A TWISTED BURNSIDE THEOREM FOR COUNTABLE GROUPS AND REIDEMEISTER NUMBERS

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Abstract. The purpose of the present paper is to prove for finitely generated groups of type I the following conjecture of A. Fel'shtyn and R. Hill [8], which is a generalization of the classical Burnside theorem.

Let G be a countable discrete group, ϕ one of its automorphisms, $R(\phi)$ the number of ϕ-conjugacy classes, and $S(\phi) = \#\operatorname{Fix}(\widehat{\phi})$ the number of ϕ-invariant equivalence classes of irreducible unitary representations. If one of $R(\phi)$ and $S(\phi)$ is finite, then it is equal to the other.

This conjecture plays a important role in the theory of twisted conjugacy classes (see [12], [6]) and has very important consequences in Dynamics, while its proof needs rather sophisticated results from Functional and Non-commutative Harmonic Analysis.

We begin a discussion of the general case (which needs another definition of the dual object). It will be the subject of a forthcoming paper.

Some applications and examples are presented.

Contents

1. Introduction and formulation of results

Definition 1.1. Let G be a countable discrete group and $\phi : G \to G$ an endomorphism. Two elements $x, x' \in G$ are said to be *ϕ-conjugate* or *twisted conjugate* iff there exists $g \in G$ with

$$x' = gx\phi(g^{-1}).$$

We shall write $\{x\}_\phi$ for the *ϕ-conjugacy* or *twisted conjugacy* class of the element $x \in G$. The number of ϕ-conjugacy classes is called the *Reidemeister number* of an endomorphism

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ϕ and is denoted by $R(\phi)$. If ϕ is the identity map then the ϕ-conjugacy classes are the usual conjugacy classes in the group G.

If G is a finite group, then the classical Burnside theorem (see e.g. [13, p. 140]) says that the number of classes of irreducible representations is equal to the number of conjugacy classes of elements of G. Let $\widehat{G}$ be the *unitary dual* of G, i.e. the set of equivalence classes of unitary irreducible representations of G.

Remark 1.2. If $\phi : G \to G$ is an epimorphism, it induces a map $\widehat{\phi} : \widehat{G} \to \widehat{G}$, $\widehat{\phi}(\rho) = \rho \circ \phi$ (because a representation is irreducible if and only if the scalar operators in the space of representation are the only ones which commute with all operators of the representation). This is not the case for a general endomorphism ϕ, because $\rho\phi$ can be reducible for an irreducible representation ρ, and $\widehat{\phi}$ can be defined only as a multi-valued map. But nevertheless we can define the set of fixed points $\operatorname{Fix}\widehat{\phi}$ of $\widehat{\phi}$ on $\widehat{G}$.

Therefore, by the Burnside's theorem, if ϕ is the identity automorphism of any finite group G, then we have $R(\phi) = \#\operatorname{Fix}(\widehat{\phi})$.

To formulate our theorem for the case of a general endomorphism we first need an appropriate definition of the $\operatorname{Fix}(\widehat{\phi})$.

Definition 1.3. Let $\operatorname{Rep}(G)$ be the space of equivalence classes of finite dimensional unitary representations of G. Then the corresponding map $\widehat{\phi}_R : \operatorname{Rep}(G) \to \operatorname{Rep}(G)$ is defined in the same way as above: $\widehat{\phi}_R(\rho) = \rho \circ \phi$.

Let us denote by $\operatorname{Fix}(\widehat{\phi})$ the set of points $\rho \in \widehat{G} \subset \operatorname{Rep}(G)$ such that $\widehat{\phi}_R(\rho) = \rho$.

Theorem 1.4 (Main Theorem)**.** *Let G be a finitely generated discrete group of type* I*, ϕ one of its endomorphism, $R(\phi)$ the number of ϕ-conjugacy classes, and $S(\phi) = \#\operatorname{Fix}(\widehat{\phi})$ the number of ϕ-invariant equivalence classes of irreducible unitary representations. If one of $R(\phi)$ and $S(\phi)$ is finite, then it is equal to the other.*

Let $\mu(d)$, $d \in \mathbb{N}$, be the *Möbius function*, i.e.

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ (-1)^k & \text{if } d \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{if } d \text{ is not square} - \text{free.} \end{cases}$$

Theorem 1.5 (Congruences for the Reidemeister numbers)**.** *Let $\phi : G \to G$ be an endomorphism of a countable discrete group G such that all numbers $R(\phi^n)$ are finite and let H be a subgroup of G with the properties*

$$\phi(H) \subset H$$

$$\forall x \in G\ \exists n \in \mathbb{N} \text{ such that } \phi^n(x) \in H.$$

If the pair (H, ϕ^n) satisfies the conditions of Theorem 1.4 for any $n \in \mathbb{N}$, then one has for all n,

$$\sum_{d|n} \mu(d) \cdot R(\phi^{n/d}) \equiv 0 \mod n.$$

These theorems were proved previously in a special case of Abelian finitely generated plus finite group [8, 9].

The interest in twisted conjugacy relations has its origins, in particular, in the Nielsen-Reidemeister fixed point theory (see, e.g. [12, 6]), in Selberg theory (see, eg. [14, 1]), and Algebraic Geometry (see, e.g. [11]).

Concerning some topological applications of our main results, they are already obtained in the present paper (Theorem 8.5). The congruences give some necessary conditions for the realization problem for Reidemeister numbers in topological dynamics. The relations with Selberg theory will be presented in a forthcoming paper.

Let us remark that it is known that the Reidemeister number of an endomorphism of a finitely generated Abelian group is finite iff 1 is not in the spectrum of the restriction of this endomorphism to the free part of the group (see, e.g. [12]). The Reidemeister number is infinite for any automorphism of a non-elementary Gromov hyperbolic group [5].

To make the presentation more transparent we start from a new approach (E.T.) for Abelian (Section 2) and compact (Section 3) groups. Only after that we develop this approach and prove the main theorem for finitely generated groups of type I in Section 5. A discussion of some examples leading to conjectures is the subject of Section 6. Then we prove the congruences theorem (Section 7) and describe some topological applications (Section 8).

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2. Abelian case

Let ϕ be an automorphism of an Abelian group G.

Lemma 2.1. *The twisted conjugacy class H of e is a subgroup. The other ones are cosets gH.*

Proof. The first statement follows from the equalities

$$h\phi(h^{-1})g\phi(g^{-1}) = gh\phi((gh)^{-1}, \quad (h\phi(h^{-1}))^{-1} = \phi(h)h^{-1} = h^{-1}\phi(h).$$

For the second statement suppose $a \sim b$, i.e. $b = ha\phi(h^{-1})$. Then

$$gb = gha\phi(h^{-1}) = h(ga)\phi(h^{-1}), \qquad gb \sim ga.$$

□

Lemma 2.2. *Suppose, $u_1, u_2 \in G$, χ_H is the characteristic function of H as a set. Then*
$$\chi_H(u_1 u_2^{-1}) = \begin{cases} 1, & \text{if } u_1, u_2 \text{ are in one coset },\\ 0, & \text{otherwise }. \end{cases}$$

Proof. Suppose, $u_1 \in g_1 H$, $u_2 \in g_2 H$, hence, $u_1 = g_1 h_1$, $u_2 = g_2 h_2$. Then
$$u_1 u_2^{-1} = g_1 h_1 h_2^{-1} g_2^{-1} \in g_1 g_2^{-1} H.$$
Thus, $\chi_H(u_1 u_2^{-1}) = 1$ if and only if $g_1 g_2^{-1} \in H$ and u_1 and u_2 are in the same class. Otherwise it is 0. □

The following Lemma is well known.

Lemma 2.3. *For any subgroup H the function χ_H is of positive type.*

Proof. Let us take arbitrary elements $u_1, u_2, \dots, u_n$ of G. Let us reenumerate them in such a way that some first are in $g_1 H$, the next ones are in $g_2 H$, and so on, till $g_m H$, where $g_j H$ are different cosets. By the previous Lemma the matrix $\|p_{it}\| := \|\chi_H(u_i u_t^{-1})\|$ is block-diagonal with square blocks formed by units. These blocks, and consequently the whole matrix are positively semi-defined. □

Lemma 2.4. *In the Abelian case characteristic functions of twisted conjugacy classes belong to the Fourier-Stieltjes algebra $B(G) = (C^*(G))^*$.*

Proof. In this case the characteristic functions of twisted conjugacy classes are the shifts of the characteristic function of the class H of e. Indeed, we have the following sequence of equivalent properties:
$$a \sim b, \qquad b = ha\phi(h^{-1}) \text{ for some } h, \qquad gb = gha\phi(h^{-1}) \text{ for some } h,$$
$$gb = hga\phi(h^{-1}) \text{ for some } h, \qquad ga \sim gb.$$
Hence, by Corollary (2.19) of [4], these characteristic functions are in $B(G)$. □

Let us remark that there exists a natural isomorphism (Fourier transform)
$$u \mapsto \widehat{u}, \qquad C^*(G) = C^*_r(G) \cong C(\widehat{G}), \qquad \widehat{g}(\rho) := \rho(g),$$
(this is a number because irreducible representations of an Abelian group are 1-dimensional). In fact, it is better to look (for what follows) at an algebra $C(\widehat{G})$ as an algebra of continuous sections of a bundle of 1-dimensional matrix algebras. over $\widehat{G}$.

Our characteristic functions, being in $B(G) = (C^*(G))^*$ in this case, are mapped to the functionals on $C(\widehat{G})$ which, by the Riesz-Markov-Kakutani theorem, are measures on $\widehat{G}$. Which of these measures are invariant under the induced (twisted) action of G ? Let us remark, that an invariant non-trivial functional gives rise to at least one invariant space – its kernel.

Let us remark, that convolution under the Fourier transform becomes point-wise multiplication. More precisely, the twisted action, for example, is defined as
$$g[f](\rho) = \rho(g) f(\rho) \rho(\phi(g^{-1})), \qquad \rho \in \widehat{G}, \quad g \in G, \quad f \in C(\widehat{G}).$$

There are 2 possibilities for the twisted action of G on the representation algebra $A_\rho \cong \mathbb{C}$: 1) the linear span of the orbit of $1 \in A_\rho$ is equal to all A_ρ, 2) and the opposite case (the action is trivial).

The second case means that the space of interviewing operators between A_ρ and $A_{\widehat{\phi}\rho}$ equals $\mathbb{C}$, and ρ is a fixed point of the action $\widehat{\phi}:\widehat{G}\to\widehat{G}$. In the first case this is the opposite situation.

If we have a finite number of such fixed points, then the space of twisted invariant measures is just the space of measures concentrated in these points. Indeed, let us describe the action of G on measures in more detail.

Lemma 2.5. *For any Borel set E one has $g[\mu](E)=\int_E g[1]\,d\mu$.*

Proof. The restriction of measure to any Borel set commutes with the action of G, since the last is point wise on $C(\widehat{G})$. For any Borel set E one has

$$g[\mu](E)=\int_E 1\,dg[\mu]=\int_E g[1]\,d\mu.$$

□

Hence, if μ is twisted invariant, then for any Borel set E and any $g\in G$ one has

$$\int_E (1-g[1])\,d\mu=0.$$

Lemma 2.6. *Suppose, $f\in C(X)$, where X is a compact Hausdorff space, and μ is a regular Borel measure on X, i.e. a functional on $C(X)$. Suppose, for any Borel set $E\subset X$ one has $\int_E f\,d\mu=0$. Then $\mu(h)=0$ for any $h\in C(X)$ such that $f(x)=0$ implies $h(x)=0$. I.e. μ is concentrated off the interior of* $\operatorname{supp} f$.

Proof. Since the functions of the form fh are dense in the space of the refered to above h's, it is sufficient to verify the statement for fh. Let us choose an arbitrary $\varepsilon>0$ and a simple function $h'=\sum_{i=1}^n a_i\chi_{E_i}$ such that $|\mu(fh')-\mu(fh)|<\varepsilon$. Then

$$\mu(fh')=\sum_{i=1}^n\int_{E_i}a_i f\,d\mu=\sum_{i=1}^n a_i\int_{E_i}f\,d\mu=0.$$

Since ε is an arbitrary one, we are done. □

Applying this lemma to a twisted invariant measure μ and $f=1-g[1]$ we obtain that μ is concentrated at our finite number of fixed points of $\widehat{\phi}$, because outside of them $f\neq 0$.

If we have an infinite number of fixed points, then the space is infinite-dimensional (we have an infinite number of measures concentrated in finite number of points, each time different) and Reidemeister number is infinite as well. So, we are done.

3. Compact case

Let G be a compact group, hence $\widehat{G}$ is a discrete space. Then $C^*(G)=\oplus M_i$, where M_i are the matrix algebras of irreducible representations. The infinite sum is in the following sense:

$$C^*(G)=\{f_i\}, i\in\{1,2,3,...\}=\hat{G}, f_i\in M_i, \|f_i\|\to 0 (i\to\infty).$$

When G is finite and $\widehat{G}$ is finite this is exactly Peter-Weyl theorem.

A characteristic function of a twisted class is a functional on $C^*(G)$. For a finite group it is evident, for a general compact group it is necessary to verify only the measurability of the twisted class with the respect to Haar measure, i.e. that twisted class is Borel. For a compact G, the twisted conjugacy classes being orbits of twisted action are compact and hence closed. Then its complement is open, hence Borel, and the class is Borel too.

Under the identification it passes to a sequence $\{\varphi_i\}$, where φ_i is a functional on M_i (the properties of convergence can be formulated, but they play no role at the moment). The conditions of invariance are the following: for each $\rho_i \in \widehat{G}$ one has $g[\varphi_i] = \varphi_i$, i.e. for any $a \in M_i$ and any $g \in G$ one has $\varphi_i(\rho_i(g)a\rho_i(\phi(g^{-1}))) = \varphi_i(a)$.

Let us recall the following well-known fact.

Lemma 3.1. *Each functional on matrix algebra has form $a \mapsto \mathrm{Tr}(ab)$ for a fixed matrix b.*

Proof. One has $\dim(M(n,\mathbb{C}))' = \dim(M(n,\mathbb{C})) = n \times n$ and looking at matrices as at operators in V, $\dim V = n$, with base e_i, one can remark that functionals $a \mapsto \langle ae_i, e_j\rangle$, $i, j = 1, \dots, n$, are linearly independent. Hence, any functional takes form

$$a \mapsto \sum_{i,j} b^i_j \langle ae_i, e_j\rangle = \sum_{i,j} b^i_j a^j_i = \mathrm{Tr}(ba), \qquad b := \|b^i_j\|.$$

□

Now we can study invariant ones:

$$\mathrm{Tr}(b\rho_i(g)a\rho_i(\phi(g^{-1}))) = \mathrm{Tr}(ba), \qquad \forall\, a, g,$$
$$\mathrm{Tr}((b - \rho_i(\phi(g^{-1}))b\rho_i(g))a) = 0, \qquad \forall\, a, g,$$

hence,

$$b - \rho_i(\phi(g^{-1}))b\rho_i(g) = 0, \qquad \forall\, g.$$

Since ρ_i is irreducible, the dimension of the space of such b is 1 if ρ_i is a fixed point of $\widehat{\phi}$ and 0 in the opposite case. So, we are done.

Remark 3.2. In fact we are only interested in finite discrete case. Indeed, for a compact G, the twisted conjugacy classes being orbits of twisted action are compact and hence closed. If there is a finite number of them, then are open too. Hence, the situation is more or less reduced to a discrete group: quotient by the component of unity.

4. Extensions and Reidemeister classes

Consider a group extension respecting homomorphism ϕ:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{i} & G & \xrightarrow{p} & G/H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \overline{\phi}} & & \\ 0 & \longrightarrow & H & \xrightarrow{i} & G & \xrightarrow{p} & G/H & \longrightarrow & 0, \end{array}$$

where H is a normal subgroup of G. The following argument has partial intersection with [10].

First of all let us notice that the Reidemeister classes of ϕ in G are mapped epimorphically on classes of $\overline{\phi}$ in G/H. Indeed,

$$p(\widetilde{g})p(g)\overline{\phi}(p(\widetilde{g}^{-1})) = p(\widetilde{g}g\phi(\widetilde{g}^{-1})).$$

Suppose, $R(\phi) < \infty$. Then the previous remark implies $R(\overline{\phi}) < \infty$. Consider a class $K = \{h\}_{\tau_g \phi'}$, where $\tau_g(h) := ghg^{-1}$, $g \in G$, $h \in H$. The corresponding equivalence relation is

$$h \sim \widetilde{h} h g \phi'(\widetilde{h}^{-1}) g^{-1}. \tag{1}$$

Since H is normal, the automorphism $\tau_g : H \to H$ is well defined. We will denote by K the image iK as well. By (1) the shift Kg is a subset of Hg is characterized by

$$hg \sim \widetilde{h}(hg)\phi'(\widetilde{h}^{-1}). \tag{2}$$

Hence it is a subset of $\{hg\}_\phi \cap Hg$ and the partition $Hg = \cup(\{h\}_{\tau_g\phi'})g$ is a subpartition of $Hg = \cup(Hg \cap \{hg\}_\phi)$.

Lemma 4.1. *Suppose, $|G/H| = N < \infty$. Then $R(\tau_g\phi') \le NR(\phi)$. More precisely, the mentioned subpartition is not more than in N parts.*

Proof. Consider the following action of G on itself: $x \mapsto gx\phi(g^{-1})$. Then its orbits are exactly classes $\{x\}_\phi$. Moreover it maps classes (2) onto each other. Indeed,

$$\widetilde{g}\widetilde{h}(hg)\phi'(\widetilde{h}^{-1})\phi(\widetilde{g}^{-1}) = \widehat{h}\widetilde{g}(hg)\phi(\widetilde{g}^{-1})\phi'(\widehat{h}^{-1})$$

using normality of the H. This map is invertible ($\widetilde{g} \leftrightarrow \widetilde{g}^{-1}$), hence bijection. Moreover, $\widetilde{g}$ and $\widetilde{g}\widehat{h}$, for any $\widehat{h} \in H$, act in the same way. Or in the other words, H is in the stabilizer of this permutation of classes (2). Hence, the cardinality of any orbit $\le N$. ☐

Hence, for any finite G/H the number of classes of the form (2) is finite: it is $\le NR(\phi)$.

Lemma 4.2. *Suppose, H satisfies the following property: for any automorphism of H with finite Reidemeister number the characteristic functions of Reidemeister classes of ϕ are linear combinations of matrix elements of some finite number of irreducible finite dimensional representations of H. Then the characteristic functions of classes* (2) *are linear combinations of matrix elements of some finite number of irreducible finite dimensional representations of G.*

Proof. Let $\rho_1, \rho_2, \dots, \rho_k$ be the above irreducible representations of H, ρ its direct sum acting on V, and π the regular (finite dimensional) representation of G/H. Let $\rho_1^I, \dots, \rho_k^I, \rho^I$ be the corresponding induced representations of G. Let the characteristic function of K be represented under the form $\chi_K(h) = \langle \rho(h)\xi, \eta\rangle$. Let $\xi^I \in L^2(G/H, V)$ be defined by the formulas $\xi^I(\overline{e}) = \xi \in V$, $\xi^I(\overline{g}) = 0$ if $\overline{g} \ne \overline{e}$. Define similarly η^I. Then for $h \in iH$ we have

$$\rho^I(h)\xi^I(\overline{g}) = \rho(s(\overline{g})hs(\overline{g}h)^{-1})\xi(\overline{g}h) = \rho(hs(\overline{g})s(\overline{g})^{-1})\xi(\overline{g}) = \begin{cases} \rho(h)\xi, & \text{if } \overline{g} = \overline{e}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $\langle \rho^I(h)\xi^I, \eta^I\rangle|_{iH}$ is the characteristic function of iK. Let $u, v \in L^2(G/H)$ be such vectors that $\langle \pi(\overline{g})u, v\rangle$ is the characteristic function of $\overline{e}$. Then

$$\langle (\rho^I \otimes \pi)(\xi^I \otimes u, \eta^I \otimes v)\rangle$$

is the characteristic function of iK. Other characteristic functions of classes (2) are shifts of this one. Hence matrix elements of the representation $\rho^I \otimes \pi$. It is finite dimensional. Hence it can be decomposed in a finite direct sum of irreducible representations. ☐

Corollary 4.3 (of previous two lemmata)**.** *Under the assumptions of the previous lemma, the characteristic functions of Reidemeister classes of ϕ are linear combinations of matrix elements of some finite number of irreducible finite dimensional representations of G.*

5. The case of groups of type I

Theorem 5.1. *Let G be a discrete group of type* I*. Then*

- [3, 3.1.4, 4.1.11] *The dual space $\widehat{G}$ is a T_1-topological space.*
- [15] *Any irreducible representation of G is finite-dimensional.*

Remark 5.2. In fact a discrete group G is of type I if and only if it has a normal, Abelian subgroup M of finite index. The dimension of any irreducible representation of G is at most $[G:M]$ [15].

Suppose $R = R(\phi) < \infty$, and let $F \subset L^\infty(G)$ be the R-dimensional space of all twisted-invariant functionals on $L^1(G)$. Let $K \subset L^1(G)$ be the intersection of kernels of functionals from F. Then K is a linear subspace of $L^1(G)$ of codimension R. For each $\rho \in \widehat{G}$ let us denote by K_ρ the image $\rho(K)$. This is a subspace of a (finite-dimensional) full matrix algebra. Let cd_ρ be its codimension.

Let us introduce the following set

$$\widehat{G}_F = \{\rho \in \widehat{G} \mid \mathrm{cd}_\rho \neq 0\}.$$

Lemma 5.3. *One has* $\mathrm{cd}_\rho \neq 0$ *if and only if ρ is a fixed point of $\widehat{\phi}$. In this case* $\mathrm{cd}_\rho = 1$.

Proof. Suppose, $\mathrm{cd}_\rho \neq 0$ and let us choose a functional φ_ρ on the (finite-dimensional full matrix) algebra $\rho(L^1(G))$ such that $K_\rho \subset \operatorname{Ker}\varphi_\rho$. Then for the corresponding functional $\varphi^*_\rho = \varphi_\rho \circ \rho$ on $L^1(G)$ one has $K \subset \operatorname{Ker}\varphi^*_\rho$. Hence, $\varphi^*_\rho \in F$ and is twisted-invariant, as well as φ_ρ. Then we argue as in the case of compact group (after Lemma 3.1).

Conversely, if ρ is a fixed point of $\widehat{\phi}$, it gives rise to a (unique up to scaling) non-trivial twisted-invariant functional φ_ρ. Let $x = \rho(a)$ be any element in $\rho(L^1(G))$ such that $\varphi_\rho(x) \neq 0$. Then $x \notin K_\rho$, because $\varphi^*_\rho(a) = \varphi_\rho(x) \neq 0$, while φ^*_ρ is a twisted-invariant functional on $L^1(G)$. So, $\mathrm{cd}_\rho \neq 0$.

The uniqueness (up to scaling) of the intertwining operator implies the uniqueness of the corresponding twisted-invariant functional. Hence, $\mathrm{cd}_\rho = 1$. ☐

Hence,

$$\widehat{G}_F = \mathrm{Fix}(\widehat{\phi}). \tag{3}$$

¿From the property $\mathrm{cd}_\rho = 1$ one obtains for this (unique up to scaling) functional φ_ρ:

$$\operatorname{Ker}\varphi_\rho = K_\rho. \tag{4}$$

Lemma 5.4. *$R = \#\widehat{G}_F$, in particular, the set $\widehat{G}_F$ is finite.*

Proof. First of all we remark that since G is finitely generated almost Abelian (cf. Remark 5.2) there is a normal Abelian subgroup H of finite index invariant under all ϕ. Hence we can apply Lemma 4.3 to G, H, ϕ. So there is a finite collection of irreducible representations of G such that any twisted-invariant functional is a linear combination of matrix elements of them, i.e. linear combination of functionals on them. If each of them gives a non-trivial contribution, it has to be a twisted-invariant functional on the corresponding matrix algebra. Hence, by the argument above, these representations belong to $\widehat{G}_F$, and the appropriate functional is unique up to scaling. Hence, $R \leq S$.

Then we use T_1-separation property. More precisely, suppose some points $\rho_1, \dots, \rho_s$ belong to $\widehat{G}_F$. Let us choose some twisted-invariant functionals $\varphi_i = \varphi_{\rho_i}$ corresponding to these points as it was described (i.e. choose some scaling). Assume that $\|\varphi_i\| = 1$, $\varphi_i(x_i) = 1$, $x_i \in \rho_i(L^1(G))$. If we can find $a_i \in L^1(G)$ such that $\varphi_i(\rho_i(a_i)) = \varphi_i^*(a_i)$ is sufficiently large and $\rho_j(a_i)$, $i \neq j$, are sufficiently small (in fact it is sufficient $\rho_j(a_i)$ to be close enough to $K_j := K_{\rho_j}$), then $\varphi_j^*(a_i)$ are small for $i \neq j$, and φ_i^* are linear independent and hence, $s < R$. This would imply $S := \#\widehat{G}_F \leq R$ is finite. Hence, $R = S$.

So, the problem is reduced to the search of the above a_i. Let $d = \max\limits_{i=1,\dots,s} \dim \rho_i$. For each i let $c_i := \|b_i\|$, where x_i is the unitary equivalence of ρ_i and $\widehat{\phi}\rho_i$ and $x_i = \rho_i(b_i)$.

Let $c := \max\limits_{i=1,\dots,s} c_i$ and $\varepsilon := \frac{1}{2 \cdot s^2 \cdot d \cdot c}$.

One can find a positive element $a_i' \in L^1(G)$ such that $\|\rho_i(a_i')\| \geq 1$ and $\|\rho_j(a_i')\| < \varepsilon$ for $j \neq i$. Indeed, since ρ_i can be separated from one point, and hence from the finite number of points: ρ_j, $j \neq i$. Hence, one can find an element v_i such that $\|\rho_i(v_i)\| > 1$, $\|\rho_j(v_i)\| < 1$ for $j \neq i$ [3, Lemma 3.3.3]. The same is true for the positive element $u_i = v_i^* v_i$. (Due to density we do not distinguish elements of L^1 and C^*). Now for a sufficiently large n the element $a_i' := (u_i)^n$ has the desired properties.

Let us take $a_i := a_i' b_i^*$. Then

$$\begin{aligned} \varphi_i^*(a_i) &= \mathrm{Tr}(x_i \rho_i(a_i)) = \mathrm{Tr}(x_i \rho_i(a_i') \rho_i(b_i)^*) = \mathrm{Tr}(x_i \rho_i(a_i') x_i^*) = \\ &= \mathrm{Tr}(x_i \rho_i(a_i')(x_i)^{-1}) = \mathrm{Tr}(\rho_i(a_i')) \geq \frac{1}{\dim \rho_i} \geq \frac{1}{d}. \end{aligned} \tag{5}$$

For $j \neq i$ one has

$$\|\varphi_j^*(a_i)\| = \|\varphi_j(\rho_j(a_i' b_i^*))\| \leq c_i \cdot \varepsilon. \tag{6}$$

Then the $s \times s$ matrix $\Phi = \varphi_j^*(a_i)$ can be decomposed into the sum of the diagonal matrix Δ and off-diagonal Σ. By (5) one has $\Delta \geq \frac{1}{d}$. By (6) one has

$$\|\Sigma\| \leq s^2 \cdot c_i \cdot \varepsilon \leq s^2 \cdot c \cdot \frac{1}{2 \cdot s^2 \cdot d \cdot c} = \frac{1}{2d}.$$

Hence, Φ is non-degenerate and we are done. □

Lemma 5.4 together with (3) completes the proof of Theorem 1.4 for automorphisms.

We need the following additional observations for the proof of Theorem 1.4 for a general endomorphism (in which (3) is false for infinite-dimensional representations).

Lemma 5.5. (1) *If ϕ is an epimorphism, then $\widehat{G}$ is $\widehat{\phi}_R$-invariant.*
(2) *For any ϕ the set $\mathrm{Rep}(G) \setminus \widehat{G}$ is $\widehat{\phi}_R$-invariant.*
(3) *The dimension of the space of intertwining operators between $\rho \in \widehat{G}$ and $\widehat{\phi}_R(\rho)$ is equal to 1 if and only if $\rho \in \mathrm{Fix}(\widehat{\phi})$. Otherwise it is 0.*

Proof. (1) and (2): This follows from the characterization of irreducible representation as that one for which the centralizer of $\rho(G)$ consists exactly of scalar operators.

(3) Let us decompose $\widehat{\phi}_R(\rho)$ into irreducible ones. Since $\dim H_\rho = \dim H_{\widehat{\phi}(\rho)}$ one has only 2 possibilities: ρ does not appear in $\widehat{\phi}(\rho)$ and the intertwining number is 0, otherwise $\widehat{\phi}_R(\rho)$ is equivalent to ρ. In this case $\rho \in \operatorname{Fix}(\widehat{\phi})$. □

The proof of Theorem 1.4 can be now repeated for the general endomorphism with the new definition of $\operatorname{Fix}(\widehat{\phi})$. The item (3) supplies us with the necessary property.

6. Examples and their discussion

The natural candidate for the dual object to be used instead of $\widehat{G}$ in the case when the different notions of the dual do not coincide (i.e. for groups more general than type I one groups) is the so-called quasi-dual $\widehat{G}$, i.e. the set of quasi-equivalence classes of factor-representations (see, e.g. [3]). This is a usual object when we need a sort of canonical decomposition for regular representation or group C^*-algebra. More precisely, one needs the support $\widehat{G}_p$ of the Plancherel measure.

Unfortunately the following example shows that this is not the case.

Example 6.1. Let G be a non-elementary Gromov hyperbolic group. As it was shown by Fel'shtyn [5] with the help of geometrical methods, for any automorphism ϕ of G the Reidemeister number $R(\phi)$ is infinite. In particular this is true for free group in two generators F_2. But the support $(\widehat{F_2})_p$ consists of one point (i.e. regular representation is factorial).

The next hope was to exclude from this dual object the II_1-points assuming that they always give rise to an infinite number of twisted invariant functionals. But this is also wrong:

Example 6.2. (an idea of Fel'shtyn realized in [10]) Let $G = (Z \oplus Z) \rtimes_\theta Z$ be the semi-direct product by a hyperbolic action $\theta(1) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Let ϕ be an automorphism of G whose restriction to Z is $-id$ and restriction to $Z \oplus Z$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $R(\phi) = 4$, while the space $\widehat{G}_p$ consists of a single II_1-point once again (cf. [2, p. 94]).

These examples show that powerful methods of the decomposition theory do not work for more general classes of groups.

On the other hand Example 6.2 disproves the old conjecture of Fel'shtyn and Hill [8] who supposed that the Reidemeister numbers of a injective endomorphism for groups of exponential growth are always infinite. More precisely, this group is amenable and of exponential growth. Together with some calculations for concrete groups which are too routine to be included in the present paper, this allow us to formulate the following question.

Question. Is the Reidemeister number $R(\phi)$ infinite for any automorphism ϕ of (non-amenable) finitely generated group G containing F_2 ?

In this relation the following example seems to be interesting.

Example 6.3. [7] For amenable and non-amenable Baumslag-Solitar groups Reidemeister numbers are always infinite.

For Example 6.2 recently we have found 4 fixed points of $\widehat{\phi}$ being finite dimensional irreducible representations. They give rise to 4 linear independent twisted invariant functionals. These functionals can also be obtained from the regular factorial representation. There also exist fixed points (at least one) that are infinite dimensional irreducible representations. The corresponding functionals are evidently linear dependent with the first 4. This example will be presented in detail in a forthcoming paper.

7. Congruences for Reidemeister numbers of endomorphisms

Lemma 7.1 ([12]). *For any endomorphism ϕ of a group G and any $x \in G$ one has $\phi(x) \in \{x\}_\phi$.*

Proof. $\phi(x) = x^{-1}x\phi(x)$. □

The following lemma is useful for calculating Reidemeister numbers.

Lemma 7.2. *Let $\phi : G \to G$ be any endomorphism of any group G, and let H be a subgroup of G with the properties*

$$\phi(H) \subset H$$
$$\forall x \in G\ \exists n \in \mathbb{N} \text{ such that } \phi^n(x) \in H.$$

Then

$$R(\phi) = R(\phi_H),$$

where $\phi_H : H \to H$ is the restriction of ϕ to H.

Proof. Let $x \in G$. Then there is n such that $\phi^n(x) \in H$. By Lemma 7.1 it is known that x is ϕ-conjugate to $\phi^n(x)$. This means that the ϕ-conjugacy class $\{x\}_\phi$ of x has non-empty intersection with H.

Now suppose that $x, y \in H$ are ϕ-conjugate, i.e. there is a $g \in G$ such that

$$gx = y\phi(g).$$

We shall show that x and y are ϕ_H-conjugate, i.e. we can find a $g \in H$ with the above property. First let n be large enough that $\phi^n(g) \in H$. Then applying ϕ^n to the above equation we obtain

$$\phi^n(g)\phi^n(x) = \phi^n(y)\phi^{n+1}(g).$$

This shows that $\phi^n(x)$ and $\phi^n(y)$ are ϕ_H-conjugate. On the other hand, one knows by Lemma 7.1 that x and $\phi^n(x)$ are ϕ_H-conjugate, and y and $\phi^n(y)$ are ϕ_H conjugate, so x and y must be ϕ_H-conjugate.

We have shown that the intersection with H of a ϕ-conjugacy class in G is a ϕ_H-conjugacy class in H. Therefore, we have a map

$$\begin{array}{rccc} Rest: & \mathcal{R}(\phi) & \to & \mathcal{R}(\phi_H) \\ & \{x\}_\phi & \mapsto & \{x\}_\phi \cap H \end{array}$$

It is evident that it has the two-sided inverse

$$\{x\}_{\phi_H} \mapsto \{x\}_\phi.$$

Therefore *Rest* is a bijection and $R(\phi) = R(\phi_H)$. □

Corollary 7.3. *Let $H = \phi^n(G)$. Then $R(\phi) = R(\phi_H)$.*

Now we pass to the proof of Theorem 1.5.

Proof. ¿From Theorems 1.4 and Lemma 7.2 it follows immediately that for every n

$$R(\phi^n) = \#\,\mathrm{Fix}\left[\widehat{\phi_H}^n : \widehat{H} \to \widehat{H}\right].$$

Let P_n denote the number of periodic points of $\widehat{\phi_H}$ of least period n. One obtains immediately

$$R(\phi^n) = \#\,\mathrm{Fix}\left[\widehat{\phi_H}^n\right] = \sum_{d|n} P_d.$$

Applying Möbius' inversion formula, we have,

$$P_n = \sum_{d|n} \mu(d) R(\phi^{\frac{n}{d}}).$$

On the other hand, we know that P_n is always divisible by n, because P_n is exactly n times the number of $\widehat{\phi_H}$-orbits in $\widehat{H}$ of cardinality n.

Now we pass to the proof of Theorem 1.5 for general endomorphisms.

¿From Theorem 1.4, Lemma 7.2 it follows immediately that for every n

$$R(\phi^n) = R(\phi_H^n) = \#\,\mathrm{Fix}(\widehat{(\phi_H^n)_R}|_{\widehat{H}})$$

Let P_n denote the number of periodic points of $(\widehat{\phi_H})_R|_{\widehat{H}}$ of least period n. One obtains by Lemma 5.5 (2)

$$R(\phi^n) = \#\,\mathrm{Fix}(\widehat{(\phi_H^n)_R}|_{\widehat{H}}) = \sum_{d|n} P_d.$$

The proof can be completed as in the case of automorphisms. $\square$

8. Congruences for Reidemeister numbers of a continuous map

Now we pass to the formulation of the topological counterpart of the main theorems. Let X be a connected, compact polyhedron and $f : X \to X$ be a continuous map. Let $p : \widetilde{X} \to X$ be the universal cover of X and $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ a lifting of f, i.e. $p \circ \widetilde{f} = f \circ p$. Two liftings $\widetilde{f}$ and $\widetilde{f}'$ are called *conjugate* if there is a elementγ in the deck transformation group $\Gamma \cong \pi_1(X)$ such that $\widetilde{f}' = \gamma \circ \widetilde{f} \circ \gamma^{-1}$. The subset $p(\mathrm{Fix}(\widetilde{f})) \subset \mathrm{Fix}(f)$ is called *the fixed point class of f determined by the lifting class* $[\widetilde{f}]$. Two fixed points x_0 and x_1 of f belong to the same fixed point class iff there is a path c from x_0 to x_1 such that $c \cong f \circ c$ (homotopy relative to endpoints). This fact can be considered as an equivalent definition of a non-empty fixed point class. Every map f has only finitely many non-empty fixed point classes, each a compact subset of X.

The number of lifting classes of f (and hence the number of fixed point classes, empty or not) is called the *Reidemeister number* of f, which is denoted by $R(f)$. This is a positive integer or infinity.

Let a specific lifting $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ be fixed. Then every lifting of f can be written in a unique way as $\gamma \circ \widetilde{f}$, with $\gamma \in \Gamma$. So the elements of Γ serve as "coordinates" of liftings with respect to the fixed $\widetilde{f}$. Now, for every $\gamma \in \Gamma$, the composition $\widetilde{f} \circ \gamma$ is a lifting of f too; so there is a unique $\gamma' \in \Gamma$ such that $\gamma' \circ \widetilde{f} = \widetilde{f} \circ \gamma$. This correspondence $\gamma \to \gamma'$ is determined by the fixed $\widetilde{f}$, and is obviously a homomorphism.

Definition 8.1. The endomorphism $\widetilde{f}_*:\Gamma\to\Gamma$ determined by the lifting $\widetilde{f}$ of f is defined by

$$\widetilde{f}_*(\gamma)\circ\widetilde{f}=\widetilde{f}\circ\gamma.$$

We shall identify $\pi=\pi_1(X,x_0)$ and Γ in the following way. Choose base points $x_0\in X$ and $\widetilde{x}_0\in p^{-1}(x_0)\subset\widetilde{X}$ once and for all. Now points of $\widetilde{X}$ are in 1-1 correspondence with homotopy classes of paths in X which start at x_0: for $\widetilde{x}\in\widetilde{X}$ take any path in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}$ and project it onto X; conversely, for a path c starting at x_0, lift it to a path in $\widetilde{X}$ which starts at $\widetilde{x}_0$, and then take its endpoint. In this way, we identify a point of $\widetilde{X}$ with a path class $\langle c\rangle$ in X starting from x_0. Under this identification, $\widetilde{x}_0=\langle e\rangle$ is the unit element in $\pi_1(X,x_0)$. The action of the loop class $\alpha=\langle a\rangle\in\pi_1(X,x_0)$ on $\widetilde{X}$ is then given by

$$\alpha=\langle a\rangle:\langle c\rangle\to\alpha\cdot c=\langle a\cdot c\rangle.$$

Now we have the following relationship between $\widetilde{f}_*:\pi\to\pi$ and

$$f_*:\pi_1(X,x_0)\longrightarrow\pi_1(X,f(x_0)).$$

Lemma 8.2. *Suppose $\widetilde{f}(\widetilde{x}_0)=\langle w\rangle$. Then the following diagram commutes:*

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r, "f_*"] \arrow[dr, "\widetilde{f}_*"'] & \pi_1(X,f(x_0)) \arrow[d, "w_*"] \\
 & \pi_1(X,x_0)
\end{tikzcd}$$

where w_* is the isomorphism induced by the path w.

In other words, for every $\alpha=\langle a\rangle\in\pi_1(X,x_0)$, we have

$$\widetilde{f}_*(\langle a\rangle)=\langle w(f\circ a)w^{-1}\rangle.$$

Remark 8.3. In particular, if $x_0\in p(\mathrm{Fix}(\widetilde{f}))$ and $\widetilde{x}_0\in\mathrm{Fix}(\widetilde{f})$, then $\widetilde{f}_*=f_*$.

Lemma 8.4 (see, e.g. [12])**.** *Lifting classes of f (and hence fixed point classes, empty or not) are in 1-1 correspondence with $\widetilde{f}_*$-conjugacy classes in π, the lifting class $[\gamma\circ\widetilde{f}]$ corresponding to the $\widetilde{f}_*$-conjugacy class of γ. We therefore have $R(f)=R(\widetilde{f}_*)$.*

We shall say that the fixed point class $p(\mathrm{Fix}(\gamma\circ\widetilde{f}))$, which is labeled with the lifting class $[\gamma\circ\widetilde{f}]$, *corresponds* to the $\widetilde{f}_*$-conjugacy class of γ. Thus $\widetilde{f}_*$-conjugacy classes in π serve as "coordinates" for fixed point classes of f, once a fixed lifting $\widetilde{f}$ is chosen.

Using Lemma 8.4 we may apply the Theorem 1.5 to the Reidemeister numbers of continuous maps.

Theorem 8.5. *Let $f:X\to X$ be a continuous map of a compact polyhedron X such that all numbers $R(f^n)$ are finite. Let $f_*:\pi_1(X)\to\pi_1(X)$ be an induced endomorphism of the group $\pi_1(X)$ and let H be a subgroup of $\pi_1(X)$ with the properties*

(1) $f_*(H)\subset H$,
(2) $\forall x\in\pi_1(X)\ \exists n\in\mathbb{N}$ *such that* $f_*^n(x)\in H$.

If the couple (H, f_*^n) *satisfies the conditions of Theorem* 1.4 *for any* $n \in \mathbb{N}$*, then one has for all* n*,*

$$\sum_{d|n} \mu(d) \cdot R(f^{n/d}) \equiv 0 \mod n.$$

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