The image of the regulator map for complete intersections of three quadrics

J. Nagel

0 Introduction

It is known that the image of the Abel–Jacobi map is a torsion group for very general complete intersections in projective space of sufficiently high multidegree [6]. In [12] I computed precise bounds and showed that these bounds are optimal. There was one exceptional case, complete intersections of four quadrics in odd–dimensional projective spaces, that had not been studied before; this case was worked out in [13, Chapter 3] using a generalisation of a theorem of Bardelli.

There is a similar result concerning the image of the regulator map

$$c_{m+1,1} : \mathrm{CH}^{m+1}(X,1) \to H^{2m+1}_{\mathcal{D}}(X, \mathbb{Z}(m+1))$$

for an even–dimensional complete intersection $X \subset \mathbb{P}^N$.

Theorem 0.1 *Let $X = V(d_0, \ldots, d_r) \subset \mathbb{P}^{2m+r+1}$ be a smooth complete intersection of dimension $2m$ ($m \geq 1$, $d_0 \geq \ldots \geq d_r$), $i : X \to \mathbb{P}^{2m+r+1}$ the inclusion map. If X is very general and if*

(C_0) $\sum_{i=0}^{r} d_i + (m-1)d_r \geq 2m + r + 3$

(C_1) $\sum_{i=1}^{r} d_i + m\, d_r \geq 2m + r + 2,$

the image of

$$c_{m+1,1} : \mathrm{CH}^{m+1}(X,1) \to H^{2m+1}_{\mathcal{D}}(X, \mathbb{Z}(m+1))/i^* H^{2m+1}_{\mathcal{D}}(\mathbb{P}^N, \mathbb{Z}(m+1))$$

is a torsion group.

In this note we study the image of the regulator map for one of the exceptional cases, complete intersections of three quadrics in odd–dimensional projective spaces. We show that the conclusion of Theorem 0.1 also holds in this case.

The two main ingredients of the proof are 1) an isomorphism between the cohomology of complete intersections of three quadrics and the cohomology of double coverings of $\mathbb{P}^2$ due to O'Grady, which is discussed in section 1; 2) a version of Nori's connectivity theorem for cyclic coverings of projective space, which is proved in section 2. Using these two results, we show in section 3 that the conclusion of Theorem 0.1 holds for very general complete intersections of three quadrics.

Notation. We shall work over the complex numbers. Cohomology is with coefficients in $\mathbb{Q}$, unless stated otherwise. Given two smooth and irreducible varieties X and Y and an integer p, the group of correspondences of degree p between X and Y is

$$\mathrm{Corr}^p(X,Y) = \mathrm{CH}^{\dim X+p}(X\times Y).$$

More generally, if X and Y are defined over a base scheme S we use the relative Chow groups $\mathrm{CH}^*(X\times Y/S)$ of cycles that are flat over S to define the group of relative correspondences

$$\mathrm{Corr}^p_S(X,Y) = \mathrm{CH}^{\dim X+p}(X\times Y/S).$$

Given an abelian group G, $\underline{G}$ denotes the constant sheaf associated to G.

1 Complete intersections of three quadrics

Let $X = V(Q_0,Q_1,Q_2) \subset \mathbb{P}^{2m+3}$ be a smooth $2m$–dimensional complete intersection of three quadrics. The variety X is the zero locus of a global section of the vector bundle $E = \bigoplus^3 \mathcal{O}_{\mathbb{P}^{2m+3}}(2)$. Set $W = \mathbb{C}[\lambda_0,\lambda_1,\lambda_2]$ and let $\mathbb{P}(W) = \mathbb{P}^2$ be the associated projective space. The universal family

$$\mathcal{X} = \{(\lambda,x)\in\mathbb{P}^2\times\mathbb{P}^{2m+3} | \lambda_0Q_0(x)+\lambda_1Q_1(x)+\lambda_2Q_2(x)=0\}$$

fits into a diagram

$$\begin{array}{ccc}\mathcal{X} & \xrightarrow{q} & \mathbb{P}^{2m+3}\\ \Big\downarrow{\scriptstyle p} & & \\ \mathbb{P}^2. & & \end{array}$$

Note that $q^{-1}(X) = \mathbb{P}^2 \times X$. Let $i : \mathbb{P}^2 \times X \to \mathcal{X}$ be the inclusion. There are natural maps

$$H^{2m}(X) \xrightarrow{q^*} H^{2m}(\mathbb{P}^2 \times X) \xrightarrow{i_*} H^{2m+4}(\mathcal{X}).$$

The map $i_* \circ q^* : H^{2m}(X) \to H^{2m+4}(\mathcal{X})$ is induced by a correspondence $\Gamma_1 \in \mathrm{Corr}^2(X, \mathcal{X})$. Let $\mathrm{Gen}\,\mathcal{X} = F_{m+1}(\mathcal{X}/\mathbb{P}^2)$ be the relative Fano scheme of $(m+1)$–planes, and let $\mathrm{Gen}\,\mathcal{X} \to S \xrightarrow{f} \mathbb{P}^2$ be the Stein factorisation of the map $\mathrm{Gen}\,\mathcal{X} \to \mathbb{P}^2$. Let $Z_S \to S$ be the universal family of $(m+1)$–planes. Put $\Gamma_2 = [Z_S] \in \mathrm{Corr}^{m+1}(S, \mathcal{X})$. Let $j : X \to \mathbb{P}^{2m+3}$ and $k : \mathcal{X} \to \mathbb{P}^2 \times \mathbb{P}^{2m+3}$ be the inclusion maps. We define

$$\begin{aligned} H^{2m}_{\mathrm{pr}}(X) &= \ker(j_* : H^{2m}(X) \to H^{2m+6}(\mathbb{P}^{2m+3})) \\ H^{2m+4}_{\mathrm{pr}}(\mathcal{X}) &= \ker(k_* : H^{2m+4}(\mathcal{X}) \to H^{2m+6}(\mathbb{P}^2 \times \mathbb{P}^{2m+3})) \\ H^2_{\mathrm{pr}}(S) &= \ker(f_* : H^2(S) \to H^2(\mathbb{P}^2)). \end{aligned}$$

The following result is due to O'Grady [16].

Theorem 1.1 (O'Grady) *The correspondence* $\Gamma = \Gamma_2 \circ {}^t\Gamma_1 \in \mathrm{Corr}^{m-1}(X, \mathcal{X})$ *induces an isomorphism of Hodge structures*

$$H^2_{\mathrm{pr}}(S, \mathbb{Q}) \cong H^{2m}_{\mathrm{pr}}(X, \mathbb{Q})(m-1).$$

The surface S is a double covering of $\mathbb{P}^2$ branched along the discriminant curve of the net of quadrics defining X. The linear map

$$\begin{aligned} H : H^0(\mathbb{P}^{2m+3}, \mathcal{O}_{\mathbb{P}}(E)) &\to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(2m+4)) \\ (Q_0, Q_1, Q_2) &\mapsto \det(\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2) \end{aligned}$$

induces a rational map

$$h : \mathbb{P}H^0(\mathbb{P}^{2m+3}, \mathcal{O}_{\mathbb{P}}(E)) - - \to \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(2m+4)).$$

The image of a set of three linearly independent quadrics (Q_0, Q_1, Q_2) is the discriminant curve $C \subset \mathbb{P}^2$ of the net of quadrics $\langle Q_0, Q_1, Q_2 \rangle$. Let $U \subset \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(2m+4))$ be the open subset that parametrises smooth curves of degree $2m+4$ in $\mathbb{P}^2$.

Lemma 1.2 *There exists a Zariski open subset* $T \subset \mathbb{P}H^0(\mathbb{P}^{2m+3}, \mathcal{O}_{\mathbb{P}}(E))$ *such that* $h : T \to U$ *is a smooth morphism.*

Proof: By a classical theorem of Dixon [4], a general smooth plane curve of even degree is the discriminant curve of a net of quadrics; cf. [1, Prop. 4.2 and Remark 4.4] for a modern proof. Hence the map h is dominant, and the assertion follows from [7, III, Lemma 10.5]. □

Over U there is a family of surfaces S_U whose fiber over $[F]$ is the double covering of $\mathbb{P}^2$ that is ramified along the curve $V(F)$. Let S_T be the pullback of this family to T. Over T we have a relative quadric bundle $\mathcal{X}_T \subset \mathbb{P}^2 \times \mathbb{P}^{2m+3} \times T$ and a family of complete intersections of quadrics $X_T \subset \mathbb{P}^{2m+3} \times T$. We can carry out the constructions preceding Theorem 1.1 in the relative setting to obtain relative correspondences $\Gamma_{T,1} \in \mathrm{Corr}^2_T(X_T, \mathcal{X}_T)$ and $\Gamma_{T,2} \in \mathrm{Corr}^{m+1}_T(S_T, \mathcal{X}_T)$ whose restrictions to a fiber coincide with the correspondences Γ_1 and Γ_2 defined before.

Let $f_U : S_U \to U$ and $g_T : X_T \to T$ be the projection maps, and let

$$\mathbb{V} = R^2(f_U)_*\mathbb{Q}_{\mathrm{pr}}, \quad \mathbb{W} = R^{2m}(g_T)_*\mathbb{Q}_{\mathrm{pr}}$$

be the associated local systems of primitive cohomology. By the proper base change theorem, the map $h : T \to U$ induces an isomorphism of local systems $h^*\mathbb{V} \cong R^2(f_T)_*\mathbb{Q}_{\mathrm{pr}}$.

Lemma 1.3 *Set $\mathbb{V}_T = h^*\mathbb{V}$. The relative correspondence $\Gamma_T = \Gamma_{T,2} \circ {}^t\Gamma_{T,1}$ induces an isomorphism of local systems $\mathbb{V}_T \cong \mathbb{W}$.*

Proof: As $\Gamma_{T,1}$ and $\Gamma_{T,2}$ are smooth over T, there exists a covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of T by contractible open sets U_α such that we have compatible C^∞ trivialisations

$$X_T|_{U_\alpha} \cong X_{t(\alpha)} \times U_\alpha, \quad \Gamma_{T,i}|_{U_\alpha} \cong \Gamma_{t(\alpha),i} \times U_\alpha$$

for $i = 1, 2$. (These are obtained by taking the flow associated to a lifting of a vector field on T.) By Theorem 1.1 we have isomorphisms of constant sheaves

$$h^*\mathbb{V}|_{U_\alpha} \cong \underline{H}^2_{\mathrm{pr}}(S_{t(\alpha)}, \mathbb{Q}) \xrightarrow{\Gamma_{t(\alpha),*}} \underline{H}^{2m}_{\mathrm{pr}}(X_{t(\alpha)}, \mathbb{Q}) \cong \mathbb{W}|_{U_\alpha}$$

for all $\alpha \in A$, such that the restrictions of $\Gamma_{t(\alpha)}$ and $\Gamma_{t(\beta)}$ to $U_\alpha \cap U_\beta$ coincide. Hence we obtain an isomorphism of sheaves $h^*\mathbb{V} \cong \mathbb{W}$. $\square$

2 Connectivity theorem for cyclic coverings

Recall the following theorem of Lazarsfeld [9, Thm. 2.1].

Theorem 2.1 (Lazarsfeld) *Let E be an ample vector bundle of rank r over a smooth projective variety Z of dimension n, and let Y be a smooth projective variety of dimension n embedded in the total space of E. Then $H^k(Y,\mathbb{Q}) \cong H^k(Z,\mathbb{Q})$ for all $k \le n-r$.*

Let $L = \mathcal{O}_{\mathbb{P}}(d)$ be a very ample line bundle on $\mathbb{P}^n$, and let N be a positive integer. Given a section $s \in H^0(\mathbb{P}^n, L^N)$, let $f : Y \to \mathbb{P}^n$ be the cyclic covering of $\mathbb{P}^n$ of degree N that is branched along the divisor $D = V(s)$. Suppose that D is smooth. Then Y is a smooth subvariety of the total space of L, and Lazarsfeld's theorem shows that

$$f^* : H^k(\mathbb{P}^n,\mathbb{Q}) \xrightarrow{\sim} H^k(Y,\mathbb{Q}) \text{ for all } k \le n-1. \tag{1}$$

By Poincaré duality, Y only has interesting cohomology in degree n. Set

$$H^n_{\mathrm{pr}}(Y,\mathbb{Q}) = \ker(f_* : H^n(Y,\mathbb{Q}) \to H^n(\mathbb{P}^n,\mathbb{Q})).$$

We have (cf. [5, Lemma 1.9] and [17, Lemmas 1.4 and 1.7])

$$\begin{aligned} f_*\mathcal{O}_Y &\cong \textstyle\bigoplus_{i=0}^{N-1} L^{-i} \\ f_*T_Y &\cong T_{\mathbb{P}} \otimes L^{-N+1} \oplus \textstyle\bigoplus_{i=1}^{N-1} T_{\mathbb{P}^n}(-\log D) \otimes L^{i-N+1} \\ f_*\Omega^p_Y &\cong \Omega^p_{\mathbb{P}^n} \oplus \textstyle\bigoplus_{i=1}^{N-1} \Omega^p_{\mathbb{P}^n}(\log D) \otimes L^{-i}. \end{aligned}$$

Using the isomorphism $H^q(Y,\Omega^p_Y) \cong H^q(\mathbb{P}^n, f_*\Omega^p_Y)$ we obtain

$$H^{p,q}_{\mathrm{pr}}(Y) \cong \textstyle\bigoplus_{i=1}^{N-1} H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}}(\log D) \otimes L^{-i}).$$

The groups on the right hand side admit an algebraic description using Jacobi rings. This leads to an isomorphism

$$H^{p,q}_{\mathrm{pr}}(Y) \cong \textstyle\bigoplus_{i=1}^{N-1} R(K_{\mathbb{P}} \otimes L^{(q+1)N-i})$$

where $R(K^a_{\mathbb{P}} \otimes L^b)$ is the quotient of $H^0(\mathbb{P}^n, K^a_{\mathbb{P}} \otimes L^b)$ by a suitable Jacobi ideal [8, Prop. 3.7].

Let $U \subset H^0(\mathbb{P}^n, L^N)$ be the subset that parametrises smooth N–fold cyclic coverings of $\mathbb{P}^n$ branched along a divisor $D \in |L|$. Let $Y_U \to U$ be the universal family and let $T \to U$ be a smooth morphism. There is a natural finite morphism $f_T : Y_T \to \mathbb{P}^N_T$. Using the connectivity result (1) and the Leray spectral sequence, we obtain

$$f_T^* : H^k(\mathbb{P}^n_T, \mathbb{Q}) \xrightarrow{\sim} H^k(Y_T, \mathbb{Q}) \text{ for all } k \leq n-1.$$

As in [15] it is possible to improve this result using the theory of infinitesimal variations of Hodge structures. The following theorem is a version of Nori's connectivity theorem for cyclic coverings.

Theorem 2.2 *Let $c \leq n$ be a natural number, and let $T \to U$ be a smooth morphism. Set $\mu = [\frac{n+c}{2}]$. The map*

$$f_T^* : H^{n+k}(\mathbb{P}^n_T, \mathbb{Q}) \to H^{n+k}(Y_T, \mathbb{Q})$$

is an isomorphism for all $k \leq c-1$ if $((\mu - c)N + 1)d \geq n + c$.

Before we give the proof, we introduce some notation.

Definition 2.3 *Let b be a nonnegative integer, and let $\pi : \mathbb{P}^n_T \to T$ be the projection map.*

(i) $\Omega^\bullet_{\mathbb{P}^n_T, Y_T} = \mathrm{Cone}(\Omega^\bullet_{\mathbb{P}^n_T} \to (f_T)_*\Omega^\bullet_{Y_T})[-1]$.

(ii) $\Omega^b_{(\mathbb{P}^n_T, Y_T)/T} = \mathrm{Cone}(\Omega^b_{\mathbb{P}^n_T/T} \to (f_T)_*\Omega^b_{Y_T/T})[-1]$;

(iii) The Leray filtration $L^\bullet$ on $\Omega^\bullet_{\mathbb{P}^n_T}$, $(f_T)_*\Omega^\bullet_{Y_T}$ and $\Omega^\bullet_{\mathbb{P}^n_T, Y_T}$ is defined by

$$\begin{aligned}
L^p\Omega^b_{\mathbb{P}^n_T} &= \mathrm{im}(\pi^*\Omega^p_T \otimes \Omega^{b-p}_{\mathbb{P}^n_T} \to \Omega^b_{\mathbb{P}^n_T}); \\
L^p(f_T)_*\Omega^b_{Y_T} &= \mathrm{im}(\pi^*\Omega^p_T \otimes (f_T)_*\Omega^{b-p}_{Y_T} \to (f_T)_*\Omega^b_{Y_T}); \\
L^p\Omega^b_{\mathbb{P}^n_T, Y_T} &= \mathrm{Cone}(L^p\Omega^b_{\mathbb{P}^n_T} \to L^p(f_T)_*\Omega^b_{Y_T})[-1].
\end{aligned}$$

Remark 2.4 As the map $\Omega^\bullet_{\mathbb{P}^n_T} \to (f_T)_*\Omega^\bullet_{Y_T}$ is injective, the complex $\Omega^\bullet_{\mathbb{P}^n_T, Y_T}$ is quasi–isomorphic to the complex coker $(\Omega^\bullet_{\mathbb{P}^n_T} \to (f_T)_*\Omega^\bullet_{Y_T})[-1]$.

By construction we have

$$\mathrm{Gr}^p_L \Omega^b_{\mathbb{P}^n_T, Y_T} \cong \pi^*\Omega^p_T \otimes \Omega^{b-p}_{(\mathbb{P}^n_T, Y_T)/T}.$$

Given a point $t \in T$, let $i_t : Y_t \to Y_T$ be the inclusion map. Taking cohomology in the short exact sequence

$$0 \to i_t^* \mathrm{Gr}^p_L f_*\Omega^{b-1}_{Y_T} \to i_t^* \mathrm{Gr}^p_L \Omega^b_{Y_T, \mathbb{P}^n_T} \to i_t^* \mathrm{Gr}^p_L \Omega^b_{\mathbb{P}^n_T} \to 0$$

we obtain an isomorphism

$$H^q(Y_t, i_t^* \operatorname{Gr}_L^p \Omega^b_{Y_T,\mathbb{P}^n_T}) \cong \Omega^p_{T,t} \otimes H_{\mathrm{pr}}^{b-p-1,q}(Y_t).$$

After these preparations, we turn to the proof of Theorem 2.2.

Proof: The proof is analogous to the proof of Nori's theorem given in [14]; cf. also [18]. We only indicate the main points and omit the details. Using mixed Hodge theory and the Leray spectral sequence one reduces to the following condition of Nori.

$$(N_c) \quad R^a\pi_*\Omega^b_{\mathbb{P}^n_T,Y_T} = 0 \text{ if } a+b=n+k, b \geq [\frac{n+k}{2}], k \leq c.$$

Using homological algebra techniques one proves that condition (N_c) holds if the complex

$$\begin{aligned}\Omega^{k-1}_{T,t} \otimes H_{\mathrm{pr}}^{b-k+1,n-b+k-1}(Y_t) \to \Omega^k_{T,t} \otimes H_{\mathrm{pr}}^{b-k,n-b+k}(Y_t) \to \\ \to \Omega^{k+1}_{T,t} \otimes H_{\mathrm{pr}}^{b-k-1,n-b+k+1}(Y_t)\end{aligned}$$

is exact for all $t \in T$ and for all (b,k) such that $k \leq c-1$ and $b \geq [\frac{n+k}{2}]$. We dualise this complex to obtain

$$\begin{aligned}\bigwedge^{k+1}T_t \otimes H_{\mathrm{pr}}^{n-b+k+1,b-k-1}(Y_t) \to \bigwedge^k T_t \otimes H_{\mathrm{pr}}^{n-b+k,b-k}(Y_t) \to \\ \to \bigwedge^{k-1}T_t \otimes H_{\mathrm{pr}}^{n-b+k-1,b-k+1}(Y_t).\end{aligned}$$

Since condition (N_c) is preserved under smooth base change [15, Lemma 2.2], it suffices to verify condition (N_c) for $T = U \subset H^0(\mathbb{P}^n, L^N)$. In this case, the tangent space to T at a point t is isomorphic to $V = H^0(\mathbb{P}^n, L^N)$. Using the Jacobi ring description we see that it suffices to prove the exactness of the complex

$$\begin{aligned}\bigwedge^{k+1}V \otimes H^0(\mathbb{P}^n, K_{\mathbb{P}} \otimes L^{(b-k)N-i}) \to \bigwedge^k V \otimes H^0(\mathbb{P}^n, K_{\mathbb{P}} \otimes L^{(b-k+1)N-i}) \to \\ \to \bigwedge^{k-1}V \otimes H^0(\mathbb{P}^n, K_{\mathbb{P}} \otimes L^{(b-k+2)N-i})\end{aligned} \tag{2}$$

for $i = 0, \ldots, N-1$. (There are additional conditions, but these lead to weaker conditions on d.) Let M_N be the vector bundle defined by the exact sequence

$$0 \to M_N \to V \otimes \mathcal{O}_{\mathbb{P}} \to L^N \to 0.$$

One readily verifies that $H^k(\mathbb{P}^n, M_N(1-k)) = 0$ for all $k > 0$. Hence the vector bundle M_N is 1–regular in the sense of Castelnuovo–Mumford. The complex (2) is exact if

$$H^1(\mathbb{P}^n, \bigwedge^{k+1} M_N \otimes K_{\mathbb{P}} \otimes L^{(b-k)N-i}) = 0.$$

The strongest condition is obtained by taking $k = c-1$, $b = \mu$ and $i = N-1$. By regularity methods we find that

$$H^1(\mathbb{P}^n, \bigwedge^c M_N \otimes K_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}((\mu - c)N + 1)d)) = 0$$

if $((\mu - c)N + 1)d \geq n + c$. $\square$

3 The image of the regulator map

Let X be a smooth projective variety of dimension n. Recall that the decomposable and indecomposable part of Bloch's higher Chow group $\mathrm{CH}^p(X, 1)$ are defined by

$$\begin{aligned} \mathrm{CH}^p_{\mathrm{dec}}(X,1) &= \mathrm{im}(\mathrm{CH}^{p-1}(X) \otimes \mathbb{C}^* \to \mathrm{CH}^p(X,1)) \\ \mathrm{CH}^p_{\mathrm{ind}}(X,1) &= \mathrm{CH}^p(X,1)/\mathrm{CH}^p_{\mathrm{dec}}(X,1). \end{aligned}$$

Let $c_{p,1} : \mathrm{CH}^p(X,1) \to H^{2p-1}_{\mathcal{D}}(X, \mathbb{Z}(p))$ be the regulator map; cf. [11, 2.3]. Put

$$\begin{aligned} H^{2p-1}_{\mathcal{D},\mathrm{dec}}(X, \mathbb{Z}(p)) &= \mathrm{im}(H^{2p-2}_{\mathcal{D}}(X, \mathbb{Z}(p-1)) \otimes \mathbb{C}^* \to H^{2p-1}_{\mathcal{D}}(X, \mathbb{Z}(p))) \\ H^{2p-1}_{\mathcal{D},\mathrm{ind}}(X, \mathbb{Z}(p)) &= H^{2p-1}_{\mathcal{D}}(X, \mathbb{Z}(p))/H^{2p-1}_{\mathcal{D},\mathrm{dec}}(X, \mathbb{Z}(p)). \end{aligned}$$

Following [3] we say that $z \in \mathrm{CH}^p(X,1)$ is regulator decomposable if $c_{p,1}(z) \in H^{2p-1}_{\mathcal{D},\mathrm{dec}}(X, \mathbb{Z}(p))$.

Remark 3.1 Given a $\mathbb{Z}$–Hodge structure H of weight $2p-2$, write

$$J(H) = \mathrm{Ext}^1_{MHS}(\mathbb{Z}(-p), H) \cong H_{\mathbb{C}}/(F^p H_{\mathbb{C}} + H_{\mathbb{Z}}).$$

Let $N^\bullet$ be the coniveau filtration on $H^*(X)$. As

$$c_{p,1}(\mathrm{CH}^p_{\mathrm{dec}}(X,1)) \subseteq J(N^{p-1}H^{2p-2}(X)) \subseteq \mathrm{Hdg}^{p-1}(X) \otimes \mathbb{C}^*$$

and the image of $\bar{c}_{p,1} : \mathrm{CH}^p_{\mathrm{ind}}(X,1) \to H^{2p-1}_{\mathcal{D},\mathrm{ind}}(X, \mathbb{Z}(p))$ is a countable group [11, Corollary 3.6], it follows that $c_{p,1}$ cannot be surjective if $H^{2p-2}(X) \neq N^{p-1}H^{2p-2}(X)$.

Theorem 2.2 gives the following result on the image of the regulator map for cyclic coverings.

Corollary 3.2 *Let $f : Y \to \mathbb{P}^{2m}$ $(m \geq 1)$ be a very general cyclic covering of degree N. Then the image of the regulator map*

$$c_{m+1,1} : \mathrm{CH}^{m+1}(Y,1) \to H^{2m+1}_{\mathcal{D}}(Y, \mathbb{Z}(m+1))/f^* H^{2m+1}_{\mathcal{D}}(\mathbb{P}^{2m}, \mathbb{Z}(m+1))$$

is a torsion group if $((m-1)N+1)d \geq 2m+2$.

Proof: Argue as in [11, Thm. 6.2], using Theorem 2.2 with $c = 2$. □

Remark 3.3 There is a similar result for the image of the Abel–Jacobi map for odd–dimensional cyclic coverings of projective space; see [10, Thm. 2]. Our proof of Theorem 2.2 was inspired by the results in Müller–Stach's paper.

A standard construction in algebraic topology allows us to define an analogue of relative cohomology for the pair $(\mathbb{P}^2_T, S_T)$, even though the map $f_T : S_T \to \mathbb{P}^2_T$ is not an inclusion. Let $M(f_T)$ be the mapping cylinder of $f_T : S_T \to \mathbb{P}^2_T$. The map f_T factors as $S_T \hookrightarrow M(f_T) \xrightarrow{\sim} \mathbb{P}^2_T$ where the second map is a homotopy equivalence. Set

$$H^k(\mathbb{P}^2_T, S_T) = H^k(M(f_T)).$$

These groups fit into a long exact sequence

$$\ldots \to H^{k-1}(S_T) \to H^k(\mathbb{P}^2_T, S_T) \to H^k(\mathbb{P}^2_T) \xrightarrow{f^*} H^k(S_T) \to \ldots$$

of cohomology groups.

Lemma 3.4 *With the above notation, we have* $H^{k+2}(\mathbb{P}^2_T, S_T) \cong H^{k+2m}(\mathbb{P}^{2m+3}_T, X_T)$ *for all* $k \geq 0$.

Proof: Let $\pi : \mathbb{P}^{2m+3}_T \to T$, $f : S_T \to T$ and $g : X_T \to T$ be the projection maps. As before, we define

$$\mathbb{V} = R^2 f_* \mathbb{Q}_{\mathrm{pr}}, \quad \mathbb{W} = R^{2m} g_* \mathbb{Q}_{\mathrm{pr}}.$$

By the Lefschetz hyperplane theorem we have

$$\bigoplus_{\substack{i+j=2m+k \\ j \neq 2m}} H^i(T, R^j \pi_* \mathbb{Q}) \cong \bigoplus_{\substack{i+j=2m+k \\ j \neq 2m}} H^i(T, R^j g_* \mathbb{Q}).$$

If we quotient out $H^{2m+k}(\mathbb{P}_T^{2m+3})$ and $H^{2m+k}(X_T)$ by this subgroup in the long exact cohomology sequence of the pair $(\mathbb{P}_T^{2m+3}, X_T)$ for all $k \geq 0$, exactness is preserved and we obtain a long exact sequence

$$\ldots \to H^{2m+k}(\mathbb{P}_T^{2m+3}, X_T) \to H^k(T, R^{2m}\pi_*\mathbb{Q}) \to H^k(T, R^{2m}g_*\mathbb{Q}) \to \ldots$$

of cohomology groups. Using the splitting $R^{2m}g_*\mathbb{Q} \cong i^*R^{2m}\pi_*\mathbb{Q} \oplus \mathbb{W}$ and the five lemma, we obtain

$$H^{2m+k}(\mathbb{P}_T^{2m+3}, X_T) \cong H^{k-1}(T, \mathbb{W}).$$

In a similar way one shows that $H^{k+2}(\mathbb{P}_T^2, S_T) \cong H^{k-1}(T, \mathbb{V}_T)$, and we conclude using Lemma 1.3.

$\square$

Theorem 3.5 *Let $X \subset \mathbb{P}^{2m+3}$ be a complete intersection of three quadrics, $i : X \to \mathbb{P}^{2m+3}$ the inclusion map. If X is very general, the image of*

$$c_{m+1,1} : \mathrm{CH}^{m+1}(X, 1) \to H_{\mathcal{D}}^{2m+1}(X, \mathbb{Z}(m+1))/i^*H_{\mathcal{D}}^{2m+1}(\mathbb{P}^{2m+3}, \mathbb{Z}(m+1))$$

is a torsion group if $m > 1$. In particular, $\mathrm{CH}^{m+1}(X, 1)$ is regulator decomposable if $m > 1$.

Proof: Apply Theorem 2.2 with $Y_T = S_T$. We find that

$$H^{k+2}(\mathbb{P}_T^2, S_T) = 0 \text{ if } k \leq 2 \text{ and } m > 1.$$

By Lemma 3.4 we obtain $H^{2m+k}(\mathbb{P}_T^{2m+3}, X_T) = 0$ if $k \leq 2$ and $m > 1$. Hence the map

$$H^{2m+1}(\mathbb{P}_T^{2m+3}, \mathbb{Q}) \to H^{2m+1}(X_T, \mathbb{Q})$$

is an isomorphism if $m > 1$. We then argue as in [11, Thm. 6.2]. $\square$

Remark 3.6

(i) We have a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^2(S,1) & \xrightarrow{\Gamma_*} & \mathrm{CH}^{m+1}(X,1) \\ \downarrow{\scriptstyle c_{2,1}} & & \downarrow{\scriptstyle c_{m+1,1}} \\ H_{\mathcal{D}}^3(S, \mathbb{Z}(2)) & \xrightarrow{\Gamma_*} & H_{\mathcal{D}}^{2m+1}(X, \mathbb{Z}(m+1)). \end{array}$$

The lower horizontal map is an isomorphism by O'Grady's theorem. If the conjectural Bloch–Beilinson filtration on (higher) Chow groups exists, then Γ_* induces an isomorphism $\mathrm{im}(c_{2,1}) \xrightarrow{\sim} \mathrm{im}(c_{m+1,1})$, and Theorem 3.5 follows from Corollary 3.2.

(ii) A parameter count shows that a general surface of even degree $d \geq 4$ in $\mathbb{P}^3$ is not the discriminant locus of a web of quadrics; cf. [1, (1.9)]. This explains the difference between the behaviour of the regulator map for complete intersections of three quadrics in even–dimensional projective space and the behaviour of the Abel–Jacobi map for complete intersections of four quadrics in odd–dimensional projective space.

Remark 3.7 The main point in the proof of Theorem 3.5 is to establish the exactness of the complex

$$0 \to H^{m+1,m-1}_{\mathrm{pr}}(X) \to \Omega^1_{T,t} \otimes H^{m,m}_{\mathrm{pr}}(X) \to \Omega^2_{T,t} \otimes H^{m-1,m+1}_{\mathrm{pr}}(X). \tag{3}$$

Define

$$\begin{aligned} E &= \bigoplus^3 \mathcal{O}_{\mathbb{P}^{2m+3}}(2), \quad P = \mathbb{P}(E) = \mathbb{P}^2 \times \mathbb{P}^{2m+3} \\ S_{a,b} &= H^0(P, \mathcal{O}_{\mathbb{P}^2}(a) \boxtimes \mathcal{O}_{\mathbb{P}^{2m+3}}(b)), \quad V = H^0(\mathbb{P}^{2m+3}, E) \cong S_{1,2}. \end{aligned}$$

The description of the dual complex using the Jacobi ring for complete intersections shows that a sufficient condition for exactness of the complex (3) is the exactness of the complex

$$\bigwedge^2 V \otimes S_{m-1,0} \to V \otimes S_{m,2} \to S_{m+1,4} \to 0$$

plus the surjectivity of the map $V \otimes J_{m,2} \to J_{m+1,4}$ on the Jacobi ideal. The latter condition leads to weaker degree conditions, so we concentrate on the first condition. The method used in the proof of Theorem 0.1 to establish exactness is the use of Castelnuovo–Mumford regularity. In this case, this method does not apply because the second index in the vector space on the left is nonpositive. (A dimension computation shows that exactness fails for $m \leq 2$. It is possible that one can prove exactness for large values of m by a more careful study of the complex, using representation theory.)

In this note, we have replaced the complex (3) by the isomorphic complex

$$0 \to H^{2,0}(S) \to \Omega^1_T \otimes H^{1,1}(S) \to H^{0,2}(S). \tag{4}$$

Put $R = \mathbb{C}[X_0, X_1, X_2]$ and $W = R_{2m-4}$. The Jacobi ring description of the cohomology of cyclic coverings of the plane, combined with Nori's base change result, shows that the complex (4) is exact if the complex

$$\bigwedge^2 W \otimes R_{m-1} \otimes W \otimes R_{2m-5} \to R_{4m-6} \to 0$$

is exact. It follows from the proof of Theorem 2.2 that this condition is satisfied if $m > 1$. Hence the description of the cohomology of X by a different Jacobi ring provides more precise information about the complex (3).

Remark 3.8 We briefly discuss the remaining possible exceptions to Theorem 0.1.

(i) $(r = 0)$ $X = V(2) \subset \mathbb{P}^{2m+1}$, $m \geq 2$, $X = V(3) \subset \mathbb{P}^3$;

(ii) $(r = 0)$ $X = V(4) \subset \mathbb{P}^3$, $X = V(3) \subset \mathbb{P}^5$;

(iii) $(r = 1)$ $X = V(d, 2) \subset \mathbb{P}^{2m+2}$, $d \geq 2$, $m \geq 1$.

The conclusion of Theorem 0.1 fails in case (i) and in case (iii) for $d = 2$ by the Noether–Lefschetz theorem. Note that in both cases $H^{2m+1}_{\mathcal{D}}(X, \mathbb{Z}(m+1)) \cong H^{2m}_{\mathcal{D}}(X, \mathbb{Z}) \otimes \mathbb{C}^*$, hence $\mathrm{CH}^{m+1}(X, 1)$ is regulator decomposable. It is known that rational nodal curves on K3 surfaces give rise to regulator indecomposable higher Chow cycles [2], hence $X = V(4) \subset \mathbb{P}^3$, $X = V(3, 2) \subset \mathbb{P}^4$ and $X = V(2, 2, 2) \subset \mathbb{P}^5$ are counterexamples. (For $X = V(4)$ see also [3],[11].) In [3] Collino shows the existence of regulator indecomposable cycles on cubic fourfolds. Hence the remaining cases are (iii) with $d > 2$ and $(d, m) \neq (3, 1)$. It is not clear what happens in these cases.

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Université Lille 1, Mathématiques - Bât. M2, F-59655 Villeneuve d'Ascq Cedex, France.

email: nagel@agat.univ-lille1.fr

Max–Planck–Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany.