ON THE GEOMETRY OF SCHUBERT VARIETIES ATTACHED TO KAC-MOODY LIE ALGEBRAS

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ABSTRACT: Let G be a group attached to a Kac-Moody Lie algebra with not necessarily symmetrizable Cartan matrix. We define Schubert varieties for G by means of a Demazure-Hansen resolution and we prove that these varieties are nonsingular in codimension one. We also determine the restriction of homogeneous line bundles on generalized flag manifolds to Schubert subvarieties.

0. INTRODUCTION: In this paper we study generalized Schubert varieties attached to Kac-Moody groups G with arbitrary Cartan matrix. Such groups contain a Tits system (B,N) providing a Bruhat decomposition

$$G = \bigcup_{w \in W} BwB$$

and a classification of parabolic subgroups P, i.e. of subgroups of G containing a conjugate of B. Set-theoretically a Schubert variety is a subset $\overline{X}_w$ of a homogeneous space G/P, $B \subset P \subset G$, of the form

$$\overline{X}_w = \bigcup_{v \leq w} (BwP)/P \quad ,$$

where $v \leq w$ denotes the Bruhat ordering on the Weyl group W. The homogeneous space G/P may be embedded into the projective space $\mathbb{P}(\omega)$ of an irreducible highest weight module $L(\omega)$ of G. We endow $\overline{X}_w$ with the structure of a complex algebraic variety by identifying it with the closure in $\mathbb{P}(\omega)$ of $X_w = (BwP)/P$ (cf. 2.2 - 2.4). Our procedure here agrees essentially with the one scetched by Tits in [25] in that we use a "Demazure-Hansen resolution" of $\overline{X}_w$. On the technical level we exploit heavily the fact that several subgroups of G stabilize finite-dimensional subspaces in the modules $L(\omega)$, on which they act regularly by algebraic quotient groups (cf. 1.11). Though we are not able to show that the algebraic geometric structure on $\overline{X}_w$ is independent of

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1980 Mathematics Subject Classification 14M15, 17B65

the module $L(\omega)$ (for symmetrizable Cartan matrices at least, this is ascertained in [25]) we prove (2.5) that the topological structure is uniquely defined. This is sufficient for topological applications as described in [11]. As further results we show that all Schubert varieties are nonsingular in codimension one and we determine the restrictions of homogeneous line bundles on G/P to Schubert varieties (2.8). The last result is important for extending part of the Schubert calculus to the framework of Kac-Moody groups as announced in our joint note [11] with E. Gutkin. Whereas this paper provides detailed proofs for the geometric results stated there, a separate paper by E. Gutkin will be occupied with the homological and cohomological applications.

We finally want to point out the technical character of this paper. Most of the objects we deal with are easily defined on a set-theoretical level by exploiting the analogy with the finite-dimensional situation. The main problem therefore consists in defining correctly the underlying algebraic geometric or topological structures and in justifying classical arguments in the infinite-dimensional context.

Our thanks go to J. Tits who communicated to us the idea for the construction of the Schubert varieties long ago (March 1981) and to E. Gutkin who started the collaboration with us on these topics and who urged us to write down the details in this paper.

1. KAC-MOODY LIE ALGEBRAS AND ASSOCIATED GROUPS. The purpose of this part is to recollect the necessary definitions and results needed in the second part. Thus we review properties of Weyl groups, Kac-Moody Lie algebras, associated groups and representations. We also add some simple lemmas of a more technical nature.

1.1 ROOT BASES. Let I be a finite set. A (generalized) Cartan matrix on I is a matrix

$$A = ((A_{ij}))_{i,j \in I}$$

satisfying

$$A_{ij} \in \mathbb{Z} \quad , \quad A_{ii} = 2 \quad ,$$

$$A_{ij} \leq 0 \quad \text{for all} \quad i \neq j \in I \quad ,$$

$$A_{ij} = 0 \iff A_{ji} = 0 \quad \text{for all} \quad i,j \in I \quad .$$

A $\mathbb{Z}$-realization of such a matrix or a root base for A is a triplet (H,∇,Δ) consisting of a free $\mathbb{Z}$-module $H \cong \mathbb{Z}^r$ for some $r \in \mathbb{N}$, a subset

$\nabla = \{h_i | i \in I\}$ of H, and a subset $\Delta = \{\alpha_i | i \in I\}$ of the dual lattice $H^* = \mathrm{Hom}_{\mathbb{Z}}(H,\mathbb{Z})$ such that

$$\alpha_i(h_j) = A_{ji} \quad \text{for all} \quad i,j \in I \quad .$$

We call Δ (resp. ∇) the set of simple or fundamental roots (resp. coroots) of (H,∇,Δ).

Let $\alpha = \alpha_i \in \Delta$. Then we also write h_α instead of h_i.

Let Γ (resp. L) be the free $\mathbb{Z}$-module generated by Δ (resp. ∇):

$$\Gamma = \bigoplus_{\alpha \in \Delta} \mathbb{Z}\alpha \quad , \quad L = \bigoplus_{h \in \nabla} \mathbb{Z}h \quad .$$

We call Γ (resp. L) the formal root lattice (resp. formal coroot lattice). Corresponding to the natural maps $\Gamma \to H^*$, $L \to H$ we have obvious pairings $\Gamma \times H \to \mathbb{Z}$, $L \times H^* \to \mathbb{Z}$.

Note that the map $\Gamma \to H^*$ (resp. $L \to H$) is injective if and only if Δ (resp. ∇) is linearly free.

1.2 WEYL GROUPS.

Let (H,∇,Δ) be a root base for a generalized Cartan matrix $A = ((A_{ij}))_{i,j \in I}$ and Γ its formal root lattice. The Weyl group W of (H,∇,Δ) is the subgroup of $\mathrm{Aut}(\Gamma)$ generated by the fundamental reflections

$$s_\alpha : \Gamma \to \Gamma \quad , \quad \alpha \in \Delta \quad , \quad s_\alpha(\gamma) = \gamma - \gamma(h_\alpha)\alpha \quad .$$

It is known (cf. [13]) that the system (W,S), $S = \{s_\alpha | \alpha \in \Delta\}$, is a Coxeter system, i.e. that W has a presentation of the form

$$s_\alpha^2 = 1 \qquad \text{all } \alpha \in \Delta ,$$

$$s_\alpha s_\beta s_\alpha \cdots = s_\beta s_\alpha s_\beta \cdots , \quad \text{all } \alpha \neq \beta \in \Delta , \ (m_{\alpha\beta} \text{ factors on each side})$$

where the numbers $m_{\alpha\beta}$ are given by the following table (we write $A_{\alpha\beta} = A_{ij}$ if $\alpha = \alpha_i$, $\beta = \alpha_j$)

$A_{\alpha\beta}A_{\beta\alpha}$	0	1	2	3	≥ 4
$m_{\alpha\beta}$	2	3	4	6	∞

The action of W on Γ extends to an action on H^* by the prescription

$$s_\alpha(\omega) = \omega - \omega(h_\alpha)\alpha \quad , \quad \omega \in H^* \quad ,$$

for the generators $s_\alpha \in S$ (this action is faithful if ∇ or Δ are linearly free). The contragredient action of s_α on H is now given by

$$s_\alpha(h) = h - \alpha(h)h_\alpha$$

for all $h \in H$.

Let $w = s_1 \cdot \ldots \cdot s_n$ be an expression of an element $w \in W$ as a product of elements $s_j \in S$. This expression is called <u>reduced</u> if n is the least number for which such an equality holds. In that case n is called the <u>length</u> $\ell(w)$ of w (cf. [5] IV).

<u>1.3 WEYL ROOTS</u>. We consider the same situation as in § 1.2. The union of orbits of $\{\alpha | \alpha \in \Delta\} \subset \Gamma$ under W is the set $\sum^R$ of <u>real</u> or <u>Weyl roots</u>. The bijection $\Delta \to \nabla$, $\alpha \mapsto h_\alpha$, can be extended to a W-equivariant bijection

$$^\vee : \textstyle\sum^R \to W\{h_\alpha \,|\, \alpha \in \Delta\} \subset L$$

given by $\gamma \mapsto \gamma^\vee = h_\gamma$.

If $\gamma \in \sum^R$ and $\gamma = w(\alpha)$ for some $\alpha \in \Delta$, $w \in W$, then $s_\gamma = ws_\alpha w^{-1}$ is called the <u>reflection belonging to the root</u> γ . We have $s_\gamma(h) = h - \gamma(h)h_\gamma$ for all $h \in H$ (h_γ interpreted as its image in H), and $s_\gamma(\omega) = \omega - \omega(h_\gamma)\gamma$ for all $\omega \in \Gamma$ (or $\omega \in H^*$, the element γ being interpreted as its image in H^*).

Any element $\gamma \in \sum^R$ lies in $\sum^{R,+} = \sum^R \cap \mathbb{N}\Delta$ or in $\sum^{R,-} = -\sum^{R,+}$. Correspondingly γ is called a <u>positive</u> or a <u>negative</u> real root.

<u>1.4 BRUHAT ORDER</u>. The definitions and statements of 1.3 make sense for arbitrary Coxeter groups. This is also true for the following proposition, a proof of which may be either found, in full generality, in [7], [8] or obtained by mimicking the proof for the finite-type situation (see for example [1] § 2).

Let w_1 , $w_2 \in W$ and $\gamma \in \sum^{R,+}$. When the conditions

$$s_\gamma w_1 = w_2 \quad \text{and} \quad \ell(w_2) = \ell(w_1) + 1$$

hold we write

$$w_1 \xrightarrow{\gamma} w_2 \quad \text{or} \quad w_1 \longrightarrow w_2 \quad .$$

If there is a chain

$$w = w_1 \to w_2 \to \ldots \to w_k = w'$$

we write $w < w'$ and say that w is <u>smaller</u> than w' .

For any $w \in W$ we let $\sum(w)$ denote the intersection

$$\textstyle\sum^{R,+} \cap w(\sum^{R,-}) = \{\gamma \in \sum^{R,+} \,|\, w^{-1}(\gamma) \in \sum^{R,-}\} \quad .$$

PROPOSITION: Let $w \in W$ and let $w = s_1 \cdot \ldots \cdot s_n$ be any reduced decomposition of w. Let $\alpha_i \in \Delta$ be such that $s_i = s_{\alpha_i}$ for $i = 1,\ldots,n$.

(i) $\sum(w) = \{\alpha_1, s_1(\alpha_2), \ldots, s_1 \cdot \ldots \cdot s_{n-1}(\alpha_n)\}$,
in particular $\ell(w) = n = \text{card} \sum(w)$.

(ii) Let $\gamma \in \sum^{R,+}$. Then
$\ell(s_\gamma w) > \ell(w) \iff \gamma \notin \sum(w)$
$\ell(s_\gamma w) < \ell(w) \iff \gamma \in \sum(w)$.

(iii) Let $w' \in W$ be such that $w' \xrightarrow{\gamma} w$. Then there is a unique index $i, 1 \le i \le n$, such that $\gamma = s_1 \cdot \ldots \cdot s_{i-1}(\alpha_i)$, and
$w' = s_1 \cdot \ldots \cdot s_{i-1} \cdot s_{i+1} \cdot \ldots \cdot s_n$.

(iv) Let $w' \in W$. Then $w' < w$ if and only if there exists a subsequence $1 \le i_1 < i_2 < \ldots < i_k \le r$, $k < n$, such that

$$w' = s_{i_1} \cdot \ldots \cdot s_{i_k} .$$

The order "<" on W is called the Bruhat order. We write $w' \le w$ if $w' = w$ or $w' < w$.

As an immediate consequence of (i) above we obtain the following result.

COROLLARY: Let $w = w_1 \cdot w_2$ be a product in W such that $\ell(w) = \ell(w_1) + \ell(w_2)$. Then

$$\sum(w) = \sum(w_1) \cup w_1 \sum(w_2) .$$

1.5 KAC-MOODY LIE ALGEBRAS.

Let (H,∇,Δ) be a root base as in § 1.1. A Kac-Moody algebra $\underline{g}$ associated with (H,∇,Δ) is a complex Lie algebra generated as a complex Lie algebra by

1) the vector space $\underline{h} = H \otimes_{\mathbb{Z}} \mathbb{C}$

2) elements e_α, f_α, $(\alpha \in \Delta)$

with the following relations which hold for any $h, h' \in \underline{h}$ and $\alpha, \beta \in \Delta$.

$$\text{(R)}\qquad \begin{aligned} [h,h'] &= 0 \\ [h,e_\alpha] &= \alpha(h)e_\alpha \\ [h,f_\alpha] &= -\alpha(h)f_\alpha \\ [e_\alpha,f_\alpha] &= h_\alpha \in H \subset \underline{h} \\ (\text{ad } e_\alpha)^{1-A_{\alpha\beta}}(e_\beta) &= 0 , \quad \alpha \ne \beta , \\ (\text{ad } f_\alpha)^{1-A_{\alpha\beta}}(f_\beta) &= 0 , \quad \alpha \ne \beta . \end{aligned}$$

We also require that $\underline{h} \cup \{e_\alpha, f_\alpha | \alpha \in \Delta\}$ injects into $\underline{g}$.

Note that when A is symmetrizable, i.e. when there exists a diagonal matrix $D \in M_I(\mathbb{Z})$ such that DA is symmetric, then there is a unique Lie algebra $\underline{g}$ with the properties above (cf. [13]). It is conjectured that the result is true for non-symmetrizable A as well.

1.6 PROPERTIES OF THE ROOT SYSTEM. We recall the root decomposition

$$\underline{g} = \bigoplus_{\gamma \in \Sigma \cup \{0\}} \underline{g}_\gamma$$

where Σ denotes the system of all roots in the root lattice Γ. When Δ is linearly free in H^* we may consider Σ as a subset of $H^* \subset \underline{h}^*$.

Then we have $\underline{g}_\gamma = \{X \in \underline{g} \mid [h,X] = \gamma(h)X \text{ for all } h \in \underline{h}\}$ for all $\gamma \in \Sigma \cup \{0\}$, and $\underline{g}_0 = \underline{h}$. Also $\underline{g}_\gamma$ is finite-dimensional for all $\gamma \in \Sigma \cup \{0\}$.

The set Σ is stable under the action of the Weyl group and $\Delta \subset \Sigma$, thus $\Sigma^R \subset \Sigma$. The complement $\Sigma^I = \Sigma \setminus \Sigma^R$ is called the set of imaginary roots. We have $\dim \underline{g}_\gamma = 1$ for all $\gamma \in \Sigma^R$.

Let $\Sigma^+ = \Sigma \cap \mathbb{N}\Delta$ denote the set of positive roots and $\Sigma^- = -\Sigma^+$ the set of negative roots. Then $\Sigma = \Sigma^+ \cup \Sigma^-$. According to this decomposition we have subalgebras of $\underline{g}$

$$\underline{u}^\pm = \bigoplus_{\gamma \in \Sigma^\pm} \underline{g}_\gamma$$

and the direct sum $\underline{g} = \underline{u}^+ \oplus \underline{h} \oplus \underline{u}^-$.

The action of W stabilizes $\Sigma^I \cap \Sigma^+$. Therefore, we also have $\Sigma(w) = \{\gamma \in \Sigma^+ \mid w^{-1}(\gamma) \in \Sigma^-\}$ for all $w \in W$ (cf. 1.4).

For any subset $S' \subset S = \{s_\alpha \mid \alpha \in \Delta\}$ we define the S'-height $ht_{S'}(\gamma)$ of a root $\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$ by

$$ht_{S'}(\gamma) = \sum_{\substack{\alpha \in \Delta \\ s_\alpha \notin S'}} c_\alpha .$$

For the $\emptyset$-height we set

$$ht(\gamma) := ht_\emptyset(\gamma) = \sum_{\alpha \in \Delta} c_\alpha .$$

1.7 KAC-MOODY LIE GROUPS. Let (H,V,Δ) be a root base as in 1.1. To avoid unnecessary complications we assume that V or Δ is linearly free. Let $\underline{g}$ be a Kac-Moody algebra associated to (H,V,Δ). In this situation one can define a group G with subgroups B and N satisfying the following properties:

1) The pair (B,N) is a Tits system in G, i.e.

i) G is generated by B and N

ii) the intersection $T = B \cap N$ is normal in N

iii) the quotient $W = N/T$ is generated by a set S of involutions such that

$$sBw \subset BwB \cup BswB$$

and

$$sBs \neq B$$

for all $s \in S$, $w \in W$.

2) The group T is isomorphic to $H \otimes_{\mathbb{Z}} \mathbb{C}^*$.

3) The system $(W = N/T, S)$ is isomorphic to the Coxeter system associated with (H,∇,Δ) in 1.2. Under this isomorphism the action of W on T is induced by the action of W on H.

4) The group T acts naturally on the subalgebra $\underline{u}^+$ of $\underline{g}$ as well as on the completion $\bar{\underline{u}}^+$ of $\underline{u}^+$ with respect to the filtration $(\underline{u}^i)_{i \in \mathbb{N}}$, $\underline{u}^i = \bigoplus_{ht(\gamma) > i} \underline{g}_\gamma$. Thus there is a natural action of T on the prounipotent proalgebraic group

$$U = \varprojlim U_i$$

corresponding to $\bar{\underline{u}}^+$. Here U_i is the unipotent algebraic group with Lie algebra $\underline{u}^+/\underline{u}^i$. The group B is now the semidirect product $B = T \ltimes U$.

REMARKS: 1) By exploiting the Tits system in G it is easily seen that the above properties characterize the group G up to isomorphism. For a construction of G cf. [23], [24], or [20].

2) Similar groups associated to $\underline{g}$ have been constructed and investigated by Garland, Kac, Peterson, Marcuson, Moody, and Teo (cf. [9], [18], [14], [15], [16], [17]). In some cases these groups differ from ours in a "smaller" subgroup U. Instead of U as above one might use the subgroup U^{min} of U generated by the additive one-parameter subgroups U_γ corresponding to the positive real roots $\gamma \in \sum^{+,R}$ (essentially this group is the one studied by Kac and Peterson).

4) Our later results on the structure of the homogeneous space G/B will not depend on the particular choice of the group G we are dealing with. For reasons of technical convenience one may (and is allowed to) prefer different

versions of these groups depending on circumstances. We will stick to our definition.

One property of the group G is that for any real root $\gamma \in \sum^R$ (not only $\sum^{+,R}$) there is a unique additive one-parameter subgroup U_γ and a homomorphism

$$u_\gamma : \mathbb{C} \to G \quad ,$$

such that

$$U_\gamma := u_\gamma(\mathbb{C})$$

and such that

$$tu_\gamma(c)t^{-1} = u_\gamma(\gamma(t)c)$$

for all $t \in T$, $c \in \mathbb{C}$. Furthermore, for all $n \in N$ and $\gamma \in \sum^R$ we have

$$n\, U_\gamma n^{-1} = U_{w(\gamma)} \quad ,$$

where w denotes the image of n in W .

We let U^- denote the subgroup of G generated by the subgroups U_γ for $\gamma \in \sum^{-,R}$. From the representation theory of G one easily sees (cf. 2.1)

$$U^- \cap B = \{1\} \quad .$$

Note that U^- is not isomorphic to U . In fact it is only isomorphic to the subgroup U^+ of U generated by the U_γ , $\gamma \in \sum^{+,R}$.

1.8 BRUHAT DECOMPOSITION. We now recall some consequences of the existence of the Tits system (B,N) in G . First we have the Bruhat decomposition, i.e. G is the disjoint union of the double classes $C(w) = BwB$ (cf. [5] IV, § 2)

$$G = \dot{\bigcup_{w \in W}} C(w) \quad .$$

For any positive real root $\gamma \in \sum^{R,+}$ let $U_\gamma \subset U$ denote the additive one-parameter group corresponding to γ . Let $w \in W$. Then the product

$$U_w = U_{\gamma_1} \cdot U_{\gamma_2} \cdot \ldots \cdot U_{\gamma_k}$$

(taken in U with respect to any fixed ordering $\gamma_1,\ldots,\gamma_k$ of the roots in $\sum(w), k = \ell(w)$) is a closed subgroup of U , isomorphic as an algebraic variety to the product $U_{\gamma_1} \times \ldots \times U_{\gamma_k}$, hence to the affine space $\mathbb{A}^k$ of dimension $k = \ell(w)$ (note that U is a proalgebraic group). Moreover, U decomposes as a product $U = U_w \cdot U_{(w)}$ where $U_{(w)} = U \cap \dot{w} U \dot{w}^{-1}$ (for any representative $\dot{w}$ of w in N). Any element x in the double class $C(w)$ admits a representation $x = u n u'$ with uniquely determined elements $u \in U_w$, $n \in N$ (such that n maps to $w \in W$), $u' \in U$, i.e. for any fixed representative $\dot{w}$ of w in N the product map

$$U_w \times B \to C(w)$$
$$(u,b) \mapsto u \dot{w} b$$

induces a bijection (for details cf. [20] Ch. 5).

For the multiplication of double classes we have (cf. [5] IV § 2)

$$C(s) \cdot C(w) = \begin{cases} C(sw) \\ C(w) \cup C(sw) \end{cases} \text{if and only if} \begin{cases} \ell(sw) = \ell(w) + 1 \\ \ell(sw) = \ell(w) - 1 \end{cases}$$

for all $s \in S$, $w \in W$. For a decomposition $w = w_1 \ldots w_q$ of an element $w \in W$ with $w_i \in W$, $i = 1,\ldots,q$ and $\ell(w) = \sum_{i=1}^{q} \ell(w_i)$ this gives (cf. loc. cit.)

$$C(w_1) \cdot \ldots \cdot C(w_q) = C(w) .$$

Let $\leq$ denote the Bruhat order on W (cf. 1.4), and define

$$\overline{C}(w) := \bigcup_{w' \leq w} C(w')$$

for any $w \in W$. For $s \in S$ we then have $\overline{C}(s) = C(s) \cup B$.

The multiplication formulae above and the characterization of $\leq$ in terms of reduced expressions yield the following result.

PROPOSITION: Let $w = s_1 \cdot \ldots \cdot s_k$ be a reduced expression of an element $w \in W$. Then

$$\overline{C}(s_1) \cdot \ldots \cdot \overline{C}(s_k) = \overline{C}(w) .$$

1.9 PARABOLIC SUBGROUPS. Let G be a group as in 1.7 and let S be the generating set for the Weyl group W. For any subset $S' \subset S$ we let $W' = W_{S'}$ denote the subgroup of W generated by S'. Let $P_{S'}$ denote the subgroup of G generated by B and by the representatives of $s \in S'$ in the group N. Then the map

$$S' \longmapsto P_{S'}$$

induces an isomorphism from the lattice of subsets of S to the lattice of subgroups of G containing B. Moreover (cf. [5] IV § 2),

$$P_{S'} = \bigcup_{w \in W'} C(w) \quad \text{and} \quad P_{S'} \cap P_{S''} = P_{S' \cap S''} \quad ,$$

and, of course, $P_\phi = B$ and $P_S = G$. The conjugates of the groups $P_{S'}$, $S' \subset S$, in G are called parabolic subgroups. The conjugates of $B = P_\phi$ are also called Borel subgroups.

For any subgroup $P_{S'}$ there is a Levi decomposition

$$P_{S'} = L_{S'} \ltimes U_{(S')} \quad ,$$

where $L_{S'}$ is a Kac-Moody Lie group attached to the root base (H, ∇', Δ') with

$$\Delta' = \{\alpha \in \Delta \mid s_\alpha \in S'\} \quad , \quad \nabla' = \{h_\alpha \in \nabla \mid \alpha \in \Delta'\} \quad ,$$

and where $U_{(S')}$ is a suitable proalgebraic subgroup of U (cf. [20] 5.9 for details).

A subset $S' \subset S$, the corresponding Weyl subgroup $W_{S'} \subset W$, and the associated parabolic subgroup $P_{S'} \subset G$ are called of finite type if $W_{S'}$ is a finite group. In this case the group $P_{S'}$ carries the structure of a proalgebraic group which is compatible with inclusions $P_{S''} \subset P_{S'}$, $S'' \subset S'$. More precisely, the Levi factor $L_{S'}$ is now a finite-dimensional reductive group, and the radical $U_{(S')}$ is the projective limit of the finite-dimensional algebraic quotients

$$U_{(S')}/U^i_{(S')} \quad , \quad i \in \mathbb{N} \quad ,$$

where $U^i_{(S')}$ is the normal subgroup of $U_{(S')}$ generated topologically as a normal subgroup of $P_{S'}$ by the root subgroups U_γ with $ht_{S'}(\gamma) > i$ (cf. [20] 5.7 for details).

Let $S' \subset S$ be of finite type. Then the quotient $P_{S'}/B$ inherits a natural structure of a projective algebraic variety. In the rank 1 case $S' = \{s\}$, $s \in S$, where the semisimple part of $L_{S'}$ is SL_2 or PGL_2 the quotient $P_{S'}/B$ is the projective line.

Let P be an arbitrary proalgebraic group and let $P \times Y \to Y$ be an action of P on an algebraic variety Y. We say that this action is regular if it factorizes over an algebraic action of an algebraic quotient group P' of P, i.e.

$$\begin{array}{ccc} P \times Y & \longrightarrow & Y \\ & \searrow \quad \nearrow & \\ & P' \times Y & \end{array}$$

Let now $P \subset G$ be a parabolic subgroup of finite type containing B, and let $B \times Y \to Y$ be a regular action of the proalgebraic group B on an algebraic variety Y . We denote by $P \times^B Y$ the bundle associated to the principal fibration $P \to P/B$ and the action of B on Y .

LEMMA: The bundle $P \times^B Y$ carries a natural structure of an algebraic variety and the natural left action of P on $P \times^B Y$ is regular.

PROOF: Since the proalgebraic structures on P and B coincide, there is a normal subgroup $U' \subset P$ such that $U' \subset B$, P/U' (hence B/U') is algebraic, and $B \times Y \to Y$ factors over an algebraic action $(B/U') \times Y \to Y$. Thus

$$P \times^B Y \cong (P/U') \times^{(B/U')} Y$$

which equips $P \times^B Y$ with the structure of an algebraic variety (obviously independent of the choice of U') and shows that the natural left action of P factors over an algebraic left action of P/U' .

1.10 PARABOLIC BRUHAT DECOMPOSITION. We fix a subset S' of S , the corresponding Weyl subgroup $W' = W_{S'} \subset W$, and the parabolic subgroup

$$P = P_{S'} = \bigcup_{w' \in W'} C(w') = \bigcup_{w' \in W'} U_{w'} w' B \quad .$$

The following lemma is well known (cf. [5] IV § 1, Ex. 3).

LEMMA 1: Any coset of W by $W_{S'}$ contains a unique element $\tilde{w}$ of minimal length, and for any element $w' \in W_{S'}$ we have $\ell(\tilde{w}w') = \ell(\tilde{w}) + \ell(w')$.

We shall denote the set of elements $\tilde{w}'$ defined in Lemma 1 by $W^{S'}$. Thus $W^{S'}$ is a system of representatives of $W/W_{S'}$ in W .

LEMMA 2: Let $w = \tilde{w} \cdot w'$ be a product in W such that $\ell(w) = \ell(\tilde{w}) + \ell(w')$, and let $\dot{\tilde{w}}$ be a representative of $\tilde{w}$ in $N \subset G$. Then the map

$$(u,u') \longmapsto u\, \dot{\tilde{w}}\, u'\, \dot{\tilde{w}}^{-1}$$

induces an isomorphism of varieties

$$U_{\tilde{w}} \times U_{w'} \xrightarrow{\sim} U_w \quad .$$

PROOF: From the Corollary in § 1.4 we get $\sum(w) = \sum(\tilde{w}) \cup \tilde{w} \sum(w')$. The claim follows now from the structure of the groups $U_{\tilde{w}}$, $U_{w'}$, U_w (cf. 1.8) and the property $\dot{w}\, U_\gamma \dot{w}^{-1} = U_{w(\gamma)}$ for all $\gamma \in \sum^R$ (cf. 1.7).

In what follows we fix a system $\{\dot{w} \mid w \in W\}$ of representatives in N of the elements of W .

PROPOSITION: Let $g \in G$. Then there is a unique element $\tilde{w} \in W^{S'}$ and there are unique elements $u \in U_{\tilde{w}}$, $p \in P$, such that $g = u\dot{\tilde{w}}p$.

PROOF: Let $g \in C(w) = U_w \dot{w} B$, and let $\tilde{w}$ be the element of minimal length in the coset wW'. By Lemma 1, we have $w = \tilde{w} \cdot w'$ with $w' \in W'$ and $\ell(w) = \ell(\tilde{w}) + \ell(w')$. Thus we have $C(w) = C(\tilde{w}) \cdot C(w')$ (cf. 1.8). Because of $C(w') \subset P$ we get

$$g \in C(w) \cdot P = U_{\tilde{w}} \dot{\tilde{w}} P .$$

To prove uniqueness let

$$g = u_i \dot{\tilde{w}}_i p_i , \quad u_i \in U_{\tilde{w}_i} , \quad p_i \in P , \quad i = 1,2,$$

be two decompositions of the desired kind. Let

$$p_i = u_i' \dot{w}_i' b_i , \quad w_i' \in W' , \quad u_i \in U_{w_i'} , \quad b_i \in B ,$$

be the Bruhat decomposition of p_i, $i = 1,2$. Then

$$\ell(\tilde{w}_i w_i') = \ell(\tilde{w}_i) + \ell(w_i') \qquad i = 1,2,$$

and by Lemma 2, we have

$$g = (u_i \dot{\tilde{w}}_i u_i' \dot{\tilde{w}}_i^{-1}) \cdot (\dot{\tilde{w}}_i \dot{w}_i') \cdot b_i \in U_{\tilde{w}_i w_i'} \tilde{w}_i w_i' B , \quad i = 1,2 .$$

From the uniqueness assertions in the usual Bruhat decomposition of g we now get

$$\tilde{w}_1 w_1' = \tilde{w}_2 w_2' ,$$

thus

$$\tilde{w}_1 = \tilde{w}_2 \quad (\tilde{w}_i \text{ is of minimal length in } \tilde{w}_i W')$$

and

$$u_1 \dot{\tilde{w}}_1 u_1' \dot{\tilde{w}}_1^{-1} = u_2 \dot{\tilde{w}}_1 u_2' \dot{\tilde{w}}_1^{-1} \quad \text{in} \quad U_{\tilde{w}_1 w_1'} .$$

Lemma 2 implies $u_1 = u_2$. From this we finally obtain $p_1 = p_2$ which proves our assertion.

1.11 REPRESENTATIONS. Let (H,∇,Δ) be a root base. We define

$$H^*_+ := \{\omega \in H^* \mid \omega(h) > 0 \text{ for all } h \in \nabla\} \quad ,$$

$$H^*_{++} := \{\omega \in H^* \mid \omega(h) > 0 \text{ for all } h \in \nabla\} \quad ,$$

and we call H^*_+ (resp. H^*_{++}) the set of dominant (resp. regular dominant) weights of the root base. Let $\underline{g}$ be a Kac-Moody Lie algebra associated to (H,∇,Δ) (cf. 1.5) and G the corresponding group (cf. 1.7). For any element $\omega \in H^*_+$ one can construct a unique irreducible $\underline{g}$-module $L(\omega)$ which can be integrated to a module of G such that the following properties hold (for details cf. [13] Ch. 3, or [20] 5.10, 5.11):

1) With respect to the torus T the module $L(\omega)$ decomposes as a direct sum of finite-dimensional eigenspaces

$$L(\omega) = \bigoplus_{\mu \in H^*} L(\omega)_\mu \quad ,$$

where

$$L(\omega)_\mu = \{v \in L(\omega) \mid t\cdot v = \mu(t)v \text{ for all } t \in T\} \quad .$$

The elements $\mu \in H^*$ with $L(\omega)_\mu \neq 0$ are called the weights of $L(\omega)$, and $L(\omega)_\mu$ is called the weight space of weight μ.

2) Any weight of $L(\omega)$ is of the form

$$\mu = \omega - \sum_{\alpha \in \Delta} c_\alpha \alpha \quad \text{for suitable} \quad c_\alpha \in \mathbb{N} \quad .$$

3) The dimension of the highest weight space $L(\omega)_\omega$ is one.

The modules $L(\omega)$ have other properties which can be deduced from the above. For example, for all $n \in N$ we have

$$n\, L(\omega)_\mu = L(\omega)_{w(\mu)}$$

where w is the image of n in W. With respect to the Levi part $L_{S'}$ of a parabolic subgroup $P_{S'}$ of finite type the module $L(\omega)$ decomposes as a direct sum of finite-dimensional modules. Also, any element $p \in P_{S'}$, S' of finite type, acts locally finitely on $L(\omega)$. More precisely, let $d_{S'}(\mu) = \sum_{s_\alpha \in S \setminus S'} c_\alpha$ for any weight $\mu = \omega - \sum_{\alpha \in \Delta} c_\alpha \alpha$ of $L(\omega)$ and put

$$L(\omega)_n := \bigoplus_{d_{S'}(\mu) = n} L(\omega)_\mu$$

for any $n \in \mathbb{N}$. Then

LEMMA 1: $L(\omega)_n$ is finite-dimensional for all $n \in \mathbb{N}$.

PROOF: Note that any $L(\omega)_n$ is stable under the Levi group $L_{S'}$ and thus decomposes into a direct sum of finite-dimensional $L_{S'}$-modules. Now $L(\omega)_0$ is generated as an $L_{S'}$-module by $L(\omega)_\omega$, and for $n > 0$, $L(\omega)_n$ is generated as an $L_{S'}$-module by the spaces $f_\alpha \cdot L(\omega)_{n-1}$, $\alpha \in \Delta$, such that $s_\alpha \notin S'$. Hence, by induction on n, we see that $L(\omega)_n$ is finite-dimensional.

Let $L(\omega)_{\leq n}$ denote the direct sum

$$\bigoplus_{i \leq n} L(\omega)_i \quad .$$

Then $L(\omega)_{\leq n}$ is finite-dimensional by the Lemma,

$$L(\omega) = \bigcup_{n \in \mathbb{N}} L(\omega)_{\leq n} \quad ,$$

and $P_{S'}$ stabilizes each $L(\omega)_{\leq n}$.

LEMMA 2: The action of $P_{S'}$ on any subspace $L(\omega)_{\leq n}$, $n \in \mathbb{N}$, is regular in the sense of 1.9.

PROOF: It suffices to look at the factors of $P_{S'} = L_{S'} \ltimes U_{(S')}$ separately. The action of $L_{S'}$ is regular since it is the integral of a finite-dimensional representation of its Lie algebra. Similarly, the action of $U_{(S')}$ is the integral of a linear representation of its Lie algebra $\bar{\underline{u}}_{(S')}$. The kernel of this representation contains the finite-codimensional ideal of $\bar{\underline{u}}_{(S')}$ generated topologically by the root spaces $\mathfrak{g}_\gamma$ with $ht_{S'}(\gamma) > n$. According to the definition of the proalgebraic structure on $U_{(S')} \subset U$, the action of $U_{(S')}$ factors over an algebraic quotient.

1.12 LIMIT TOPOLOGIES ON REPRESENTATION SPACES. Let L be a $\mathbb{C}$-vector space (in the applications, L will be an irreducible highest weight module $L(\omega)$ as introduced in 1.11). We equip any finite-dimensional complex vector space E with the usual Hausdorff topology which we also call the analytical topology. On L, which will be infinite-dimensional in general, we define the analytical limit topology as the finest topology rendering continuous all embeddings of finite-dimensional vector spaces

$$E \hookrightarrow L$$

Almost by definition, the following properties hold:

<u>LEMMA 1</u>: (i) A subset $U \subset L$ is open (resp. closed) $\iff$ For all finite-dimensional subspaces $E \subset L$ the intersection $U \cap E$ is open (resp. closed) in E .

(ii) Any linear subspace $L' \subset L$ is closed.

(iii) Let X be a topological space. Then $f : L \to X$ is continuous $\iff$ For all finite-dimensional subspaces $E \subset L$ the restriction $f_{|E} : E \to X$ is continuous.

<u>LEMMA 2</u>: Let $(E_i)_{i \in I}$ be any system of finite-dimensional subspaces of L which is cofinal with the system of all finite-dimensional subspaces of L . Then, as topological spaces, we have

$$L \cong \varinjlim_{i \in I} E_i \quad .$$

In particular, we have

$$L \cong \varinjlim_{E \subset L\,,\,\dim E < \infty} E \quad .$$

<u>LEMMA 3</u>: The analytical limit topology on L is Hausdorff.

<u>LEMMA 4</u>: Let $\phi : L \to M$ be a linear map of complex vector spaces. Then ϕ is continuous with respect to the analytical limit topologies.

To any $\mathbb{C}$-vector space L we can associate its projective space $\mathbb{P}(L) = (L \setminus \{0\})/\mathbb{C}^*$ and equip it with the corresponding quotient topology. It is easy to check that this topology is the finest topology on $\mathbb{P}(L)$ which renders continuous all linear embeddings of finite-dimensional projective spaces

$$\mathbb{P}(E) \hookrightarrow \mathbb{P}(L) \quad .$$

We thus call this topology the <u>analytical limit topology</u>, too. It is obvious now, that analogues, Lemma 1' and Lemma 2', of Lemma 1 and 2 hold. We also note that the natural projection $p : L \setminus \{0\} \to \mathbb{P}(L)$ is open, thus we obtain

<u>LEMMA 3'</u>: The analytical limit topology on $\mathbb{P}(L)$ is Hausdorff.

<u>LEMMA 4'</u>: Let $\phi : L \to M$ be a linear map of complex vector spaces and $K = \phi^{-1}(0)$ its kernel. Then the induced map

$$\mathbb{P}(\phi) \quad : \quad \mathbb{P}(L) \setminus \mathbb{P}(K) \quad \to \quad \mathbb{P}(M)$$

is continuous with respect to the analytical limit topologies.

PROOF: This follows from the definition of quotient topology and the commutative diagram

$$\begin{array}{ccc} L \setminus K & \xrightarrow{\quad \phi \quad} & M \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ \mathbb{P}(L) \setminus \mathbb{P}(K) & \xrightarrow{\quad \mathbb{P}(\phi) \quad} & \mathbb{P}(M) \end{array}$$

in which p , p' , and ϕ are continuous.

Instead of starting from the usual topology on finite-dimensional vector spaces we could have based our definitions on the Zariski topology on finite-dimensional vector spaces. We call the corresponding limit topologies on L and $\mathbb{P}(L)$ the Zariski limit topologies. We have obvious analogues of Lemma 1, 1', 2, 2', 4, and 4' (of course Lemma 3 and 3' are no longer true).

However, when speaking of the Zariski topology we should also emphasize the algebraic geometric structure on the finite-dimensional spaces E and $\mathbb{P}(E)$. This leads us to consider L resp. $\mathbb{P}(L)$ as inductive limit of the algebraic varieties E resp. $\mathbb{P}(E)$. The ring of regular functions on L will consist of all functions $L \to \mathbb{C}$ whose restriction to a finite-dimensional subspace E is regular in the usual sense. The structure sheaf on $\mathbb{P}(L)$ is defined analogously.

Let now $L = L(\omega)$ be a representation space of a Kac-Moody group G as considered in 1.11. Then any $g \in G$ acts on $L(\omega)$ and $\mathbb{P}(L(\omega))$ as an automorphism of the topological as well as the algebraic geometric structures.

For later applications we now want to prove an auxiliary result about products of projective spaces. Let L and M be complex vector spaces. The product map $\otimes : L \times M \to L \otimes M$, $(v,w) \longmapsto v \otimes w$, induces a map $(L \setminus \{0\}) \times (M \setminus \{0\}) \to L \otimes M \setminus \{0\}$, and by passing to quotients, a map

$$\mathbb{P}(\otimes) \ : \ \mathbb{P}(L) \times \mathbb{P}(M) \hookrightarrow \mathbb{P}(L \otimes M) \quad .$$

In the following we consider the analytical limit topologies on $\mathbb{P}(L)$, $\mathbb{P}(M)$, $\mathbb{P}(L \otimes M)$, and we equip $\mathbb{P}(L) \times \mathbb{P}(M)$ with the product topology.

PROPOSITION: $P(\otimes)$ induces a homeomorphism of $\mathbb{P}(L) \otimes \mathbb{P}(M)$ onto a closed subset of $\mathbb{P}(L \otimes M)$.

PROOF: The result is well known (and trivial) in case L and M are finite-dimensional. To extend it to the general situation we observe that

1) $\mathbb{P}(L) \times \mathbb{P}(M)$ is homeomorphic to the direct limit $\varinjlim \mathbb{P}(E) \times \mathbb{P}(F)$,

taken over all finite-dimensional subspaces $E \subset L$ and $F \subset M$,

2) the system of the $E \otimes F$, E,F as in 1), is cofinal with the sytem of all finite-dimensional subspaces of $L \otimes M$,

3) $\mathbb{P}(\otimes)(\mathbb{P}(L) \times \mathbb{P}(M)) \cap \mathbb{P}(E \otimes F) = \mathbb{P}(\otimes)(\mathbb{P}(E) \times \mathbb{P}(F))$ for all E,F as in 1).

Because of 1) and 2) the closed immersions

$$\mathbb{P}(E) \times \mathbb{P}(F) \hookrightarrow \mathbb{P}(E \otimes F)$$

induce a continuous injection of the corresponding direct limits

$$\mathbb{P}(\otimes) : \mathbb{P}(L) \times \mathbb{P}(M) \hookrightarrow \mathbb{P}(L \otimes M) .$$

Property 3) now implies that $\mathbb{P}(\otimes)$ is a closed map. This proves our claim.

REMARK: To get an analogue of the above proposition in the Zariski topology one has to take 1) as the definition of the Zariski product topology on $\mathbb{P}(L) \times \mathbb{P}(M)$ (since, in general, the Zariski topology on a product of varieties differs from the product of the Zariski topologies). With this definition and using the corresponding finite-dimensional result (cf. [19] I § 5), the proof above shows that $\mathbb{P}(\otimes)$ embeds $\mathbb{P}(L) \times \mathbb{P}(M)$ as a Zariski-limit-closed subset of $\mathbb{P}(L \otimes M)$.

2. FLAG MANIFOLDS AND SCHUBERT VARIETIES

2.1 EMBEDDINGS OF THE HOMOGENEOUS SPACES G/P. Let G be a Kac-Moody Lie group as in 1.7 and $P = P_{S'}$, $S' \subset S$, a parabolic subgroup. In this chapter we study the homogeneous space G/P. Since the torus T is contained in B we may choose T arbitrarily large, i.e. we may assume that the set Δ of simple roots and the set ∇ of simple coroots are linearly free. In this case the sets H^*_+ and H^*_{++} of dominant and regular dominant weights are nontrivial.

LEMMA 1: Let $\omega \in H^*_+$. Then the stabilizer of ω in W equals $W_{S'}$, where $S' = \{s_\alpha \in S \mid \omega(h_\alpha) = 0\}$.

PROOF: The statement follows for example from the properties of the W-action on the Tits cone in $H^* \otimes \mathbb{R}$ (cf. [13] Ch. 3, or [20] 6.1).

For $\omega \in H^*_+$ let $\mathbb{P}(\omega)$ denote the projective space of the module $L(\omega)$. Let $f_1 \in \mathbb{P}(\omega)$ be the point corresponding to the line $L(\omega)_\omega$ in $L(\omega)$ and $X(\omega) \subset \mathbb{P}(\omega)$ the G-Orbit of f_1 under the natural action of G on $\mathbb{P}(\omega)$.

LEMMA 2: Let $\omega \in H^*_+$ and $S' = \{s_\alpha \in S \mid \omega(h_\alpha) = 0\}$. Then the map

$$\varepsilon : G \to \mathbb{P}(\omega) \quad , \quad \varepsilon(g) = g \cdot f_1 \quad ,$$

induces a bijection $G/P_{S'} \tilde{\to} X(\omega)$.

PROOF: By Lemma 1, the stabilizer in N of the point f_1 consists of the preimage of $W_{S'}$ under the projection $N \to W$. On the other hand f_1 is also stabilized by B . Since the subgroups of G containing B are exactly of the form $P_{S''} = \langle B, \dot{W}_{S''} \rangle$, $S'' \subset S$, we see that the stabilizer of f_1 is the group $P_{S'}$.

Let $\Delta' = \{\alpha \in \Delta \mid s_\alpha \in S'\}$ and let $U^{-,R}_{(S')}$ denote the subgroup of G generated by the one-parameter subgroups U_γ where $\gamma \in \sum^{-,R} \setminus \mathbb{Z} \cdot \Delta'$.

LEMMA 3: We have $U^{-,R}_{(S')} \cap P_{S'} = \{1\}$.

PROOF: For $\lambda \in H^*_+$ and $n \in \mathbb{N}$ let $L(\lambda)_n$ denote the direct sum of the weight spaces $L(\lambda)_\mu$ with $d_{S'}(\mu) = n$ (cf. 1.11). Then for all $g \in P_{S'}$ we have

$$g(L(\lambda)_n) \subset \bigoplus_{m \leq n} L(\lambda)_m \quad ,$$

whereas for all $u \in U^{-,R}_{(S')}$ we have

$$(Id - g)(L(\lambda)_n) \subset \bigoplus_{m > n} L(\lambda)_m \quad .$$

Applying this to a direct sum V of modules $L(\lambda)$ such that G acts faithfully on V we get our assertion.

COROLLARY: The restriction of ε to $U^{-,R}_{(S')}$ induces an injection into $X(\omega)$.

REMARK: The group $U^{-,R}_{(S')}$ need not be normalized by $L_{S'}$. This is true only for the larger group $U^-_{(S')}$ generated by all $L_{S'}$ - conjugates of $U^{-,R}_{(S')}$. Then $U^-_{(S')}$ is also normalized by $U^- := U^{-,R}_{(\emptyset)}$. Since $L_{S'}$ preserves the $L(\lambda)_n$, the proof of Lemma 3 still shows $U^-_{(S')} \cap P_{S'} = \{1\}$.

We now deal with an "infinitesimal" analogue of the corollary above. Let v_1 be a non-zero element in $f_1 = L(\omega)_\omega$ and let

$$d\varepsilon : \underline{g} \to L(\omega)/\mathbb{C} v_1$$

be defined by $d\varepsilon(x) = x \cdot v_1 \bmod \mathbb{C} v_1$. We put

$$\underline{u}^{-,R}_{(S')} := \bigoplus_{\gamma \in \sum^{-,R} \setminus \mathbb{Z}\cdot\Delta'} \underline{g}_\gamma .$$

LEMMA 4: The restriction of $d\varepsilon$ to $\underline{u}^{-,R}_{(S')}$ is injective.

PROOF: Since the infinitesimal stabilizer $\underline{p} = \{x \in \underline{g} \mid x\cdot v_1 \subset \mathbb{C}\, v_1\}$ of the line $\mathbb{C}\, v_1$ is normalized by T (and $\underline{h}$ = Lie T) we have a decomposition

$$\underline{p} = \bigoplus_{\alpha \in \sum \cup \{0\}} \underline{g}_\alpha \cap \underline{p} .$$

Thus the lemma follows if we can show that $\underline{g}_\gamma \cap \underline{p} = \{0\}$ for all $\gamma \in \sum^{-,R} \setminus \mathbb{Z}\cdot\Delta'$. However, for all $x \in \underline{g}_\gamma\setminus\{0\}$ we have $\exp(x) \in U_\gamma\setminus\{1\}$. By Lemma 3 we know $U_\gamma \cap P_{S'} = \{1\}$. Thus for all $x \in \underline{g}_\gamma\setminus\{0\}$, $x \notin \underline{p}$ (one may argue in a finite-dimensional U_γ-stable subspace of $L(\omega)$ containing v_1 !).

2.2 SCHUBERT VARIETIES. Let us fix a dominant weight $\omega \in H^*_+$ and the corresponding parabolic subgroup $P = P_{S'}$ of G, where $S' = \{s_\alpha \in S \mid \omega(h_\alpha) = 0\}$. By restriction, the analytical (resp. Zariski) limit topology on $\mathbb{P}(\omega)$ (cf. 1.12) induces a topology on $X(\omega)$ which we simply call the analytical (resp. Zariski) topology on $X(\omega)$. With respect to both topologies G acts as a group of homeomorphism of $X(\omega)$. In particular, $X(\omega)$ is homogeneous as a topological space.

Let $\pi : N \to W$ be the natural projection. Since $n(L(\omega)_\omega) = L(\omega)_{\pi(n)}$ for all $n \in N$, and $\dim L(\omega)_\omega = 1$ the point $n\cdot f_1 \in X(\omega) \subset \mathbb{P}(\omega)$ depends only on $\pi(n)$. We therefore define for all $w \in W$

$$f_w := n \cdot f_1 \quad \text{if} \quad \pi(n) = w .$$

We denote by $X(\omega)_w$ the B-orbit of f_w and by $\overline{X}(\omega)_w$ its closure in the Zariski limit topology on $\mathbb{P}(\omega)$. By 1.11, Lemma 2 we can find a finite-dimensional B-invariant subspace $\mathbb{P}' \subset \mathbb{P}(\omega)$ on which B acts regularly. Then $\overline{X}(\omega)_w$ equals the Zariski closure of $X(\omega)_w$ in $\mathbb{P}'$. As the orbit of an algebraic quotient of B acting on $\mathbb{P}'$, $X(\omega)_w$ is Zariski open in its closure $\overline{X}(\omega)_w$ (cf. [2] I, 1.8). By [19] VII, § 2, Lemma 1, it follows that $\overline{X}(\omega)_w$ coincides with the closure of $X(\omega)_w$ in the analytical topology on $\mathbb{P}'$. Hence $\overline{X}(\omega)_w$ also agrees with the closure of $X(\omega)_w$ in the analytical limit topology on $\mathbb{P}(\omega)$.

Since the point $f_1 \in \mathbb{P}(\omega)$ is stabilized by P, and since $f_w = \dot{w}f_1$ for any $\dot{w} \in W$ such that $\pi(\dot{w}) = w$, we have

$$B\cdot f_w = B\dot{w}f_1 = U_w\dot{w}\,f_1 = U_w\cdot f_w ,$$

and the map

$$U_w \longrightarrow X(\omega)_w \quad ,$$

$$u \longmapsto u \cdot f_w \quad ,$$

is bijective if and only if w is of minimal length in its $W_{S'}$ - coset $w \cdot W_{S'}$. In this case, this map is in fact an isomorphism of algebraic varieties since it is a bijective morphism of an algebraic group U_w onto an orbit $X(\omega)_w$ and since we are in a characteristic zero situation. We call $\overline{X(\omega)}_w$ a <u>Schubert variety</u> and $X(\omega)_w$ its <u>open cell</u>.

By the parabolic Bruhat decomposition (cf. 1.10) we have

$$X(\omega) = \dot{\bigcup_{w \in W^{S'}}} X(\omega)_w \quad .$$

Since we do not know yet whether $X(\omega)$ is closed in $\mathbb{P}(\omega)$, we cannot decide whether the Schubert varieties are contained in $X(\omega)$. In 2.4 we will first prove this last fact and then derive the closedness of $X(\omega)$ in $\mathbb{P}(\omega)$.

<u>2.3 BOTT-SAMELSON-DEMAZURE-HANSEN VARIETIES</u>. Let $(s_1,\dots,s_k)$ be a sequence of elements $s_i \in S \subset W$, let $\alpha_i \in \Delta$ be the root corresponding to s_i and $P_i = P_{\{s_i\}} = C(s_i) \cup B$ the rank-1 parabolic subgroup of G generated by B and a representative $\dot{s}_i$ of s_i. We denote by $\overline{Z}(s_1,\dots,s_k)$ the iterated associated bundle

$$P_1 \times^B (P_2 \times^B (\dots \times^B (P_k/B))\dots)$$

which may also be considered as the quotient of $P_1 \times P_2 \times \dots \times P_k$ by the right B^k - action

$$(p_1,\dots,p_k)\cdot(b_1,\dots,b_k) = (p_1b_1, b_1^{-1}p_2b_2,\dots,b_{k-1}^{-1}p_kb_k) \quad .$$

We denote the projection of $P_1 \times \dots \times P_k$ onto $\overline{Z}(s_1,\dots,s_k)$ by $q(s_1,\dots,s_k)$ or simply by q if there is no danger of confusion.

According to 1.9., Lemma, $\overline{Z}(s_1,\dots,s_k)$ is a smooth complete algebraic variety of dimension k. In fact, it is an iteration of k $\mathbb{P}^1$-bundles with section (starting over a point base):

The homogeneous space P_i/B is a projective line $\mathbb{P}^1_i$ which decomposes under the left B-action into two orbits

$$\mathbb{P}^1_i = U_i(0) \cup \{\infty_i\} \quad ,$$

where

$$U_i(0) = B\,\dot{s}_i\,B/B = U_{\alpha_i}\,\dot{s}_i\,B/B \cong \mathbb{C} ,$$

$$\infty_i = B/B \in P_i/B .$$

From the above we get a decomposition of $\overline{Z}(s_1,\ldots,s_k)$ into affine spaces. Let J denote a subsequence $(i_1,\ldots,i_j)$ of $(1,\ldots,k)$. We put

$$Z_J = q(C_1 \times C_2 \times\ldots\times C_k) = q(\tilde{U}_1 \times\ldots\times \tilde{U}_k) ,$$

where

$$C_i = \begin{cases} C(s_i) \\ B \end{cases} \quad \text{and} \quad \tilde{U}_i = \begin{cases} U_{\alpha_i}\,\dot{s}_i \\ e \end{cases} \quad \text{if} \quad \begin{cases} i \in J \\ i \notin J \end{cases} .$$

Then

$$Z_J = U_1 \times\ldots\times U_k$$

where

$$U_i = \begin{cases} U_i(0) \\ \infty_i \end{cases} \quad \text{if} \quad \begin{cases} i \in J \\ i \notin J \end{cases} ,$$

and q induces an isomorphism

$$\tilde{U}_1 \times\ldots\times \tilde{U}_k \xrightarrow{\sim} U_1 \times\ldots\times U_k .$$

In particular, Z_J is a locally closed algebraic submanifold of $\overline{Z}(s_1,\ldots,s_k)$ isomorphic to the affine space $\mathbb{A}^j$, $j = \mathrm{card}(J)$.

The Zariski closure $\overline{Z}_J$ of Z_J in $\overline{Z}(s_1,\ldots,s_k)$ is the image of $G_1 \times\ldots\times G_k$ under q, where

$$G_i = \begin{cases} \overline{C}(s_i) = C(s_i) \cup B \\ B \end{cases} \quad \text{if} \quad \begin{cases} i \in J \\ i \notin J \end{cases} .$$

We note that $\overline{Z}_j$ is isomorphic to the iterated associated bundle

$$G_1 \times^B G_2 \times\ldots\times {}^B G_k/B ,$$

which itself is isomorphic to $\overline{Z}(s_{i_1},\ldots,s_{i_j})$.

We call $\overline{Z}(s_1,\ldots,s_k)$ the <u>Bott-Samelson-Demazure-Hansen variety</u> associated to the sequence $(s_1,\ldots,s_k)$. They were first introduced in a differential geometric and topological context by Bott and Samelson ([4]). Demazure and Hansen adapted the construction to the algebraic geometric situation to use it for the desingularization of Schubert varieties of finite-dimensional algebraic groups G as well as for the determination of the Chow ring of the corresponding homogeneous space G/B (cf. [6], [12]). In the present situation, the varieties $\overline{Z}(s_1,\ldots,s_k)$ were first considered by Tits ([25]) using a slightly different formulation (his formulation, in terms of galleries, is however intimately related to the original construction of Bott and Samelson in terms of piecewise geodesic paths, cf. [4], I, 5).

2.4 A DESINGULARISATION OF SCHUBERT VARIETIES. We fix a dominant weight $\omega \in H^*_+$ and $S' = \{s_\alpha \in S \mid \omega(h_\alpha) = 0\}$. Let $w \in W^{S'}$ be an element of minimal length in its $W_{S'}$-coset, $w = s_1 \cdot \ldots \cdot s_k$ a reduced decomposition of w , and $\overline{Z} = \overline{Z}(s_1,\ldots,s_k)$ the Bott-Samelson-Demazure-Hansen variety associated to the sequence $(s_1,\ldots,s_k)$. Let

$$m \;:\; P_1 \times\ldots\times P_k \to G$$

denote the multiplication map, $m(p_1,\ldots,p_k) = p_1 \cdot\ldots\cdot p_k$. Then the composition of m with $\varepsilon : G \to X(\omega) \subset \mathbb{P}(\omega)$ obviously factors over the quotient map q :

$$\begin{array}{ccccc} & & G & & \\ & \overset{m}{\nearrow} & & \overset{\varepsilon}{\searrow} & \\ P_1 \times\ldots\times P_k & & & & \mathbb{P}(\omega) \\ & \underset{q}{\searrow} & & \underset{\delta}{\nearrow} & \\ & & \overline{Z} & & \end{array} .$$

LEMMA: The image of $\varepsilon \circ m$ is contained in a finite-dimensional subspace $\mathbb{P}'$ of $\mathbb{P}(\omega)$, and the map $\varepsilon \circ m : P_1 \times\ldots\times P_k \to \mathbb{P}'$ factors over an algebraic morphism $\mu : P' \to \mathbb{P}'$ of an algebraic quotient P' of $P_1 \times\ldots\times P_k$.

PROOF: Using 1.11, Lemma 2, we see inductively that there is a sequence

$$\{f_1\} = \mathbb{P}_{k+1} \subset \mathbb{P}_k \subset \ldots \subset \mathbb{P}_2 \subset \mathbb{P}_1 = \mathbb{P}'$$

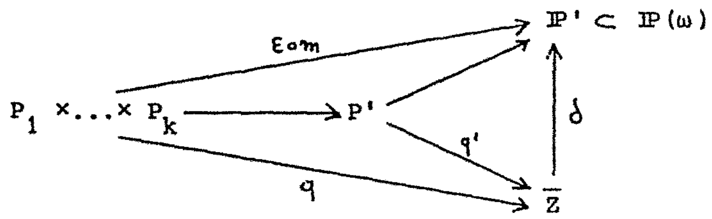
of finite-dimensional linear subspaces $\mathbb{P}_i \subset \mathbb{P}(\omega)$ such that $\mathbb{P}_i$ is P_i-stable and the action of P_i on $\mathbb{P}_i$ is regular in the sense of 1.9. Our claim follows from that.

THEOREM (compare [25] 8.1, 8.2):
(i) The map $\delta : \overline{Z} \to \mathbb{P}(\omega)$ induces a birational morphism of $\overline{Z}$ onto the Schubert variety $\overline{X}(\omega)_w \subset \mathbb{P}(\omega)$.
(ii) The Schubert variety $\overline{X}(\omega)_w$ decomposes as a disjoint union

$$\overline{X}(\omega)_w = \dot{\bigcup_{\substack{w' \leq w \\ w' \in W^{S'}}}} X(\omega)_{w'} .$$

In particular, $\overline{X}(\omega)_w$ is contained in $X(\omega)$.

PROOF: By Lemma 1, the definition of the algebraic structure on $\overline{Z}$ (cf. 2.3, 1.9), and the definition of proalgebraic group (cf. [20] 5.2), we obtain that $\varepsilon \circ m$ and q factor over a common algebraic quotient P' of $P_1 \times \ldots \times P_k$



where $q' : P' \to \overline{Z}$ is the quotient of P' by the algebraic action of a suitable algebraic quotient of B^k . Thus δ is also a morphism of algebraic varieties. Since $\overline{Z}$ is complete and irreducible its image $\delta(\overline{Z})$ under δ is an irreducible and closed subvariety of $\mathbb{P}'$. Since $\delta(\overline{Z}) = \varepsilon \circ m(P_1 \times \ldots \times P_k)$ and $m(P_1 \times \ldots \times P_k) = \overline{C}(s_1) \cdot \ldots \cdot \overline{C}(s_k) = \overline{C}(w)$ (cf. 1.8, Proposition) we obtain

$$\delta(\overline{Z}) = \varepsilon(\overline{C}(w)) = \dot{\bigcup_{\substack{w' \leq w \\ w' \in W^{S'}}}} X(\omega)_{w'} .$$

Thus $\overline{X}(\omega)_w \subset \delta(\overline{Z}) \subset X(\omega)$. On the other hand, by dimensional reasons, $X(\omega)_w$ is open in $\delta(\overline{Z})$. Because of the irreducibility of $\delta(\overline{Z})$ we thus get $\overline{X}(\omega)_w = \delta(\overline{Z})$. It remains to be shown that $\delta : \overline{Z} \to \overline{X}(\omega)_w$ is birational. For that we observe that m induces an isomorphism $(U_{\alpha_1} \dot{s}_1) \times \ldots \times (U_{\alpha_k} \dot{s}_k) \tilde{\to} U_w \dot{w}$, (where $\dot{w} = \dot{s}_1 \ \ldots \ \dot{s}_k$, cf. also 1.10, Lemma 2), ε induces an isomorphism $U_w \dot{w} \tilde{\to} X(\omega)_w$, and q induces an isomorphism from $(U_{\alpha_1} \dot{s}_1) \times \ldots \times (U_{\alpha_k} \dot{s}_k)$ onto the open subset $Z_{(1,\ldots,k)}$ of $\overline{Z}$ (in the notations of 2.3). Thus δ induces an isomorphism of $Z_{(1,\ldots,k)}$ onto $X(\omega)_w$. Since $Z_{(1,\ldots,k)}$ is dense in $\overline{Z}$ this proves the birationality of δ .

REMARKS: 1) The open subset $Z_{(1,\ldots,k)}$ of $\overline{Z}$ is in fact the precise preimage of $X(\omega)_w$ under δ . This follows from the fact that $\varepsilon \circ m$ maps the complement of $q^{-1}(Z_{(1,\ldots,k)}) = C(s_1) \times \ldots \times C(s_k)$ onto the complement of $X(\omega)_w$ in $\overline{X}(\omega)_w$.

2) Part (i) of the theorem generalizes a result of Demazure and Hansen for finite-dimensional algebraic groups ([6], [12]). Part (ii) in that case is due to Chevalley (unpublished, ~ 1958). Proofs may be found in [22] Th. 23, [3] Th. 3.13, [1] Th. 2.11. The generalization to the present situation was

first made by Tits following a suggestion of Deligne ([24], [25]). Apart from a difference on the technical level our proof follows the ideas in [25]. Part (ii) was also proved by Peterson and Kac in case the underlying Cartan matrix is symmetrizable ([18]).

Let us add the following consequence which we pointed out already in 2.2:

<u>COROLLARY</u>: $X(\omega)$ is a closed subspace of $\mathbb{P}(\omega)$ with respect to the Zariski and analytical limit topology.

<u>PROOF</u>: We have to show that the intersection of $X(\omega)$ with an arbitrary finite-dimensional linear subspace $\mathbb{P}' \subset \mathbb{P}(\omega)$ is Zariski closed in $\mathbb{P}'$. By 1.11, Lemma 1, we may assume without loss of generality that $\mathbb{P}'$ is B-stable. Then $X(\omega) \cap \mathbb{P}'$ decomposes as a finite union of B-orbits $X(\omega)_w$, $w \in W(\mathbb{P}')$ (the cardinality of $W(\mathbb{P}')$ is limited by the number of $w \in W$ with $\ell(w) \leq \dim \mathbb{P}'$). Obviously, their closures $\overline{X}(\omega)_w$ are contained in $\mathbb{P}'$. By the theorem, these $\overline{X}(\omega)_w$ are contained in $X(\omega)$ as well. Therefore

$$X(\omega) \cap \mathbb{P}' \subset \bigcup_{w \in W(\mathbb{P}')} \overline{X}(\omega)_w \subset X(\omega) \cap \mathbb{P}'$$

which shows what we claimed.

<u>REMARK</u>: In case the underlying Cartan matrix is symmetrizable this result also follows from the fact that $X(\omega)$ can be described in $\mathbb{P}(\omega)$ by means of "strongly regular" equations in the sense of [14] 3A (cf. [18]). Conversely, the corollary is equivalent only to the weaker statement that $X(\omega)$ can be defined by regular equations in the sense of loc. cit. and of. 1.12.

<u>2.5 INDEPENDENCE OF THE TOPOLOGY</u>. Let us call two dominant weights $\omega, \lambda \in H^*_+$ <u>parabolically equivalent</u> if for all $h \in \nabla$: $\omega(h) = 0 \iff \lambda(h) = 0$. In this section we want to show that the topology on $X(\omega)$ and the Schubert varieties $\overline{X}(\omega)_w$ depends only on the equivalence class of ω . All subsequent statements concern the analytical as well as the Zariski topology.

Note that for all $w' \leq w$ we have natural embeddings $\overline{X}(\omega)_{w'} \hookrightarrow \overline{X}(\omega)_w$ (2.4, Theorem).

<u>LEMMA</u>: $X(\omega)$ is homeomorphic to the direct limit $\varinjlim_{w \in W} \overline{X}(\omega)_w$.

<u>PROOF</u>: We have to prove that the natural continuous bijection $\varinjlim \overline{X}(\omega)_w \to X(\omega)$ is closed. For that, let $A \subset X(\omega)$ be a subset such that for all $w \in W$ the intersection $A \cap \overline{X}(\omega)_w$ is closed. We have to show that $A \cap \mathbb{P}'$ is closed in $\mathbb{P}'$ for any finite-dimensional subspace $\mathbb{P}' \subset \mathbb{P}(\omega)$. Without loss of generality we may assume $\mathbb{P}'$ to be B-stable (1.11, Lemma 2).

Then the intersection $X(\omega) \cap \mathbb{P}'$ is a finite union of Schubert varieties $\overline{X}(\omega)_w$, $w \in W(\mathbb{P}') \subset W$, $\operatorname{card}(W(\mathbb{P}')) < \infty$. Since $A \cap \overline{X}(\omega)_w$ is closed in $\overline{X}(\omega)_w$ and hence in $\mathbb{P}'$, the finite union $A \cap \mathbb{P}' = \bigcup_{w \in W(\mathbb{P}')} A \cap \overline{X}(\omega)_w$ is closed in $\mathbb{P}'$.

PROPOSITION: Let $\omega, \lambda \in H_+^*$ be parabolically equivalent. Then there is a G-equivariant homeomorphism $X(\omega) \to X(\lambda)$. In particular, for any $w \in W$ there is a B-equivariant homeomorphism $\overline{X}(\omega)_w \to \overline{X}_w(\lambda)$.

PROOF: We will first deal with the case that the Cartan matrix of G is symmetrizable (1). Then we will explain the necessary modifications needed in the non-symmetrizable case (2).

1) By 1.12 Proposition (cf. also the remarks appended for the case of the Zariski topology) we obtain a G-equivariant embedding $\mathbb{P}(\otimes)$ of $X(\omega) \times X(\lambda) \subset \mathbb{P}(\omega) \times \mathbb{P}(\lambda)$ onto a closed subset of $\mathbb{P}(L(\omega) \otimes L(\lambda))$. On the other hand, the module $L(\omega+\lambda)$ embeds into the tensor product $L(\omega) \otimes L(\lambda)$, a highest weight vector $v(\omega+\lambda)$ of $L(\omega+\lambda)$ being mapped to the product $v(\omega) \otimes v(\lambda)$ of highest weight vectors $v(\omega) \in L(\omega)$, $v(\lambda) \in L(\lambda)$ (cf. [13] § 10.8). Thus we get a G-equivariant embedding $\iota : X(\omega+\lambda) \to \mathbb{P}(\omega+\lambda) \to \mathbb{P}(L(\omega) \otimes L(\lambda))$. Since the image of ι is contained in the image of $\mathbb{P}(\otimes)$ we may now consider $X(\omega+\lambda)$ as a G-stable closed subset of $X(\omega) \times X(\lambda)$ giving rise to two G-equivariant continuous projections

$$X(\omega) \xleftarrow{pr_1} X(\omega+\lambda) \xrightarrow{pr_2} X(\lambda) \quad .$$

Since ω , $\omega+\lambda$, λ are parabolically equivalent, pr_1 and pr_2 are bijective. When restricted to the compact (resp. complete) Schubert varieties, these projections become closed. Hence we get homeomorphisms

$$\overline{X}(\omega)_w \xleftarrow{\simeq} \overline{X}(\omega+\lambda)_w \xrightarrow{\simeq} \overline{X}(\lambda)_w$$

for all $w \in W$ which are compatible with the natural inclusions existing for $w' \leq w$. Using the lemma we see that pr_1 and pr_2 are homeomorphisms, too.

2) In case of a non-symmetrizable Cartan matrix one does not know whether the integrable highest weight submodule $L'(\omega+\lambda) \subset L(\omega) \otimes L(\lambda)$ generated by $v(\omega) \otimes v(\lambda)$ is irreducible, i.e. isomorphic to $L(\omega+\lambda)$. However, redoing the theory of sections 2.1 to 2.4 for $L'(\omega+\lambda)$ is no problem. We thus get an embedding of G/P ($P = P_{S'}$, $S' = \{s_\alpha \in S \mid (\omega+\lambda)(h_\alpha) = 0\}$) onto a closed subset $X'(\omega+\lambda) \subset \mathbb{P}(L'(\omega+\lambda))$ with Schubert subvarieties $\overline{X}'(\omega+\lambda)_w$. In the proof above we only have to replace $X(\omega+\lambda)$ by $X'(\omega+\lambda)$ and $\overline{X}(\omega+\lambda)_w$ by $\overline{X}'(\omega+\lambda)_w$ to end up with the same result.

REMARKS: 1) As a result of the arguments in part 2) of the proof above one gets that $X(\omega)$ is G-homeomorphic to $X'(\omega)$ for any integrable highest weight module $L'(\omega)$. This can also be seen by directly investigating the natural map $L'(\omega) \to L(\omega)$.

2) In the case of a symmetrizable Cartan matrix, Tits has announced that the algebraic-geometric structure of the Schubert varieties $\overline{X}(\omega)_w$ depends only on the equivalence class of ω (cf. [25] 8). In case $\overline{X}(\omega)_w$ and $\overline{X}(\lambda)_w$ are normal (e.g. smooth) varieties this can also be deduced from our proof. However, normality of the Schubert varieties is still an open problem in the context of Kac-Moody groups. For non-singularity in codimension one, cf. 2.6.

Since the topology on $X(\omega) \cong G/P$, $P = P_{S'}$, does not depend on the weight ω inside the equivalence class determined by P, we have equipped G/P with a well defined topology (analytical or Zariski). We call this topological space the flag manifold of G of type P (or of type S', or of type Δ'). If $P \subset Q$ are parabolic subgroups of G, then the proof above shows that the natural map $G/P \to G/Q$ is continuous.

2.6. ON THE SINGULAR LOCUS OF SCHUBERT VARIETIES. In this section we fix $\omega \in H^*_+$, $\Delta' = \{\alpha \in \Delta \mid \omega(h_\alpha) = 0\}$, $S' = \{s_\alpha \in S \mid \alpha \in \Delta'\}$, and we simply write X for the flag manifold $X(\omega) \subset \mathbb{P}(\omega)$ of type Δ'. Similarly we write X_w resp. $\overline{X}_w$ for $X(\omega)_w$ resp. $\overline{X}(\omega)_w$. Our main objective will be to show that the singular locus of any Schubert variety $\overline{X}_w$, $w \in W^{S'}$, has codimension ≥ 2 in $\overline{X}_w$. Since $\overline{X}_w$ can be embedded in a finite-dimensional B-stable subspace $\mathbb{P}' \subset \mathbb{P}(\omega)$ on which B acts regularly (cf. 1.11) we see that the singular locus of $\overline{X}_w$ consists of a union of B-orbits, i.e. of X_v for some $v \in W^{S'}$ such that $v < w$. Thus we will show

THEOREM: For all $v,w \in W^{S'}$ with $v \xrightarrow{\gamma} w$ for some real positive root γ, the points of X_v are nonsingular points of $\overline{X}_w$.

We will achieve the proof of this result by a series of auxiliary results. The main idea is the same as the one in [1] Proposition 4.3, where the same result is proved for the flag manifolds G/B in the finite-dimensional case. However, by the generalization to arbitrary parabolic P and by using neither a topological nor an algebraic-geometric structure on the group G we have to deal with some extra technical difficulties.

From now on, we shall also fix $v,w \in W^{S'}$ such that $v \xrightarrow{\gamma} w$ for some real positive root $\gamma \in \sum^{+,R}$. Then we have

$$w = s_\gamma v \quad \text{and} \quad \gamma \in \textstyle\sum(w) \quad , \quad \text{i.e.} \quad w^{-1}(\gamma) \in \textstyle\sum^- \quad .$$

LEMMA 1: For all $x \in W^{S'}$ we have

$$x^{-1}(\textstyle\sum(x)) \subset \textstyle\sum^{-,R} \setminus \mathbb{Z}\cdot\Delta'$$

or, equivalently

$$x^{-1}U_x x \subset U^{-,R}_{(S')} .$$

PROOF: This follows for example from the uniqueness of the refined Bruhat decomposition (1.10 Proposition) since for $\beta \in \sum(x)$ with $x^{-1}(\beta) \in \mathbb{Z}\cdot\Delta'$ we would get $U_\beta \dot{x} \subset \dot{x}P_{S'}$.

Let now $\dot{v} \in N$ be a representative of v . Consider the map

$$\kappa : U_v \times U_{-\gamma} \to G \quad , \quad \kappa(u_1,u_2) = \dot{v}^{-1}u_1u_2\dot{v} .$$

LEMMA 2: The map κ is injective and its image is contained in $U^{-,R}_{(S')}$.

PROOF: Since $U \cap U_{-\gamma} = \{1\}$ (cf. 2.1, Lemma 3) and $U_v \subset U$ the product map $(u_1,u_2) \longmapsto u_1u_2$ injects $U_v \times U_{-\gamma}$ into G . Lemma 1, applied to $x = v$, gives $\dot{v}^{-1}U_{\dot{v}}\dot{v} \subset U^{-,R}_{(S')}$. The similar statement for $U_{-\gamma}$ follows from Lemma 1, applied to $x = w$:

$$v^{-1}(-\gamma) = w^{-1}(\gamma) \in w^{-1}(\textstyle\sum(w)) \subset \textstyle\sum^{-,R} \setminus \mathbb{Z}\cdot\Delta' .$$

In the following we let $\mathbb{P}' \subset \mathbb{P}(\omega)$ denote the finite-dimensional subspace which corresponds to the linear subspace

$$L(\omega)_{\leq d(w(\omega))} = \bigoplus_{d(\mu) \leq d(w(\omega))} L(\omega)_\mu$$

of $L(\omega)$. Here $d(\mu)$ denotes the depth of μ , i. e.

$$d(\mu) = \sum_{\alpha \in \Delta} c_\alpha \quad \text{for} \quad \mu = \omega - \sum_{\alpha \in \Delta} c_\alpha \alpha .$$

Let G_γ denote the rank-1-semisimple subgroup of G generated by the one-parameter groups U_γ and $U_{-\gamma}$. We denote $T \cap G_\gamma$ by T_γ and we put $B_{\pm\gamma} := T_\gamma \ltimes U_{\pm\gamma}$. Let $\dot{s}_\gamma \in N_{G_\gamma}(T_\gamma)$ be a representative of s_γ and let $\mathbb{P} \subset X$ denote the G_γ-orbit of f_w . Since $f_v = \dot{s}_\gamma f_w$, we have $f_v \in \mathbb{P}$. Note that $f_w \neq f_v$ (by 2.1 Lemma 2 and $v,w \in W^{S'}$).

LEMMA 3: Let $d = v(\omega)(h_\gamma)$.

(i) The line f_w (resp. f_v) is of lowest weight $-d$ (resp. of highest weight d) in the smallest G_γ-submodule of $L(\omega)$ containing f_w (or, equivalently, f_v),

(ii) $d > 0$,

(iii) $\mathbb{P}$ is contained in $\mathbb{P}'$,

(iv) $\mathbb{P}$ is isomorphic to the projective line $\mathbb{P}^1$,

(v) $\mathbb{P}$ is embedded in $\mathbb{P}'$ (and thus in $\mathbb{P}(\omega)$) as a subvariety of degree d (i.e. any hyperplane of $\mathbb{P}'$ (or $\mathbb{P}(\omega)$) not containing $\mathbb{P}$ cuts $\mathbb{P}$ in d points, counted with multiplicity).

PROOF: Since $w^{-1}(\gamma) \in \sum^-$ and since ω is the highest weight in $L(\omega)$ we get that $w(\omega) - \gamma$ is not a weight of $L(\omega)$. Thus $f_w = L(\omega)_w$ is a lowest weight space for (G_γ, B_γ) of weight $w(\omega)(h_\gamma) = -v(\omega)(h_\gamma)$. Since $f_v = \dot{s}_\gamma f_w$, the line f_v is a highest weight space in the (irreducible) G_γ-module of $L(\omega)$ generated by f_w and G_γ . Thus (i). Assertion (ii) follows from the fact that $f_w \neq f_v$. Statement (iii) follows from (i). For (iv) we note that f_w is fixed by T and by $U_{-\gamma}$, since $w^{-1}(-\gamma) \in \sum^+$. Since $\dim \mathbb{P} > 0$ we get $G_\gamma/B_{-\gamma} \cong \mathbb{P}$, $g \mapsto gf_w$, and $G_\gamma/B_{-\gamma} \cong \mathbb{P}^1$. Finally, (v) follows from a classical result about the embedding of $G_\gamma/B_\gamma = \mathbb{P}^1$ into $\mathbb{P}(V)$, where V is the (d+1)-dimensional irreducible module of G_γ (cf. [1] Lemma 2.10 and proof of Proposition 4.4, for example).

Since the finite-dimensional subspace $\mathbb{P}' \subset \mathbb{P}(\omega)$ is stable under the action of U (by construction) and since $\mathbb{P}$ is contained in $\mathbb{P}'$, the image of the map

$$\xi : U_v \times \mathbb{P} \to X \quad , \quad \xi(u,z) = uz \quad ,$$

is contained in $\mathbb{P}'$. Thus ξ induces a U_v-equivariant morphism of algebraic varieties

$$U_v \times \mathbb{P} \to \mathbb{P}' \cap X$$

which we shall also denote by ξ .

LEMMA 4: The following properties hold:

(i) $\xi(U_v \times f_v) = X_v$,

(ii) $\xi(U_v \times (\mathbb{P} \setminus \{f_v\})) \subset X_w$,

(iii) the restriction of ξ to $U_v \times (\mathbb{P}\setminus\{f_w\})$ is injective,

(iv) there is a Zariski open neighborhood $\mathcal{G}$ of f_v in $\mathbb{P}\setminus\{f_w\}$ such that

ξ is of maximal rank $\ell(w)$ at all points of $U_v \times \mathcal{P}$.

(v) $\xi(U_v \times \mathcal{P})$ is open in $\overline{X}_w$.

PROOF: Assertion (i) follows immediately from the definitions. To see (ii) note that $\mathbb{P}\setminus\{f_v\} = U_\gamma \cdot f_w$ by the "translated" Bruhat decomposition $G_\gamma = \dot{s}_\gamma B_{-\gamma} \cup U_\gamma B_{-\gamma}$ of G_γ with respect to $B_{-\gamma}$. Thus

$$\xi(U_v \times (\mathbb{P}\setminus\{f_v\})) = U_v U_\gamma f_w \subset U f_w = X_w \quad .$$

For the remaining assertions we consider the composition

$$\eta = \dot{v}^{-1} \circ \xi : U_v \times \mathbb{P} \to X \quad .$$

Since $\dot{v}^{-1}$ is linear, it is sufficient to prove statements (iii) and (iv) for η . From the Bruhat decomposition $G_\gamma = B_{-\gamma} \cup U_{-\gamma}\dot{s}_\gamma B_{-\gamma}$ we see that $U_{-\gamma}$ bijects onto $\mathbb{P}\setminus\{f_w\} : u \longmapsto uf_v$. Thus

$$\eta(U_v \times (\mathbb{P}\setminus\{f_w\})) = \dot{v}^{-1}(U_v U_{-\gamma} f_v) = (\dot{v}^{-1} U_v U_{-\gamma} \dot{v}) f_1 = \varepsilon \circ \kappa(U_v \times U_{-\gamma})$$

where $\kappa : U_v \times U_{-\gamma} \to G$ is as in Lemma 2 and where $\varepsilon : G \to X$ is the orbit map $g \mapsto gf_1$. By Lemma 2 we know that κ is injective with image contained in $U^{-,R}_{(S')}$. By 2.1, Corollary, the restriction of ε to this group is injective, too. Hence $\varepsilon \circ \kappa$ and thus η are injective.

To prove (iv) it suffices to show that the differential of $\varepsilon \circ \kappa$ at the neutral element $(e,e) \in U_v \times U_{-\gamma}$ is injective (semicontinuity of rank and U_v-equivariance). This follows from 2.1, Lemma 4 and the following factorization of $d_{(e,e)}\varepsilon \circ \kappa$:

$$\Big(\bigoplus_{\beta \in \sum(v)} \mathfrak{g}_\beta\Big) \oplus \mathfrak{g}_{-\gamma} \xrightarrow{\ \mathrm{Ad}\,\dot{v}^{-1}\ } \Big(\bigoplus_{\alpha \in \sum^{-,R}\setminus \mathbb{Z}\cdot\Delta'} \mathfrak{g}_\alpha\Big) = \underline{u}^{-,R}_{(S')} \xrightarrow{\ d\varepsilon\ } L(\omega)/f_1 .$$

We finally prove (v). By (ii) and (iv), the restriction of ξ to $U_v \times (\mathcal{P}\setminus\{f_v\})$ is an etale morphism into X_w . Thus the image $\xi(U_v \times (\mathcal{P}\setminus\{f_v\}))$ is Zariski open in X_w . We have to show that the complement $A = X_w \setminus \xi(U_v \times (\mathcal{P}\setminus\{f_v\}))$ is closed in $X_w \cup X_v$. Note that A is Zariski closed in X_w and thus of dimension $< \dim X_w = \ell(w)$. Assume that A is not closed in $X_w \cup X_v$. Then there is an irreducible component A_o of A such that the Zariski closure $\overline{A}_o$ of A_o in $\overline{X}_w$ meets X_v . By the U_v-stability of A , and thus of $\overline{A}_o$, we get $X_v \subset \overline{A}_o$. Thus $\ell(v) = \dim X_v \leq \dim \overline{A}_o \leq \ell(v)$ and $\dim \overline{A}_o = \ell(v)$. Since $\overline{A}_o$ is irreducible, X_v is Zariski dense in $\overline{A}_o$, in particular $\overline{X}_v = \overline{A}_o$. But this implies $A_o \cap X_w \subset \overline{A}_o \cap X_w = \emptyset$, a contradiction.

To prove our theorem, let us consider the germ $(\overline{X}_w, x)$ of $\overline{X}_w$ at a point $x \in X_v$ (in the analytical or etale topology) and decompose it into its irreducible components

$$(\overline{X}_w, x) = \bigcup_{i=1}^{n} (V_i, x) \quad .$$

From Lemma 4(iv) we know that at least one component, say V_1, is smooth. To prove that $(\overline{X}_w, x)$ is smooth we have to show that $n = 1$. This can be derived from the following Lemma 5 whose proof will be given later.

By a <u>neighborhood of</u> X_v <u>in</u> X_w we will understand the intersection of X_w with a neighborhood (in the analytical or etale topology) of X_v in $\overline{X}_w$.

<u>LEMMA 5</u>: Any U_v-stable neighborhood of X_v in X_w contains a connected such neighborhood.

Let us now deduce the irreducibility of $(\overline{X}_w, x)$. Because of the transitive U_v-action on X_v, the procedure of attaching to any $y \in X_v$ the set C_y of irreducible components of $(\overline{X}_w, y)$ defines a U_v-equivariant unramified n-fold covering $C \to X_v$ of X_v. Since X_v is simply connected, this covering is trivial. On the other hand, the smoothness of X_w implies that different irreducible components (V_i, x) of $(\overline{X}_w, x)$ intersect only along X_v. Thus $(\overline{X}_w \setminus X_v, x)$ decomposes into n connected components $(V_i \setminus X_v, x)$, $i = 1, \ldots, n$. From the triviality of the covering $C \to X_v$ we now deduce that any sufficiently small U_v-stable neighborhood of X_v in X_w decomposes into n connected components. Now Lemma 5 forces $n = 1$ which had to be shown.

We now have to furnish a proof of Lemma 5. Let $L(\omega)^*$ be the dual space of $L(\omega)$ on which G acts by the contragredient representation. For any $x \in W^{S'}$ we choose $\phi_x \in L(\omega)^*$ with the properties

$$\phi_x\Big|_{L(\omega)_\mu} \equiv 0 \text{ if } \mu \neq x(\omega) \quad , \text{ and } \quad \phi_x\Big|_{L(\omega)_{x(\omega)}} \neq 0 \quad .$$

Then ϕ_x is well determined up to a non-zero scalar, in particular ϕ_x and $\dot{x}\,\phi_1$ are proportional. Let U^- denote the subgroup of G generated by all U_α, $\alpha \in \sum^{-,R}$.

<u>LEMMA 6</u>: For any $x \in W^{S'}$ we have

(i) ϕ_x is invariant under U_x,

(ii) the restriction of ϕ_x to $L(\omega)_{\leq d(x(\omega))}$ is invariant under U.

<u>PROOF</u>: One easily checks that ϕ_1 is invariant under U^-. Thus ϕ_x is invariant under $\dot{x}U^-\dot{x}^{-1}$. Now $x^{-1}(\sum(x)) \subset \sum^-$ implies $U_x \subset \dot{x}U^-\dot{x}^{-1}$, thus (i).

For the second assertion note that U acts trivially on the quotient $L(\omega)_{\leq d(x(\omega))}/L(\omega)_{< d(x(\omega))}$.

Let us now consider the specific situation studied before.

LEMMA 7: The quotient ϕ_w/ϕ_v defines a U_v-invariant meromorphic function $\phi : X_w \cup X_v \to \mathbb{P}^1$. We have $\phi^{-1}(0) = X_v$. The restriction $\overline{\phi} : \mathbb{P} \to \mathbb{P}^1$ of ϕ to $\mathbb{P}$ is a map of degree $d(= v(\omega)(h_\gamma))$.

PROOF: Note that ϕ_w (resp. ϕ_v) vanishes nowhere on X_w (resp. X_v). Moreover ϕ_w vanishes on X_v. Thus ϕ_w/ϕ_v defines a meromorphic function $\phi : X_w \cup X_v \to \mathbb{P}^1$ which vanishes exactly on X_v. The U_v-invariance of ϕ follows from Lemma 6, (i) applied to ϕ_v, and (ii) applied to ϕ_w. The map $\overline{\phi}$ has degree d since the fiber $\overline{\phi}^{-1}(a)$ consists of the d points (counted with multiplicity) in the intersection of $\mathbb{P}$ with the hyperplane $\phi_w - a\phi_v = 0$ of $\mathbb{P}(\omega)$ (cf. Lemma 3(v)).

Now we have collected the means to prove Lemma 5: We consider the restriction of the meromorphic function ϕ to the open subset $\Omega = \xi(U_v \times \mathcal{S}) \subset X_w \cup X_v$ (cf. Lemma 4). Because of its U_v-invariance, the composition $\phi \circ \xi$ factors as $\overline{\phi} \circ pr_2$:

$$\begin{array}{ccc} U_v \times \mathcal{S} & \xrightarrow{\ \xi\ } & \Omega \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle \phi} \\ \mathcal{S} & \xrightarrow{\ \overline{\phi}\ } & \mathbb{P}^1 \end{array}$$

Now let Ω' be an arbitrary U_v-stable open neighborhood of X_v in $\overline{X}_w$. After intersecting with Ω we may assume $\Omega' \subset \Omega$. Then $\xi^{-1}(\Omega')$ is of the form $U_v \times \mathcal{S}'$ for some open neighborhood $\mathcal{S}'$ of f_v in $\mathcal{S}$. Since $\overline{\phi}^{-1}(0) = X_v \cap \mathbb{P} = \{f_v\}$ we may find a connected open neighborhood $\mathcal{S}''$ of f_v in $\mathcal{S}'$ such that $\mathcal{S}'' = \overline{\phi}^{-1}(\overline{\phi}(\mathcal{S}''))$. Since $\overline{\phi}$ is open, the image $\overline{\phi}(\mathcal{S}'')$ is an open neighborhood of $0 \in \mathbb{P}^1$. Thus $\phi^{-1}(\overline{\phi}(\mathcal{S}'')) = \xi(U_v \times \mathcal{S}'')$ is an open neighborhood of X_v in $\overline{X}_w$. Because $\mathcal{S}''\setminus\{f_v\}$ is connected, the image $\xi(U_v \times (\mathcal{S}''\setminus\{f_v\})) = \phi^{-1}(\overline{\phi}(\mathcal{S}'')) \cap X_w$ is connected, too. Thus Lemma 5 and the Theorem are proved.

REMARK: The proof above could be simplified a lot if we had available a good theory providing an algebraic geometric structure of G compatible with the corresponding structures on $X(\omega)$. Some results in that direction are found or announced in [14].

2.7. HOMOGENEOUS LINE BUNDLES ON FLAG MANIFOLDS. In this section we want to define topological homogeneous line bundles on the flag manifolds $X(\omega)$. Everything can be interpreted in the analytical or Zariski topology.

First we have to study tautological line bundles on projective spaces. For that let L be a complex vector space with basis $(e_i)_{i\in I}$ and dual linear forms $\phi_i \in L^*$, $\phi_i(e_j) = \delta_{ij}$ for all $i,j \in I$. By $[v]$ we shall denote the equivalence class in $\mathbb{P}(L) = (L\setminus\{0\})/\mathbb{C}^*$ of an element $v \in L\setminus\{0\}$. The following result is immediate:

LEMMA 1: For any $i \in I$ the map

$$L^{(i)} = \bigoplus_{j\in I\setminus\{i\}} \mathbb{C}e_j \longrightarrow \mathbb{P}(L) \quad ,$$

$$v \longmapsto [e_i+v] \quad ,$$

induces a homeomorphism of $L^{(i)}$ onto the open subset $\mathbb{P}(L)_{\phi_i} = \{[x] \in \mathbb{P}(L) \mid \phi_i(x) \neq 0\}$.

Consider now

$$\mathcal{L}(L) = \{(\ell,v) \in \mathbb{P}(L) \times L \mid v \in \ell\} \quad .$$

Then $\mathcal{L}(L)$ is a closed subset, and the projection

$$pr_1 : \mathcal{L}(L) \to \mathbb{P}(L)$$

realizes $\mathcal{L}(L)$ as a set-theoretic line bundle on $\mathbb{P}(L)$.

LEMMA 2:

(i) $\mathcal{L}(L)$ is a topological line bundle,

(ii) any linear automorphism of L induces a continuous automorphism of $\mathcal{L}(L)$,

(iii) for any subspace $L' \subset L$, the restriction $\mathcal{L}(L)|_{\mathbb{P}(L')}$ is isomorphic to $\mathcal{L}(L')$,

(iv) for any finite-dimensional subspace $E \subset L$, the restriction $\mathcal{L}(L)|_{\mathbb{P}(E)} = \mathcal{L}(E)$ is algebraic.

PROOF: We only have to show (i), the other claims are (then) obvious. It is clear that the projection $\mathcal{L}(L) \to \mathbb{P}(L)$, the addition $\mathcal{L}(L) \times_{\mathbb{P}(L)} \mathcal{L}(L) \to \mathcal{L}(L)$, and the scalar multiplication $\mathbb{C} \times \mathcal{L}(L) \to \mathcal{L}(L)$ are continuous. It remains to show that $\mathcal{L}(L)$ is locally trivial. This results from the existence of the following continuous sections

$$\sigma_i : \mathbb{P}(L)_{\phi_i} \cong V^{(i)} \longrightarrow \mathcal{L}(L)|_{\mathbb{P}(L)_{\phi_i}} \quad ,$$

$\sigma_i(v) = ([e_i+v], e_i+v)$, $v \in V^{(i)}$, which are nowhere vanishing on $V^{(i)}$. Note that $\bigcup_{i \in I} \mathbb{P}(L)_{\phi_i} = \mathbb{P}(L)$.

We call $\mathcal{L}(L)$ the <u>tautological line bundle on</u> $\mathbb{P}(L)$.

Let us fix $\Delta' \subset \Delta$, $\nabla' = \{h_\alpha \in \nabla \mid \alpha \in \Delta'\}$, $S' = \{s_\alpha \in S \mid \alpha \in \Delta'\}$. We put $H^*(\Delta') = \{\omega \in H' \mid \omega(h) = 0$ for $h \in \nabla'\}$ and $H^*_+(\Delta') = H^*(\Delta') \cap H^*_+$. Then $H^*(\Delta')$ is the $\mathbb{Z}$-dual of $H(\Delta') = H/(H \cap \mathbb{Q} \cdot \nabla')$. Let $P = P_{S'}$ and DP its derived subgroup; then $P/DP \cong H(\Delta') \otimes_{\mathbb{Z}} \mathbb{C}^*$ (cf. [20] 7.7).

For any $\omega \in H^*_+(\Delta')$ we have a continuous map

$$\delta_\omega : G/P \to \mathbb{P}(\omega) \quad , \quad \delta_\omega(gP) = g \cdot L(\omega)_\omega \quad .$$

This map induces a bijection onto its image $X(\omega)$ only when $\omega \in H^*_{++}(\Delta') = \{\omega \in H^*(\Delta') \mid \omega(h) > 0$ for all $h \in \nabla \setminus \nabla'\}$. We let

$$\mathcal{L}_{G/P}(\omega) = \delta^*_\omega(\mathcal{L}(L(\omega)))$$

denote the pull back to G/P of the tautological line bundle on $\mathbb{P}(\omega)$. Since G acts on $\mathcal{L}(L(\omega))$ and G/P by continuous automorphisms we get a natural action of G on $\mathcal{L}_{G/P}(\omega)$ by continuous automorphisms. In particular, we can write $\mathcal{L}_{G/P}(\omega)$ as an associated bundle

$$\mathcal{L}_{G/P}(\omega) \cong G \times^P \mathbb{C}_\omega \qquad \text{(as G-sets)} \quad ,$$

where $G \times^P \mathbb{C}_\omega$ is the quotient of $G \times \mathbb{C}$ by the P-action $p(g,z) = (gp^{-1}, \omega(p)z)$, $p \in P$, $g \in G$, $z \in \mathbb{C}$. Here $\omega \in H^*_+(\Delta')$ is lifted to a character of P

$$P \to P/DP = H(\Delta') \otimes_{\mathbb{Z}} \mathbb{C}^* \xrightarrow{\omega} \mathbb{C}^* \quad .$$

We may thus view our definition of $\mathcal{L}_{G/P}(\omega)$ as providing $G \times^P \mathbb{C}_\omega$ with a G-invariant topology. To extend this procedure to more general bundles we first have to prove a compatibility property (which is trivial on the G-set-theoretical level).

<u>LEMMA 3</u>: Let $\omega, \lambda \in H^*_+(\Delta')$. Then $\mathcal{L}_{G/P}(\omega+\lambda)$ is G-line-bundle-homeomorphic to the tensor product $\mathcal{L}_{G/P}(\omega) \otimes \mathcal{L}_{G/P}(\lambda)$.

PROOF: This is a corollary of 1.12 Proposition, the proof of 2.5 Proposition, and the following cartesian diagrams:

$$\begin{array}{ccccc} \mathcal{L}(\omega+\lambda) & \longrightarrow & \mathcal{L}(L(\omega+\lambda)) & \hookrightarrow & \mathcal{L}(L(\omega)\otimes L(\lambda)) \\ \downarrow & & \downarrow & & \downarrow \\ G/P & \xrightarrow{\delta_{\omega+\lambda}} & \mathbb{P}(\omega+\lambda) & \hookrightarrow & \mathbb{P}(L(\omega)\otimes L(\lambda)) \end{array}$$

$$\begin{array}{ccccccc} \mathcal{L}(\omega)\otimes\mathcal{L}(\lambda) & \hookrightarrow & pr_1^*\mathcal{L}(\omega)\otimes pr_2^*\mathcal{L}(\lambda) & \longrightarrow & pr_1^*\mathcal{L}(L(\omega))\otimes pr_2^*\mathcal{L}(L(\lambda)) & \hookrightarrow & \mathcal{L}(L(\omega)\otimes L(\lambda)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G/P & \xrightarrow{\text{diag.}} & G/P\times G/P & \xrightarrow{\delta_\omega\times\delta_\lambda} & \mathbb{P}(\omega)\times\mathbb{P}(\lambda) & \hookrightarrow & \mathbb{P}(L(\omega)\otimes L(\lambda)). \end{array}$$

Note that $Im((\delta_\omega\times\delta_\lambda)\circ \text{diag.}) \cong X(\omega+\lambda)\subset X(\omega)\times X(\lambda)$, cf. 2.5 .

For any $\omega\in H^*_+(\Delta')$ let $\mathcal{L}_{G/P}(-\omega)$ denote the dual bundle of $\mathcal{L}_{G/P}(\omega)$. Since any $\lambda\in H^*(\Delta')$ can be written as $\omega-\omega'$ for suitable $\omega,\omega'\in H^*_+(\Delta')$, we define

$$\mathcal{L}_{G/P}(\lambda) = \mathcal{L}_{G/P}(\omega)\otimes\mathcal{L}_{G/P}(-\omega') \quad .$$

Because of Lemma 3, this definition is free from ambiguities, and we have

$$\mathcal{L}_{G/P}(\lambda+\mu) = \mathcal{L}_{G/P}(\lambda)\otimes\mathcal{L}_{G/P}(\mu)$$

for all $\lambda,\mu\in H^*(\Delta')$.

2.8. HOMOGENEOUS LINE BUNDLES ON SCHUBERT VARIETIES. Let $\Delta'\subset\Delta$, $\nabla'\subset\nabla$, S' , P be as in 2.7. We first want to study the homology and cohomology of the Schubert varieties $\overline{X}(\omega)_w$ for all $w\in W^{S'}$ and $\omega\in H^*_{++}(\Delta')$. From 2.5 we know that the topology on $\overline{X}(\omega)_w$ is independent of the choice of ω in $H^*_{++}(\Delta')$.

PROPOSITION 1: For any $w\in W^{S'}$ and $\omega\in H^*_{++}(\Delta')$ we have

$$H_{2q+1}(\overline{X}(\omega)_w,\mathbb{Z}) = 0 = H^{2q+1}(\overline{X}(\omega)_w,\mathbb{Z})$$

$$H_{2q}(\overline{X}(\omega)_w,\mathbb{Z}) \cong \mathbb{Z}^{n(w,q)} \cong H^{2q}(\overline{X}(\omega)_w,\mathbb{Z}) \quad ,$$

for all $q\in\mathbb{N}$. Here $n(w,q)$ is the number of $w'\in W^{S'}$ such that $w'\leq w$ and $\ell(w')=q$. Moreover, a basis of $H_{2q}(\overline{X}(\omega)_w,\mathbb{Z})$ is given by the fundamental classes of the Schubert varieties $\overline{X}(\omega)_{w'}$, for $w'\in W^{S'}$ with $w'\leq w$ and $\ell(w')=q$.

PROOF: Let $X_i \subset \overline{X}(\omega)_w$ be the union of all Schubert cells $X(\omega)_w$, with $w' \in W^{S'}$, $w' \leq w$, and $\ell(w') \leq i$. Then X_i is closed in $\overline{X}(\omega)_w$ and $X_i \setminus X_{i-1}$ is the disjoint union of those $X(\omega)_{w'}$ for which $\ell(w') = i$. We shall prove analogous claims for the X_i by induction on i, the start $i = 0$ being trivial. By [10] I 5.4.2 it is sufficient to prove the claim for cohomology. For that we use the long exact sequence for cohomology with compact support and integral coefficients (cf. [10] II. 4.10.1):

$$\ldots \to H_c^k(X_i \setminus X_{i-1}) \to H^k(X_i) \to H^k(X_{i-1}) \to H_c^{k+1}(X_i \setminus X_{i-1}) \to \ldots .$$

Using that $H_c^k(X_i \setminus X_{i-1})$ is nonzero only for $k = 2i$, where it is freely spanned by the "duals" of the fundamental classes $[\overline{X}(\omega)_{w'}]$, $w' \in W^{S'}$, $w' \leq w$, $\ell(w) = i$, and by the induction hypothesis we arrive at the desired result.

Since $X(\omega)$ is the inductive limit of the $\overline{X}(\omega)_w$, we directly obtain:

COROLLARY 1: For all $q \in \mathbb{N}$ we have

$$H_{2q+1}(X(\omega), \mathbb{Z}) = 0 = H^{2q+1}(X(\omega), \mathbb{Z})$$

$$H_{2q}(X(\omega), \mathbb{Z}) \cong \mathbb{Z}^{n(q)} \cong H^{2q}(X(\omega), \mathbb{Z}) ,$$

where $n(q)$ is the number of $w \in W^{S'}$ with $\ell(w) = q$. Moreover, $H_{2q}(X(\omega), \mathbb{Z})$ is freely spanned by the fundamental classes of the Schubert varieties $\overline{X}(\omega)_w$, $w \in W^{S'}$, $\ell(w) = q$.

The following conclusion is also well known (cf. [26] 19.1.11). For a complex variety Y let $A_*(Y) = \bigoplus_{k \in \mathbb{N}} A_k(Y)$ denote the graded group of algebraic cycles on Y modulo rational equivalence (k denoting the dimension).

COROLLARY 2: For any $w \in W^{S'}$ and $\omega \in H^*_{++}(\Delta')$ we have

$$A_q(\overline{X}(\omega)_w) = \bigoplus \mathbb{Z} \cdot [\overline{X}(\omega)_v] ,$$

where the direct sum extends over all $v \in W^{S'}$, $v \leq w$, such that $\ell(v) = q$, and where $[\overline{X}(\omega)_v]$ denotes the cycle class of the variety $\overline{X}(\omega)_v$.

Recall that on an irreducible variety Y any algebraic line bundle is isomorphic to a line bundle of the form $\mathcal{O}_Y(D)$ for a locally principal divisor D on Y. In fact, the association $D \mapsto \mathcal{O}_Y(D)$ passes to an isomorphism

$$\mathcal{C}\ell(Y) \xrightarrow{\sim} \mathrm{Pic}(Y)$$

between the group $\mathcal{C}\ell(Y)$ of classes of Cartier divisors on Y and the group $\mathrm{Pic}(Y)$ of isomorphism classes of line bundles on Y (cf. [19] VI § 1.4).

PROPOSITION 2: Let $\omega \in H^*_{++}(\Delta')$ and $w \in W^{S'}$. Then

$$\mathcal{L}_{G/P}(\omega)|\overline{X}(\omega)_w \cong \mathcal{O}_{\overline{X}(\omega)_w}(-D_{\omega,w})$$

where $D_{\omega,w}$ equals the cycle

$$\sum_{v \in W^{S'},\, v \xrightarrow{\gamma} w} v(\omega)(h_\gamma)\,[\overline{X}(\omega)_v] \quad .$$

PROOF: Recall the functional $\phi_w : L(\omega) \to \mathbb{C}$ from 2.6, Lemma 6 and 7. The composition

$$\mathcal{L}_{G/P}(\omega)|\overline{X}(\omega)_w \hookrightarrow \mathbb{P}(\omega) \times L(\omega) \xrightarrow{\phi_w \circ pr_2} \mathbb{C}$$

defines a regular section $s_w \in H^0(\overline{X}(\omega)_w, \mathcal{L}_{G/P}(-\omega))$ which vanishes nowhere on $X(\omega)_w$ (cf. 2.6, Lemma 7). To prove our assertion, we have to show that s_w vanishes with multiplicity $v(\omega)(h_\gamma)$ along $X(w)_v$, $v \xrightarrow{\gamma} w$. This follows from the second assertion of 2.6, Lemma 7.

In the following, we consider $\overline{X}(\omega)_w$ as a topological space. Thus we can restrict all line bundles $\mathcal{L}_{G/P}(\lambda)$ to $\overline{X}(\omega)_w$ as topological bundles. Using that any weight $\lambda \in H^*(\Delta')$ can be written as a difference $\omega - \omega'$ of weights $\omega, \omega' \in H^*_{++}(\Delta')$ and exploiting the homeomorphisms $\overline{X}(\omega)_w \cong \overline{X}(\omega')_w$ we obtain (cf. [1] Lemma 4.2, [26] 19.1.2):

COROLLARY 3: Let $\lambda \in H^*(\Delta')$ and let $c_1(\lambda) \in H^2(\overline{X}(\omega)_w, \mathbb{Z})$ be the first Chern class of $\mathcal{L}_{G/P}(\lambda)|\overline{X}(\omega)_w$. Then

$$c_1(\lambda) \cap [\overline{X}(\omega)_w] = - \sum_{v \in W^{S'},\, v \xrightarrow{\gamma} w} v(\lambda)(h_\gamma) \cdot [\overline{X}(\omega)_v]$$

in $H_{2\ell(w)-2}(\overline{X}(\omega)_w, \mathbb{Z})$. (Here $[Y]$ denotes the fundamental class of a variety Y, and $\cap$ denotes the cap product.)

REMARKS: 1) The equivalent of Corollary 3 (for $\Delta' = \emptyset$) in the Chow ring $A(G/B)$ for finite-dimensional groups G was first established by Chevalley (~ 1958, unpublished, cf. [6] 4.4 for a proof). The homological form is also proved in [1] § 4, Proposition 3, Lemma 4.2, by which we were guided.

2) In [1] and [6], Corollary 3 or its algebraic equivalent are used to evaluate arbitrary polynomials in the Chern classes $c_1(\lambda)$, $\lambda \in H^*$, on the Schubert cycles $[\overline{X}_w]$ of G/B. This can also be done in the present context,

cf. [11] Theorème 3. A detailed elaboration of that point will be published by E. Gutkin (for part of it cf. [27]).

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