{1}{2}}$ on a compact Riemann manifold.

But (a) is carried out without difficulty in our case (see [Au]) and (b) is clearly true for some constant $c > 0$. We thus get:

Lemma. There is a C^0-bound for U in terms of the C^0-bound for F.

Since C^2 and $C^{2,\alpha}$ estimates are carried out by local calculations and the classical maximum principle, exactly the same arguments in [Ya] give us the C^2 and $C^{2,\alpha}$ bounds for U in our case, i.e., in the orbifold category. Now the proof of Theorem 1 is complete.

In particular, any generalized K3 surfaces X with a Fujiki-Moišezon-Kähler form ϕ admits a unqiue Einstein-Kähler orbifold-metric form $\tilde{\phi}$ such that $[\tilde{\phi}]=[\phi]$ in $H^2(Y;\mathbb{R})$.

3. Isometric deformations

Theorem 2. Suppose that X is a generalized K3 surface and $g_{\alpha\bar{\beta}}$ is an Einstein-Kähler orbifold-metric (necessarily Ricci-flat) on X. Then

(1) $X \times S^2$ has a complex structure $\mathfrak{X}$ such that

a) the projection $\pi : \mathfrak{X} \longrightarrow S^2 \cong P_1(\mathbb{C})$ is a holomorphic map and fibers are generalized K3 surfaces. From $\mathfrak{X} \longrightarrow P_1(\mathbb{C})$ we can obtain a family of non-singular K3 surfaces $\tilde{\mathfrak{X}} \longrightarrow P_1(\mathbb{C})$,

b) if (X,α) is a marked generalized K3 surface, then α induces an isomorphism of local systems (in fact trivial systems) $\alpha : R^2\pi_*\mathbb{Z}_{\tilde{\mathfrak{X}}} \xrightarrow{\sim} P_1(\mathbb{C}) \times L$

c) for each $t \in P_1(\mathbb{C})$ the periods in Ω of $X_t = \pi^{-1}(t)$ is an oriented two-plane in the three dimensional space $E \subset L_{\mathbb{R}}$ spanned by $(\mathrm{Re}\ \omega_X,\ \mathrm{Im}\ \omega_X,\ \mathrm{Im}\ g_{\alpha\bar{\beta}})$ or more intrincicaly three linearly independent parallel self-dual two forms with respect to the Ricci-flat orbifold metric.

(2) For each $t \in P_1(\mathbb{C})$ the Ricci-flat Riemannian orbifold-metric (determined by $g_{\alpha\bar{\beta}}$) on X_t is orbifold-Kählerian with respect to the corresponding complex structure.

(3) The base space $P_1(\mathbb{C})$ parametrizes all complex structures with respect to which g is Kähler.

Proof. The proof is based on the following two lemmas and the Andreotti-Weil remark.

Lemma 1. The Kähler orbifold-metric $g_{\alpha\overline{\beta}}$ is Ricci-flat if and only if for a positive constant $c \in \mathbb{R}_+$ we have an equality

$$\phi \wedge \phi = c\ \omega_X \wedge \overline{\omega_X}$$

of differentiable 4-forms, where $\phi = \operatorname{Im} g_{\alpha\overline{\beta}}$ is the Kähler form of g and ω_X is the holomorphic 2-form on X without zeros and poles.

Proof. It is clear. Q.E.D.

Lemma 2. Let (X,g) be as in Theorem. Let ρ be a (closed) 2-form written as

$$\rho = a\ \omega_X + b\ \overline{\omega_X} + e\ \phi$$

where $a,b,e \in \mathbb{C}$. Then $\rho \wedge \rho = 0$ as a form if and only if $[\rho] \wedge [\rho] = 0$ as a cohomology class in $H^4(X;\mathbb{C})$ where $[\rho]$ denotes the cohomology class of ρ in $H^2(X;\mathbb{C})$.

Proof. Since $\rho \wedge \rho = (2ab + e^2 c)\omega_X \wedge \overline{\omega_X}$,
$\rho \wedge \rho = 0 \iff 2ab + e^2 c = 0 \Longleftrightarrow [\rho] \wedge [\rho] = 0$. The last

equivalence is because $\int_X \omega_X \wedge \overline{\omega_X} > 0$. Q.E.D.

<u>Andreotti-Weil remark</u> [W]. Let X be an oriented differentiable manifold of dimension 4. If there is a $\mathbb{C}$-valued 2-form ρ on X such that a) $\rho \wedge \rho = 0$, b) $\rho \wedge \bar{\rho} > 0$ everywhere, and c) $d\rho = 0$, then X admits a unique complex structure such that ρ is a holomorphic 2-form.

<u>Proof of Theorem 2.</u> S^2 parametrizes all oriented two-planes in the three dimensional space $E \subset L_{\mathbb{R}}$ spanned by $\{\mathrm{Re}\ \omega_X, \mathrm{Im}\omega_X, \mathrm{Im}\ g_{\alpha\bar{\beta}}\}$. We may assume that $\{\mathrm{Re}\ \omega_X, \mathrm{Im}\ \omega_X, \mathrm{Im}\ g_{\alpha\bar{\beta}}\}$ is an orthonormal basis with respect to $<,>|_E$. Let E_t be any oriented two-plane in E and let α, β be an orthonormal basis in E_t, then we define

$$\omega_t = \alpha + i\beta.$$

Clearly $\omega_t \wedge \omega_t = 0$ and $\omega_t \wedge \overline{\omega_t} > 0$, since α, β is an orthonormal basis in E_t. So $\omega_t = a\omega_X + b\bar{\omega}_X + e\ \mathrm{Im}\ g_{\alpha\bar{\beta}}$ and defines a new complex structure on X. It is clear that if x is a simple singular point on X and U is a pseudo-convex neighborhood of x and $U = V/G$, where $V \subset \mathbb{C}^2$ and $G \subset SU(2)$. Let $\pi: V \setminus \{0\} \to V\setminus\{0\}/G \cong U\setminus\{x\}$. Then $\pi^*(\omega_t|V\setminus\{x\})$ can be prolonged to a 2-form on V invariant under the action of G. So in such a way we get a complex analytic family $\mathfrak{X} = \bigcup_{t \in S^2} X_t$. In fact, the complex structure $\mathfrak{X}$ is nothing but that the twistor space for a half-conformally flat Riemannian 4-manifold [AtHS]. (Recall that any Kähler metric with vanishing Ricci tensor is anti-self-dual.) Now let ℓ_t be a vector orthogonal to E_t in E. Let $\alpha,\beta,\gamma \in \mathbb{R}$ be such that $\ell_t = \alpha \operatorname{Re} \omega_X + \beta \operatorname{Im} \omega_X + \gamma \operatorname{Im} g_{\alpha\bar{\beta}}$. Suppose that $\{\operatorname{Re} \omega_t, \operatorname{Im} \omega_t, \ell_t\}$ defines the same orientation on $E \subset \Gamma(X,\Lambda^+)$ as $\{\operatorname{Re} \omega_X, \operatorname{Im} \omega_X, \operatorname{Im} g_{\alpha\bar{\beta}}\}$. ℓ_t is a closed form of type $(1,1)$. $\det(\ell_t)$ vanishes nowhere on $X \setminus \operatorname{Sing} X$ and $\det(\ell_t) = c\, \omega_t \wedge \overline{\omega_t} = c\omega_X \wedge \overline{\omega_X}$. On the other hand, for each $x \in X\setminus \operatorname{Sing} X$ we can find $A \in SO(4)$ such that

$$d\ell^+(A)(g_{\alpha\bar{\beta}})\,{}^t d\ell^+(A) = \ell_t \quad \text{in} \quad T_x X ,$$

where ℓ^+ is the homomorphism of $SO(4)$ to $SO(3)$ determined by the decomposition of the second exterior representation of $SO(4)$ into irreducible spaces. So ℓ_t is positive definite and is an Einstein-Kähler metric on X-Sing X Riemannian equivalent to $g_{\alpha\bar{\beta}}$. Since $\operatorname{Re} \omega_X$, $\operatorname{Im} \omega_X$ and $\operatorname{Im} g_{\alpha\bar{\beta}}$ are smooth differential forms on X in the sense of orbifold, ℓ_t defines an Einstein-Kähler orbifold-metric with respect to the new complex structure corresponding to ω_t. Notice that in the family $\mathfrak{X} \longrightarrow S^2 \cong P_1(\mathbb{C})$ we can resolve the singularities and get a family of non-singular K3 surfaces $\tilde{\mathfrak{X}} \longrightarrow P_1(\mathbb{C})$. This is so because the singularities of

X are of type $\mathbb{C}^2/\Gamma \times P_1(\mathbb{C})$, where $\Gamma \subset SU(2)$. The desingularization can be done by successive blow ups. So the marking is well defined on $\tilde{X} \longrightarrow P_1(\mathbb{C})$, since all fibers are non-singular K3 surfaces and $\tilde{X} \cong P_1(\mathbb{C}) \times Y$ as C^∞ manifolds, where Y is the minimal solution of X. See also the arguments in section 5. Q.E.D.

In [Va] , Varouchas proved the following:

Fact. Let X be an analalytic variety admitting an open covering $\{U_i\}$ and a family of functions which are continuous and strictly plurisubharmonic $\psi_j : U_j \longrightarrow \mathbb{R}$ such that $\psi_j - \psi_k$ is pluriharmonic on $U_j \cap U_k$. Then X is Kählerian in the sense of Fujiki-Moišezon.

Let (X,ϕ) be a generalized K3 surface with a Ricci-flat Kähler orbifold-metric form ϕ. In this situation the proof of Varouchas shows that we can find a Fujiki-Moišezon-Kähler form $\tilde{\phi}$ in the same cohomology class as ϕ.

4. Surjectivity of polarized period map for generalized K3 surfaces

In this section, we prove the strong version of Morrison's Surjectivity Theorem (Theorem C). Namely, we show that every polarization ϕ in Theorem C contains a Kähler form on X in the sense of Fujiki-Moišezon ([F],[Mo]). Let M be the set of all isomorphism classes of marked polarized generalized K3 surfaces under the following equivalence: $(X,\phi,\alpha) \sim (X',\phi',\alpha')$ if and only if there is an isomorphism $f:Y' \longrightarrow Y$ which induces isomorphisms on the exceptional sets, such that $f^*(\phi) = \phi'$ and the diagram

$$\begin{tikzcd}
H^2(Y;\mathbb{Z}) \arrow[dr, "\overline{\alpha}"] \arrow[dd, "f^*"'] & \\
 & L \\
H^2(Y',\mathbb{Z}) \arrow[ur, "\overline{\alpha}'"'] &
\end{tikzcd}$$

is commutative. Thus Theorems A and C are unified in the following:

<u>Theorem A+C.</u> The polarized period map $p(X,\phi,\alpha) = (\alpha_{\mathbb{R}}(\phi),[\alpha_{\mathbb{C}}(\omega_X)])$ descends to a bijection $\tau: M \longrightarrow K\Omega$.

Now we define a subset M_1 of M in the following way: the equivalence class of (X,ϕ,α) is an element of M_1 if and only if ϕ is a Kähler class on X in the sense of Fujiki-Moišezon. The main result in this paper is:

<u>Strong Surjectivity Theorem</u>. The map $\tau: M_1 \longrightarrow K\Omega$ is surjective.

Combining this with Theorem A, we have:

<u>Theorem 3.</u> $\mathcal{M} = \mathcal{M}_1$, i.e., every polarization ϕ for any generalized K3 surface contains a Kähler metric in the sense of Fujiki-Moišezon, and the map $\tau: \mathcal{M}_1 = \mathcal{M} \to K\Omega$ is bijective.

<u>Proof of Strong Surjectivity Theorem.</u> Let

$$\pi : K\Omega \longrightarrow G_3^+(L_{\mathbb{R}})$$

is defined by $\pi(\kappa,[\omega]) = P_\omega + \mathbb{R} \cdot \kappa$, where P_ω is an oriented positive 2-plane in $L_{\mathbb{R}}$ whose oriented basis is $\{\mathrm{Re}\,\omega, \mathrm{Im}\,\omega\}$. $G_3^+(L_{\mathbb{R}})$ is the moduli space for oriented positive definite 3-planes in $L_{\mathbb{R}}$ which turns out to be the Riemannian symmetric space

$$SO_0(3,19)/SO(3) \times SO(19) \ .$$

Using the isometric deformation of generalized K3 structures with respect to the Ricci-flat orbifold metric, we get the following:

<u>Lemma.</u> The image of $\tau: \mathcal{M}_1 \longrightarrow K\Omega$ consists of fibers of $\pi : K\Omega \longrightarrow G_3^+(L_{\mathbb{R}})$.

<u>Proof of Lemma.</u> Let $(\kappa[\omega]) \in K\Omega$ be in the image of τ. We find a generalized K3 surface X and the Kähler form

ϕ such that $\alpha_{\mathbb{R}}([\phi]) = \kappa$ and $[\alpha_{\mathbb{C}}(\omega_X)] = [\omega]$ for some marking α. From theorem 1, we can find a unqiue Ricci-flat Einstein-Kähler orbifold-metric in the form of $\phi + \sqrt{-1}\partial\bar{\partial}U$ for an orbifold smooth function U. The cohomology class of $\phi + \sqrt{-1}\partial\bar{\partial}U$ (in the sense of current) is the same as $[\phi]$. Now by Theorem 2, there is an isometric family of generalized K3 structures parametrized by $P_1(\mathbb{C})$ and the period of these structures is exactly the fiber $\pi^{-1}(\pi(\kappa,[\omega])) \cong P_1(\mathbb{C})$. By Varouchas [Va], the cohomology class of each Einstein-Kähler orbifold-metric form contains a Kähler form in the sense of Fujiki-Moišezon. This completes the proof of Lemma.

Just as in the proof of Surjectivity Theorem for smooth Einstein-Kähler K3 surfaces [L], the remaining part of Theorem 1 is divided into three steps.

Step 1. Suppose $(\kappa,[\omega]) \in K\Omega$ is such that $(P_\omega + \mathbb{R}\cdot\kappa) \cap L$ contains a primitive rank 2 lattice M. By Lemma, we may replace $(\kappa,[\omega])$ by any other elements in $\pi^{-1}\pi(\kappa,[\omega])$. We may thus assume that $M \subset P_\omega$. By the weak version of Surjectivity Theorem due to Morrison, we can find a marked polarized generalized K3 surface (X,ϕ,α) such that $\alpha_{\mathbb{R}}(\phi) = \kappa$ and $[\alpha_{\mathbb{C}}(\omega_X)] = [\omega]$. Since $I^2(X)$ is an orthogonal complement of integral classes, $I^2(X) \otimes_{\mathbb{Z}} \mathbb{R} = I(X)_{\mathbb{R}}$ is a linear subspace of $H^2(Y;\mathbb{R})$ defined over $\mathbb{Q}$. Since $M(\subset P_\omega \subset I^2(X)_{\mathbb{R}})$ is defined over $\mathbb{Z}$, the orthogonal complement of P_ω in $I^2(X)_{\mathbb{R}}$ is defined over $\mathbb{Q}$. So the elements ℓ which are defined over $\mathbb{Q}$ are dense in $C_P^+(X)$. By the theorem of Mayer [Ma], such ℓ contains a Kähler metric on X in the sense of Fujiki-Moišezon. Since $C_P^+(X)$ is a convex cone, ϕ is a linear combination of rational points in $C_P^+(X)$ with positive coefficients. So, ϕ is a Kähler class on X in the sense of Fujiki-Moišezon.

<u>Step</u> 2. Suppose $(\kappa,[\omega]) \in K\Omega$ is such that $(P_\omega+\mathbb{R}\cdot\kappa)\cap L$ contains a primitive rank 1 lattice L. $I^2([\omega]) = L_{\mathbb{C}}^{W([\omega])}$ is defined over $\mathbb{Q}$. $V^+([\omega])$ is partitioned into chambers by reflection hypersurfaces H_δ for $\delta \in \Delta([\omega])$. Let K be the chamber containing κ. If $\eta \in K$ is such that $(P_\omega + \mathbb{R}\cdot\eta)\cap L$ contains a primitive rank 2 lattice, then $(\eta,[\omega]) \in \operatorname{Im}\tau$, i.e., there is a $(X_\eta,\phi_\eta,\alpha_\eta)$ with $\alpha_{\eta\mathbb{R}}(\phi_\eta) = \eta$ and $[\alpha_{\eta\mathbb{C}}(\omega_{X_\eta})] = [\omega]$. It is shown in the proof of the weak version of Surjectivity Theorem (see pp. 326-327 of [Mr]) that the isomorphism class of X_η is independent of $\eta \in K$. Such η with the property as above are dense in an open convex subcone K of $V^+([\omega])$. So, we can find a marked polarized generalized K3 surface (X,ϕ,α) such that $\alpha_{\mathbb{R}}(\phi) = \kappa, [\alpha_{\mathbb{C}}(\omega_X)] = [\omega]$ and ϕ contains a Kähler metric in the sense of Fujiki-Moišezon.

<u>Step</u> 3. Let $(\kappa,[\omega]) \in K\Omega$ be an arbitrary point, and K the chamber of $V^+([\omega])$ with respect to the action

of the Weyl group $W([\omega])$ containing κ. Since $I^2([\omega])$ is defined over $\mathbb{Q}$, the η such that $(P_\omega + \mathbb{R}\cdot\eta) \cap L$ contains a primitive rank 1 lattice are dense in K. For such η, we can find a $(X_\eta, \phi_\eta, \alpha_\eta)$ such that $\alpha_{\eta_{\mathbb{R}}}(\phi_\eta) = \eta$, $[\alpha_{\eta\mathbb{C}}(\omega_{X_\eta})] = [\omega]$ and ϕ_η contains a Kähler metric, by Step 2. The isomorphism class of X_η is independent of the choice of $\eta \in K$. Since K is a convex cone, κ contains a Kähler metric in the sense of Fujiki-Moišezon. Q.E.D.

It is shown in [Vn] that the action of the automorphism group Γ of L on $K\Omega \cong SO_0(3,19)/SO(2)\times SO(19)$ is discrete and properly discontinuous. We thus have a moduli space for the isomorphism classes of polarized generalized K3 surfaces:

<u>Corollary</u> 4. The coarse moduli space for the following objects are all isomorphic to

$$\Gamma\backslash K\Omega = \Gamma\backslash(SO_0(3,19)/SO(2)\times SO(19))$$

under the correspondence induced by the polarized period map,

(i) the isomorphism classes of polarized generalized K3 surfaces,

(ii) the isomorphism classes of polarized generalized K3 surfaces whose polarization comes from a Kähler form in the sense of Fujiki-Moišezon,

(iii) the isomorphism classes of Einstein-Kähler generalized K3 surfaces with volume 1.

Proof. The bijection (ii) $\rightleftarrows$ (iii) is given by Theorem. There is a natural injection (ii) $\to$ (i). Theorem A means that there is an injection (i) $\longrightarrow \Gamma\backslash K\Omega$ induced from the period map. Theorem 1 means (ii) $\longrightarrow \Gamma\backslash K\Omega$ is surjective. Q.E.D.

Remark. Einstein-Kähler generalized K3 surfaces with simple singularities correspond to the fixed points Fix (W) of the group $W \subset \Gamma$ generated by all reflections

$$S_\delta(v) = v + \langle v,\delta\rangle , \quad \text{where} \quad \delta \in L \quad \text{and} \quad \langle\delta,\delta\rangle=-2.$$

Fix (W) is a countable union of submanifolds of real codimension 3 ([Mr]).

5. Moduli of Einstein metrics on a K3 surface.

In this section we define the period map for Ricci-flat orbifolds diffeomrophic to generalized K3 surfaces and study its properties. We begin with some standard facts from 4-dimensional Riemannian geometry [AtHS]. Let (M,g) be a 4-dimensional Riemannian manifold with a metric g and $\Lambda^2=\Lambda^+ \oplus \Lambda^-$ the decomposition of 2-forms into self-dual and anti-self-dual parts. The Riemannian curvature tensor defines a self-adjoint transformation $R:\Lambda^2 \longrightarrow \Lambda^2$ expressed as $R(e_i \wedge e_j) = \frac{1}{2} \sum_{i,j,k,\ell} R_{ijk\ell} e_k \wedge e_\ell$, where $\{e_i\}$ is a local orthonormal basis of 1-forms. If we write $R = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ relative to the decomposition $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, the decomposition of the curvature tensor into irreducible pieces under $SO(4)$ is given by

$$R \longrightarrow (\mathrm{tr}\, A, B, W_+, W_-)$$

where $\mathrm{tr}\, A = \mathrm{tr}\, C = \frac{1}{4}$ scalar curvature, B = the traceless Ricci tensor, and $W_+ = A - \frac{1}{3}\mathrm{tr}\, A$, $W_- = C - \frac{1}{3}\mathrm{tr}\, C$, the Weyl tensors. If the metric is Kähler with vanishing Ricci-tensor, then $R \wedge \omega \equiv 0$, where ω is the Kähler form. This means that R is anti-self-dual with vanishing Ricci tensor: $R = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}$, $\mathrm{tr}\, C = 0$. For any Einstein metric over 4-manifolds, Hitchin [H] showed an inequality $2e(g) \geq -P_1(g)$ between the Euler form $e(g)$ and the Pontrjagin form $P_1(g)$. The equality occurs if and only if the curvature R is anti-self-dual and Ricci-flat. In particular any Ricci-flat Riemannian metric on a K3 surface is anti-self-dual.

Let X be a real four dimensional differentiable orbifold which is orbifold-diffeomorphic to a generalized K3 surface X'. Suppose X admits a Ricci-flat-metric g. Then we have:

Theorem 5. Let (X,g) be as above. Then the bundle of self-dual 2-forms (in the sense of orbifolds) is a flat trivial bundle with respect to the Levi-Civita connection.

Proof. As in the proof of Lemma 12 in [Kb] we get

$$(*) \qquad \int_X e(X,g) = e(Y) - \sum_{p\in \mathrm{Sing}\, X} \left(e(Ep) - \frac{1}{|Gp|}\right),$$

where $e(X,g)$ is the Euler form for the Levi-Civita connection of g, $e(Ep)$ is the Euler number of the exceptional set Ep for the simple singularitiy $p \in X$ and $|Gp|$ is the order of the corresponding finite subgroup Gp of $SU(2)$. Let g_1 and g_2 be two Riemannian orbifold metrics on X and $P_1(g_1)$, $P_1(g_2)$ the corresponding Pontrjagin forms respectively. Then $P_1(g_1) - P_1(g_2) = d\eta$, where η is a orbifold-3-form on X. So, we have $\int_X P_1(g_1) - \int_X P_1(g_2) = \int_X d\eta = 0$. Now in a small neighborhood of simple singularities of X we can introduce the canonical orbifold-complex structure, such as $\mathbb{B}^2/G$ where G is a finite subgroup of $SU(2)$ and $\mathbb{B}^2$ is an open ball in $\mathbb{C}^2$. We can thus deform g to be Kähler-orbifold metric around simple singularities with respect to the above complex structure. If the metric g is Kähler then $P_1(g) = c_1(g)^2 - 2c_2(g)$. Just as in the proof of Lemma 12 in [Kb] we have

$$(**) \qquad \frac{1}{3}\int_X P_1(g) = \text{sign}\ (Y) + \frac{2}{3}\sum_{p\in \text{Sing}\ X}(e(Ep) - \frac{1}{|Gp|}).$$

Formulas (*) and (**) are valid for any Riemannian orbifold metric. Now we assume that g is an orbifold-metric with vanishing Ricci tensor. From (*) and (**) we have

$$\int_X 2e(X,g) + P_1(X,g) = 0.$$

Applying the same argument as in [H] we see that the Ricci-flat orbifold-metric g is anti-self-dual: $R = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}$ with $\operatorname{tr} C = 0$. For any oriented Riemannian four-manifold the curvature of the induced connection on the bundle Λ^+ of self-dual 2-forms from the Levi-Civita connection is given by $A + B^* \in \operatorname{Hom}(\Lambda^+, \Lambda^2)$. In fact, the bundle Λ^2 of 2-forms is the adjoint bundle associated with the orthonormal fram bundle and the second exterior power representation λ^2 of $SO(4)$ splits into two irreducible subspaces $\lambda^2 = \lambda^+ \oplus \lambda^-$. The representation λ^+ defines a homomorphism $\ell^+ : SO(4) \to SO(3)$ which gives rise a principal $SO(3)$-bundle whose adjoint bundle is Λ^+. So, in our case, the bundle Λ^+ with the induced connection is flat. Since the metric g is an orbifold-metric and the minimal resolution of X is simply connected, the bundle Λ^+ is falt and trivial, i.e., Λ^+ has three linearly independent parallel sections. Q.E.D.

<u>Remark.</u> From the above proof one sees that there exists an orbifold-complex structure J on X such that the metric g is a Kähler orbifold-metric. Since the metric is an Einstein-Kähler orbifold metric with vanishing Ricci tensor and the canonical bundle on the minimal resolution Y descends to an orbifold-holomorphic line bundle on holomorphic orbifold-2-forms, Y must have trivial canonical bundle. So, Y is a K3 surface with the given complex structure J.

Let (X,g) be as in Theorem 2 and $\alpha: I^2(X) \to L$ a marking, i.e., a metric injection which extends to an isometry $\overline{\alpha}: H^2(Y) \longrightarrow L$. The triple (X,g,α) is a marked K3-orbifold with a Ricci-flat metric g. We define the period map p of all equivalence classes of marked K3-orbifold with a Ricci-flat metric to $G_3^+(L_{\mathbb{R}}) \cong SO_0^{\cdot}(3,19)/SO(3) \times SO(19)$ in the following way: $p(X,g,\alpha)$ is the oriented three-plane in $L_{\mathbb{R}}$ generated by the $\alpha_{\mathbb{R}}$-image of the oriented basis (three ordered linearly independent parallel self-dual 2-forms on X) of the space of parallel self-dual 2-forms. Here, if $\{e_1,e_2,e_3,e_4\}$ is an oriented basis for $\mathbb{R}^4$ then $\{e_1 \wedge e_2 + e_3 \wedge e_4,\ e_1 \wedge e_3 + e_4 \wedge e_2,\ e_1 \wedge e_4 + e_2 \wedge e_3\}$ gives the induced orientation on $\Lambda^+(\mathbb{R}^4) \subset \Lambda^2(\mathbb{R}^4)$. Two marked Ricci-flat K3-orbifolds (X,g,α) and (X',g',α') are said to be equivalent if there exists a diffeomorphism $f: Y' \longrightarrow Y$ which descends to an orbifold-diffeomorphism of X' to X and $f^*g = g'$, $\overline{\alpha'} \circ f^* = \overline{\alpha}$. Write N for the set of all equivalence classes of marked K3-orbifold with a Ricci-flat metric.

<u>Theorem</u> 6. The period map $p(X,g,\alpha) \in G_3^r(L_{\mathbb{R}})$ descends to the bijection $\sigma : N \longrightarrow G_3^+(L_{\mathbb{R}})$.

<u>Proof.</u> Suppose $p(X,g,\alpha) = p(X',g',\alpha') = \chi \in G_3^+(L_{\mathbb{R}})$. Pick a point $(\kappa,[\omega]) \in K\Omega$ in the fiber $\pi^{-1}(\chi)$, where $\pi : K\Omega \longrightarrow G_3^+(L_{\mathbb{R}})$ is the natural projection with the fiber S^2. There are marked generalized K3 surfaces (X,ϕ,α) and (X',ϕ',α') such that g and g' are Einstein-Kähler orbifold-metric. By using the isometric deformation, with respect to the Ricci-flat orbifold-metric, we may assume that the polarized periods are the same for (X,ϕ,α) and (X',ϕ',α'). From Theorem A, there exists a unique isomorphism $\overline{\Phi} : Y' \longrightarrow Y$ which descends to a unique isomorphism $\Phi : X' \longrightarrow X$ such that $\Phi^* = \overline{\gamma} = \overline{\alpha'} \circ \overline{\alpha}$ $(\Phi^* = \alpha'^{-1} \circ \alpha$ on $I^2(X)$ and $I^2(X'))$. Φ, which is an orbifold-diffeomorphism of X' to X, is an isometry with respect to g and g'. So, (X,g,α) and (X',g',α') are equivalent, i.e., σ is injective. To show the surjectivity of σ we pick a point $x \in G_3^+(L_{\mathbb{R}})$. Choose any $(\kappa,[\omega])$ in the fiber $\pi^{-1}(x) \subset K\Omega$. From the strong version of Surjectivity Theorem, there exists a marked generalized Einstein-Kähler K3 surface (X,ϕ,α) with its period $(\kappa,[\omega])$. If we forget the complex structure of (X,ϕ,α) and look at it only as a Ricci-flat

marked K3-orbifold, then its period is $\pi(\kappa,[\omega]) = x$.

Q.E.D.

Γ acts on both N and $G_3^+(L_{\mathbb{R}})$. The action of Γ on $G_3^+(L_{\mathbb{R}})$ is discrete and properly discontinuous [Vn]. The following is a generalization of the corresponding results in [Bo1] and [T2].

Theorem 7. The set of all isomorphism classes of Ricci-flat K3-orbifolds is isomorphic to

$$\Gamma \backslash (SO_0(3,19)/SO(3) \times SO(19)) .$$

The Ricci-flat K3-orbifold with simple singularities correspond to the fixed points $\mathrm{Fix}(W)$ of W. $\mathrm{Fix}(W)$ is a countable union of submanifold of codimension 3.

Proof. The last statement follows from the arguments in pp. 311-317 of [Mr].

Q.E.D.

For the convergence of non-singular Ricci-flat metrics to an orbifold-metric we can show the following:

Theorem 8. Let $\{E_t\}$ be a sequence of three dimensional subspaces in $L_{\mathbb{R}}$ such that

a) $<,>$ on each E_t is positive definite,

b) for every $\delta \in L$ with $<\delta,\delta> = -2$, $s_\delta(E_t) \neq E_t$,

c) $\lim_{t\to 0} E_t = E_0$, where $<,>$ on E_0 is positive definite and there exists $\delta \in L$ such that $\langle\delta,\delta\rangle = -2$ and $s_\delta(E_0) = E_0$,

d) let $\{g_{ij}(t)\}$ be a sequence of Einstein metrics that corresponds to E_t and suppose that vol $(g_{ij}(t) = 1$ for all t.

Then $\lim_{t\to 0} g_{ij}(t) = g_{ij}(0)$ exists and $g_{ij}(0)$ is an Einstein-Kähler orbifold-metric with respect to a complex structure on a generalized K3 surface X corresponding to some two dimensional oriented subsapce $F_0 \subset E_0$.

Proof. Let $F_t \subset E_t$ be a sequence of two dimensional subspaces in E_t such that $\lim_{t\to 0} F_t = F_0$ exists and F_0 is a two dimensional subspace in E_0. From Surjectivity Theorem and Global Torelli Theorem, we see that the sequence $\{F_t\}$ corresponds to a unique sequence of K3 surfaces (X_t,α) with a fixed marking α such that $\lim_{t\to 0} (X_t,\alpha) = (X_0,\alpha)$. Let ω_t be a unique holomorphic 2-form such that on X_t we have $\int_{X_t} \omega_t \wedge \overline{\omega_t} = 1$. Clearly $\lim_{t\to 0} \omega_t = \omega_0$ exists and $\int_{X_t} \omega_0 \wedge \overline{\omega_0} = 1$. Now let F'_t be the two dimensional subspace in E_t defined by Re ω_t and Im $g_{\alpha\overline{\beta}}(t)$, where $g_{\alpha\overline{\beta}}$ is the Einstein-Kähler metric on X_t corresponding to $\alpha^{-1}(\kappa_t)$, where $\kappa_t \perp F_t$ in E_t. Since we may suppose that $\langle\kappa_t,\kappa_t\rangle = 1$ we get from $E_t \longrightarrow E_0$ and $F_t \longrightarrow F_0$ that

$$\lim_{t\to 0} \kappa_t = \kappa_0 \in E_0 \quad \text{and} \quad \kappa_0 \perp F_0.$$

So $\lim_{t\to 0} F'_t = F_0$ exists and repeating the same arguments as for F_t we get that there exists a unique family of K3 surfaces $(X_t^{\prime},\alpha)$ with a fixed marking α such that $\lim_{t\to 0}(X_t^{\prime},\alpha) = (X_0^{\prime},\alpha)$. From the theory of isometric deformation of K3 structures with respect to tha Calabi-Yau metric (see section 3), we get that if $\omega_t^{\prime}$ is a holomorphic two form on $X_t^{\prime}$ such that $\int_{X_t^{\prime}} \omega_t^{\prime} \wedge \overline{\omega_t^{\prime}} = 1$, then

$$\omega_t^{\prime} = \operatorname{Re} \omega_t + i \operatorname{Im} g_{\alpha\overline{\beta}}(t).$$

Since $\lim_{t\to 0} \omega_t^{\prime} = \omega_0^{\prime}$ exists we get that $\lim_{t\to 0} \operatorname{Im} g_{\alpha\overline{\beta}}(t)$ exists. Now it is easy to see that $\lim_{t\to 0} \operatorname{Im} g_{\alpha\overline{\beta}}(t)$ is an Einstein-Kähler orbifold-metric form on X_0. This is so because for each point $x \in X$ vol $(g_{\alpha\overline{\beta}}(t)) = \omega_t \wedge \overline{\omega_t}$ and $\omega_0^{\prime}$ is an orbifold-holomorphic 2-form in a neighborhood of some root systems of (-2)-curves in X_0.

Q.E.D.

6. The number of quotient singularities

In section 2 we have proved the existence of a Ricci-flat Einstein-Kähler orbifold-metric on some orbifolds. This metric is used to estimate the maximal possible number of quotient singularities on a certain orbifold and to determine what occurs in case the maximal number is attained. The following is a generalization of Thm. 1 in [N].

<u>Theorem</u> 9. Let X be a compact complex surface with at worst isolated quotient singularities which admits a Kähler form in the sense of Fujiki-Moišezon. Let $\overline{X}$ be the minimal resolution for X and D its exceptional sets. Let μ_i be the nonnegative rational numbers such that $K_{\overline{X}} + \sum_i \mu_i D_i \sim 0$ near D as $\mathbb{Q}$-divisors (such μ_i's are uniquely determined), where $D = \sum_i D_i$ is the decomposition into irreducible components. Suppose that some tensor power of $K_{\overline{X}} + \sum_i \mu_i D_i$ is a trivial line bundle. Then we have the following inequality:

$$e(\overline{X}) - \sum_{p \in \operatorname{Sing} X} \left(e(D_p) - \frac{1}{|G_p|}\right) \geq 0,$$

where D_p is the exceptional set for the minimal resolution of p, G_p is the corresponding local fundamental group around p. The equality occurs if and only if $X = \Gamma \backslash T^2$, where T^2 is a complex 2-torus and Γ is a group of Euclidean motions acting on T^2 discretely and properly discontinuously with only isolated fixed points.

<u>Proof</u>. From Theorem 1, there exists an Ricci-flat Einstein-Kähler orbifold-metric on X. Using the same arguments as in [Kb] we see that the integral of the Euler form with respect to the Levi-Civita connection of the orbifold-metric is equal to $e(\overline{X}) - \sum_{p \in \operatorname{Sing} X} \left(e(D_p) - \frac{1}{|G_p|}\right)$. On the other hand, since our metric is Ricci-flat Einstein-Kähler, only the anti-self-dual Weyl tensor W_- remains in the decomposition

of the curvature tensor (see section 5). So, the Euler form is equal to $\frac{1}{8\pi^2}|W_-|^2 *1$ and thus we get

$$0 \le \frac{1}{8\pi^2}\int_X |W_-|^2 *1 = e(\overline{X}) - \sum_{p \in \mathrm{Sing}\,X} \left(e(D_p) - \frac{1}{|G_p|}\right).$$

The equality occurs if and only if $W_- \equiv 0$ for our Ricci-flat Einstein-Kähler orbifold-metric. Since every compact flat orbifold is uniformized by a torus, the equality occurs if and only if X is uniformized by a torus with the covering transformation group consisting of Euclidean motions.

Q.E.D.

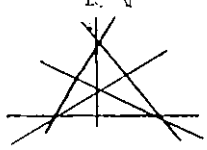
<u>Corollary</u> 10. For generalized K3 surfaces X,

$$24 - \sum_{p \in \mathrm{Sing}\,X} \left(e(D_p) - \frac{1}{|G_p|}\right) \le 0,$$

where equality occurs if and only if $X = \Gamma \backslash T^2$, with T^2 a complex 2-torus and Γ a group of Euclidean motions.

The Kummer surface with (-2)-curves collapsed is the simplest example of the above equality: $24 - 16 \times \frac{3}{2} = 0$. Ivinskis [I] found a non-trivial example

which is as follows. Consider the double covering branched over a sexitic curve in $P_2(\mathbb{C})$ with simple singularities. The double covering X is a generalized K3 surface. If $\sum_{p \in \mathrm{Sing}\,X} \left(e(D_p) - \frac{1}{|G_p|}\right) = 24$, then $X = \Gamma \backslash T^2$. For the sexitic curve



, the double cover X

has $4D_4$ and $3\ A_1$ singularities. So,

$$\sum_{p\in\mathrm{Sing}\ X} (e(D_p) - \frac{1}{|G_p|}) = 4\times(5 - \frac{1}{8}) + 3\times(2 - \frac{1}{2}) = 24.$$

For the dual sexitic curve of a smooth one, X has $9\ A_2$ singularities. So,

$$\sum_{p\in\mathrm{Sing}\ X} (e(D_p) - \frac{1}{|G_p|}) = 9\times(3 - \frac{1}{3}) = 24.$$

The sexitics with the above property are classified in [I]. For the classification of complex crystallographic groups, see [Yo·KT].

These examples show that the equality case in Theorem 9 is not void. As a final remark, we mention the degeneration of Riemannian metrics. The convergence in Theorem 8 is the simplest example of the degeneration of Riemannian metrics with bounded Ricci curvature and volume. Namely, the following occurs: there exists certain submanifolds ((-2)-rational curves) such that the "area" goes to zero and the Riemannian sectional curvature concentrates along these, and the formal Euler number $\int$(Euler form) decreases (in a "quantized" way in our case) at the limit. In the above examples, the curvature tensor concentrates so completely that the limit metric is a <u>flat</u> orbifold-metric.

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