ON WEAK MAPS BETWEEN 2-GROUPS

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ABSTRACT. We give an explicit handy (and cocycle-free) description of the groupoid of weak maps between two crossed-modules using what we call a *papillon*; Theorem 8.3. We define composition of papillons and this way find a bicategory that is naturally biequivalent to the 2-category of pointed homotopy 2-types. This has applications in the the study of 2-group actions (say, on stacks).

1. INTRODUCTION

With the emergence of higher-categorical structures in mathematics and physics in the past few years, 2-groups have become ever more present in everyday mathematics. They often appear as symmetries of objects in 2-categories. This way they make their ways into the theory of stacks (they act on stacks, [BeNo]), gerbes with connections or local systems on 2-vector bundles (via monodromy, [PoWa, BDR]), 2-representation theory (in which vector spaces are replaced by 2-vector spaces [El, GaKa]), and so on. Besides, 2-groups sometimes also appear abstractly, without being a priori the symmetries of an object in a 2-category; for instance, they encode the 2-homotopy-type of a topological space. They also appears as a special class of monoidal categories.

The 2-groups that arise in nature are often *weak*, in the sense that, either the multiplication is only associative up to higher coherences, or inverses exists only in a weak sense (or both). Furthermore, interesting morphisms between 2-groups are also usually weak, in the sense that they preserve products only up to higher coherences. It is a standard fact that one can strictify weak 2-groups, but not the weak functors. Put differently, the 2-category of *strict* 2-groups and *weak* functors between them is the "homotopically correct" habitat for 2-groups.

The 2-groups by themselves can be codified conveniently using crossed-modules. Weak morphisms between 2-groups, however, are hairy gadgets and to write them down one ends up with messy cocycles. It is therefore desirable to find a clean way to deal with weak morphisms between 2-groups so as to avoid all the annoying cocycles which clutter up computations.

The aim of this paper is to do exactly this. Namely, we give a concrete and manageable cocycle-free model for the space of weak morphisms between two crossed-modules. (It is easy to see that this space is a 1-type, so its homotopy type is described by a groupoid.) **Theorem 8.3** (also see **Theorem 10.1**) gives us a functorial model for this groupoid in terms of what we call *papillons*. Papillons indeed provide a neat bicategory structure on crossed-modules (**Theorem 8.14**), therefore giving rise to a model for the homotopy category of pointed connected 2-types.

One might argue that the classification of homotopy classes of weak morphisms between 2-groups is, in a sense, a well-known subject, because 2-groups are essentially the same as pointed connected homotopy 2-types, and obstruction theory is an effective tool to give us classification results via cohomological invariant. These classification results (e.g., Theorem 8.6) are, however, not explicit enough for most applications (one such application being the study of 2-group actions on stacks – or more generally, on objects in 2-categories – as we explain below).

A typical application. As an example of the usefulness of our cocycle-free approach to weak morphisms, let us explain in some detail how it can be used in studying group actions on stacks, and how it leads to completely geometric consequences, which a priori have nothing to do with 2-groups. Given a stack $\mathcal{X}$ (say, topological, differentiable, analytic, algebraic, etc.), the set $\operatorname{Aut}\mathcal{X}$ of self-equivalences of $\mathcal{X}$ is naturally a weak 2-group. (It is weak because equivalences are not strictly invertible; the associativity, however, remains strict.) To have a 2-group $\mathfrak{H}$ act on $\mathcal{X}$ is now the same thing as to have a weak map from $\mathfrak{H}$ to $\operatorname{Aut}\mathcal{X}$. If two such maps are related by a (pointed) transformation, they should be regarded as giving the "same" action of $\mathfrak{H}$ on $\mathcal{X}$. So the question is to classify such equivalence classes of weak maps $f\colon \mathfrak{H} \to \operatorname{Aut}\mathcal{X}$. The weakness of $\operatorname{Aut}\mathcal{X}$, and of the map f, are, however, disturbing and we would like to make things as strict as possible. Using a bit of homotopy theory and some standard strictification procedures, it can be shown that what we are looking for is $[\mathfrak{H},\mathfrak{G}]_{\mathbf{2Gp}}$, where $\mathfrak{G} = \operatorname{Aut}\mathcal{X}$. Here, $[\mathfrak{H},\mathfrak{G}]_{\mathbf{2Gp}}$ stands for the set of morphisms from $\mathfrak{H}$ to $\mathfrak{G}$ in the homotopy category $\operatorname{Ho}(\mathbf{2Gp})$ of the category of strict 2-groups and strict maps. Equivalence of 2-groups and crossed-modules (§ 3.3) now enables us to translate the problem to the language of crossed-modules, using which we give an explicit description of $[\mathfrak{H},\mathfrak{G}]_{\mathbf{2Gp}}$ in very simple group theoretic terms (Theorem 8.3; also see Theorem 10.1 and Corollary 10.2 for the case of group actions).

Thanks to the very explicit nature of the above procedure, we are able to give solid constructions with stacks, circumventing a lot of "weaknesses" and coherence conditions that arise in studying group actions on stacks. An application of this strictification method is given in [BeNo], where the covering theory of stacks is used to classify smooth Deligne-Mumford analytic curves, and also to give an explicit description of them as quotient stacks. For instance, using these 2-group theoretic techniques we obtain the following completely geometric result: every smooth analytic Deligne-Mumford stack of dimension one is the quotient stack for the action of either a finite group or a central finite extension of $\mathbb{C}^*$ on a Riemann surface.

Structure of the paper.

The first six sections are devoted to recalling some standard facts about 2-groups and crossed-modules and fixing notation. Essential for more easily reading the paper is the fact that the category of 2-groups is equivalent to the category of crossed-modules (§3.3). The reader will find it beneficial to keep in mind how this equivalence works, as we will freely switch back and forth between 2-groups and crossed-modules throughout the paper (sometimes even using the two terms synonymously). For us, 2-groups are the *conceptual* side of the story, whereas crossed-modules provide the *computational* framework.

Viewed as 2-groupoids with one object, 2-groups (hence, also crossed-modules) can be treated via the Moerdijk-Svensson model structure [MoSe]. This is briefly recalled in Section 6. We point out that, all we need from closed model categories is the notion of fibrant/cofibrant resolution and the way it can be used to compute hom-sets in the homotopy category. Taking this for granted, the reader unfamiliar with closed model categories can proceed with no difficulty.

Section 7 concerns some elementary constructions from group theory. We introduce a push-out construction for crossed-modules and work out its basic properties. This push-out construction is used in an essential way in the next section.

Section 8 is the core of the paper. In it we state and prove our main result (Theorem 8.3). This is based on the notion of *papillon* (Definition 8.1). Using our classification theorem we give an explicit model for the 2-category of crossed-modules and weak morphisms (Section 8.5) in terms of papillons.

Section 9 discusses the "abelian" versions of the results of Section 8. Namely, we state the corresponding results for the derived category of complexes of length 2 in an abelian category A.

In Section 10 we consider a special case of Theorem 8.3 in which the source 2-group is an honest group (Theorem 10.1) and discuss its connection with results of Dedecker and Blanco-Bullejos-Faro. In Section 11 we discuss a cohomological version of this (Theorem 11.6). This cohomological classification is not a new result (with some diligence, the reader sees that it is a special case of the work of [AzCe]), and our emphasis is only to make precise the way it relates to Theorem 10.1, as it was used in [BeNo].

In Sections 12 and 13 we explain the homotopical meaning of Theorem 10.1 in terms the Postnikov decomposition of the classifying space of a crossed-module; again, this is folklore. We make this precise using the notion of *difference fibration* which is an obstruction theoretic construction introduced in [Ba] used in studying the liftings of a map into a fibration (from the base to the total space).

In the appendix we review basic general facts about 2-categories and 2-groupoids.

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Contents

2. Notation and terminology

We list some of the notations and conventions used throughout the paper.

The structure map of a crossed-module $\mathfrak{G} = [G_2 \to G_1]$ is usually denoted by $\alpha \mapsto \underline{\alpha}$. The components of a morphism $P : \mathfrak{H} \to \mathfrak{G}$ of crossed-modules are denoted by $p_2 \colon H_2 \to G_2$ and $p_1 \colon H_1 \to G_1$. The action of G_1 on G_2 is denoted by $-^a$, and so is the conjugation action of G_1 on itself.

In a semi-direct product $A \rtimes B$, respectively $B \ltimes A$, the group B acts on A on the right, respectively left.

We tend to use the term "map" where it is perhaps more appropriate to use the term "morphism".

For 2-groups $\mathfrak{G}$ and $\mathfrak{H}$, when we say *the homotopy class of a weak map from $\mathfrak{H}$ to $\mathfrak{G}$*, we mean a map in $\mathrm{Ho}(\mathbf{2Gp})$ from $\mathfrak{H}$ to $\mathfrak{G}$; this terminology is justified by Remark 14.13.

For objects A and B in a category $\mathbf{C}$ with a notion of weak equivalence, we denote the the set of morphisms in the homotopy category from B to A, that is $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(B, A)$, by $[B, A]_{\mathbf{C}}$.

We usually abuse notation and denote a short exact sequence

$$1 \to N \to E \to \Gamma \to 1$$

simply by E. Quotient maps, such as $E \to \Gamma$ in the above sequence, or $G_1 \to \pi_1\mathfrak{G}$, are usually denoted by $x \mapsto \bar{x}$.

3. Quick review of 2-groups and crossed-modules

3.1. **Quick review of 2-groups.** We recall some basic facts about 2-groups and crossed-modules. Ourmain references are [Br, Wh, McWh, MoSe, Lo, BaLa].

A **2-group** $\mathfrak{G}$ is a group object in the category of groupoids. Alternatively, we can define a 2-group to be a groupoid object in the category of groups, or also, as a (strict) 2-category with one object in which all 1-morphisms and 2-morphisms are invertible (in the strict sense). We will try to adhere the 'group in groupoids' point of view throughout the paper, but occasionally switching back and forth between different points of view is inevitable. Therefore, the reader will find it rewarding to master how the equivalence of these three point of views works.

A (strict) *morphism* $f\colon \mathfrak{G} \to \mathfrak{H}$ of 2-groups is a map of groupoids that respects the group operation. If we view $\mathfrak{G}$ and $\mathfrak{H}$ as 2-categories with one object, such f is nothing but a strict 2-functor. The category of 2-groups is denoted by **2Gp**.

To a 2-group $\mathfrak{G}$ we associate the groups $\pi_1\mathfrak{G}$ and $\pi_2\mathfrak{G}$ as follows. The group $\pi_1\mathfrak{G}$ is the set of isomorphism classes of object of the groupoid $\mathfrak{G}$. The group structure on $\pi_1\mathfrak{G}$ is induces from the group operation of $\mathfrak{G}$. The group $\pi_2\mathfrak{G}$ is the group of automorphisms of the identity object $e \in \mathfrak{G}$. This is an abelian group. A morphism $f\colon \mathfrak{G} \to \mathfrak{H}$ of 2-groups is called an *equivalence* if it induces isomorphisms on π_1 and π_2. The *homotopy category* of 2-groups is the category obtained by inverting all the equivalences in **2Gp**. We denote it by $\mathrm{Ho}(\mathbf{2Gp})$.

Caveat: an equivalence between 2-groups need not have an inverse. Also, two equivalent 2-groups may not be related by an equivalence, but only a zig-zag of equivalences.

3.2. **Quick review of Crossed-modules.** A **crossed-module** $\mathfrak{G} = [\varphi\colon G_2 \to G_1]$ is a pair of groups G_1, G_2, a group homomorphism $\varphi\colon G_2 \to G_1$, and a (right) action of G_1 on G_2, denoted $-^a$, which lifts the conjugation action of G_1 on the image of φ and descends the conjugation action of G_2 on itself. The kernel of φ is a central (in particular abelian) subgroup of G_2 and is denoted by $\pi_2\mathfrak{G}$. The image of φ is a normal subgroup of G_1 whose cokernel is denoted by $\pi_1\mathfrak{G}$. A (strict) morphism of crossed-modules is a pair of group homomorphisms which commute with the φ maps and respect the actions. A morphism is called an *equivalence* if it induces isomorphisms on π_1 and π_2.

Notation. Elements of G_2 are usually denoted by Greek letters and those of G_1 by lower case Roman letters. The components of a map $P : \mathfrak{H} \to \mathfrak{G}$ of crossed-modules are denoted by $p_2\colon H_2 \to G_2$ and $p_1\colon H_1 \to G_1$. We sometimes suppress φ from the notation and denote $\varphi(\alpha)$ by $\underline{\alpha}$. For elements g and a in a group G we sometimes denote $a^{-1}ga$ by g^a. The compatibility assumptions built in the definition of a crossed-module make this unambiguous. With this notation, the two compatibility axioms of a crossed-modules can be written in the following way:

CM1. $\forall \alpha, \beta \in G_2,\ \beta^{\underline{\alpha}} = \beta^{\alpha}$;
CM2. $\forall \beta \in G_2, \forall a \in G_1,\ \underline{\beta^a} = \underline{\beta}^a$.

3.3. **Equivalence of 2-groups and crossed-modules.** There is a natural pair of inverse equivalences between the category **2Gp** of 2-groups and the category **CrossedMod** of crossed-modules. Furthermore, these functors preserve π_1 and π_2. They are constructed as follows.

Functor from 2-groups to crossed-modules. Let $\mathfrak{G}$ be a 2-group. Let G_1 be the group of objects of $\mathfrak{G}$, and G_2 the set of arrows emanating from the identity object e; the latter is also a group (namely, it is a subgroup of the group of arrows of $\mathfrak{G}$).

Define the map $\varphi\colon G_2 \to G_1$ by sending $\alpha \in G_2$ to $t(\alpha)$.

The action of G_1 on G_2 is given by conjugation. That is, given $\alpha \in G_2$ and $g \in G_1$, the action is given by $g^{-1}\alpha g$. Here were are thinking of g as an identity arrow and multiplication takes place in the group of arrows of of $\mathfrak{G}$. It is readily checked that $[\varphi\colon G_2 \to G_1]$ is a crossed-module.

Functor from crossed-modules to 2-groups. Let $[\varphi\colon G_2 \to G_1]$ be a crossed-module. Consider the groupoid $\mathfrak{G}$ whose underlying set of objects is G_1 and whose set of arrows is $G_1 \ltimes G_2$. The source and target maps are given by $s(g,\alpha) = g$, $t(g,\alpha) = g\varphi(\alpha)$. Two arrows (g,α) and (h,β) such that $g\varphi(\alpha) = h$ are composed to $(g,\alpha\beta)$. Now, taking into account the group structure on G_1 and the semi-direct product group structure on $G_1 \ltimes G_2$, we see that $\mathfrak{G}$ is indeed a groupoid object in the category of groups, hence a 2-group.

The above discussion shows that there is a pair of inverse functors inducing an equivalence between **CrossedMod** and **2Gp**. These functors respect π_1 and π_2. Therefore, we have an equivalence

$$\mathrm{Ho}(\mathbf{CrossedMod}) \rightleftarrows \mathrm{Ho}(\mathbf{2Gp}).$$

4. Transformations between morphisms of crossed-modules

We go over the notions of transformation and pointed transformation between maps of crossed-modules. These are adaptations of the usual 2-categorical notions, translated to the crossed-module language (see Appendix, §14.1). The idea is to think of a crossed-module as a 2-group (§ 3.3), which is itself thought of as a 2-groupoid with one object.

Definition 4.1. Let $\mathfrak{G} = [\varphi\colon G_2 \to G_1]$ and $\mathfrak{H} = [\psi\colon H_2 \to H_1]$ be crossed-modules, and let $P, Q\colon \mathfrak{H} \to \mathfrak{G}$ be morphisms between them. A *transformation* $T\colon P \Rightarrow Q$ consists of a pair (a,θ) where $a \in G_1$ and $\theta\colon H_1 \to G_2$ is a crossed homomorphism for the induced action, via p_1^a, of H_1 on G_2 (that is, $\theta(hh') = \theta(h)^{p_1(h')^a}\theta(h')$). We require the following:

T1. $p_1(h)^a\underline{\theta(h)} = q_1(h)$, for every $h \in H_1$;

T2. $p_2(\beta)^a\theta(\underline{\beta}) = q_2(\beta)$, for every $\beta \in H_2$.

We say T is *pointed* if $a = 1$; in this case, we denote T simply by θ. When θ is the trivial map, the transformation T is called *conjugation by* a; in this case, we use the notation $Q = P^a$ or $Q = a^{-1}Pa$.

Remark 4.2. Given a 2-group $\mathfrak{G}$ and an element a in G_1 (the group of objects) we define the morphism $c_a\colon \mathfrak{G} \to \mathfrak{G}$, called *conjugation by* a, to be the map that sends an object g (respectively, an arrow α) to $a^{-1}ga$ (respectively, $a^{-1}\alpha a$). If we consider the corresponding crossed-module $[\varphi\colon G_2 \to G_1]$, the conjugation morphism c_a sends $g \in G_1$ to $a^{-1}ga$ and $\alpha \in G_2$ to α^a. In the notation of Definition 4.1, it is easy to see that $P^a = c_a \circ P$.

Lemma 4.3. *Let $P, Q\colon \mathfrak{H} \to \mathfrak{G}$ be maps of 2-groups and (a,θ) a transformation from P to Q. Then $\pi_i Q = (\pi_i P)^a$, $i = 1,2$. In particular, if P and Q are related by a pointed transformation, then they induce the same map on homotopy groups.*

Proof. Obvious. □

Let $P, Q, R\colon \mathfrak{H} \to \mathfrak{G}$ be maps of crossed-modules. Given homotopies $(a,\theta)\colon P \Rightarrow Q$ and $(b,\sigma)\colon Q \Rightarrow R$, consider the pointwise product $\theta\sigma\colon H_1 \to G_2$. It is easily checked that $(ab, \theta\sigma)$ is a transformation from P to R.[1] This construction, of course, corresponds to the usual composition of weak 2-transformation between 2-functors.

A transformation $(a,\theta)\colon P \Rightarrow Q$ has an inverse $(a^{-1}, \theta^{-1})\colon Q \Rightarrow P$, where $\theta^{-1}\colon H_1 \to G_2$ is defined by $\theta^{-1}(h) := \theta(h)^{-1}$.

Definition 4.4. Let $\mathfrak{G}$ and $\mathfrak{H}$ be crossed-modules. We define the *mapping groupoid* $\underline{\mathcal{H}om}_*(\mathfrak{H}, \mathfrak{G})$ to be the groupoid whose objects are crossed-module maps $\mathfrak{H} \to \mathfrak{G}$ and whose morphisms are pointed transformations.

Remark 4.5. Observe that in the definition above of $\underline{\mathcal{H}om}_*(\mathfrak{H}, \mathfrak{G})$ we have not used 'modifications' and, in particular, the outcome is a groupoid and not a 2-groupoid. This is because between two pointed transformations there is no non-trivial *pointed* modification.

Lemma 4.6. *The groupoid $\underline{\mathcal{H}om}_*(\mathfrak{H}, \mathfrak{G})$ is functorial in both variables. That is, given maps $P\colon \mathfrak{H}' \to \mathfrak{H}$ and $Q\colon \mathfrak{G} \to \mathfrak{G}'$, we get a natural map of groupoids*

$$(P^*, Q_*)\colon \underline{\mathcal{H}om}_*(\mathfrak{H}, \mathfrak{G}) \to \underline{\mathcal{H}om}_*(\mathfrak{H}', \mathfrak{G}').$$

The groupoid $\underline{\mathcal{H}om}_*(\mathfrak{H}, \mathfrak{G})$ is an algebraic version of the space of *simplicial* (pointed) homotopies between maps of simplicial sets. (This can be made precise using the notion of nerve a of a 2-group; see Appendix, Proposition 14.11). The space of simplicial homotopies is, however, not usually the "correct" mapping space, as it lacks certain desired homotopy invariance properties. For instance, an equivalence $\mathfrak{H}' \to \mathfrak{H}$ of crossed-modules does not necessarily induce an equivalence $\underline{\mathcal{H}om}_*(\mathfrak{H}, \mathfrak{G}) \to \underline{\mathcal{H}om}_*(\mathfrak{H}', \mathfrak{G})$ of groupoids. We explain in Section 6 how this failure can be fixed by making use of cofibrant replacements in the category of crossed-modules (especially, see Definition 6.6).

5. Kernel and cokernel of a map of 2-groups

Let $\mathfrak{H} = [H_2 \to H_1]$ and $\mathfrak{G} = [G_2 \to G_1]$ be crossed-modules and $P\colon \mathfrak{H} \to \mathfrak{G}$ a map of crossed-modules. Consider the fiber product $H_1 \times_{G_1} G_2$. By definition of fiber product, there is a natural group homomorphism $H_2 \to H_1 \times_{G_1} G_2$.

Lemma 5.1. *Consider the action of $H_1 \times_{G_1} G_2$ on H_2 in which (h, α) acts as on β by $\beta \mapsto \beta^h$. With this action $[H_2 \to H_1 \times_{G_1} G_2]$ is a crossed-module.*

Proof. Easy. □

Definition 5.2. The crossed-module $[H_2 \to H_1 \times_{G_1} G_2]$ of Lemma 5.1 is called the *kernel* of $P\colon \mathfrak{H} \to \mathfrak{G}$, and is denoted by $\mathrm{Ker}(P)$. The *cokernel* of P, denoted $\mathrm{Coker}(P)$, is defined to be the cokernel of the induced map $\pi_1 P\colon \pi_1\mathfrak{H} \to \pi_1\mathfrak{G}$. (The cokernel is just a pointed set, in general.)

[1]When verifying this, keep in mind that θ is a crossed homomorphism for the action on H_1 on G_2 induced from P^a, whereas σ is so for the action induced from Q^b.

Lemma 5.3. *Let $P, Q\colon \mathfrak{H} \to \mathfrak{G}$ be maps of crossed-modules, and $T\colon P \to Q$ a transformation (Definition 4.1) between them. Then, there is a natural isomorphism $T^*\colon \operatorname{Ker}(Q) \to \operatorname{Ker}(P)$.*

Proof. Let $T = (a, \theta)$. Define $t_1\colon H_1 \times_{G_1,Q} G_2 \to H_1 \times_{G_1,P} G_2$ by $t_1(h,\alpha) = (h, \alpha^a\theta(h)^{-1})$, and let $t_2\colon H_2 \to H_2$ be the identity map. It is easily checked that $T^* = (t_2, t_1)\colon \operatorname{Ker}(Q) \to \operatorname{Ker}(P)$ is the desired isomorphism. $\square$

Proposition 5.4. *Let $P\colon \mathfrak{H} \to \mathfrak{G}$ be a map of crossed-modules. There is a natural map of crossed-modules $\operatorname{Ker}(P) \to \mathfrak{H}$. This map gives rise to a long exact sequence:*

$$1 \to \pi_2 \operatorname{Ker}(P) \to \pi_2\mathfrak{H} \to \pi_2\mathfrak{G} \to \pi_1 \operatorname{Ker}(P) \to \pi_1\mathfrak{H} \to \pi_1\mathfrak{G} \to \operatorname{Coker}(P) \to 1.$$

Proof. Exercise. $\square$

Corollary 5.5. *A map $P\colon \mathfrak{H} \to \mathfrak{G}$ of crossed-modules is an equivalence if an only if $\operatorname{Ker}(P)$ and $\operatorname{Coker}(P)$ are trivial (i.e. equivalent to a point).*

Remark 5.6. Consider the induced map $NP\colon N\mathfrak{H} \to N\mathfrak{G}$ on the nerves (see Appendix). Let F be the homotopy fiber of NP. Then, $\operatorname{Ker}(P)$ is equivalent the Whitehead 2-group of the connected component of F containing the base point, and $\operatorname{Coker}(P)$ is the set of connected components of F. The above exact sequence is nothing but the fiber homotopy exact sequence. See Remark 8.8.

6. Moerdijk-Svensson closed model structure and crossed-modules

It has been known since [Wh] that crossed-modules model pointed connected homotopy 2-types. That is, the pointed homotopy type of a connected pointed CW-complex with $\pi_i X = 0$, $i \geq 3$, is determined by (the equivalence class of) a crossed-module. In particular, the homotopical invariants of such a CW-complex can be read off from the corresponding crossed-module.

The approach in [Wh] and [McWh] to the classification of 2-types is, however, not functorial. To have a functorial classification of homotopy 2-types (i.e. one that also accounts for maps between such objects), it is best to incorporate closed model categories. Of course, it is unreasonable to expect to have a closed model structure on the category of crossed-modules, as we need to allow non-connected objects too. To do so, recall that a crossed-module can be regarded as a 2-group, and a 2-group is simply a 2-groupoid with one object. So the natural candidates for non-connected 2-types are 2-groupoids.

In [MoSe], Moerdijk and Svensson introduce a closed model structure on the (strict) category of (strict) 2-groupoids, and show that there is a Quillen pair between the closed model category of 2-groupoids and the closed model category of CW-complexes, which induce an equivalence between the homotopy category of 2-groupoids and the homotopy category of CW-complexes with vanishing π_i, $i \geq 3$. We use this model structure to deduce some results about crossed-modules using the fact that crossed-modules can be thought of as 2-groupoids with one object.

We emphasize that, in working with crossed-modules, what we are using is the *pointed* homotopy category. So we need to adopt a pointed version of the Moerdijk-Svensson structure. But this does not cause any additional difficulty as everything in [MoSe] carries over to the pointed case. For a quick review of the Moerdijk-Svensson structure see Appendix.

It is easy to see that a weak equivalence between 2-groups in the sense of Moerdijk-Svensson is the same as a weak equivalence between 2-groups (crossed-modules) in the sense of Section 3. Let us see how the fibrations look like.

Definition 6.1. A map $(f_2, f_1)\colon [H_2 \to H_1] \to [G_2 \to G_1]$ of crossed-modules is called a *fibration* if f_2 and f_1 are both surjective. It is called *a trivial fibration* if, furthermore, the map $H_2 \to H_1 \times_{G_1} G_2$ is an isomorphism.

We leave it to the reader to translate these to the language of 2-groups and see that they coincide with Moerdijk-Svensson definition of (trivial) fibration.

Let us now look at cofibrations. In fact, we will only describe what the cofibrant objects are, because that's all we need in this paper.

Definition 6.2. A crossed-module $[G_2 \to G_1]$ is *cofibrant* if G_1 is a free group.

Observe that this is much weaker than Whitehead's notion of a free crossed-module. However, this is the one that corresponds to Moerdijk-Svensson's definition.

Proposition 6.3. *A crossed-module $\mathfrak{G} = [G_2 \to G_1]$ is cofibrant in the sense of Definition 6.2 if and only if its corresponding 2-group is cofibrant in the Moerdijk-Svensson structure.*

Proof. This follows immediately from the Remark on page 194 of [MoSe], but we give a direct proof. A 2-group $\mathfrak{G}$ is cofibrant in Moerdijk-Svensson structure, if and only if every trivial fibration $\mathfrak{H} \to \mathfrak{G}$, where $\mathfrak{H}$ is a 2-group*oid*, admits a section. But, we can obviously restrict ourselves to 2-groups $\mathfrak{H}$. So, we can work entirely within crossed-modules, and use the notion of trivial fibration as in Definition 6.1.

Assume G_1 is free. Let $(f_2, f_1)\colon [H_2 \to H_1] \to [G_2 \to G_1]$ be a trivial fibration. Since G_1 is free and f_1 is surjective, there is a section $s_1\colon G_1 \to H_1$. Using the fact that $H_2 \cong H_1 \times_{G_1} G_2$, we also get a natural section $s_2\colon G_2 \to H_2$ for the projection $H_1 \times_{G_1} G_2 \to G_2$, namely, $s_2(\alpha) = \big(s_1(\underline{\alpha}), \alpha\big)$. It is easy to see that $(s_2, s_1)\colon [G_2 \to G_1] \to [H_2 \to H_1]$ is a map of crossed-modules.

To prove the converse, choose a free group F_1 and a surjection $f_1\colon F_1 \to G_1$. Form the pull back crossed-module $[F_2 \to F_1]$ by setting $F_2 = F_1 \times_{G_1} G_2$. Then, we have a trivial fibration $[F_2 \to F_1] \to [G_2 \to G_1]$. By assumption, this has a section, so in particular we get a section $s_1\colon G_1 \to F_1$ which embeds G_1 as a subgroup of F_1. It follows from Nielsen's theorem that G_1 is free. $\square$

Remark 6.4. It is easy to see that a 2-group $\mathfrak{G}$ is cofibrant in the Moerdijk-Svensson structure if an only if the inclusion $* \to \mathfrak{G}$ is a cofibration. So the definition of cofibrant is the same in the pointed category. Also, in the pointed category, all 2-groupoids are fibrant.

Example 6.5.

1. Let $\mathfrak{G} = [G_2 \to G_1]$ be an arbitrary crossed-module. Let $F_1 \to G_1$ be a surjective map from a free group F_1, and set $F_2 := F_1 \times_{G_1} G_2$. Consider the crossed-module $\mathfrak{F} = [F_2 \to F_1]$. Then $\mathfrak{F}$ is cofibrant, and the natural map $\mathfrak{F} \to \mathfrak{G}$ is a trivial fibration (Definition 6.1). In other words, $\mathfrak{F} \to \mathfrak{G}$ is a cofibrant replacement for $\mathfrak{G}$.
2. Let Γ be a group, and $F/R \cong \Gamma$ be a presentation of Γ as a quotient of a free group F. Then the map of crossed-modules $[R \to F] \to [1 \to \Gamma]$ is a cofibrant replacement for Γ.

Definition 6.6. Let $\mathfrak{H}$ and $\mathfrak{G}$ be 2-groups (or crossed-modules). Choose a cofibrant replacement $\mathfrak{F} \to \mathfrak{H}$ for $\mathfrak{H}$, as in Example 6.5.**1**. The *derived mapping groupoid* $\underline{\mathcal{RH}om}_*(\mathfrak{H}, \mathfrak{G})$ is defined to be $\underline{\mathcal{H}om}_*(\mathfrak{F}, \mathfrak{G})$, where $\underline{\mathcal{H}om}_*$ is as in Definition 4.4.

Observe that in the Moerdijk-Svensson structure all 2-groups are automatically fibrant, so in the above definition we do not need a fibrant replacement for $\mathfrak{G}$.

The derived mapping groupoid $\underline{\mathcal{RH}om}_*(\mathfrak{H}, \mathfrak{G})$ depends on the choice of the cofibrant replacement $\mathfrak{F} \to \mathfrak{H}$, but it is unique up to an equivalence of groupoids (which is itself unique up to transformation). Another way of thinking about the derived mapping groupoid $\underline{\mathcal{RH}om}_*(\mathfrak{H}, \mathfrak{G})$ is that it gives a model for the groupoid of *weak morphisms* from $\mathfrak{H}$ to $\mathfrak{G}$ and pointed weak transformations between them.[2] For more on this, and also the interpretation of the notion of weak morphism in the crossed-module language, we refer the reader to [No]; also see Remark 14.13.

The fact that derived mapping groupoids are the correct models for homotopy invariant mapping spaces is justified by the following

Proposition 6.7. *Let $\mathfrak{H}$ and $\mathfrak{G}$ be 2-groups (or crossed-modules). We have a natural bijection*

$$\pi_0\underline{\mathcal{RH}om}_*(\mathfrak{H}, \mathfrak{G}) \cong [\mathfrak{H}, \mathfrak{G}]_{\mathbf{2Gp}}.$$

Proof. By Proposition 14.12, we have

$$\pi_0\underline{\mathcal{RH}om}_*(\mathfrak{H}, \mathfrak{G}) \cong \pi_0\mathbf{Hom}_*(N\mathfrak{H}, N\mathfrak{G}) \cong [N\mathfrak{H}, N\mathfrak{G}]_{\mathbf{SSet}_*}.$$

By Proposition 14.9, $[N\mathfrak{H}, N\mathfrak{G}]_{\mathbf{SSet}_*} \cong [\mathfrak{H}, \mathfrak{G}]_{\mathbf{2Gp}}$. ∎

In Section 8 we give a canonical explicit model for $\underline{\mathcal{RH}om}_*(\mathfrak{H}, \mathfrak{G})$.

7. Some group theory

In this section we introduce some basic group theoretic. These will be used in the proof of Theorem 8.3.

7.1. **Generalized semi-direct products.** We define a generalized notion of semi-direct product of groups, and use that to introduce a push-out construction for crossed-modules.

Let H, G and K be groups, each equipped with a right action of K, the one on K itself being conjugation. We denote all the actions by $-^k$ (even the conjugation one). Assume we are given a K-equivariant diagram

$$\begin{tikzcd} H \arrow[r, "p"] \arrow[d, "l"'] & G \\ K & \end{tikzcd}$$

in which we require the compatibility condition $g^{l(h)} = g^{p(h)} \ \big(:= p(h)^{-1}gp(h)\big)$ is satisfied for every $h \in H$ and $g \in G$.

[2]Since we are working in the *pointed* category, the modification are trivial. This is due to he fact that, whenever X and Y are pointed connected homotopy 2-types, the pointed mapping space $\mathbf{Hom}_*(X, Y)$ is a 1-type.

Definition 7.1. The *semi-direct product* $K \ltimes^H G$ of K and G along H is defined to be $K \ltimes G/N$, where

$$N = \big\{\big(l(h)^{-1}, p(h)\big),\ h \in H\big\}.$$

For this definition to make sense, we have to verify that N is a normal subgroup of $K \ltimes G$. This is left to the reader. Hint: show that G centralizes N, and an element $k \in K$ acts by

$$\big(l(h)^{-1}, p(h)\big) \mapsto \big(l(h^k)^{-1}, p(h^k)\big).$$

There are natural group homomorphisms $p' \colon K \to K \ltimes^H G$ and $l' \colon G \to K \ltimes^H G$, making the following diagram commute

$$\begin{tikzcd} H \arrow[r, "p"] \arrow[d, "l"'] & G \arrow[d, "l'"] \\ K \arrow[r, "p'"'] & K \ltimes^H G \end{tikzcd}$$

There is also an action of $K \ltimes^H G$ on G which makes the above diagram equivariant. An element $(k, g) \in K \ltimes^H G$ acts on $x \in G$ by sending it to $g^{-1}x^k g$. Indeed, $[l' \colon G \to K \ltimes^H G]$ is a crossed-module.

Lemma 7.2. *In the above square, the induced map* $\mathrm{Coker}(l) \to \mathrm{Coker}(l')$ *is an isomorphism and the induced map* $\ker(l) \to \ker(l')$ *is surjective. The kernel of the latter is equal to* $\ker(p) \cap \ker(l)$.

Proof. Straightforward. □

The relative semi-direct product construction satisfies the obvious universal property. Namely, to give a homomorphism $K \ltimes^H G \to T$ to an arbitrary group T is equivalent to giving a pair of homomorphisms $\lambda \colon G \to T$ and $\varpi \colon K \to T$ such that

- $$\begin{tikzcd} H \arrow[r, "p"] \arrow[d, "l"'] \arrow[dr, phantom, "\circlearrowleft"] & G \arrow[d, "\lambda"] \\ K \arrow[r, "\varpi"'] & T \end{tikzcd}$$
- $\lambda(g^k) = \lambda(g)^{\varpi(k)}$, for every $g \in G$ and $k \in K$.

Remark 7.3. Consider the the subgroup $I = p\big(\ker(l)\big) = \ker l' \subseteq G$. It is a K-invariant central subgroup of G. If in the relative semi-direct product construction we replace G by G/I the outcome will be the same.

In the following lemma we slightly modify the notation and denote $K \ltimes^H G$ by $K \ltimes^{H,p} G$.

Lemma 7.4. *Notation being as above, let* $\theta \colon K \to G$ *be a crossed homomorphism (i.e.* $\theta(kk') = \theta(k)^{k'}\theta(k)$*). Consider the group homomorphism* $q \colon H \to G$ *defined by* $q(h) = p(h)\theta(l(h))$*, and use it to form* $K \ltimes^{H,q} G$*. Also consider the new action* *of K on G given by $g^{*k} := \theta(k)^{-1}g^k\theta(k)$. (The $*$ is used just to differentiate the new action from the old one.) Then, there is a natural isomorphism*

$$\begin{array}{rccc} \theta^*\colon & K \ltimes^{H,q} G & \xrightarrow{\sim} & K \ltimes^{H,p} G \\ & (k,g) & \mapsto & \big(k,\theta(k)g\big). \end{array}$$

The map θ^* makes the following triangle commute:

$$\begin{tikzcd} & G \arrow[dl, "l'_q"'] \arrow[dr, "l'_p"] & \\ K \ltimes^{H,q} G \arrow[rr, "\theta^*"'] & & K \ltimes^{H,p} G \end{tikzcd}$$

Furthermore, we have the following commutative triangle of isomorphisms

$$\begin{tikzcd} \mathrm{Coker}(l'_q) \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & \mathrm{Coker}(l'_p) \arrow[dl, "\sim"] \\ & \mathrm{Coker}\, l & \end{tikzcd}$$

where the top row is induced by θ^*.

Proof. We use the universal property of the relative semi-direct product. To give a map from $K \ltimes^{H,p} G$ to a group T is equivalent to giving a pair (λ,ϖ) of maps $\lambda\colon G \to T$ and $\varpi\colon K \to T$ satisfying the two conditions described in the paragraph just before the lemma. To such a pair, we can associate a new pair (λ',ϖ'), with $\lambda' := \lambda$ and $\varpi'(k) := \varpi(k)\lambda(\theta(k))$. It is easy to see that the pair (λ',ϖ') satisfies the two conditions required by the universal property of $K \ltimes^{H,q} G$. Similarly, we can go backwards from a pair (λ',ϖ') for $K \ltimes^{H,q} G$ to a pair (λ,ϖ) for $K \ltimes^{H,p} G$. It is easy to see that this correspondence is realized by θ^*. This proves that θ^* is an isomorphism.

Commutativity of the triangles is obvious. (Also see Lemma 7.2.) $\square$

We have the following converse for Lemma 7.4

Lemma 7.5. *Consider the semi-direct product diagrams*

$$\begin{tikzcd} H \arrow[r, "p"] \arrow[d, "l"'] & G \\ K & \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "q"] \arrow[d, "l"'] & G \\ K & \end{tikzcd}$$

where the K-actions on H are the same. Assume we are given an isomorphism of groups $\vartheta\colon K \ltimes^{H,q} G \xrightarrow{\sim} K \ltimes^{H,p} G$ such that the triangles of Lemma 7.4 commute. Denote $p(\ker l) \subseteq G$ by I.

- **i.** *If $\ker l \subseteq \ker p$, then there is a unique crossed homomorphism $\theta\colon K \to G$ such that $\vartheta = \theta^*$ (see Lemma 7.4, and Remark 7.3).*
- **ii.** *If K is a free group, then there exists a (not necessarily unique) crossed homomorphism $\theta\colon K \to G$ such that $\vartheta = \theta^*$.*
- **iii.** *If for crossed-homomorphisms θ and θ' we have $\theta^* = \theta'^*$, then the difference of θ and θ' factors through $I \subseteq G$. That is, for every $k \in K$, $\theta^{-1}(k)\theta'(k)$ lies in I).*

Proof of. (**i**). Pick an element $(k,g) \in K \ltimes^{H,q} G$, and let $\theta(k,1) = (k',g')$. (Note that $(k,1)$ and (k',g') are just representatives for actual elements in the corresponding relative semi-direct product groups). By the commutativity of the above triangle, the images of k and k' are the same in $\mathrm{Coker}(l)$; that is, there exists $h \in H$ such that $kl(h) = k'$. So, after adjusting (k',g') by the $\big(l(h)^{-1}, p(h)\big) \in N$ (see Definition 7.1), we may assume $k' = k$; that is $\vartheta(k,1) = (k,g')$. Define $\theta(k)$ to be g'. It is easily verified that θ is a crossed homomorphism, and that $\theta^* = \vartheta$.

Proof of (**ii**). Replace G by G/I and apply (**i**) to obtain $\theta \colon K \to G/I$. Then use freeness of K to lift θ to G.

Proof of (**iii**). Easy. □

7.2. **A push-out construction for crossed-modules.** Continuing with the set-up of the previous section, we now bring crossed-modules into the picture. Namely, we assume that $[l \colon H \to K]$ is a crossed-module. (Note that the condition **CM2** of crossed-modules (§ 3.2) is already part of the hypothesis.) To be compatible with our crossed-module notation, let us denote H, K, G and l by H_2, H_1, G_2 and $\underline{\ \ }$ respectively. Recall that $[G_2 \to H_1 \ltimes^{H2} G_2]$ is again a crossed-module.

Definition 7.6. Let $\mathfrak{H} = [H_2 \to H_1]$ be a crossed-module. Let G_2 be a group, and $p \colon H_2 \to G_2$ a group homomorphism. We call the crossed-module $[G_2 \to H_1 \ltimes^{H2} G_2]$ the *push-out* of $\mathfrak{H}$ along p, and denote it by $p_*\mathfrak{H}$.

Lemma 7.7. *There is a natural induced map of crossed-modules $p_\diamond \colon \mathfrak{H} \to p_*\mathfrak{H}$. Furthermore, $\pi_1 p_\diamond$ is an isomorphism and $\pi_2 p_\diamond$ is surjective. The kernel of $\pi_2 p_\diamond$ is equal to*

$$\{\beta \in H_2 \mid \underline{\beta} = 1,\ p(\beta) = 1\}.$$

In particular, if $p \colon H_2 \to G_2$ is injective, then $p_\diamond \colon [H_2 \to H_1] \to [G_2 \to H_1 \ltimes^{H2} G_2]$ is an equivalence of crossed-modules.

Proof. Straightforward. □

Assume now that we are given two crossed-modules $\mathfrak{G} = [G_2 \to G_1]$, $\mathfrak{H} = [H_2 \to H_1]$ and a morphism $P \colon \mathfrak{H} \to \mathfrak{G}$ between them. This gives us a diagram

$$\begin{tikzcd} H_2 \arrow[r, "p_2"] \arrow[d, "\underline{\ \ }"'] & G_2 \\ H_1 & \end{tikzcd}$$

like the one in the beginning of this section. We also have an action of H_1 on G_2 with respect to which p_2 is H_1-equivariant. Namely, for $\alpha \in G_2$ and $h \in H_1$, we define α^h to be $\alpha^{p_1(h)}$, the latter being the action in $\mathfrak{G}$. So, we can form the crossed-module $[G_2 \to H_1 \ltimes^{H2} G_2]$.

Define the map $\rho \colon H_1 \ltimes^{H2} G_2 \to G_1$ by $\rho(h,\alpha) := p_1(h)\underline{\alpha}$. It is easily seen to be well-defined. We obtain the following commutative diagram of crossed-modules:

$$
\begin{tikzcd}
H_2 \arrow[rr, bend left, "p_2"] \arrow[dr] \arrow[dd] & & G_2 \arrow[dd] \\
& G_2 \arrow[ur, "="'] \arrow[dd] & \\
H_1 \arrow[rr, "p_1"'] \arrow[dr] & & G_1 \\
& H_1 \ltimes^{H2} G_2 \arrow[ur, "\rho"'] &
\end{tikzcd}
$$

Note that the front-left square is almost an equivalence of crossed-modules (Lemma 7.7); it is an actual equivalence if and only if $\pi_2 P\colon \pi_2\mathfrak{H} \to \pi_2\mathfrak{G}$ is injective. If this is the case, the above diagram means that, up to equivalence, we have managed to replace our crossed-module map $P\colon \mathfrak{H} \to \mathfrak{G}$ with one, i.e. $P_*\mathfrak{H} \to \mathfrak{G}$, i n which p_2 is the identity map (the front-right square).

Notation. If $P\colon \mathfrak{H} \to \mathfrak{G}$ is a map of crossed-modules, we use the notation $P_*\mathfrak{H}$ instead of $p_{2,*}\mathfrak{H}$.

The next thing we consider is, how the push-out construction for crossed-modules behaves with respect to pointed transformations between maps.

Lemma 7.8. *Let $P, Q\colon \mathfrak{H} \to \mathfrak{G}$ be maps of crossed-modules and $\theta\colon P \Rightarrow Q$ a pointed transformation between them (Definition 4.1). Then, we have the following commutative diagram of maps of crossed-modules:*

$$
\begin{tikzcd}
H_2 \arrow[rr, "q_2"] \arrow[dd] & & G_2 \arrow[rr, "="] \arrow[dr, "="'] \arrow[dd] & & G_2 \arrow[dd] \\
& H_2 \arrow[rr, "p_2"] \arrow[dd] & & G_2 \arrow[ur, "="'] \arrow[dd] & \\
H_1 \arrow[rr, "q_1"'] & & H_1 \ltimes^{H_2,Q} G_2 \arrow[rr, "\rho_Q"] \arrow[dr, "\theta^*"', "\cong"] & & G_1 \\
& H_1 \arrow[rr, "p_1"'] & & H_1 \ltimes^{H_2,P} G_2 \arrow[ur, "\rho_P"'] &
\end{tikzcd}
$$

in which the front faces compose to P and the back faces compose to Q. Here θ^* is obtained by the construction of Lemma 7.4 applied to $\theta\colon H_1 \to G_2$. Furthermore, the following triangle commutes:

$$
\begin{tikzcd}
\pi_1(Q_*\mathfrak{H}) \arrow[rr, "\pi_1(\theta^*)", "\sim"'] \arrow[dr, "\sim"'] & & \pi_1(P_*\mathfrak{H}) \arrow[dl, "\sim"] \\
& \pi_1(\mathfrak{H}) &
\end{tikzcd}
$$

Proof. This is basically a restatement of Lemma 7.4. Only proof of the equality $\rho_Q = \rho_P \circ \theta^*$ is missing. To prove this, pick $(h,\alpha) \in H_1 \ltimes^{H_2,Q} G_2$. Since $\theta^*(h,\alpha) = \big(h, \theta(h)\alpha\big)$, we have

$$\rho_P(\theta^*\big(h,\alpha)\big) = p_1(h)\underline{\theta(h)}\,\underline{\alpha} = q_1(h)\underline{\alpha} = \rho_Q(h,\alpha).$$

$\square$

Lemma 7.9. *Consider the commutative diagrams of Lemma 7.8, but with ϑ instead of θ^*. Assume $\pi_2 P \colon \pi_2\mathfrak{H} \to \pi_2\mathfrak{G}$ is the zero homomorphism. Then, there is a unique transformation $\theta \colon P \Rightarrow Q$ such that $\vartheta = \theta^*$.*

Proof. This is more or less a restatement of Lemma 7.5.**i**, with H_1, H_2 and G_2 playing the roles of K, H and G, respectively. More explicitly, construct $\theta \colon H_1 \to G_2$ as in Lemma 7.5. It automatically satisfies condition **T2** of Definition 4.1. The fact that it satisfies **T1** follows from the definition of θ^*; see Lemma 7.4. □

8. The classification theorem

In this section, we give a description for the groupoid of weak maps between two crossed-modules (Theorem 8.3). The key is the following definition.

Definition 8.1. Let $\mathfrak{G} = [\varphi \colon G_2 \to G_1]$ and $\mathfrak{H} = [\psi \colon H_2 \to H_1]$ be crossed-modules. By a *papillon* from $\mathfrak{H}$ to $\mathfrak{G}$ we mean a commutative diagram of groups

$$\begin{tikzcd}
H_2 \arrow[dd, "\psi"'] \arrow[dr, "\kappa"] & & G_2 \arrow[dl, "\iota"'] \arrow[dd, "\varphi"] \\
& E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
H_1 & & G_1
\end{tikzcd}$$

in which both diagonal sequences are complexes, and the NE-SW sequence, that is, $G_2 \to E \to H_1$, is short exact. We require ρ and σ satisfy the following compatibility with actions. For every $x \in E$, $\alpha \in G_2$, and $\beta \in H_2$,

$$\iota(\alpha^{\rho(x)}) = x^{-1}\iota(\alpha)x, \quad \kappa(\beta^{\sigma(x)}) = x^{-1}\kappa(\beta)x.$$

We denote the above papillon by the tuple $(E, \rho, \sigma, \iota, \kappa)$. A *morphism* between two papillons $(E, \rho, \sigma, \iota, \kappa)$ and $(E', \rho', \sigma', \iota', \kappa')$ is an isomorphism $f \colon E \to E'$ commuting with all four maps. We define $\mathcal{M}(\mathfrak{H}, \mathfrak{G})$ to be the groupoid of papillons from $\mathfrak{H}$ to $\mathfrak{G}$.

Lemma 8.2. *In a papillon $(E, \rho, \sigma, \iota, \kappa)$, every element in the image of ι commutes with every element in the image of κ.*

Proof. Easy. □

The following theorem explains why we papillons are interesting objects.

Theorem 8.3. *There is a natural equivalence of groupoids, unique up to homotopy,*

$$\Omega \colon \underline{\mathcal{RH}om}_*(\mathfrak{H}, \mathfrak{G}) \to \mathcal{M}(\mathfrak{H}, \mathfrak{G}).$$

Proof. We begin with the following observation. Consider the product crossed-module $[H_2 \times G_2 \to H_1 \times G_1]$. Then, to give a papillon from $\mathfrak{H}$ to $\mathfrak{G}$ is equivalent to giving a triangle

$$\begin{tikzcd}
& H_2 \times G_2 \arrow[dl] \arrow[dr, "(\psi,\varphi)"] & \\
E \arrow[rr, "\rho"'] & & H_1 \times G_1
\end{tikzcd}$$

which has the property that ρ intertwines the action on $H_2 \times G_2$ of E (by conjugation) with the action of $H_1 \times G_1$ (from the crossed-module structure) and that the projection map $[H_2 \times G_2 \to E] \to [H_2 \to H_1]$ is an equivalence of crossed-modules. (For this we use Lemma 8.2.)

Let us now construct the functor $\Omega\colon \underline{\mathcal{RHom}}_*(\mathfrak{H}, \mathfrak{G}) \to \mathcal{M}(\mathfrak{H}, \mathfrak{G})$. Choose a cofibrant replacement $\mathfrak{F} = [F_2 \to F_1]$ for $\mathfrak{H}$. Then, $\underline{\mathcal{RHom}}_*(\mathfrak{H}, \mathfrak{G}) \cong \underline{\mathcal{H}om}_*(\mathfrak{F}, \mathfrak{G})$. So, we have to construct $\underline{\mathcal{H}om}_*(\mathfrak{F}, \mathfrak{G}) \to \mathcal{M}(\mathfrak{H}, \mathfrak{G})$. We use the push-out construction of Section 7.1. Namely, given a morphisms $P\colon \mathfrak{F} \to \mathfrak{G}$, we push-out $\mathfrak{H}$ along the map $F_2 \to H_2 \times G_2$ to obtain the crossed module $H_2 \times G_2 \to E$. This is defined to be $\Omega(P)$. This way we obtain a triangle

$$\begin{tikzcd} & H_2 \times G_2 \arrow[dl] \arrow[dr, "(\psi,\varphi)"] & \\ E \arrow[rr, "\rho"'] & & H_1 \times G_1 \end{tikzcd}$$

Lemma 7.7 implies that the induced map $\mathfrak{F} \to [H_2 \times G_2 \to E]$ is a weak equivalence of crossed modules. Therefore, $[H_2 \times G_2 \to E] \to [H_2 \to H_1]$ is also an equivalence of crossed-modules. Thus, we see that $\Omega(P) := [H_2 \times G_2 \to E]$ is really a papillon from $\mathfrak{H}$ to $\mathfrak{G}$. This defines the effect of Ω on objects. The effect on morphisms is defined in the obvious way (see Lemma 7.8).

We need to show that Ω is essentially surjective, full and faithful. Essential surjectivity follows from the fact that $\mathfrak{F}$ is cofibrant, and fullness follows from Lemma 7.5.**ii**. Let us prove the faithfulness. Let $\theta, \theta'\colon F_1 \to G_2$ be transformations between $P, Q\colon \mathfrak{F} \to \mathfrak{G}$ such that $\Omega(\theta) = \Omega(\theta')$. Define $\hat{\theta}\colon F_1 \to H_2 \times G_2$ by $\hat{\theta} = (1, \theta)$. Define $\hat{\theta}'\colon F_1 \to H_2 \times G_2$ in the similar way. By hypothesis, we have $\hat{\theta}^* = \hat{\theta'}^* : E_Q \to E_P$. By Lemma 7.5.**iii**, for every $x \in F_1$, the element $\hat{\theta}^{-1}(x)\hat{\theta}'(x)$ lies in the image of $\pi_2\mathfrak{F}$ under the map $F_2 \to H_2 \times G_2$. Since $\mathfrak{F} \to \mathfrak{H}$ induces an isomorphism on π_2, the only element in this image which is of the form $(1, \alpha)$ is $(1, 1)$. On the other hand, every element $\hat{\theta}^{-1}(x)\hat{\theta}'(x)$ is of the form $(1, \alpha)$. We conclude that $\hat{\theta}^{-1}(x)\hat{\theta}'(x) = (1, 1)$. Therefore, $\theta(x) = \theta'(x)$. This completes the proof. □

Remark 8.4. Roman Mikhailov has pointed out to me that there might be a direct proof of the theorem using the notion of weak map of crossed-modules ([No], Definition 8.4). By definition, a weak map from $[H_2 \to H_1]$ to $[G_2 \to G_1]$ consists of a triple (p_1, p_2, ε), where $p_1\colon H_1 \to G_1$ is a set map, $p_2\colon H_2 \to G_2$ is a group homomorphism, and $\varepsilon\colon H_1 \times H_1 \to G_2$ is a set map. The axioms these maps should satisfy are spelled out in *loc. cit.* Given such a triple, one defines the group E to have $H_1 \times G_2$ as underlying set, endowed with a product which is twisted by the 2-cocycle ε. More precisely,

$$(h, g) \cdot (h', g') := \big(hh', gg'^{p_1(h)}\varepsilon(h, h')\big).$$

The group homomorphisms $\rho\colon E \to G_1$ is defined by

$$\rho(h, g) = p_1(h)\underline{g}.$$

The reader can verify that this gives rise to a papillon, and that this construction takes a (pointed) transformation of weak maps of crossed-modules (*loc. cit* Definition 8.5) to a morphism of papillons.

Example 8.5. Let X be a topological space, and A an abelian group. We can use the above theorem to give a description of the second cohomology $H^2(X, A)$. Recall that this cohomology group is in bijection with the set of homotopy classes of maps $X \to K(A, 2)$, where $K(A, 2)$ stands for the Eilenberg-MacLane space. Since $K(A, 2)$ is simply connected, we can, equivalently, work with pointed homotopy classes. Assume the 2-type of X is represented by the crossed-module $\mathfrak{H} = [H_2 \to H_1]$. Then, there is a natural bijection $H^2(X, A) \cong [\mathfrak{H}, A]_{\mathbf{2Gp}}$, where we think of A as the crossed module $[A \to 1]$. From Theorem 8.3 we conclude that $H^2(X, A)$ is in natural bijection with the set of isomorphism classes of pairs (E, κ), where E is a central extension of H_1 by A, and $\kappa\colon H_2 \to E$ is an E-equivariant homomorphism. Here, E acts on itself by conjugation and on H_2 via the projection $E \to H_1$.

8.1. **Cohomological point of view.** We quote the following cohomological description of the set of pointed homotopy classes of weak maps from $\mathfrak{H}$ to $\mathfrak{G}$, that is, $[\mathfrak{H}, \mathfrak{G}]_{\mathbf{CrossedMod}}$ from [El] (which is apparently due to Joyal and Street; see *loc. cit.* §3.5).

Theorem 8.6 ([El], Theorem 3.14)**.** *Let $\mathfrak{G}$ and $\mathfrak{H}$ be crossed-modules, and assume they are represented by triples $(\pi_1\mathfrak{G}, \pi_2\mathfrak{G}, [\alpha])$ and $(\pi_1\mathfrak{H}, \pi_2\mathfrak{H}, [\beta])$, respectively, where α and β are 3-cocycles representing the corresponding Postnikov invariants (see Section 12). Then, there is a natural bijection between the set of pointed homotopy classes of weak maps from $\mathfrak{H}$ to $\mathfrak{G}$, that is, $[\mathfrak{H}, \mathfrak{G}]_{\mathbf{CrossedMod}}$, and the set of triples $(\chi, \lambda, [c])$, where $\chi\colon \pi_1\mathfrak{H} \to \pi_1\mathfrak{G}$ is a group homomorphism, $\lambda\colon \pi_2\mathfrak{H} \to \pi_2\mathfrak{G}$ is a χ-equivariant homomorphism such that $[\chi^*(\alpha)] = [\lambda_*(\beta)]$ in $H^3(\pi_1\mathfrak{H}, \pi_2\mathfrak{G}^\chi)$, and $[c]$ is the class, modulo coboundary, of a 2-cochain on $\pi_1\mathfrak{H}$ with values in $\pi_2\mathfrak{G}^\chi$, such that $\partial c = \chi^*(\alpha) - \lambda_*(\beta)$. Here, $\pi_2\mathfrak{G}^\chi$ stands for $\pi_2\mathfrak{G}$ made into a $\pi_1\mathfrak{H}$-module via χ.*

Presumably, with correct modifications of the arguments of Section 13, one can prove that the above classification is compatible with the classification in terms of papillons (Theorem 8.3), and also with the obstruction theoretic classification of pointed homotopy classes of maps $N\mathfrak{H} \to N\mathfrak{G}$. We will not, however, get into this.

8.2. **The induced map on homotopy groups.** By Theorem 8.3, a papillon $\mathcal{P} = (E, \rho, \sigma, \iota, \kappa)$ from $\mathfrak{H}$ to $\mathfrak{G}$ induces a well-defined map $N\mathfrak{H} \to N\mathfrak{G}$ in the *homotopy category* of simplicial sets, which should be thought of as the "nerve of the weak map $\mathcal{P}$". Indeed, any choice of a set-theoretic section s for the map $\sigma\colon E \to H_1$ naturally gives rise a simplicial map $N_s\mathcal{P}\colon N\mathfrak{H} \to N\mathfrak{G}$. Furthermore, if s' is another choice of a section, then there is a natural simplicial homotopy between $N_s\mathcal{P}$ and $N_{s'}\mathcal{P}$.

In particular, a papillon $\mathcal{P}$ induces natural homomorphisms on homotopy groups π_1 and π_2. We can in fact describe these maps quite explicitly.

To define $\pi_1\mathcal{P}\colon \pi_1\mathfrak{H} \to \pi_1\mathfrak{G}$, let x be an element in $\pi_1\mathfrak{H}$ and choose $y \in E$ such that $\sigma(y) = x$. Define $\pi_1\mathcal{P}(x)$ to be the class of $\rho(y)$ in $\pi_1\mathfrak{G}$. This is easily seen to be well-defined. To define $\pi_2\mathcal{P}\colon \pi_2\mathfrak{H} \to \pi_2\mathfrak{G}$, let β be an element in $\pi_2\mathfrak{H}$ and consider $y = \kappa(\beta) \in E$. Since $\sigma(y) = \psi(\beta) = 1$, there exists a unique $\alpha \in G_2$ such that $\iota(\alpha) = y$. We define $\pi_2\mathcal{P}(\beta)$ to be α. Note that $\varphi(\alpha) = \rho(y) = 1$, so α is indeed in $\pi_2\mathfrak{G}$.

8.3. **Homotopy fiber of a papillon.** Let $\mathcal{P}\colon (E, \rho, \sigma, \iota, \kappa)$ be a papillon from $\mathfrak{H}$ to $\mathfrak{G}$ as above. We define its *homotopy fiber* $\mathfrak{F} = \mathfrak{F}_{\mathcal{P}}$ to be the following 2-groupoid.

The set of objects of $\mathfrak{F}$ is G_1. Given $g, g' \in G_1$, the set of 1-morphisms in $\mathfrak{F}$ from g to g' is the set of all $x \in E$ such that $g\rho(x) = g'$. For every two such $x, y \in E$, the set of 2-morphisms from x to y is the set of all $\gamma \in H_2$ such that $x\kappa(\gamma) = y$. We depict this 2-cell by

$$\begin{tikzcd} g \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow\gamma"] & g'. \end{tikzcd}$$

It is clear how to define compositions rules in $\mathfrak{F}$, except perhaps for horizontal composition of 2-morphisms. Consider two 2-morphisms

$$\begin{tikzcd} g \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow\gamma"] & g' \arrow[r, bend left, "z"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\delta"] & g''. \end{tikzcd}$$

We define their composition to be

$$\begin{tikzcd} g \arrow[r, bend left, "xz"] \arrow[r, bend right, "yt"'] \arrow[r, phantom, "\Downarrow\gamma^{\sigma(z)}\delta"] & g''. \end{tikzcd}$$

There is a natural (strict) morphism of 2-groupoids $\Phi\colon \mathfrak{F} \to \mathfrak{H}$. To describe this map, we need to think of $\mathfrak{H}$ as a 2-groupoid with one object, as in Section 3.3. We recall how this works. The unique object of $\mathfrak{H}$ is denoted $\bullet$. The set of 1-morphisms of $\mathfrak{H}$ is H_1. Given two 1-morphisms $h, h' \in H_1$, the set of 2-morphisms from h to h' is the set of all $\gamma \in H_2$ such that $h\psi(\gamma) = h'$.

The natural map $\Phi\colon \mathfrak{F} \to \mathfrak{H}$ is described as follows:

$$\begin{tikzcd} g \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow\gamma"] & g' & \mapsto & \bullet \arrow[r, bend left, "\sigma(x)"] \arrow[r, bend right, "\sigma(y)"'] \arrow[r, phantom, "\Downarrow\gamma"] & \bullet. \end{tikzcd}$$

Theorem 8.7. *The sequence $N\mathfrak{F} \xrightarrow{N\Phi} N\mathfrak{H} \xrightarrow{N\mathcal{P}} N\mathfrak{G}$, which is well-defined in the homotopy category of simplicial sets, is a homotopy fiber sequence.*

In order to prove Theorem 8.7, we recall a few facts about homotopy fibers in **2Gpd**. Given a strict morphism $P\colon \mathfrak{H} \to \mathfrak{G}$ of 2-groupoids, and a base point $\bullet$ in $\mathfrak{G}$, there is a standard model for the homotopy fiber of P which is given by the following strict fiber product:

$$\mathrm{Fib}_P := * \times_{\bullet,\mathfrak{G},s} \mathfrak{G}^I \times_{t,\mathfrak{G},P} \mathfrak{H}.$$

Here $\mathfrak{G}^I := \underline{\mathcal{H}om}(0 \to 1, \mathfrak{G})$ is the 2-groupoid of 1-morphisms of $\mathfrak{G}$, and $s, t\colon \mathfrak{G}^I \to \mathfrak{G}$ are the source and target functors. (For a more precise definition of $\mathfrak{G}^I$ see Definition 14.1.)

In the case where $\mathfrak{G}$ and $\mathfrak{H}$ are 2-groups associated to crossed-modules $[G_2 \to G_1]$ and $[H_2 \to H_1]$, the 2-groupoid Fib_P is described more explicitly as follows:

- Objects are elements of G_1.

- 1-morphisms from g to g' are pairs $(x, \alpha) \in H_1 \times G_2$, as in the 2-cell

$$\begin{tikzcd} \bullet \arrow[rr, "p_1(x)"] & \arrow[d, phantom, "\alpha"] & \bullet \\ & \bullet \arrow[ul, "g"] \arrow[ur, "g'"'] & \end{tikzcd}$$

  This means, $gp_1(x)\underline{\alpha} = g'$.
- 2-morphisms from (x, α) to (y, β) are elements $\gamma \in H_2$ making the following diagram commutative:

$$\begin{tikzcd} \bullet \arrow[rr, bend left, "p_1(y)"] \arrow[rr, bend right, "p_1(x)"'] \arrow[rr, phantom, "p_2(\gamma)\quad \beta"] & & \bullet \\ & \bullet \arrow[ul, "g"] \arrow[ur, "g'"'] \arrow[u, phantom, "\alpha"] & \end{tikzcd}$$

  More precisely, we want $x\underline{\gamma} = y$ and $p_2(\gamma)\beta = \alpha$.

Remark 8.8. Observe that, in the above situation, Fib_P is naturally a pointed 2-groupoid, the base point being the identity element of G_1. It is easy to check that the 2-group of automorphisms of the base point is naturally isomorphic to the 2-group associated to $\ker P$; see Definition 5.2.

There is a natural projection functor $pr\colon \mathrm{Fib}_P = * \times_{\bullet,\mathfrak{G},s} \mathfrak{G}^I \times_{t,\mathfrak{G},f} \mathfrak{H} \to \mathfrak{H}$ which fits in the following fiber homotopy sequence:

$$\mathrm{Fib}_P \xrightarrow{pr} \mathfrak{H} \xrightarrow{P} \mathfrak{G}.$$

In the case where the map $p_2\colon H_2 \to G_2$ is surjective, there is a smaller model for the homotopy fiber of P which we now describe. Let Fib'_P be the full sub-2-category of Fib_P in which for 1-morphisms we only take the ones for which α is the identity. It is easily checked that, in this case, the inclusion $\mathrm{Fib}'_P \subset \mathrm{Fib}_P$ is an equivalence of 2-groupoids.

Let us record this as a lemma.

Lemma 8.9. *Let $\mathfrak{G} = [G_2 \to G_1]$ and $\mathfrak{H} = [H_2 \to H_1]$ be crossed-modules, viewed as 2-groups. Let $P\colon \mathfrak{H} \to \mathfrak{G}$ be a strict morphism of 2-groups, and let Fib'_P be the 2-groupoid defined in the previous paragraph. If $p_2\colon H_2 \to G_2$ is surjective, then the sequence*

$$\mathrm{Fib}'_P \xrightarrow{pr} \mathfrak{H} \xrightarrow{P} \mathfrak{G}$$

is a homotopy fiber sequence in the category of 2-groupoids.

We can now prove Theorem 8.7.

Proof of Theorem 8.7. We will apply Lemma 8.9 to the morphism of crossed-modules $P\colon [H_2 \times G_2 \to E] \to [G_2 \to G_1]$. Observe that the crossed-module on the left is a model for $\mathfrak{H}$ (via the natural weak equivalence $(pr_1, \sigma)\colon [H_2 \times G_2 \to E] \to [H_2 \to H_1]$), so P is a strict model for $\mathcal{P}$.

It is easily verified that the 2-groupoid $\mathfrak{F}$ of the theorem is canonically *isomorphic* to the 2-groupoid Fib'_P of Lemma 8.9. So, the homotopy fiber sequence of

Lemma 8.9 is naturally isomorphic, in the homotopy category of 2-groupoids, to the sequence

$$\mathfrak{F} \xrightarrow{\Phi} \mathfrak{H} \xrightarrow{\mathcal{P}} \mathfrak{G}.$$

Taking the nerves gives the homotopy fiber sequence of Theorem 8.7. ☐

We now derive some corollaries of Theorem 8.7.

Proposition 8.10. *Let $\mathcal{P}\colon (E,\rho,\sigma,\iota,\kappa)$ be a papillon from $\mathfrak{H}$ to $\mathfrak{G}$. Consider the nerve $N\mathcal{P}\colon N\mathfrak{H} \to N\mathfrak{G}$, which is well-defined in the homotopy category of simplicial sets, and let F be its homotopy fiber (note that F is naturally pointed). Let*

$$C: \quad H_2 \xrightarrow{\kappa} E \xrightarrow{\rho} G_1$$

be the NW-SE sequence of $\mathcal{P}$. Then, there are natural isomorphisms $H_i(C) \cong \pi_i F$, $i = 0,1,2$. (Of course, for $i = 0$ this means an isomorphism of pointed sets.)

Proof. By Theorem 8.7, F is naturally homotopy equivalent to $N\mathfrak{F}$. It is easily verified that $H_i(C) \cong \pi_i \mathfrak{F}$, $i = 0,1,2$. The result follows. ☐

Remark 8.11. Note that the truncated sequence $H_2 \to E$ is indeed naturally a crossed-module (take the action of E on H_2 induced via σ). The 2-groupoid $\mathfrak{F}$ of Theorem 8.7 can be recovered from the sequence C together with this extra structure.

Corollary 8.12. *Notation being as in Proposition 8.10, there is a long exact sequence*

$$1 \to H_2(C) \to \pi_2\mathfrak{H} \to \pi_2\mathfrak{G} \to H_1(C) \to \pi_1\mathfrak{H} \to \pi_1\mathfrak{G} \to H_0(C) \to 1.$$

Proof. Immediate. ☐

Proposition 8.13. *A papillon $(E,\rho,\sigma,\iota,\kappa)$ from $\mathfrak{H}$ to $\mathfrak{G}$ is a weak equivalence (i.e. induces a homotopy equivalence on the nerves), if and only if the NW-SE sequence*

$$H_2 \xrightarrow{\kappa} E \xrightarrow{\rho} G_1$$

is short exact. A weak inverse for this papillon in obtained by simply flipping the papillon, as in the following diagram

$$\begin{tikzcd}
G_2 \arrow[dd, "\varphi"'] \arrow[dr, "\iota"] & & H_2 \arrow[dd, "\psi"] \arrow[dl, "\kappa"'] \\
& E \arrow[dl, "\rho"] \arrow[dr, "\sigma"'] & \\
G_1 & & H_1
\end{tikzcd}$$

Proof. Immediate. ☐

8.4. **Weak morphisms versus strict morphisms.** We saw in the Theorem 8.3 that $\mathcal{M}(\mathfrak{H},\mathfrak{G})$ is a model for the groupoid of weak morphism from $\mathfrak{H}$ to $\mathfrak{G}$. Strict morphisms $\mathfrak{H} \to \mathfrak{G}$ form a subgroupoid of $\mathcal{M}(\mathfrak{H},\mathfrak{G})$. The objects of this subgroupoid are precisely the papillons for which the NE-SW short exact sequence is split.

More precisely, given a strict morphisms $P\colon \mathfrak{H} \to \mathfrak{G}$, the corresponding papillon looks as follows:

$$\begin{tikzcd}
H_2 \arrow[dd, "\psi"'] \arrow[dr, "\kappa"] & & G_2 \arrow[dl, "\iota"'] \arrow[dd, "\varphi"] \\
& H_1 \ltimes G_2 \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
H_1 & & G_1
\end{tikzcd}$$

where $\iota = (1, \mathrm{id})$, $\sigma = pr_1$, $\kappa(\beta) = \big(\psi(\beta), p_2(\beta^{-1})\big)$, and $\rho(x, \alpha) = p_1(x)\varphi(\alpha)$. Here, the action of H_1 on G_2 is obtained via p_1 from that of G_1 on G_2.

Observe that, any papillon coming from a strict morphism has a canonical splitting, and any pointed transformation between such strict transformations preserves this splitting. So, the groupoid $\underline{\mathcal{H}om}_*(\mathfrak{H}, \mathfrak{G})$ is naturally equivalent to the groupoid of pairs $(\mathcal{P}, s)$, where $\mathcal{P}$ is a split papillon, and s is a choice of splitting for it (the morphisms in the latter groupoid are required to preserve splitting). Under this identification, the functor $\underline{\mathcal{H}om}_*(\mathfrak{H}, \mathfrak{G}) \to \mathcal{M}(\mathfrak{H}, \mathfrak{G})$ is simply the forgetful functor that forgets the splitting.

8.5. **Bicategory of crossed-modules and weak maps.** Theorem 8.3 enables us to give a concrete model for the 2-category of crossed-modules, weak morphisms and weak transformation. More precisely, define $\mathcal{CM}$ to be the bicategory whose objects are crossed-modules and whose morphism groupoids are $\mathcal{M}(\mathfrak{H}, \mathfrak{G})$. The composition functors $\mathcal{M}(\mathfrak{K}, \mathfrak{H}) \times \mathcal{M}(\mathfrak{H}, \mathfrak{G}) \to \mathcal{M}(\mathfrak{K}, \mathfrak{G})$ are defined as follows. Given papillons

$$\begin{tikzcd}
K_2 \arrow[dd, "\xi"'] \arrow[dr] & & H_2 \arrow[dl, "\iota'"'] \arrow[dd, "\psi"] & H_2 \arrow[dd, "\psi"'] \arrow[dr, "\kappa"] & & G_2 \arrow[dl] \arrow[dd, "\varphi"] \\
& F \arrow[dl] \arrow[dr, "\rho'"'] & & & E \arrow[dl, "\sigma"] \arrow[dr] & \\
K_1 & & H_1 & H_1 & & G_1
\end{tikzcd}$$

we define their composite to be the papillon

$$\begin{tikzcd}
K_2 \arrow[dd, "\xi"'] \arrow[dr] & & G_2 \arrow[dl] \arrow[dd, "\varphi"] \\
& F \overset{H_2}{\underset{H_1}{\times}} E \arrow[dl] \arrow[dr] & \\
K_1 & & G_1
\end{tikzcd}$$

where $F \overset{H_2}{\underset{H_1}{\times}} E$ is the quotient of the group L of pairs $(y, x) \in F \times E$ such that $\rho'(y) = \sigma(x) \in H_1$, modulo the subgroup $I = \{\big(\iota'(\beta), \kappa(\beta)\big) \in F \times E \mid \beta \in H_2\}$. (This is a variant of Definition 11.3.)

The following theorem follows immediately.

Theorem 8.14. *With morphism groupoids being $\mathcal{M}(\mathfrak{H}, \mathfrak{G})$, crossed-modules form a bicategory $\mathcal{CM}$. There is a natural weak functor* **CrossedMod** $\to \mathcal{CM}$ *that is a bijection on objects and is faithful on morphism groupoids.*

Remark 8.15. The bicategory $\mathcal{CM}$ is a model for (i.e., is naturally biequivalent to) the 2-category of 2-groups and *weak* morphisms between them; see [No], especially Propositions 8.1 and 7.8. More precisely, the latter is the 2-category whose objects are 2-groups, whose 1-morphisms are weak morphisms between 2-groups, and whose 2-morphisms are pointed transformations between them.

The advantage of working with $\mathcal{CM}$ is that 1-morphisms and 2-morphisms in $\mathcal{CM}$ are quite explicit and in order to describe mapping groupoids one does not need to get bugged down with things like coherence conditions (say, the ones coming from the hexagon axiom) or complicated cocycles, which in practice make explicit computations intractable.

9. Interlude: the derived category of complexes of length two

Let A be an abelian category. The results of the previous section are also valid for the category of complexes of length 2 in A. More precisely, let $\mathcal{C}_2(\mathsf{A})$ be the bicategory whose objects are complexes $X^{-1} \to X^0$, whose morphisms are abelian[3] papillons

$$\begin{tikzcd}
X^{-1} \arrow[dd, "\psi"'] \arrow[dr, "\kappa"] & & Y^{-1} \arrow[dd, "\varphi"] \arrow[dl, "\iota"'] \\
& E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
X^0 & & Y^0
\end{tikzcd}$$

and whose 2-morphisms are morphisms (necessarily invertible) of papillons. The composition of papillons is defined as in the case of usual papillons (Section 8.5), and the criterion for strictness of a papillon is also valid (Section 8.4). Let us describe the additive structure.

Given two morphisms P, P' in $\mathcal{C}_2(\mathsf{A})$

$$\begin{tikzcd}
X^{-1} \arrow[dd] \arrow[dr, "\kappa"] & & Y^{-1} \arrow[dd] \arrow[dl, "\iota"'] \\
& E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
X^0 & & Y^0
\end{tikzcd}
\qquad
\begin{tikzcd}
X^{-1} \arrow[dd] \arrow[dr, "\kappa'"] & & Y^{-1} \arrow[dd] \arrow[dl, "\iota'"'] \\
& E' \arrow[dl, "\sigma'"] \arrow[dr, "\rho'"'] & \\
X^0 & & Y^0
\end{tikzcd}$$

with the same source and target, we define $P + P'$ to be

$$\begin{tikzcd}
X^{-1} \arrow[dd] \arrow[dr, "{(\kappa,\kappa')}"] & & Y^{-1} \arrow[dd] \arrow[dl, "{(0,\iota)}"'] \\
& E \overset{Y^{-1}}{\underset{X^0}{\times}} E' \arrow[dl, "\sigma=\sigma'"] \arrow[dr, "\rho+\rho'"'] & \\
X^0 & & Y^0
\end{tikzcd}$$

[3]Abelian means that, since there are no actions, we are dropping the requirement for compatibility of actions.

We define $-P$ by

$$\begin{tikzcd} X^{-1} \arrow[d, "d"'] \arrow[dr, "\kappa"] & & Y^{-1} \arrow[dl, "-\iota"'] \arrow[d, "d"] \\ & E \arrow[dl, "\sigma"] \arrow[dr, "-\rho"'] & \\ X^0 & & Y^0 \end{tikzcd}$$

The category obtained by identifying 2-isomorphic morphisms in $\mathcal{C}_2(\mathsf{A})$ is naturally equivalent to the full subcategory of the derived category $\mathcal{D}(\mathsf{A})$ consisting of complexes sitting in degrees $[-1,0]$. Under this equivalence, the diagonal sequence

$$X^{-1} \xrightarrow{\kappa} E \xrightarrow{\rho} Y^0,$$

viewed as a complex sitting in degrees $[-2,0]$, corresponds to the mapping cone.

10. A special case of Theorem 8.3

In this section we look at the special case of Theorem 8.3 with $\mathfrak{H} = [1 \to \Gamma]$, where Γ is a group. The groupoid $\mathcal{M}(\Gamma, \mathfrak{G})$ looks as follows. The object of $\mathcal{M}(\Gamma, \mathfrak{G})$ are diagrams of the form

$$\begin{tikzcd} & G_2 \arrow[dl] \arrow[dr, "\varphi"] & \\ E \arrow[rr, "\rho"'] & & G_1 \end{tikzcd}$$

Here E is an extension of Γ by G_2, and for every $x \in E$ and $\alpha \in G_2$ we require that $\alpha^{\rho(x)} = x^{-1}\alpha x$. In fact, the short exact sequence

$$1 \to G_2 \to E \to \Gamma \to 1$$

is also part of the data, but we suppress it from the notation and denote such a diagram simply by (E, ρ).

A morphism in $\mathcal{M}(\Gamma, \mathfrak{G})$ from (E, ρ) to (E', ρ') is an isomorphism $f\colon E \to E'$ of extensions (so it induces identity on G_2 and Γ) such that $\rho = \rho' \circ f$.

Theorem 10.1. *The functor*

$$\Omega\colon \underline{\mathcal{RH}om}_*(\Gamma, \mathfrak{G}) \to \mathcal{M}(\Gamma, \mathfrak{G})$$

is an equivalence of groupoids.

Proof. Follows immediately from Theorem 8.3. □

Corollary 10.2 (Dedecker, [De])**.** *There is a bijection*

$$\pi_0\colon \operatorname{Hom}_{\mathrm{Ho}(\mathbf{2Gp})}(\Gamma, \mathfrak{G}) \xrightarrow{\sim} \pi_0\mathcal{M}(\Gamma, \mathfrak{G}).$$

In other words, the homotopy classes of weak maps from Γ to $\mathfrak{G}$ are in a natural bijection with isomorphism classes of diagrams of the form

$$\begin{tikzcd} & G_2 \arrow[dl] \arrow[dr, "\varphi"] & \\ E \arrow[rr, "\rho"'] & & G_1 \end{tikzcd}$$

where the E is an extension of Γ by G_2, and for every $x \in E$ and $\alpha \in G_2$ the equality $\alpha^{\rho(x)} = x^{-1}\alpha x$ is satisfied.

Remark 10.3. If we write the long exact sequence of Proposition 5.4 for the above situation, where P is now given by the triangle of Corollary 10.2 and $\mathfrak{H} = [G_2 \to E]$, we get

$$1 \to \pi_2\mathfrak{G} \to \pi_1 \operatorname{Ker} P \to \Gamma \to \pi_1\mathfrak{G} \to \operatorname{Coker} P \to 1.$$

Let $I \subseteq \Gamma$ be the image of $\pi_1 \operatorname{Ker}(P)$ under the map $\pi_1 \operatorname{Ker}(P) \to \Gamma$. This gives us the short exacts sequence

$$1 \longrightarrow \pi_2\mathfrak{G} \longrightarrow \pi_1 \operatorname{Ker}(P) \longrightarrow I \longrightarrow 1 .$$

This short exact sequence is identical to the sequence

$$1 \longrightarrow \pi_2\mathfrak{G} \longrightarrow \operatorname{Ker}\rho \longrightarrow I \longrightarrow 1 ,$$

where ρ is as in the lemma.

Example 10.4. Let K be a group. Let $\mathfrak{Aut}(K)$ be the crossed-module $[K \to \operatorname{Aut}(K)]$.[4] The groupoid of extensions of Γ by K is naturally equivalent to the groupoid $\mathcal{M}(\Gamma, \mathfrak{Aut}(K))$, which is itself naturally equivalent to the derived mapping groupoid $\underline{\mathcal{RH}om}_*(\Gamma, \mathfrak{Aut}(K))$, by Theorem 10.1. In particular, by Corollary 10.2, the set of homotopy classes of weak maps from Γ to $\mathfrak{Aut}(K)$ is in natural bijection with isomorphism classes of extensions of Γ by K.

Remark 10.5. The above example is identical to Theorem 2 (and Proposition 3) of [BBF] in the case where $\mathcal{G}$ (notation as in *loc. cit.*) has only one object. Observe also that, indeed, the general case of Theorem 2 (and also Proposition 3) of [BBF], with $\mathfrak{G}$ an arbitrary groupoid, is equivalent to this special case. To see this, note that we may assume $\mathcal{G}$ is connected. On the other hand, since both sides of the equality in Theorem 2 (and also Proposition 3) of [BBF] are functorial in $\mathcal{G}$, we may assume $\mathcal{G}$ has only one object.

Example 10.6. Let A be an abelian group, and consider the crossed-module $\mathfrak{A} = [A \to 1]$. The groupoid of central extensions of Γ by A is naturally equivalent to the groupoid $\mathcal{M}(\Gamma, \mathfrak{A})$, which is itself naturally equivalent to the derived mapping groupoid $\underline{\mathcal{RH}om}_*(\Gamma, \mathfrak{A})$, by Theorem 10.1. In particular, by Corollary 10.2, the set of homotopy classes of weak maps from Γ to $\mathfrak{A}$ is in natural bijection with isomorphism classes of central extensions of Γ by A, which is itself in natural bijection with $H^2(\Gamma, A)$.[5]

It is interesting to compute $\pi_0\underline{\mathcal{RH}om}_*(\Gamma, \mathfrak{A})$ straight from definition of the derived mapping groupoid (Definition 6.6). For this, we choose a presentation $F/R \cong \Gamma$ of Γ, where F is free. The crossed-module maps $[R \to F] \to [A \to 1]$ are precisely the group homomorphism $g \colon R \to A$ that are constant on the conjugacy classes (under the F-action) of elements in R. In other words, $g(x) = 1$ for every $x \in [F, R]$. Two such homomorphisms g and g' are homotopic, if there is a group homomorphism $h \colon F \to A$ such that $h|_R = g'g^{-1}$. So, $\pi_0\underline{\mathcal{RH}om}_*(\Gamma, \mathfrak{A})$ is in natural bijection with

$$\operatorname{Coker}\{\operatorname{Hom}(F, A) \to \operatorname{Hom}(R/[F, R], A)\}.$$

[4]The corresponding 2-group is the 2-group of self-equivalences of K, where K is viewed as a category with one object.

[5]This is of course not surprising, since $\mathcal{A}$ is an algebraic model for the Eilenberg-MacLane space $K(A, 2)$.

This is Hopf's famous formula for $H^2(\Gamma, A)$.

We end this section with a cute (and presumably well-known) application of Corollary 10.2.

A 2-group $\mathfrak{G}$ is called *split* if it is completely determined by $\pi_1\mathfrak{G}$, $\pi_2\mathfrak{G}$, and the action of the former on the latter. More precisely, $\mathfrak{G}$ is split if it is isomorphic in $\mathrm{Ho}(\mathbf{CrossedMod})$ to the crossed-module $[\varphi\colon \pi_2\mathfrak{G} \to \pi_1\mathfrak{G}]$, where φ is the trivial homomorphism. From the homotopical point of view, the following proposition is straightforward (see the beginning of Section 12). However, to give a purely algebraic proof seems to be tricky. We give a proof that makes use of Corollary 10.2.

Proposition 10.7. *Let $\mathfrak{G} = [G_2 \to G_1]$ be a crossed-module, and assume that the map $\mathfrak{G} \to \pi_1\mathfrak{G}$, viewed in $\mathrm{Ho}(\mathbf{CrossedMod})$, admits a section. Then $\mathfrak{G}$ is split.*

Proof. By Corollary 10.2, there exists an extension

$$1 \longrightarrow G_2 \longrightarrow E \xrightarrow{f} \pi_1\mathfrak{G} \longrightarrow 1$$

and a map $\rho\colon E \to G_1$ satisfying the conditions stated therein. Consider the semi-direct product $G_2 \ltimes \pi_2\mathfrak{G}$ where G_2 acts on $\pi_2\mathfrak{G}$ by conjugation. It fits in a crossed-module $\mathfrak{G}' = [G_2 \ltimes \pi_2\mathfrak{G} \to E]$ where the map $G_2 \times \pi_2\mathfrak{G} \to E$ is obtained from the first projection map, and the action of E on both factors is by conjugation. We have a homomorphism $\sigma\colon G_2 \ltimes \pi_2\mathfrak{G} \to G_2$ which on the first factor is just the identity and on the second factor is the inclusion map. It is easy to see that $(\sigma, \rho)\colon \mathfrak{G}' \to \mathfrak{G}$ is a crossed-module map; in fact, it is an equivalence of crossed-modules. On the other hand, $(pr_2, f)\colon [G_2 \ltimes \pi_2\mathfrak{G} \to E] \to [\pi_2\mathfrak{G} \to \pi_1\mathfrak{G}]$ is also an equivalence of crossed-modules. So we have constructed a zigzag of equivalences that connect $\mathfrak{G}$ to the trivial crossed-module $[\pi_2\mathfrak{G} \to \pi_1\mathfrak{G}]$. So $\mathfrak{G}$ is split. ∎

10.1. **Side note: extensions of a 2-group by a group.** Example 10.4 suggests that one may define the notion of *extension of a 2-group by a group* as follows. Let $\mathfrak{G}$ be a 2-group and K a group. Then we can define a *weak action of $\mathfrak{G}$ on K* (or and *extension of $\mathfrak{G}$ by K*) to be an object in the groupoid $\mathcal{M}(\mathfrak{G}, \mathfrak{Aut}(K))$. It follows from Theorem 8.3 that if $[G_2 \to G_1]$ is a crossed-module presentation for $\mathfrak{G}$, then the groupoid of extensions of $\mathfrak{G}$ by K is equivalent to the groupoid of papillons of the form

$$\begin{tikzcd}
G_2 \arrow[dd] \arrow[dr, "\kappa"] & & K \arrow[dl] \arrow[dd] \\
& E \arrow[dl, "\sigma"] \arrow[dr] & \\
G_1 & & \operatorname{Aut} K
\end{tikzcd}$$

Equivalently, an extension of $\mathfrak{G}$ by K consists of the following data:

- An extension (as groups) of G_1 by K

$$1 \longrightarrow K \longrightarrow E \xrightarrow{\sigma} G_1 \longrightarrow 1$$

- A group homomorphism $\kappa\colon G_2 \to E$ such that

$$\forall \alpha \in G_2 \ \forall x \in E, \quad \kappa(\alpha^{\sigma(x)}) = x^{-1}\kappa(\alpha)x.$$

We do not have any application for this idea and will not pursue it further.

11. Cohomological point of view

In this section we give a cohomological characterization of $\mathcal{M}(\Gamma, \mathfrak{G})$, Theorem 11.6, and compare it to Theorem 10.1. Theorem 11.6 is well-known[6], but the explicit way in which it relates to Theorem 10.1 is what we are interested in. Since this construction has been quoted in [BeNo], we feel obliged to include the precise account. We begin by recalling some standard facts about group extensions.

11.1. **Groups extensions and H^2; review.** We recall some basic facts about classification of group extensions via cohomological invariants. Our main reference is [Br].[7]

Let N and Γ be groups (not necessarily abelian). We would like to classify extensions

$$1 \to N \to E \to \Gamma \to 1,$$

up to isomorphism. First of all, notice that such an extension gives rise to a group homomorphism $\psi \colon \Gamma \to \mathrm{Out}(N)$. So we might as well fix ψ as part of the data, and classify extension which induce ψ. Denote the set of such extensions by $\mathcal{E}(\Gamma, N, \psi)$. Let C denote the center of N, made into a Γ-module through ψ. We have the following theorem.

Theorem 11.1 ([Br], Theorem 6.6)**.** *The set $\mathcal{E}(\Gamma, N, \psi)$ admits a natural free, transitive action by the abelian group $H^2(\Gamma, C)$. Hence, either $\mathcal{E}(\Gamma, N, \psi) = \emptyset$, or else there is a bijection $\mathcal{E}(\Gamma, N, \psi) \leftrightarrow H^2(\Gamma, C)$. This bijection depends on the choice of a particular element of $\mathcal{E}(\Gamma, N, \psi)$.*

Remark 11.2. If we are given a lift $\tilde{\psi} \colon \Gamma \to \mathrm{Aut}(N)$ of ψ, then we have a distinguished element in $\mathcal{E}(\Gamma, N, \psi)$, namely, the semi-direct product $N \rtimes \Gamma$. This is automatically the case if, for instance, N is abelian. Therefore, when N is abelian, we have a canonical bijection $\mathcal{E}(\Gamma, N, \psi) \leftrightarrow H^2(\Gamma, C)$.

The meaning of this theorem is that, given two elements E_0, E in $\mathcal{E}(\Gamma, N, \psi)$, one can produce their *difference* as an element in $H^2(\Gamma, C)$. Notice that C is now abelian, so, by Remark 11.2, every element in $H^2(\Gamma, C)$ gives rise to a canonical extension of Γ by C. Below we will explain how this extension can be explicitly constructed from E_0 and E.

Let us call a complex $M \to E \to \Gamma$ *semi-exact* if the left map is injective, the right map is surjective, and the kernel K of $E \to \Gamma$ is generated by M and $C_K(M)$ (the the centralizer of M in K). This last condition guarantees that there is a well-defined homomorphism $\psi \colon \Gamma \to \mathrm{Out}(M)$.

Assume we are given two semi-exact sequences

$$M \to E_0 \to \Gamma \quad , \quad M \to E \to \Gamma$$

such that $\psi_0 = \psi$. Define L to be the group of pairs $(x, y) \in E_0 \times E$ such that $\bar{x} = \bar{y} \in \Gamma$, and that conjugation by x and y induce the same automorphism of M. Observe that $I = \{(a, a) \in E_0 \times E \mid a \in M\}$ is a normal subgroup of L.

Definition 11.3. Notation being as above, define $E_0 \overset{M}{\underset{\Gamma}{\times}} E := L/I$.

[6] The diligent reader can dig it out of [AzCe], §5.

[7] Also see Example 10.4.

There is an obvious surjective homomorphism $E_0 \overset{M}{\underset{\Gamma}{\times}} E \to \Gamma$. It fits in a natural exact sequence

$$1 \to C \to C_{K_0}(M) \times C_K(M) \to E_0 \overset{M}{\underset{\Gamma}{\times}} E \to \Gamma \to 1,$$

where C is the center of M mapping diagonally to $C_K(M) \times C_{K'}(M)$. (This gives us two semi-exact sequences:

$$C_{K_0}(M) \to E_0 \overset{M}{\underset{\Gamma}{\times}} E \to \Gamma \quad , \quad C_K(M) \to E_0 \overset{M}{\underset{\Gamma}{\times}} E \to \Gamma$$

which are somehow mirror to each other.)

If the two semi-exact sequences that we started with were actually exact, we would have $C_K(M) = C_{K'}(M) = C$. So, by identifying C as the cokernel of the diagonal map $C \to C \times C$, we obtain the following exact sequence

$$1 \to C \to E_0 \overset{M}{\underset{\Gamma}{\times}} E \to \Gamma \to 1.$$

More explicitly, we define the map $C \to E_0 \overset{M}{\underset{\Gamma}{\times}} E$ by sending a to $(a, 1)$.

Definition 11.4. Let $E_0, E \in \mathcal{E}(\Gamma, N, \psi)$. Define the *difference* $D(E_0, E)$ to be the the sequence $1 \to C \to E_0 \overset{N}{\underset{\Gamma}{\times}} E \to \Gamma \to 1$ defined above.

Remark 11.5. Observe that $E_0 \overset{M}{\underset{\Gamma}{\times}} E$ is symmetric with respect to E_0 and E, but $D(E_0, E)$ is not. What determines the sign in the above construction is the map $C \to E_0 \overset{N}{\underset{\Gamma}{\times}} E$. So, if instead of $a \mapsto (a, 1)$ we used $a \mapsto (1, a)$ we would obtain $D(E, E_0)$, because in $C \to E_0 \overset{M}{\underset{\Gamma}{\times}} E$ the elements $(1, a)$ and $(a, 1)$ are inverse to each other.

Conversely, given $E_0 \in \mathcal{E}(\Gamma, N, \psi)$ and an extension $1 \to C \to H \to \Gamma \to 1$ (recall that C is the center of N), we can recover E as the difference $E := D(E_0, H)$. In other words, consider the group $E_0 \overset{C}{\underset{\Gamma}{\times}} H$. This contains $N = N \overset{C}{\underset{1}{\times}} C$ as a normal subgroup. The sequence

$$1 \to N \to E_0 \overset{C}{\underset{\Gamma}{\times}} H \to \Gamma \to 1$$

is the desired extension.

11.2. **Cohomological classification of maps into a 2-group.** In this subsection we prove the following cohomological classification of the homotopy classes of weak maps from a group Γ to a 2-group $\mathfrak{G}$. The result itself is well-known, but the way in which it relates to the classification theorem (Theorem 10.1) is what we are interested in.

Theorem 11.6. *Let $\mathfrak{G}$ be a 2-group, and let Γ be a discrete group. Fix a homomorphism $\chi \colon \Gamma \to \pi_1\mathfrak{G}$, and let $[\Gamma, \mathfrak{G}]^{\chi}_{\mathbf{2Gp}}$ be the set of homotopy classes of weak maps $\Gamma \to \mathfrak{G}$ inducing χ on π_1. Then, either $[\Gamma, \mathfrak{G}]^{\chi}_{\mathbf{2Gp}}$ is empty, or it is naturally a transitive $H^2(\Gamma, \pi_2\mathfrak{G})$-set. (Here, $\pi_2\mathfrak{G}$ is made into a Γ module via χ.)*

In Section 12 we see exactly when $[\Gamma, \mathfrak{G}]^{\chi}_{\mathbf{2Gp}}$ is non-empty (see Remark 12.1) .

Remark 11.7. If in the above proposition we take $\mathfrak{G} = \mathfrak{Aut}(N)$ we recover Theorem 11.1; see Example 10.4.

Conventions for this subsection. Throughout this section, we fix χ (and consequently an action of Γ on $\pi_2\mathfrak{G}$). We will think of $H^2(\Gamma, \pi_2\mathfrak{G})$ as the group of isomorphism classes of extensions of Γ by $\pi_2\mathfrak{G}$ for which the induced action of Γ on $\pi_2\mathfrak{G}$ is the one we have fixed. Whenever we talk about an extension of Γ by $\pi_2\mathfrak{G}$ we assume that this condition is satisfied.

Proof of Theorem 11.6. The construction of the transitive action of $H^2(\Gamma, \pi_2\mathfrak{G})$ on $[\Gamma, \mathfrak{G}]^{\chi}_{\mathbf{2Gp}}$ has the same flavor as the constructions of the previous subsection. To start, we discuss a generalized version of the difference construction introduced in the preceding subsection.

Assume $[\Gamma, \mathfrak{G}]^{\chi}_{\mathbf{2Gp}}$ is non-empty, and fix an element (E_0, ρ_0) in it as in the following diagram (see Theorem 10.1):

$$\begin{tikzcd} & G_2 \arrow[dl] \arrow[dr, "\varphi"] & \\ E_0 \arrow[rr, "\rho_0"'] & & G_1 \end{tikzcd}$$

Let (E, ρ) be another such diagram. Define the group L by

$$L = \{(x, y) \in E_0 \times E \mid \bar{x} = \bar{y},\ \rho_0(x) = \rho(y)\}.$$

There is a natural surjective group homomorphism $L \to \Gamma$ sending (x, y) to $\bar{x} = \bar{y}$. The kernel of this map is the following group:

$$\{(\alpha, \beta) \in G_2 \times G_2 \mid \alpha\beta^{-1} \in \pi_2\mathfrak{G}\} = I \times \pi_2\mathfrak{G}$$

where

$$I := \{(\beta, \beta) \in E_0 \times E \mid \beta \in G_2\},$$

and $\pi_2\mathfrak{G}$ is identified with the subgroup of elements of the form $(\alpha, 1)$, $\alpha \in \pi_2\mathfrak{G}$. It is easy to check that I is normal in L.

Definition 11.8. Define $E_0 \times_{\mathfrak{G}} E = L/I$. The map $\alpha \mapsto (\alpha, 1)$ identifies $\pi_2\mathfrak{G}$ with a normal subgroup of L with cokernel Γ. The extension

$$1 \to \pi_2\mathfrak{G} \to E_0 \times_{\mathfrak{G}} E \to \Gamma \to 1,$$

or its class in $H^2(\Gamma, \pi_2\mathfrak{G})$, is called the *difference* of (E_0, ρ_0) and (E, ρ) and is denoted by $D\big((E_0, \rho_0), (E, \rho)\big)$.[8]

We can also go backwards. Namely, given (E_0, ρ_0) and an extension

$$1 \to \pi_2\mathfrak{G} \to K \to \Gamma \to 1$$

define $E := E_0 \overset{\pi_2\mathfrak{G}}{\underset{\Gamma}{\times}} K$ (see Definition 11.3). The inclusion $G_2 \hookrightarrow E_0$ induces a natural homomorphism $G_2 \to E$ which identifies G_2 with a normal subgroup of E. The quotient is Γ. We have a natural map $\rho \colon E \to G_1$ defined by $\rho(x, a) = \rho_0(x)$. This is easily seen to be well-defined. Finally, it is easy to check that the action of

[8]Definition 11.4 is a special case of Definition 11.8 with $\mathfrak{G} = \mathfrak{Aut}(N)$; see Example 10.4.

E on G_2 induced via ρ is equal to the conjugation action of E on G_2. This gives us the desired diagram:

$$\begin{tikzcd} & G_2 \arrow[dl] \arrow[dr, "\varphi"] & \\ E \arrow[rr, "\rho"'] & & G_1 \end{tikzcd}$$

This completes the proof of Theorem 11.6. $\square$

An interesting special case is when $\chi\colon \Gamma \to \pi_1\mathfrak{G}$ can be lifted to $\tilde{\chi}\colon \Gamma \to G_1$. In the following corollary we fix such a lift.

Corollary 11.9. *With the hypothesis of the preceding paragraph, every class $f \in [\Gamma, \mathfrak{G}]^{\chi}_{\mathbf{2Gp}}$ is uniquely characterized by (the isomorphism class of) an extension*

$$1 \to \pi_2\mathfrak{G} \to K \to \Gamma \to 1.$$

More explicitly, given such an extension we obtain (E, ρ), where $E := K \ltimes^{\pi_2\mathfrak{G}} G_2$ (Definition 7.1), and $\rho(k, \alpha) := \tilde{\chi}(\bar{k})\underline{\alpha}$, for $(k, \alpha) \in E$. Here the action of K on G_2 is obtained via $\tilde{\chi}$ from that of G_1 on G_2.

Proof. Set $E_0 = \Gamma \ltimes G_2$, the action being obtained through $\tilde{\chi}$, and define $\rho_0\colon E_0 \to G_1$ by $\rho(x, \alpha) := \chi(x)\underline{\alpha}$. The result follows from the discussion leading to this lemma. (We also use the fact that $K \overset{\pi_2\mathfrak{G}}{\underset{\Gamma}{\times}} (\Gamma \ltimes G_2) = K \ltimes^{\pi_2\mathfrak{G}} G_2$.) $\square$

An important special case of the above corollary is when there exists a section for the map $G_1 \to \pi_1\mathfrak{G}$. In this case, after fixing such a section, we have an explicit description of the elements in $[\Gamma, \mathfrak{G}]_{\mathbf{2Gp}}$, for any group Γ.

12. Homotopical point of view

The results of the previous sections are best understood if viewed from the point of homotopy theory of 2-types. Recall that 2-groups (respectively, weak functors between 2-groups, weak natural transformations) are algebraic models for pointed and connected homotopy 2-types (respectively, pointed continuous maps, pointed homotopies). Using this dictionary, the problem of classification of weak maps between 2-groups translates to the problem of classification of (pointed) homotopy classes of continuous maps between (pointed) homotopy 2-types. The latter can be solved using standard technique from obstruction theory.

In this section we explain how the homotopical approach works and give a homotopical proof of Theorem 11.6. In the next section we compare the homotopical approach with the approaches of the previous section. These all are presumably folklore and well-known.

12.1. **The classifying space functor and homotopical proof of Theorem 11.6.** Consider the *classifying space* functor $B\colon \mathrm{Ho}(\mathbf{2Gp}) \to \mathrm{Ho}(\mathbf{Top}_*)$ defined by $B(\mathfrak{G}) = |N(\mathfrak{G})|$. By Corollary 14.10 of Appendix, we know that this gives rise to an equivalence between the homotopy category of 2-groups and the homotopy category of pointed connected CW-complexes with vanishing π_i, $i \geq 3$ (the so called *connected homotopy 2-types*), a result essentially due to Whitehead [Wh]. So it would be most natural to view the algebraic results of the previous sections from this homotopic perspective. To complete the picture, we recall MacLane-Whitehead characterization of pointed connected homotopy 2-types from [McWh].

Mac Lane and Whitehead show that, to give a connected pointed homotopy 2-types is equivalent to giving a triple (π_1, π_2, κ) where π_1 is an arbitrary group, π_2 is an abelian group endowed with a π_1 action, and $\kappa \in H^3(\pi_1, \pi_2)$. To see where such a triple comes from, let X be a pointed connected CW-complex such that $\pi_i X = 0$, $i \geq 3$. Consider the Postnikov decomposition of X:

$$\begin{array}{rl} K(\pi_2, 2) - & X_2 \\ & \downarrow \\ & X_1 \;\; = K(\pi_1, 1) \end{array}$$

The triple corresponding to X is $(\pi_1 X, \pi_2 X, \kappa)$, where κ is the Postnikov invariant corresponding to the above picture. Recall that this Postnikov invariant is the obstruction to existence of a section for the fibration $p\colon X_2 \to X_1$. We know from obstruction theory that, if this obstruction vanishes, then for any choice of base points $x_1 \in X_1$ and $x_2 \in p^{-1}(x_1)$, there exist a pointed section $X_1 \to X_2$. Furthermore, after fixing such a section, the pointed homotopy classes[9] of such section are in bijection with $H^2(\pi_1 X, \pi_2 X)$.

Homotopical proof of Theorem 11.6. Passing to classifying space induces a bijection between $[\Gamma, \mathfrak{G}]_{\mathbf{2Gp}}$ and pointed homotopy class of maps $B\Gamma \to B\mathfrak{G}$. Let

$$\begin{array}{rl} K(\pi_2\mathfrak{G}, 2) - & X_2 \\ & \downarrow \\ & X_1 \;\; = K(\pi_1\mathfrak{G}, 1) \end{array}$$

be the Postnikov tower of $B\mathfrak{G}$. To give a group homomorphism $\chi\colon \Gamma \to \pi_1\mathfrak{G}$ is equivalent to giving a pointed homotopy class from $B\Gamma$ to X_1. Fix such a class $F\colon B\Gamma \to X_1$. The question is now to classify the pointed (equivalently, fiberwise – because the fiber is simply connected) homotopy classes of lifts $f\colon B\Gamma \to X_2$ of F. We know from obstruction theory that the obstruction to existence of such a lift is precisely $F^*(\kappa) = \chi^*(\kappa) \in H^3(\Gamma, \pi_2 X)$, where $\pi_2 X$ is made into a Γ-module via χ. By obstruction theory, whenever this obstruction vanishes, the set of pointed (equivalently, fiberwise) homotopy classes of lifts of such lifts f is a transitive $H^2(\Gamma, \pi_2 X)$-set. □

Remark 12.1. As we saw in the above proof, for a fixed $\chi\colon \Gamma \to \pi_1\mathfrak{G}$, the set $[\Gamma, \mathfrak{G}]^{\chi}_{\mathbf{2Gp}}$ is non-empty if and only if $\chi^*(\kappa) \in H^3(\Gamma, \pi_2\mathfrak{G})$ is zero, where κ is the Postnikov invariant of $\mathfrak{G}$.

Example 12.2. Let Γ and N be discrete groups. For a given $\chi\colon \Gamma \to \mathrm{Out}(N)$, the obstruction to lifting χ to a map $\Gamma \to \mathfrak{Aut}(N)$ (equivalently, to finding an extension of Γ by N giving giving rise to χ – see Example 10.4), is the element $\chi^*(\kappa) \in H^3(\Gamma, C(N))$, where $C(N)$ is the center of N and $\kappa \in H^3(\mathrm{Out}(N), C(N))$ is the Postnikov invariant of $\mathfrak{Aut}(N)$. (In other words, the Postnikov invariant κ of $\mathfrak{Aut}(N)$ is the universal obstruction class for the existence of group extensions.)

[9] We should actually be considering *fiberwise* homotopy classes, but since in our case the fiber is simply connected we get the same thing.

When $\chi^*(\kappa) = 0$, the set of lifts of χ to $\mathfrak{Aut}(N)$ (equivalently, extensions of Γ by N giving rise to χ) admits a natural transitive action of $H^2(\Gamma, C)$, where $C = \pi_2\mathfrak{Aut}(N)$ is the center of N.

For the sake of amusement, we also include a homotopical proof for Proposition 10.7.

Homotopical proof of Proposition 10.7. Let $\mathfrak{G}$ be as in Proposition 10.7. We have to show that the 2-type $B\mathfrak{G}$ is split. That is, it is homotopy equivalent to the product $K(\pi_1\mathfrak{G}, 1) \times K(\pi_2\mathfrak{G}, 2)$ of Eilenberg-MacLane spaces. We know that $\mathfrak{G}$ is classified by the triple $(\pi_1\mathfrak{G}, \pi_2\mathfrak{G}, \kappa)$. By assumption, the action of $\pi_1\mathfrak{G}$ on $\pi_2\mathfrak{G}$ is trivial. Also, by assumption, the corresponding Postnikov tower $X_2 \to X_1$ has a section, so κ vanishes. Since the 2-type $K(\pi_1\mathfrak{G}, 1) \times K(\pi_2\mathfrak{G}, 2)$ also gives rise to the same triple, it must be (pointed) homotopy equivalent to $B\mathfrak{G}$, which is what we wanted to prove. $\square$

13. Compatibility of different approaches

Let $p\colon Y \to X$ be a fibration of CW-complexes (or simplicial sets). In this section we recall the notion of *difference fibration* for two liftings of a map $F\colon A \to X$, and use it to clarify Definition 11.8, as well as to explain why the cohomological classification of maps into a 2-group (Section 11.2) is compatible with the homotopical approach (Section 12).

A more detailed discussion of the difference fibration construction, and its application to obstruction theory, can be found in [Ba].

13.1. **Difference fibration construction.** The difference constructions of §11.1 and §11.2 are special cases of (and were originally motivated by) a general difference construction for maps of simplicial sets (or topological spaces). In this section we review this general construction and explain how it relates to the algebraic versions of it that we have already encountered in previous sections.

Let X and Y be simplicial sets and $p\colon Y \to X$ a simplicial map. Let A be another simplicial set and $F\colon A \to X$ a map of simplicial sets. We are interested in the classification of lifts of f to Y, if such lifts exist. In our case of interest, A and X are going to be 1-types and Y a 2-type, so everything is explicit and easy.

A useful tool in the study of such lifting problems is the *difference construction*, as in ([Ba], page 293).[10]

Let $f_0, f\colon A \to Y$ be liftings of F. To measure the difference between f_0 and f, we construct the simplicial set $D_p(f_0, f)$ as in the following cartesian diagram:

$$\begin{tikzcd} D_p(f_0,f) \arrow[r] \arrow[d] & Y^{\Delta^1} \arrow[d, "{(d_0,d_1,p^{\Delta^1})}"] \\ A \arrow[r, "{(f_0,f,c)}"'] & Y \times Y \times X^{\Delta^1} \end{tikzcd}$$

Here, $c\colon A \to X^{\Delta^1}$ is the map that sends $a \in A$ to the constant path at $F(a)$. We are actually interested in the case where $p\colon Y \to X$ is a fibration.

[10]We will use the simplicial version of Baues's definition though.

The following lemma justifies the terminology, but since we will not need it here we will not give the proof (except in a special case – see Proposition 13.7). In the case of topological spaces, a proof is can be found in [Ba].

Lemma 13.1. *If $p\colon Y \to X$ is a fibration, then $D_p(f_0, f) \to A$ is also a fibration.*

The usefulness of the difference construction is justified by the following proposition.

Proposition 13.2.

- **i**. *The simplicial set of sections to the map $D_p(f_0, f) \to A$ is naturally isomorphic to the simplicial set of fiberwise homotopies between f_0 and f.*
- **ii**. *The primary difference of the sections f_0 and f is precisely the primary obstruction to the existence of a section to $D_p(f_0, f) \to A$.*

Proof. Part (**i**) follows from the definition. Part (**ii**) is proved in [Ba] (see page 295 *loc. cit.*) in the case of topological spaces. The proof can be adopted to the simplicial situation. □

When all spaces and maps are pointed, $D_p(f_0, f)$ is also naturally pointed. In this case we have:

Proposition 13.3.

- **i**. *The simplicial set of pointed sections to the map $D_p(f_0, f) \to A$ is naturally isomorphic to the simplicial set of fiberwise homotopies between f_0 and f which fix the base point.*
- **ii**. *The primary difference of the sections f_0 and f is precisely the primary obstruction to the existence of a section to $D_p(f_0, f) \to A$ (everything pointed).*

Difference fibration construction can indeed be performed in any category with fiber products in which there is an interval, and we have a notion of internal hom.[11] For instance, **Cat**, **2Cat**, and **2Gpd** are examples of such a category, where for the interval we take $\mathbf{I}_1 = \{0 \to 1\}$, and the internal homs are given by $\underline{\mathcal{H}om}$, as in Definition 14.1. So, given a diagram

$$\begin{tikzcd} & \mathfrak{D} \arrow[d, "p"] \\ \mathfrak{A} \arrow[ur, bend left, "f_0"] \arrow[ur, bend right, "f"'] \arrow[r, "F"'] & \mathfrak{C} \end{tikzcd}$$

of 2-categories, we can talk about the difference $D_p(f_0, f)$ of f_0 and f. This is a 2-category with a natural functor $D_p(f_0, f) \to A$. Furthermore, if $\mathfrak{A}$, $\mathfrak{C}$ and $\mathfrak{D}$ are pointed (i.e. have a chosen object), then so is $D_p(f_0, f)$. The following 2-categorical version of Proposition 13.2 is also valid.

Proposition 13.4. *Let $\mathfrak{A}$, $\mathfrak{C}$ and $\mathfrak{D}$ be 2-categories (respectively, pointed 2-categories) as above. Then the 2-category of sections (respectively, pointed sections) to the functor $D_p(f_0, f) \to A$ is naturally isomorphic to the 2-category of fiberwise transformations (respectively, pointed fiberwise transformations) between f_0 and f.*

[11]We are not asking for a monoidal structure here.

Clearly if $\mathfrak{A}$, $\mathfrak{C}$ and $\mathfrak{D}$ are 2-groupoids, then so is $D_p(f_0, f)$. So, we can talk about difference construction for 2-groupoids. It is also true that, if p is a fibration (Appendix, Definition 14.2), then so is $D_p(f_0, f) \to A$. The latter statement, whose simplicial counterpart we did not prove (Lemma 13.1), can be proved easily by verifying the conditions of Definition 14.2.

Definition 13.5. In the above situation, assume $\mathfrak{A}$, $\mathfrak{C}$ and $\mathfrak{D}$ are 2-groupoids with one object (i.e. 2-groups), and let $* \in \mathrm{Ob}\, D_p(f_0, f)$ be the canonical base point of $D_p(f_0, f)$. We define $D_p(f_0, f)_*$ to be the 2-group of automorphisms of the object $*$.

Remark 13.6. Proposition 13.4 remains valid when $\mathfrak{A}$, $\mathfrak{C}$ and $\mathfrak{D}$ are 2-groups, and $D_p(f_0, f)$ is replaced by $D_p(f_0, f)_*$. However, the natural functor $D_p(f_0, f)_* \to \mathfrak{A}$ is not in general a fibration anymore.

Finally, observe that the nerve functor $N\colon \mathbf{2Cat} \to \mathbf{SSet}$ respects the difference construction. That is, $ND_p(f_0, f)$ is naturally homotopy equivalent to $D_{Np}(Nf_0, Nf)$. This is because N preserves fiber products (Appendix, Proposition 14.5) and path spaces (Appendix, Proposition 14.11).

13.2. **Difference fibrations for crossed-modules.** We saw in the previous subsection that we can perform difference construction for 2-groups. We will make this more explicit using the language of crossed-modules. So assume $\mathfrak{F}$, $\mathfrak{G}$ and $\mathfrak{H}$ are crossed-modules, as in the following picture:

$$\begin{tikzcd} & \mathfrak{G} \arrow[d, "p"] \\ \mathfrak{H} \arrow[ur, bend left, "f'"] \arrow[ur, "f"'] \arrow[r] & \mathfrak{F} \end{tikzcd}$$

To avoid notational complications, we have used f' instead of f_0. In what follows, it would be helpful to think of elements of G_1 (respectively, G_2) as 1-cells (respectively, 2-cells) of the nerve $N\mathfrak{G}$ (see Appendix).

Let $[D_2 \to D_1]$ be the crossed-module presentation of $D_p(f', f)_*$. We will write down exactly what D_1 and D_2 are. By definition, we have

$$D_1 = \{(h, \alpha) \mid h \in H_1, \alpha \in \mathrm{Ker}\, p_2, \text{ s.t. } f_1'(h)\underline{\alpha} = f_1(h)\}.$$

It should be clear what this means: α is 2-cell that is vertical (because it is in $\mathrm{Ker}\, p_2$) and joins the 1-cells $f_1'(h)$ to $f_1(h)$. The group multiplication is

$$(h, \alpha)(k, \beta) = (hk, \alpha^{f_1'(k)}\beta).$$

To determine D_2, we have to pick a 2-cell $\beta \in H_2$, and find all pointed vertical homotopies from $f_2'(\beta)$ to $f_2(\beta)$. A pointed homotopy from $f_2'(\beta)$ to $f_2(\beta)$ is a 2-cell $\gamma \in G_2$ such that $f_2'(\beta)\gamma = f_2(\beta)$. To ensure it is vertical, we need to have $\gamma \in \mathrm{Ker}\, p_2$. This, however, is automatic, since $p_2 f_2'(\beta) = p_2 f_2(\beta)$. The conclusion is that, for any $\beta \in H_2$, there is a unique vertical homotopy from $f_2'(\beta)$ to $f_2(\beta)$; it is given by $\gamma = f_2'(\beta)^{-1} f_2(\beta)$. Therefore,

$$D_2 = H_2.$$

The map $D_2 \to D_1$ is given by

$$H_2 = D_2 \ni \ \beta \ \mapsto \ \big(\underline{\beta},\ f_2'(\beta)^{-1} f_2(\beta)\big) \ \in D_1.$$

An element $(h,\alpha) \in D_1$ acts on $\beta \in D_2 = H_2$ by sending it to β^h.

There is a natural map of crossed-modules $q\colon D_p(f',f)_* \to \mathfrak{H}$ given by $q_1(h,\alpha) = h$ and $q_2 = \mathrm{id}$.

13.3. **The special case.** We now consider the special case of the difference construction that is relevant to Theorem 11.6, and show that it recovers Definition 11.8 (see Proposition 13.7). Suppose we are given a group Γ, a 2-group $\mathfrak{G}$, and a homomorphism $\chi\colon \Gamma \to \pi_1\mathfrak{G}$. We want to study lifts of χ to maps $\Gamma \to \mathfrak{G}$. Keep in mind that here we are talking about maps in the *homotopy category* of 2-groups. So it would definitely be false to consider 2-group maps $\Gamma \to \mathfrak{G}$. One way to handle the situation is to work with *weak maps*. But this is not very convenient. The better way would be to pick a cofibrant replacement $\mathfrak{H}$ for Γ (see Example 6.5), and use the fact that a map $\Gamma \to \mathfrak{G}$ in the homotopy category of 2-groups can be represented by a map of 2-groups $\mathfrak{H} \to \mathfrak{G}$, and that the latter is unique up to pointed transformation.

Recall (Example 6.5) how we construct a cofibrant replacement for Γ: we choose a presentation $F/R = \Gamma$, where F is a free group, and form the crossed-module $\mathfrak{H} = [R \to F]$; the natural map $\mathfrak{H} \to \Gamma$ is then our cofibrant replacement. Throughout the paper, we fix such a cofibrant replacement.

Our problem is to study the difference construction for the following situation:

$$\begin{tikzcd} & \mathfrak{G} \arrow[d, "p"] \\ \mathfrak{H} \arrow[ur, bend left, "f'"] \arrow[ur, "f"'] \arrow[r, "\chi"'] & \pi_1\mathfrak{G} \end{tikzcd}$$

All maps are now honest maps of crossed-modules. (We have abused notation and denoted the induced map $\mathfrak{H} \to \pi_1\mathfrak{G}$ also by χ.) As we saw in the previous subsection, this picture gives a map of crossed-modules $D_p(f',f)_* \to \mathfrak{H}$, which is indeed a fibration of crossed-modules in the sense of Definition 6.1. Our aim is to compare this with Definition 11.8.

Let (E,ρ), respectively (E',ρ'), be the object of $\mathcal{M}(\Gamma,\mathfrak{G})$ obtained from pushing out $\mathfrak{H}$ along f, respectively f; see Definition 7.6). Recall (Definition 11.8) that the difference $D\big((E',\rho'),(E,\rho)\big)$ is defined to be the following exact sequence:

$$1 \to \pi_2\mathfrak{G} \to E' \times_{\mathfrak{G}} E \to \Gamma \to 1.$$

We prove the following

Proposition 13.7. *Notation being as above, the map $D_p(f',f)_* \to \mathfrak{H}$ is a fibration of crossed-modules and is naturally equivalent to $E' \times_{\mathfrak{G}} E \to \Gamma$. More precisely, we have the following commutative square in which the horizontal arrows are equivalences of crossed-modules and the vertical arrows are fibrations:*

$$\begin{tikzcd} D_p(f',f)_* \arrow[r, "\sim"] \arrow[d] & E' \times_{\mathfrak{G}} E \arrow[d] \\ \mathfrak{H} \arrow[r, "\sim"] & \Gamma \end{tikzcd}$$

Proof. We use the explicit description of the crossed-module $D_p(f',f)_* = [D_2 \to D_1]$ given in § 13.2:

$$D_1 = \{(x,\alpha) \mid x \in F,\ \alpha \in G_2,\ \text{s.t. } f_1'(x)\underline{\alpha} = f_1(x)\} \quad \text{and} \quad D_2 = R.$$

The map $D_2 \to D_1$ is given by $r \mapsto \big(\underline{r},\ f_2'(r)^{-1}f_2(r)\big)$, where $\underline{r}$ stands for the image of r in F. Notice that this is an injection, so we can identify D_2 with a subgroup of D_1. Let us now give an explicit description of $E' \times_{\mathfrak{G}} E$. Recall (Definition 7.6) that

$$E = F \ltimes^{R,f} G_2 \quad \text{and} \quad E' = F \ltimes^{R,f'} G_2.$$

Using this, we get

$$E' \times_{\mathfrak{G}} E = \big\{ \big((x,\alpha),(y,\beta)\big) \mid \bar{x} = \bar{y} \in \Gamma,\ f_1'(x)\underline{\alpha} = f_1(y)\underline{\beta} \in G_1 \big\} / J,$$

where $x,y \in F$, $\alpha,\beta \in G_2$, and J is the subgroup generated by

$$\big\{ \big((\underline{r}, f_2'(r)^{-1}),(1,1);\quad r \in R \big\}, \quad \big\{ \big((1,1),(\underline{s}, f_2(s)^{-1})\big);\quad s \in R \big\},$$
$$\text{and} \quad \big\{ \big((1,\gamma),(1,\gamma)\big);\ \gamma \in G_2 \big\}.$$

Therefore,

$$J = \big\{ \big((\underline{r}, f_2'(r)^{-1}\gamma),(\underline{s}, f_2(s)^{-1}\gamma)\big);\quad r,s \in R,\ \gamma \in G_2 \big\}.$$

Define the map $\Lambda\colon D_1 \to E' \times_{\mathfrak{G}} E$ by

$$\Lambda(x,\alpha) = \big((x,\alpha),(x,1)\big).$$

We claim that Λ is surjective and its kernel is $R = D_2$. This proves that the map $D_p(f',f)_* \to E' \times_{\mathfrak{G}} E$ is an equivalence of crossed-modules.

Surjectivity. Pick an element $a = \big((x,\alpha),(y,\beta)\big)$ in $E' \times_{\mathfrak{G}} E$. Since $\bar{x} = \bar{y}$, there is $s \in R$ such that $y = x\underline{s}$. Using the fact that

$$\big((x,\alpha),(x\underline{s},\beta)\big) = \big((x,\alpha),(x, f_2(s)\beta)\big)$$

in $E' \times_{\mathfrak{G}} E$, we may assume that $x = y$, that is, $a = \big((x,\alpha),(x,\beta)\big)$. On the other hand, after multiplying on the right by $\big((1,\beta^{-1}),(1,\beta^{-1})\big) \in J$, we may assume that $\beta = 1$; that is $a = \big((x,\alpha),(x,1)\big)$. This is obviously in the image of Λ.

Kernel of Λ *is* R. Easy verification.

To show that $D_p(f',f)_* \to \mathfrak{H}$ is a fibration, we have to show that the map $D_1 \to F$ which sends (x,α) to x, and the map $\mathrm{id}\colon D_2 = R \to R$ are surjective (Definition 6.1). The latter is obvious. The former follows from the fact that, for every $x \in F$, we have the equality $\overline{f_1(x)} = \overline{f_1'(x)}$ in $\pi_1\mathfrak{G}$. This is true because both these elements are equal to $\chi(x)$.

Commutativity of the square is obvious. □

14. Appendix: 2-categories and 2-groupoids

In this appendix we quickly go over some basic facts and constructions we need about 2-categories, and fix some terminology. Most of the material in this appendix can be found in [No].

For us, a *2-category* means a strict 2-category. A *2-groupoid* is a 2-category in which every 1-morphism and every 2-morphism has an inverse (in the strict sense). Every category (respectively, groupoid) can be thought of as a 2-category (respectively, 2-groupoid) in which all 2-morphisms are identity.

A *2-functor* between 2-categories means a strict 2-functor. We sometimes refer to a 2-functor simply by a functor, or a *map of 2-categories*. A 2-functor between 2-groupoids is simply a 2-functor between the underlying 2-categories.

By *fiber product* of 2-categories we mean strict fiber product. We will not encounter *homotopy* fiber product of 2-categories in this paper.

The terms 'morphism', '1-morphism' and 'arrow' will be used synonymously. We use multiplicative notation for elements of a groupoid, as opposed to the compositional notation (it means, fg instead of $g \circ f$).

Notation. We use the German letters $\mathfrak{C}$, $\mathfrak{D}$,... for general 2-categories and $\mathfrak{G}$, $\mathfrak{H}$,... for 2-groupoids. The upper case script letters A, B, C... are used for objects in such 2-categories, lower case script letters a, b, g, h... for 1-morphisms, and lower case Greek letters α, β... for 2-morphisms. We denote the category of 2-categories by **2Cat** and the category of 2-groupoids by **2Gpd**.

14.1. **2-functors, weak 2-transformations and modifications.** We recall what weak 2-transformations between strict 2-functors are. More details can be found in [No]. We usually suppress the adjective weak.

Let $P, Q \colon \mathfrak{D} \to \mathfrak{C}$ be (strict) 2-functors. By a *weak 2-transformation* $T \colon P \Rightarrow Q$ we mean, assignment of an arrow t_A in $\mathfrak{C}$ to every object A in $\mathfrak{D}$, and a 2-morphism θ_c in $\mathfrak{C}$ to every arrow c in $\mathfrak{D}$, as in the following diagram:

$$\begin{tikzcd} P(A) \arrow[r, "t_A"] \arrow[d, "P(c)"'] & Q(A) \arrow[d, "Q(c)"] \\ P(B) \arrow[r, "t_B"'] \arrow[ur, phantom, "\overset{\theta_c}{\nearrow}"] & Q(B) \end{tikzcd}$$

We require that $\theta_{\mathrm{id}} = \mathrm{id}$, and that θ_h satisfy the obvious compatibility conditions with respect to 2-morphisms and composition of morphisms.

A transformation between two weak transformations T, S, sometimes called a *modification*, is a rule to assign to each object $A \in \mathfrak{D}$ a 2-morphism μ_A in $\mathfrak{C}$ as in the following diagram:

$$\begin{tikzcd} P(A) \arrow[r, bend left, "t_A"] \arrow[r, bend right, "s_A"'] \arrow[r, phantom, "\Downarrow\mu_A"] & Q(A) \end{tikzcd}$$

The 2-morphisms μ_A should satisfy the obvious compatibility relations with θ_c and σ_c, for every arrow $c \colon A \to B$ in $\mathfrak{H}$. (Here σ_c are for S what θ_c are for T.) This relation can be written as $\theta_c \mu_A = \mu_B \sigma_c$.

Definition 14.1. Given 2-categories $\mathfrak{C}$ and $\mathfrak{D}$, we define the mapping 2-category $\underline{\mathcal{H}om}(\mathfrak{D}, \mathfrak{C})$ to be the 2-category whose objects are strict 2-functors from $\mathfrak{D}$ to $\mathfrak{C}$, whose 1-morphisms are weak 2-transformations between 2-functors, and whose 2-morphisms are modifications. When $\mathfrak{C}$ and $\mathfrak{D}$ are 2-groupoids, then $\underline{\mathcal{H}om}(\mathfrak{D}, \mathfrak{C})$ is also a 2-groupoid.

Viewing **2Gp** as a full subcategory of **2Gpd**, we can use the same notion for 2-groups as well. In fact, in the case of 2-groups, we are more interested in the pointed versions of the above definition. Namely, a *pointed* 2-transformation is required to satisfy the extra condition $t_* = \mathrm{id}$. A pointed modification is, by definition, the identity modification!

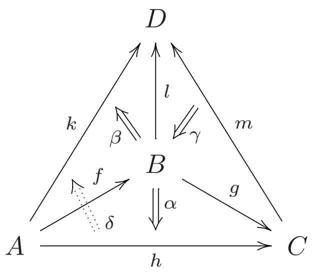
For 2-groups $\mathfrak{G}$ and $\mathfrak{H}$, we denote the 2-groupoid of *pointed* weak maps from $\mathfrak{H}$ to $\mathfrak{G}$ by $\underline{\mathcal{H}om}_*(\mathfrak{H}, \mathfrak{G})$.

14.2. **Nerve of a 2-category.** We review the nerve construction for 2-categories, and recall its basic properties [MoSe], [No].

Let $\mathfrak{G}$ be a 2-category . We define the *nerve* of $\mathfrak{G}$, denoted by $N\mathfrak{G}$, to be the simplicial set defined as follows. The set of of 0-simplices of $N\mathfrak{G}$ is the set of objects of $\mathfrak{G}$. The 1-simplices are the morphisms in $\mathfrak{G}$. The 2-simplices are diagrams of the form

$$\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd}$$

where $\alpha \colon fg \Rightarrow h$ is a 2-morphism. The 3-simplices of $N\mathfrak{G}$ are commutative tetrahedra of the form



Commutativity of the above tetrahedron means $(f\gamma)(\beta) = (\alpha m)(\delta)$. That is, the following square of transformations is commutative:

$$\begin{tikzcd} fgm \arrow[r, Rightarrow, "f\gamma"] \arrow[d, Rightarrow, "\alpha m"'] & fl \arrow[d, Rightarrow, "\beta"] \\ hm \arrow[r, Rightarrow, "\delta"'] & k \end{tikzcd}$$

For $n \geq 3$, an n-simplex of $N\mathfrak{G}$ is an n-simplex such that each of its sub 3-simplices is a commutative tetrahedron as described above. In other words, $N\mathfrak{G}$ is the coskeleton of the 3-truncated simplicial set $\{N\mathfrak{G}_0, N\mathfrak{G}_1, N\mathfrak{G}_2, N\mathfrak{G}_3\}$ defined above.

The nerve gives us a functor $N \colon \mathbf{2Cat} \to \mathbf{SSet}$, where $\mathbf{SSet}$ is the the category of simplicial sets.

14.3. **Moerdijk-Svensson closed model structure on 2-groupoids.** We quickly review the Moerdijk-Svensson closed model structure on the category of 2-groupoids. The main reference is [MoSe].

Definition 14.2. Let $\mathfrak{H}$ and $\mathfrak{G}$ be 2-groupoids, and $P \colon \mathfrak{H} \to \mathfrak{G}$ a functor between them. We say that P is a *fibration*, if it satisfies the following properties:

- **F1**. For every arrow $a \colon A_0 \to A_1$ in $\mathfrak{G}$, and every object B_1 in $\mathfrak{H}$ such that $P(B_1) = A_1$, there is an object B_0 in $\mathfrak{H}$ and an arrow $b \colon B_0 \to B_1$ such that $P(b) = a$.

F2. For every 2-morphism $\alpha\colon a_0 \Rightarrow a_1$ in $\mathfrak{G}$ and every arrow b_1 in $\mathfrak{H}$ such that $P(b_1) = a_1$, there is an arrow b_0 in $\mathfrak{H}$ and a 2-morphism $\beta\colon b_0 \Rightarrow b_1$ such that $P(\beta) = \alpha$.

Definition 14.3. Let $\mathfrak{G}$ be a 2-groupoid, and A an object in $\mathfrak{G}$. We define the following.

- $\pi_0\mathfrak{G}$ is the set of equivalence classes of objects in $\mathfrak{G}$.
- $\pi_1(\mathfrak{G}, A)$ is the group of 2-isomorphism classes of arrows from A to itself. The *fundamental groupoid* $\Pi_1\mathfrak{G}$ is the groupoid whose objects are the same as those of $\mathfrak{G}$ and whose morphisms are 2-isomorphism classes of 1-morphisms in $\mathfrak{G}$.
- $\pi_2(\mathfrak{G}, A)$ is the group of 2-automorphisms of the identity arrow $1_A\colon A \to A$.

These invariants are functorial with respect to 2-functors. A map $\mathfrak{H} \to \mathfrak{G}$ is called a *(weak) equivalence of 2-groupoids* if it induces a bijection on π_0, π_1 and π_2, for every choice of a base point.

Having defined the notions of fibration and equivalence between 2-groupoids, we define *cofibrations* using the left lifting property. There is a more explicit description of cofibrations which can be found in ([MoSe] page 194), but we skip it here.

Theorem 14.4 ([MoSe], Theorem 1.2)**.** *With weak equivalences, fibrations and cofibrations defined as above, the category of 2-groupoids has a natural structure of a closed model category.*

The nerve functor is a bridge between the homotopy theory of 2-groupoids and the homotopy theory of simplicial sets. To justify this statement, we quote the following from [MoSe].

Proposition 14.5 (see [MoSe], Proposition 2.1)**.**

i. *The functor $N\colon$ **2Cat** $\to$ **SSet** is faithful, preserves fiber products, and sends transformations between 2-functors to simplicial homotopies.*
ii. *The functor N sends a fibration between 2-groupoids (Definition 14.2) to a Kan fibration. Nerve of every 2-groupoid is a Kan complex.*
iii. *For every (pointed) 2-groupoid $\mathfrak{G}$ we have $\pi_i(\mathfrak{G}) \cong \pi_i(N\mathfrak{G})$, $i = 0, 1, 2$.*
iv. *A map $f\colon \mathfrak{H} \to \mathfrak{G}$ of 2-groupoids is an equivalence if and only if $Nf\colon N\mathfrak{H} \to N\mathfrak{G}$ is a weak equivalence of simplicial sets.*

Remark 14.6. We can think of a 2-group as a 2-groupoid with one object. This identifies **2Gp** with a full subcategory of **2Gpd**. So, we can talk about nerves of 2-groups. This is a functor $N\colon$ **2Gp** $\to$ **SSet**$_*$, where **SSet**$_*$ is the category of pointed simplicial sets. The above proposition remains valid if we replace 2-groupoids by 2-groups and **SSet** by **SSet**$_*$ throughout.

The functor $N\colon$ **2Gpd** $\to$ **SSet** has a left adjoint $W\colon$ **SSet** $\to$ **2Gpd**, called the *Whitehead 2-groupoid* whose definition can be found in ([MoSe] page 190, Example 2).

It is easy to see that W preserves homotopy groups. In particular, it sends weak equivalences of simplicial sets to equivalences of 2-groupoids. Much less obvious is the following

Theorem 14.7 ([MoSe], Section 2)**.** *The pair*

$$W\colon \mathbf{SSet} \rightleftharpoons \mathbf{2Gpd} : N$$

is a Quillen pair. It satisfies the following properties:

- **i**. *Each adjoint preserves weak equivalences.*
- **ii**. *For every 2-groupoid $\mathfrak{G}$, the counit $WN(\mathfrak{G}) \to \mathfrak{G}$ is a weak equivalence*
- **iii**. *For every simplicial set X such that $\pi_i X = 0$, $i \geq 3$, the unit of adjunction $X \to NW(X)$ is a weak equivalence.*

In particular, the functor $N\colon \mathrm{Ho}(\mathbf{2Gpd}) \to \mathrm{Ho}(\mathbf{SSet})$ induces an equivalence of categories between $\mathrm{Ho}(\mathbf{2Gpd})$ and the category of homotopy 2-types. (The latter is defined to be the full subcategory of $\mathrm{Ho}(\mathbf{SSet})$ consisting of all X such that $\pi_i X = 0$, $i \geq 3$.)

Remark 14.8. The pointed version of the above theorem is also valid. The proof is just a minor modification of the proof of the above theorem.

The following Proposition follows from Theorem 14.7 and Remark 14.8.

Proposition 14.9. *The functor $N\colon \mathrm{Ho}(\mathbf{2Gp}) \to \mathrm{Ho}(\mathbf{SSet}_*)$ induces an equivalence between $\mathrm{Ho}(\mathbf{2Gp})$ and the full subcategory of $\mathrm{Ho}(\mathbf{SSet}_*)$ consisting of connected pointed homotopy 2-types.*

It is also well-known that the geometric realization functor $|-|\colon \mathbf{SSet}_* \to \mathbf{Top}_*$ induces an equivalence of of homotopy categories. So we have the following

Corollary 14.10. *The functor $|N(-)|\colon \mathrm{Ho}(\mathbf{2Gp}) \to \mathrm{Ho}(\mathbf{Top}_*)$ induces an equivalence between $\mathrm{Ho}(\mathbf{2Gp})$ and the full subcategory of $\mathrm{Ho}(\mathbf{Top}_*)$ consisting of connected pointed homotopy 2-types.*

The following proposition says that derived mapping 2-groupoids have the correct homotopy type. We have denoted the category $\{0 \to 1\}$ by $\mathbf{I}$.

Proposition 14.11. *Let $\mathfrak{G}$ and $\mathfrak{H}$ be 2-groupoids. Then there is a natural homotopy equivalence*

$$N\underline{\mathcal{RH}om}(\mathfrak{H}, \mathfrak{G}) \simeq \mathbf{Hom}(N\mathfrak{H}, N\mathfrak{G}),$$

where the left hand side is defined to be $\underline{\mathcal{H}om}(\mathfrak{F}, \mathfrak{G})$, where $\mathfrak{F}$ is a cofibrant replacement for $\mathfrak{H}$, and the right hand side is the simplicial mapping space. In particular, $N\underline{\mathcal{RH}om}(\mathbf{I}, \mathfrak{G}) \simeq (N\mathfrak{G})^{\Delta^1}$, that is, the nerve of the path category is naturally homotopy equivalent to the path space of the nerve.

We also have the pointed version of the above proposition.

Proposition 14.12. *Let $\mathfrak{G}$ and $\mathfrak{H}$ be 2-groups. Then, there is a natural homotopy equivalence*

$$N\underline{\mathcal{RH}om}_*(\mathfrak{H}, \mathfrak{G}) \simeq \mathbf{Hom}_*(N\mathfrak{H}, N\mathfrak{G}).$$

Remark 14.13. In the definition of $\underline{\mathcal{H}om}$ we have used *strict* 2-functors as objects, but the arrows are *weak* 2-transformation. There is a variant of this in which the 2-functors are also weak, and this leads to different simplicial mapping spaces between 2-group(oid)s. If in Propositions 14.11 and 14.12 above we used this simplicial mapping space on the left hand side, we would not need to use a cofibrant replacement on $\mathfrak{H}$ to compute the derived mapping spaces. In other words, the derived mapping spaces would coincide with the actual mapping spaces.

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