The $GL(m|n)$ type quantum matrix algebras II: the structure of the characteristic subalgebra and its spectral parameterization

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Abstract

In our previous paper [GPS2] the Cayley-Hamilton identity for the $GL(m|n)$ type quantum matrix algebra was obtained. Here we continue investigation of that identity. We derive it in three alternative forms and, most importantly, we obtain it in a factorized form. The factorization leads to a separation of the spectra of the quantum supermatrix into the "even" and "odd" parts. The latter, in turn, allows us to parameterize the characteristic subalgebra (which can also be called the subalgebra of spectral invariants) in terms of the supersymmetric polynomials in the eigenvalues of the quantum supermatrix. For our derivation we use two auxiliary results which may be of independent interest. First, we calculate the multiplication rule for the linear basis of the Schur functions $s_\lambda(M)$ for the characteristic subalgebra of the Hecke type quantum matrix algebra. The structure constants in this basis are the Littlewood-Richardson coefficients. Second, we derive a series of bilinear relations in the graded ring Λ of Schur symmetric functions in countably many variables (see [Mac]).

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1 Introduction

In the present paper, we continue the investigation of the supersymmetric $GL(m|n)$ type quantum matrix (QM) algebras initiated in [GPS2]. Let us recall briefly the history of the subject.

The first examples of the QM algebras were considered in the seminal papers of V. Drinfel'd [D] and L. Faddeev, N. Reshetikhin and L. Takhtajan [RTF]. There, a particular family of QM algebras — the algebras of quantized functions on the groups, shortly called the RTT algebras, were defined. Soon after, another important subclass of QM algebras — the reflection equation (RE) algebras, were introduced into consideration (see, e.g., [KS, KSas]). The general definition of the QM algebras was found by L.Hlavaty who aimed at giving a unified description for RTT and RE algebras [Hl]. This idea might seem quite strange at first glance (the representation theories of the RTT and the RE algebras are very different). At the same time, the structure investigations carried out separately for the RTT [EOW, Zh, IOPS] and the RE algebras [NT, PS, GPS1] reveal a remarkable similarity of both the algebras to the classical matrix algebra. Namely, it turns out that the RE and the RTT families admit a noncommutative generalization of the Cayley-Hamilton theorem and for the matrices of generators in both the cases a noncommutative analogue of their spectra can be constructed. Having this in mind, the general definition of the QM algebras was independently reproduced in [IOP2] and the noncommutative version of the Cayley-Hamilton theorem was derived for the QM algebras of the general linear type (see [IOP1, IOP2, IOP3]).

The family of $GL(m)$ type QM algebras was a good case to start with. An investigation of the other classical series of the QM algebras falls into two cases — the case of the Hecke type QM algebras and the case of the Birman-Murakami-Wenzl (BMW) type QM algebras. The difference is in the choice of a quotient of the group algebra of the braid group which enters (through its R-matrix representation) into the QM algebra definition. The Hecke case contains the general linear type and its supersymmetric generalization — the $GL(m|n)$ type QM algebras. The BMW case includes orthogonal- and symplectic- type QM algebras and their supersymmetric analogues. An investigation of the BMW case was started in [OP2], where the Cayley-Hamilton identity and the spectra of the orthogonal- and symplectic- type QM algebras were identified.

The supersymmetric $GL(m|n)$ type QM algebra was studied in our previous paper [GPS2]. In that paper, we gave a proper definition of the family of the $GL(m|n)$ type QM algebras and proved the Cayley-Hamilton identity for them. Our work may be viewed as a generalization of both the results by I. Kantor and I. Trishin on the Cayley-Hamilton equation for the supermatrices [KT1, KT2] (the invariant Cayley-Hamilton equation in their terminology), and by P.D. Jarvis and H.S. Green on the characteristic identities for the general linear Lie superalgebras [JG].

Still lacking in the $GL(m|n)$ case is the identification of the spectrum of the quantum supermatrices.[1] Alternatively, one can ask for a proper parameterization of the characteristic subalgebra of the $GL(m|n)$ type QM algebra (the abelian subalgebra of the QM algebra which the coefficients of the Cayley-Hamilton identity belong to). This problem is addressed in the present work. First, we investigate in detail the structure of the characteristic subalgebra in the Hecke case. Then, we derive a series of bilinear relations in the graded ring Λ of Schur symmetric functions in countably many variables (for the definition see [Mac]). These combinatorial relations may be of independent interest.

The structure of the paper is as follows. In the next section, subsection 2.1, we derive the multiplication rule for the set of linear basic elements of the Hecke type characteristic subalgebra — the so-called Schur functions $s_\lambda(M)$ (the notation is explained below). The structure constants in this basis are just the Littlewood-Richardson coefficients. In other words, we define the homomorphic map from the ring of symmetric functions Λ to the characteristic subalgebra of the Hecke type QM algebra. To efficiently apply this map in the $GL(m|n)$ case, we need a series of bilinear relations for the Schur symmetric functions $s_\lambda \in \Lambda$.[2] They are proved in subsection 2.2. For derivation we use the Jacobi-Trudi formulas for the Schur functions s_λ and apply the Plücker relations. The same method was used in [LWZ, Kl] for the derivation of different bilinear relations for the Schur functions. We also remark that our bilinear relations certainly have common roots with the factorization formula for the supersymmetric functions [BR, PrT].

In section 3, we derive three alternative expressions for the Cayley-Hamilton identity for the $GL(m|n)$ type QM algebra. In subsection 3.1, the bilinear identities of subsection 2.2 are used to factorize the $GL(m|n)$ type characteristic identity into a product of two terms. Let us stress that the factorization is achieved without extending the algebra by the eigenvalues of the quantum supermatrix. To the best of our knowledge, this fact has not been observed before even in the classical supermatrix case. The factorization allows us to separate "even" and "odd" eigenvalues of the quantum supermatrix in a covariant manner. That is, we do not specify explicitly the $\mathbb{Z}_2$-grading for the components of the quantum supermatrix. Instead, we observe the "manifestation of even and odd variables" in the factorization property of the characteristic polynomial. Two more versions of the Cayley-Hamilton theorem are presented in subsection 3.2. They are given in terms of the (skew-)symmetric powers of the quantum matrices[3] and generalize the corresponding results of [KT2, T] to the case $q \neq 1$. Yet another series of bilinear relations for the Schur symmetric functions s_λ is used here for derivations (see lemma 6). These relations are also applied in the last section for parameterization of the Schur functions $s_\lambda(M)$.

Finally, in section 4, we compute expressions for the coefficients of the $GL(m|n)$ type Cayley-

[1]Here we put the problem for the generic QM algebra. For the subfamily of RE algebras and at the level of finite dimensional representations it was considered in [GL].

[2]There should be no confusion between the elements $s_\lambda \in \Lambda$ and their homomorphic images $s_\lambda(M)$ in the characteristic subalgebra. The argument in the latter notation is used for distinguishing purposes. It refers to the matrix of generators of the QM algebra.

[3]The notion of the skew-symmetric power of the matrix was suggested by A.M. Lopshits (see [GGB], p.342.)

Hamilton identity in terms of the quantum matrix eigenvalues. The resulting parameterization is given in terms of the supersymmetric polynomials [Stem] (see also [Mac], section 1.3, exercises 23 and 24). It is worth mentioning that the supersymmetric polynomials were originally introduced by F. Berezin [Ber] for a description of invariant polynomials on the Lie superalgebra $\mathfrak{gl}(m|n)$ (see also [Ser] and references therein).

Some auxiliary q-combinatorial formulae which we need for derivations in section 2.1 are proved in the appendix.

Throughout this text we keep the notation of the paper [GPS2]. When referring to formulae from that paper we use the shorthand quotation, e.g., symbol (I-3.21) refers to formula (21) from section 3 of [GPS2]. For reader's convenience in the rest of the introduction we collect a list of notation, definitions and results mainly from [GPS2].

Let V be a finite dimensional $\mathbb{C}$-linear space, $\dim V = N$. Consider a pair of elements $R, F \in \mathrm{Aut}(V^{\otimes 2})$. Fixing some basis $\{v_i\}_{i=1}^N$ in the space V we identify operators R and F with their matrices in that basis. We use the shorthand matrix notation of [RTF]. I.e., we write R_i (or, sometimes, more explicitly $R_{i\,i+1}$) for the matrix of the operator $\mathrm{Id}^{\otimes(i-1)} \otimes R \otimes \mathrm{Id}^{\otimes(k-i-1)}$ acting in the space $V^{\otimes k}$. Here $\mathrm{Id} \in \mathrm{Aut(V)}$ denotes the identity operator. The integer k is not shown in the matrix notation. In each particular formula the actual value of k can be easily reconstructed. Few more conventions: I is the identity matrix; $P \in \mathrm{Aut}(V^{\otimes 2})$ is the permutation automorphism ($P(u \otimes v) = v \otimes u$).

The pair of operators R and F can be used as an initial data set for the QM algebra, provided they satisfy the following conditions

i) The matrices of both operators R and F are *strict skew invertible*. The skew invertibility means, say for R, the existence of an operator $\Psi^R \in \mathrm{End}(V^{\otimes 2})$ such that $\mathrm{Tr}_{(2)} R_{12} \Psi^R_{23} = P_{13}$, where the subscript in the notation of the trace shows the number of the space V, where the trace is evaluated (here we adopt labelling $V^{\otimes k} := V_1 \otimes V_2 \otimes \ldots \otimes V_k$). The strictness condition implies invertibility of an element $D^R_1 := \mathrm{Tr}_{(2)} \Psi^R_{12}$: $D^R \in \mathrm{Aut}(V)$.

   With the matrix D^R one then defines the *R-trace* operation $\mathrm{Tr}_R : \ \mathrm{Mat}_N(W) \to W$

$$\mathrm{Tr}_R(X) := \sum_{i,j=1}^N D^{R}{}^j_i X^i_j, \quad X \in \mathrm{Mat}_N(W),$$

   where W is any linear space (in considerations below W is the space of the QM algebra).

ii) The operators R and F are the *R-matrices*, that means they satisfy the *Yang-Baxter equations*

$$R_1R_2R_1 = R_2R_1R_2\,, \qquad F_1F_2F_1 = F_2F_1F_2\,.$$

iii) The operators R and F form a *compatible pair* $\{R, F\}$ (the order of operators in this notation is essential)

$$R_1F_2F_1 = F_2F_1R_2\,, \qquad F_1F_2R_1 = R_2F_1F_2\,.$$

Given the pair $\{R, F\}$ satisfying conditions i)–iii) the *quantum matrix algebra* $\mathcal{M}(R, F)$ is defined as a unital associative algebra which is generated by N^2 components of the matrix $\|M^i_j\|_{i=1}^N$ subject to the relations[4]

$$R_1 M_{\overline{1}} M_{\overline{2}} = M_{\overline{1}} M_{\overline{2}} R_1\,. \tag{1.1}$$

[4]In [GPS2] we also demand skew invertibility of an operator $R_f := F^{-1}R^{-1}F$ in the definition of the QM algebra. As is proved in [OP2] (see lemma 3.6), the latter condition is a consequence of i)–iii).

Here we used the iterative procedure

$$M_{\overline{1}} = M, \quad M_{\overline{k+1}} = F_k M_{\overline{k}} F_k^{-1} \tag{1.2}$$

for the production of copies $M_{\overline{k}}$ of the matrix M. The defining relations (1.2) then imply the same type relations for any consecutive pair of the copies of M (see lemma I-4)

$$R_k\, M_{\overline{k}} M_{\overline{k+1}} = M_{\overline{k}} M_{\overline{k+1}}\, R_k. \tag{1.3}$$

Imposing additional conditions on the R-matrix R we then extract specific series of the QM algebras.

iv) Demanding R to be the *Hecke type* R-matrix, that means its minimal polynomial to be of the second order

$$(R + q^{-1}I)(R - qI) = 0\,, \quad q \in \{\mathbb{C} \setminus 0\}\,, \tag{1.4}$$

we specify to the *Hecke type* QM algebra. The $\mathbb{C}$-number q becomes the parameter of the algebra.

v) Given a Hecke type R-matrix (1.4), one can construct a series of *R-matrix representations* of the *Hecke algebras*[5] $\mathcal{H}_p(q)$

$$\rho_R : \mathcal{H}_p(q) \to \mathrm{End}(V^{\otimes p}), \quad p = 2, 3, \dots\,. \tag{1.5}$$

Let us impose an additional restriction on the parameter q

$$q^{2k} \neq 1, \qquad k = 2, 3, \dots\,, \tag{1.6}$$

which ensures the algebras $\mathcal{H}_p(q)$, $p = 2, 3, \dots$, to be semisimple. Then we can further specify to a series of the $GL(m|n)$ *type* QM algebras. For their definition we use a set of the primitive idempotents $E^\lambda_\alpha \in H_p(q)$ labelled by the standard Young tableaux $\left\{ {\lambda \atop \alpha} \right\}$, where $\lambda \vdash p$ is a partition of p, and index α enumerates different standard tableaux corresponding to the partition λ (see section I-2). The $GL(m|n)$ *type* QM algebra is characterized by the following conditions

a) the representations ρ_R (1.5) are faithful for all $p < (m + 1)(n + 1)$;

b) for $p \geq (m + 1)(n + 1)$ the kernel of ρ_R is generated by (any one of) the primitive idempotents $E^{((n+1)^{(m+1)})}_\alpha$ corresponding to the rectangular Young diagram $((n + 1)^{(m+1)})$;

c) the Schur function $s_{(n^m)}(M)$ (see definition below) corresponding to the rectangular Young diagram (n^m) is an invertible element of the QM algebra.[6]

Note that relation $m + n = N (= \dim V)$ between the algebra parameters m, n and N is not assumed in the above definition. Although it is indeed satisfied in standard examples (say, for the QM algebras constructed by the Drinfel'd-Jimbo R-matrix R). there exist exceptions from this rule. A series of counter-examples was constructed in [Gur].

[5] A brief description of the Hecke algebras, their R-matrix representations, the primitive idempotents and the basis of matrix units is given in [GPS2], sections 2 and 3. For a more detailed exposition of the subject the reader is referred to [R, OP1] and to references therein.

[6] This condition was not imposed in [GPS2]. We will need it now for the spectral parameterization of the characteristic subalgebra (see eqs.(3.5), (3.6)).

¿From now on we restrict ourselves to considering the Hecke type QM algebras with the parameter q satisfying condition (1.6).

The *characteristic subalgebra* $\mathrm{Char}(R,F)$ of the QM algebra $\mathcal{M}(R,F)$ is a linear span of the set of *Schur functions* $s_\lambda(M)$

$$s_0(M) := 1, \qquad s_\lambda(M) := \mathrm{Tr}_{R(1\dots k)}(M_{\overline{1}}\dots M_{\overline{k}}\,\rho_R(E^\lambda_\alpha)) \quad \lambda\vdash k,\;\; k=1,2,\dots, \tag{1.7}$$

where E^λ_α is any one of the primitive idempotents corresponding to the partition λ (the expression in (1.7) does not depend on α). As was shown in [IOP1], $\mathrm{Char}(R,F)$ is an abelian algebra with respect to the multiplication in $\mathcal{M}(R,F)$.

Consider a subspace $\mathrm{Pow}(R,F)\subset \mathrm{Mat}_N(\mathcal{M}(R,F))$ which is spanned linearly by the elements

$$I\,\mathrm{ch}(M)\,, \qquad \forall\,\mathrm{ch}(M)\in\mathrm{Char}(R,F)\,, \quad \text{and} \tag{1.8}$$

$$M^{(x^{(k)})} := \mathrm{Tr}_{R(2\dots k)}(M_{\overline{1}}\dots M_{\overline{k}}\,\rho_R(x^{(k)}))\,, \qquad \forall\,x^{(k)}\in\mathcal{H}_k(q)\,, \quad k=1,2,\dots\,. \tag{1.9}$$

In what follows elements of the space $\mathrm{Pow}(R,F)$ will be shortly called *the quantum matrices*. In [GPS2] it was shown that the space of the quantum matrices carries the structure of the right $\mathrm{Char}(R,F)$-module and as a $\mathrm{Char}(R,F)$-module it is spanned by a series of *quantum matrix powers of* M

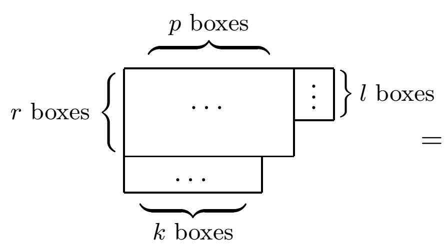
$$M^{\overline{0}} := I, \quad M^{\overline{1}} := M, \quad M^{\overline{k}} := \mathrm{Tr}_{R(2\dots k)}(M_{\overline{1}}\dots M_{\overline{k}}\,R_{k-1}\dots R_1), \quad k=2,3,\dots\,. \tag{1.10}$$

In section 4.4 of [OP2] an analogue of the matrix multiplication was introduced for the space $\mathrm{Pow}(R,F)$. It was shown there that the *quantum matrix multiplication* agrees with the right $\mathrm{Char}(R,F)$-module structure; it is associative (see proposition 4.12) and, moreover it is commutative (see propositions 4.13, 4.14). The latter result should not be surprising as all the elements of $\mathrm{Pow}(R,F)$ are descendants of the only quantum matrix M.[7] For our purposes in this paper it is enough knowing formulae

$$M^{\overline{k}} = \underbrace{M * M * \dots * M}_{k\ \text{times}}, \qquad (I\,\mathrm{ch}(M)) * M^{\overline{k}} = M^{\overline{k}} * (I\,\mathrm{ch}(M))\,, \quad \forall\,\mathrm{ch}(M)\in\mathrm{Char}(R,F)\,, \tag{1.11}$$

where by symbol "$*$" we denote the quantum matrix product. We also notice that for the family of the RE algebras the product $*$ reduces to the usual matrix product. For the detailed description of the quantum matrix multiplication the reader is referred to [OP2].

The main result of our previous paper [GPS2] is the Cayley-Hamilton theorem for the $GL(m|n)$ type QM algebras (see theorem I-10). For its compact formulation and for later convenience we introduce a shorthand notation for the following Young diagrams (partitions)



$$= \Big((p+1)^l, p^{(r-l)}, k\Big) =: [r|p]^l_k\,. \tag{1.12}$$

Here the indices k and l take values $l=0,\dots,r$, $k=0,\dots,p$. If one of the indices k or l takes zero value, we will omit it in the notation, e.g., $[r|p]^0_k = [r|p]_k$.

[7] There should be no confusion between the quantum matrix product and the multiplication in $\mathcal{M}(R,F)$. The latter one is the product of the matrix components, while the first one is the product of the quantum matrices.

Theorem 1 (Cayley-Hamilton identity) *In the setting i)–iv) and v)–a,b) the quantum matrix M composed of the generators of the $GL(m|n)$ type QM algebra $\mathcal{M}(R,F)$ fulfills the characteristic identity*

$$\sum_{i=0}^{n+m} M^{\overline{m+n-i}} \sum_{k=\max\{0,i-n\}}^{\min\{i,m\}} (-1)^k \, q^{2k-i} \, s_{[m|n]^k_{i-k}}(M) \, \equiv 0 \, . \tag{1.13}$$

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2 Structure of the characteristic subalgebra

Consider the graded ring Λ of symmetric functions in countably many variables. A $\mathbb{Z}$-basis of Λ is given by the Schur symmetric functions s_λ, $\lambda \vdash n$, for $n \geq 0$ (we adopt definitions and notation of ref.[Mac], sections 1.2 and 1.3).

It is not accidental that the similar notation $s_\lambda(M)$ is assigned to the elements (1.7) of the characteristic subalgebra of the Hecke type quantum matrix algebra $\mathcal{M}(R,F)$. Indeed, consider the additive map map

$$\Lambda \ni s_\lambda \;\mapsto\; s_\lambda(M) \in \mathrm{Char}(R,F) \subset \mathcal{M}(R,F) \;\textit{(Hecke type)}\, . \tag{2.1}$$

Our first main result is as follows.

Theorem 2 *In the setting i)–iv) and (1.6) the additive map (2.1) defines the homomorphism of rings.*

A proof of the theorem is given in the subsection 2.1.

In the subsection 2.2 we derive some bilinear relations for the Schur symmetric functions $s_\lambda \in \Lambda$. These relations are necessary for the derivations in section 3.

2.1 Littlewood-Richardson multiplication formula for $s_\lambda(M)$

We will prove the theorem 2 by a direct calculation. To this end we adopt its alternative formulation

Theorem 2 *Let $\mathcal{M}(R,F)$ be a Hecke type QM algebra generated by the components of matrix M. Assume that condition (1.6) on the algebra parameter q is satisfied. Then, the multiplication in the corresponding characteristic subalgebra* $\mathrm{Char}(R,F)$ *is described by the relations*

$$s_\lambda(M) s_\mu(M) = \sum_{\nu \vdash (k+n)} c^\nu_{\lambda\mu} s_\nu(M), \tag{2.2}$$

where $s_\lambda(M), s_\mu(M) \in$ $\mathrm{Char}(R,F)$ *are the Schur functions (1.7), and $c^\nu_{\lambda\mu}$ are the Littlewood-Richardson coefficients (see, e.g., [Mac], section 1.9).*

Proof. Since the cases $m = 0$ or $k = 0$ in (2.2) are trivial, we assume $m \geq 1$ and $k \geq 1$.

Let us first prove the relation (2.2) for the case $\mu = (1^k)$ is a single column diagram. In that case it reads

$$s_\lambda(M) s_{(1^k)}(M) = \sum_{\substack{\nu \supset \lambda \\ \nu \vdash (k+n)}}{}' s_\nu(M). \tag{2.3}$$

Here $\supset$ denotes the inclusion relation on the set of standard Young tableaux (see section I.2.1) and the summation $\sum'$ is taken only over those diagrams ν whose set theoretical difference with λ is a vertical strip (for terminology see [Mac], section 1.1).

For single column diagrams (1^k), $k = 2, 3, \dots$, their corresponding primitive idempotents $E^{(1^k)}$ satisfy the well known iterative relations (see, e.g. [TW], lemma 7.2, or [GPS1], section 2.3)

$$E^{(1)} = 1, \qquad E^{(1^k)} = \frac{(k-1)_q}{k_q}\, E^{(1^{k-1})} \Bigl(\frac{q^{k-1}}{(k-1)_q}\, 1 \, - \, \sigma_{k-1} \Bigr) E^{(1^{k-1})}, \tag{2.4}$$

where we use notation of the section I.2.2. We shall apply these relations for a derivation of eq.(2.3). Consider the following chain of transformations

$$\begin{aligned}
s_\lambda(M) s_{(1^k)}(M) \, &= \, \mathrm{Tr}_{R(1\dots n+k)} \Bigl[\rho_R(E^\lambda_\alpha)\, \rho_R(E^{(1^k)\uparrow n})\, M_{\overline{1}} \dots M_{\overline{n+k}} \Bigr] \\
= \frac{(k-1)_q}{k_q}\, &\mathrm{Tr}_{R(1\dots n+k)} \Bigl[\rho_R(E^\lambda_\alpha\, E^{(1^{k-1})\uparrow n}) \Bigl(\frac{q^{k-1}}{(k-1)_q} I - R_{n+k-1} \Bigr) \rho_R(E^{(1^{k-1})\uparrow n})\, M_{\overline{1}} \dots M_{\overline{n+k}} \Bigr] \\
= \frac{(k-1)_q}{k_q}\, &\mathrm{Tr}_{R(1\dots n+k)} \Bigl[\rho_R(E^\lambda_\alpha\, E^{(1^{k-1})\uparrow n}) \Bigl(\frac{q^{k-1}}{(k-1)_q} I - R_{n+k-1} \Bigr) M_{\overline{1}} \dots M_{\overline{n+k}} \Bigr] = \dots \\
= \frac{1}{k_q}\, &\mathrm{Tr}_{R(1\dots n+k)} \Bigl[\rho_R(E^\lambda_\alpha) \Bigl(qI - R_{n+1} \Bigr) \dots \Bigl(\frac{q^{k-1}}{(k-1)_q} I - R_{n+k-1} \Bigr) M_{\overline{1}} \dots M_{\overline{n+k}} \Bigr].
\end{aligned} \tag{2.5}$$

Here in the first line we substitute definition (1.7) for the Schur functions and use eq.(I.3.19) for $s_{(1^k)}(M)$ (the notation $E^{\mu\uparrow n}_\beta$ is described in lemma I.6). We remind that this expression is independent of the choice of index α labelling the primitive idempotents $E^\lambda_\alpha \in \mathcal{H}_n(q)$. In the second line we apply formula (2.4) (recall that $R_i = \rho_R(\sigma_i)$). In the third line we use relations (1.3) to permute the term $\rho_R(E^{(1^{k-1})\uparrow n})$ with the product of matrices M, then apply cyclic property of the R-trace to move $\rho_R(E^{(1^{k-1})\uparrow n})$ to the leftmost position, and take into account the commutativity of the idempotents $E^{(1^{k-1})\uparrow n}$ and E^λ_α. Repeating these transformations $(k-1)$ times we eventually obtain the last line expression.

Let us denote the argument of the R-traces in (2.5) as

$$Q(R) \;\; := \;\; \rho_R(E^\lambda_\alpha)\, X_{n+1}\,, \tag{2.6}$$

where

$$X_i \;\; := \;\; \Bigl(\frac{q^{i-n}}{(i-n)_q} I - R_i \Bigr) \Bigl(\frac{q^{i-n+1}}{(i-n+1)_q} I - R_{i+1} \Bigr) \dots \Bigl(\frac{q^{k-1}}{(k-1)_q} I - R_{n+k-1} \Bigr). \tag{2.7}$$

We notice that in view of relations (1.3) and the cyclic property of the R-trace one can perform cyclic permutations of factors in $Q(R)$ without altering the expression (2.5). We shall use this cyclic invariance in order to transform $Q(R)$ to a suitable form.

The strategy of the transformation is as follows. We use a sequence of resolutions of the idempotent $E^\lambda_\alpha \in \mathcal{H}_n(q)$ $(\lambda \vdash n)$ in terms of idempotents $E^\nu_\beta \in \mathcal{H}_{n+i}(q)$ $(\nu \vdash (n+i),\ i \geq 1)$ described in (I.2.21)

$$E^\lambda_\alpha = \sum_{\substack{\nu \supset \lambda \\ \nu \vdash (n+i)}} \sum_{\substack{\beta: \\ \beta \supset \alpha}} E^\nu_\beta. \tag{2.8}$$

We successively increase i in (2.8) from 2 to k and evaluate the factors $(q^{i-1}/(i-1)_q I - R_{n+i-1})$ in $Q(R)$ on the idempotents $\rho_R(E^\nu_\beta)$

$$\rho_R(E^\nu_\beta)\Bigl(\frac{q^{i-1}}{(i-1)_q}I - R_{n+i-1}\Bigr) \overset{\circlearrowleft}{=} \frac{(\ell_{n+i-1} + i - 1)_q}{(i-1)_q\,(\ell_{n+i-1})_q}\,\rho_R(E^\nu_\beta). \tag{2.9}$$

Here $\ell_j := c(j) - c(j+1)$ denotes the difference of the contents of boxes with numbers j and $(j+1)$ in the standard tableau $\left\{ {\nu \atop \beta} \right\}$ (for definitions see section I.2.1); the symbol "$\overset{\circlearrowleft}{=}$" means equality modulo cyclic permutation of factors.

The evaluation rule can be argued as follows. Observe that the relations

$$E^\nu_\beta\,\sigma_j \equiv E^\nu_\beta\Bigl(\sigma_j + \frac{q^{-\ell_j}}{(\ell_j)_q}1\Bigr) - \frac{q^{-\ell_j}}{(\ell_j)_q}\,E^\nu_\beta = \frac{(\ell_j+1)_q}{(\ell_j)_q}E^\nu_{\beta\,\pi_j(\beta)} - \frac{q^{-\ell_j}}{(\ell_j)_q}\,E^\nu_\beta, \quad 1 \le j \le n+i-1, \tag{2.10}$$

are satisfied in the algebra $\mathcal{H}_{n+i}(q)$ (see (I.2.16)). Here the symbol $E^\nu_{\beta\,\pi_j(\beta)}$ stands for the off-diagonal matrix unit labelled by the pair of standard Young tableaux $\left\{ {\nu \atop \beta} \right\}$ and $\left\{ {\nu \atop \pi_j(\beta)} \right\}$, where the tableau $\left\{ {\nu \atop \pi_j(\beta)} \right\}$ is obtained from the tableau $\left\{ {\nu \atop \beta} \right\}$ by the permutation π_j of boxes j and $(j+1)$. If $\left\{ {\nu \atop \pi_j(\beta)} \right\}$ is non-standard the term with $E^\nu_{\beta\,\pi_j(\beta)}$ is absent in (2.10).

Now, transform the expression $\rho_R(E^\nu_\beta)R_{n+i-1} = \rho_R(E^\nu_\beta\sigma_{n+i-1})$ in the left hand side of (2.9) with the use of eq.(2.10). In $Q(R)$ the contribution of the off-diagonal matrix unit $\rho_R(E^\nu_{\beta\,\pi_j(\beta)})$ vanishes by virtue of the cyclic invariance. Indeed,

$$\begin{aligned}\rho_R(E^\nu_{\beta\,\pi_j(\beta)})\,X_{n+i} \;=\; \rho_R(E^{\nu'}_{\beta'}E^\nu_{\beta\,\pi_j(\beta)})\,X_{n+i} &\overset{\circlearrowleft}{=} \rho_R(E^\nu_{\beta\,\pi_j(\beta)})\,X_{n+i}\;\rho_R(E^{\nu'}_{\beta'})\\ &= \rho_R(E^\nu_{\beta\,\pi_j(\beta)}E^{\nu'}_{\beta'})\,X_{n+i} \;=\; 0.\end{aligned} \tag{2.11}$$

Here the idempotent $E^{\nu'}_{\beta'}$ corresponds to the standard tableau $\left\{ {\nu' \atop \beta'} \right\}$ obtained from the tableau $\left\{ {\nu \atop \beta} \right\}$ by removing the box with the number $(n+i)$. The first and the last equalities in (2.11) are consequences of eq.(2.8) and the multiplication table for the matrix units (I.2.7). In the second equality we made the cyclic permutation of terms which is allowed in $Q(R)$. The factors $\rho_R(E^{\nu'}_{\beta'})$ and X_{n+i} are built of the mutually commuting R-matrices wherefrom the third equality in (2.11) follows.

Eventually, collecting the coefficients at the diagonal matrix unit $\rho_R(E^\nu_\beta)$ in $Q(R)$ results in the right hand side of eq.(2.9).

So, we begin the transformation of $Q(R)$. Setting $i = 2$ in (2.8) we come to the expression

$$Q(R) = \sum_{\substack{\nu\supset\lambda\\ \nu\vdash(n+2)}}\;\sum_{\substack{\beta:\\ \beta\supset\alpha}} \rho_R(E^\nu_\beta)\Bigl(qI - R_{n+1}\Bigr)\,X_{n+2}. \tag{2.12}$$

For our calculation we have to specify an explicit way of enumeration of the Young tableaux. For a given tableau $\left\{ {\lambda \atop \alpha} \right\}$, $\lambda \vdash n$, we take the index $\alpha := \{a_1, a_2, \dots a_n\}$ to be an ordered set of pairs of integers $a_i := \{x_i, y_i\}$, where x_i and y_i are, respectively, the number of column and row where the i-th box stands. Recall that the content of the i-th box is $c(i) = x_i - y_i$ (see sec.I.2.1).

In the summation index β in eq.(2.12) only the last two components vary. We shortly denote them as a and b, that is $\beta = \{\dots, a, b\}$. For a and b in the summation (2.12) we have following three possibilities.

i) $a = \{x, y\}, b = \{x+1, y\}$. In this case $\ell_{n+1} = c(n+1) - c(n+2) = -1$. Hence, due to relation (2.9) such tableaux do not contribute to $Q(R)$.

ii) $a=\{x,y\}, b=\{x,y+1\}$. In this case $\ell_{n+1}=c(n+1)-c(n+2)=1$. Hence, due to relation (2.9) the contributions of such tableaux in (2.12) equal

$$2_q\,\rho_R(E^{\nu}_{\{\dots,a,b\}})X_{n+2}. \tag{2.13}$$

iii) $a=\{x,y\}, b=\{\bar{x},\bar{y}\}$, *such that* $x\neq\bar{x}$ *and* $y\neq\bar{y}$. In this case we combine contributions coming from two tableaux of the same shape with indices $\beta=\{\dots a,b\}$ and $\pi_{n+1}(\beta)=\{\dots b,a\}$. Taking into account eq.(2.9) we get

$$\Big(\rho_R(E^{\nu}_{\{\dots,a,b\}})\,\frac{(\ell_{n+1}\,+\,1)_q}{(\ell_{n+1})_q}\;+\;\rho_R(E^{\nu}_{\{\dots,b,a\}})\,\frac{(\ell_{n+1}\,-\,1)_q}{(\ell_{n+1})_q}\Big)\,X_{n+2} \tag{2.14}$$

for the corresponding summands in (2.12).

Noticing that the term (2.13) fits the form (2.14) with $\ell_{n+1}=1$ we can rewrite (2.12) as

$$Q(R)\overset{\circlearrowleft}{=}\sum_{\substack{\nu\supset\lambda\\ \nu\vdash(n+2)\\ (a,b)}}{}^{'}\Big(\rho_R(E^{\nu}_{\{\dots,a,b\}})\,\frac{(\ell_{n+1}\,+\,1)_q}{(\ell_{n+1})_q}\;+\;\rho_R(E^{\nu}_{\{\dots,b,a\}})\,\frac{(\ell_{n+1}\,-\,1)_q}{(\ell_{n+1})_q}\Big)\,X_{n+2}\,, \tag{2.15}$$

where the summation goes over different shape diagrams $\nu\vdash(n+2)$ which are counted by *unordered* pairs (a,b), $a=\{x,y\}$ and $b=\{\bar{x},\bar{y}\}$. There is an additional condition $y\neq\bar{y}$ which means that in the diagram ν the boxes with numbers $(n+1)$ and $(n+2)$ can not appear in the same row. It is this restriction which the summation symbol $\sum'$ refers to (c.f. (2.3)).

For what follows it is suitable to change our notation for ℓ_{n+1}. We substitute

$$\ell_{n+1}=c(n+1)-c(n+2)\quad\longrightarrow\quad\ell_{ab}=(x-y)-(\bar{x}-\bar{y})$$

to manifest clearly the dependence on the summation variables a and b.

We now proceed to the next step of the transformation. Substituting (2.8) for $i=3$ into eq.(2.15) and noticing $\ell_{ab}=-\ell_{ba}$ we obtain

$$Q(R)\overset{\circlearrowleft}{=}\sum_{\substack{\tau\supset\lambda\\ \tau\vdash(n+2)\\ (a,b)}}{}^{'}\;\sum_{\substack{\nu\vdash(n+3):\\ c=\nu\backslash\tau}}\Big(\rho_R(E^{\nu}_{\{\dots,a,b,c\}})\,\frac{(\ell_{ab}{+}1)_q}{(\ell_{ab})_q}+\rho_R(E^{\nu}_{\{\dots,b,a,c\}})\,\frac{(\ell_{ba}{+}1)_q}{(\ell_{ba})_q}\Big)\Big(\frac{q^2}{2_q}I{-}R_{n+2}\Big)X_{n+3}, \tag{2.16}$$

where c labels all possible complements of the diagram $\tau\vdash(n+2)$ by the $(n+3)$-th box. Applying relation (2.9) we reduce this expression to the form

$$Q(R)\overset{\circlearrowleft}{=}\sum_{\substack{\tau\supset\lambda\\ \tau\vdash(n+2)\\ (a,b)}}{}^{'}\;\sum_{\substack{\nu\vdash(n+3):\\ c=\nu\backslash\tau}}\Big(\rho_R(E^{\nu}_{\{\dots,a,b,c\}})\,\frac{(\ell_{ab}{+}1)_q}{(\ell_{ab})_q}\,\frac{(\ell_{bc}{+}2)_q}{2_q(\ell_{bc})_q}\;+\;\rho_R(E^{\nu}_{\{\dots,b,a,c\}})\,\frac{(\ell_{ba}{+}1)_q}{(\ell_{ba})_q}\,\frac{(\ell_{ac}{+}2)_q}{2_q(\ell_{ac})_q}\Big)X_{n+3}, \tag{2.17}$$

Next, we observe that the idempotents $\rho_R(E^{\nu}_{\{\dots,a,b,c\}})$ and $\rho_R(E^{\nu}_{\{\dots,b,a,c\}})$ in the expression above can be identified. Indeed, denoting $\sigma_i(\ell):=(\sigma_i-q^{\ell}/\ell_q\,1)$ we have

$$\begin{aligned}\rho_R(E^{\nu}_{\{\dots,b,a,c\}})X_{n+3}&\overset{\circlearrowleft}{=}\rho_R\Big(\sigma_{n+1}(\ell_{ab})E^{\nu}_{\{\dots,b,a,c\}}\Big)X_{n+3}\,\rho_R\Big(\sigma_{n+1}(\ell_{ab})\Big)^{-1}\\ &=\rho_R\Big(E^{\nu}_{\{\dots,a,b,c\}}\sigma_{n+1}(-\ell_{ab})(\sigma_{n+1}(\ell_{ab}))^{-1}\Big)X_{n+3}\overset{\circlearrowleft}{=}\rho_R(E^{\nu}_{\{\dots,a,b,c\}})X_{n+3},\end{aligned} \tag{2.18}$$

where the cyclic invariance together with relations (I.2.13), (I.2.10) and (2.11) were taken into account. Thus, from now on the order of labels a and b makes no difference in the notation $E^{\nu}_{\{\dots,a,b,c\}}$ and we simplify it to $E^{\nu}_{\{\dots,c\}}$. Then, the expression (2.17) reduces to

$$Q(R) \overset{\circlearrowleft}{=} \sideset{}{'}\sum_{\substack{\tau\supset\lambda\\ \tau\vdash(n+2)\\ (a,b)}} \sum_{\substack{\nu\vdash(n+3):\\ c=\nu\setminus\tau}} \rho_R(E^{\nu}_{\{\dots,c\}}) \frac{(\ell_{ac}+1)_q}{(\ell_{ac})_q}\frac{(\ell_{bc}+1)_q}{(\ell_{bc})_q} X_{n+3}. \tag{2.19}$$

Here, noticing that $\ell_{ab}=\ell_{ac}-\ell_{bc}$, we have transformed the coefficients at $\rho_R(E^{\nu}_{\{\dots,c\}})$ using the q-combinatorial formula (A.3) for $k=2$ and $b_1=\ell_{ac}$, $b_2=\ell_{bc}$ (see Appendix). The double summation is carried out with the restriction that boxes $(n+1)$, $(n+2)$ and $(n+3)$ which are labelled by a, b and c must be placed in different rows of the diagram ν.

Finally, we prepare the expression (2.19) for the next step calculation by collecting the summands which correspond to tableaux of the same shape

$$\begin{aligned} Q(R) \overset{\circlearrowleft}{=} \sideset{}{'}\sum_{\substack{\nu\supset\lambda\\ \nu\vdash(n+3)\\ (a,b,c)}} \Big(\rho_R(E^{\nu}_{\{\dots,a,b,c\}}) \frac{(\ell_{ac}+1)_q}{(\ell_{ac})_q}\frac{(\ell_{bc}+1)_q}{(\ell_{bc})_q} + \rho_R(E^{\nu}_{\{\dots,b,c,a\}}) \frac{(\ell_{ba}+1)_q}{(\ell_{ba})_q}\frac{(\ell_{ca}+1)_q}{(\ell_{ca})_q} \\ + \rho_R(E^{\nu}_{\{\dots,c,a,b\}}) \frac{(\ell_{cb}+1)_q}{(\ell_{cb})_q}\frac{(\ell_{ab}+1)_q}{(\ell_{ab})_q}\Big) X_{n+3}. \end{aligned} \tag{2.20}$$

where the summation goes over different shape diagrams $\nu\vdash(n+3)$ counted by *unordered* triples (a,b,c) such that neither pair of boxes a, b and c is placed at the same row of ν.

Repeating the transformations described in eqs.(2.16)–(2.20) successively for $i=4,\dots,k$ and using q-combinatorial relations (A.3), we eventually obtain

$$Q(R) \overset{\circlearrowleft}{=} \sideset{}{'}\sum_{\substack{\tau\supset\lambda\\ \tau\vdash(n+k-1)\\ (a_1,\dots,a_{k-1})}} \sum_{\substack{\nu\vdash(n+k)\\ a_k=\nu\setminus\tau}} \rho_R(E^{\nu}_{\{\dots,a_k\}}) \prod_{i=1}^{k-1} \frac{(\ell_{a_ia_k}+1)_q}{(\ell_{a_ia_k})_q}\,. \tag{2.21}$$

Here the unordered $(k-1)$-tuples $(a_1,\dots a_{k-1})$ counting different shape diagrams $\tau\vdash(n+k-1)$ are subject to restriction that $\tau\setminus\lambda$ is a vertical strip. The summation variable a_k labels all possible complements of the diagram $\tau\vdash(n+k-1)$ by the $(n+k)$-th box.

Formula (2.21) is the $i=k$ step analogue of the relation (2.19). An important difference is the absence of the X-term in the right hand side of the expression (one can say that $X_{n+k}=1$). Therefore, in the final expression for $Q(R)$ we have no need to distinguish between the different idempotents $\rho_R(E^{\nu}_{\{\dots,a_k,\dots\}})$ (a_k taking various positions) corresponding to the same shape diagram $\nu\vdash(n+k)$. Thus, the analogue of eq.(2.20) reads

$$Q(R) \overset{\circlearrowleft}{=} \sideset{}{'}\sum_{\substack{\nu\supset\lambda\\ \nu\vdash(n+k)\\ (a_1,\dots,a_k)}} \rho_R(E^{\nu}_{\{\dots\}}) \sum_{j=1}^{k}\prod_{\substack{i=1\\ i\neq j}}^{k} \frac{(\ell_{a_ia_j}+1)_q}{(\ell_{a_ia_j})_q} \;=\; k_q \sideset{}{'}\sum_{\substack{\nu\supset\lambda\\ \nu\vdash(n+k)}} \rho_R(E^{\nu}_{\{\dots\}})\,. \tag{2.22}$$

Here by $E^{\nu}_{\{\dots\}}$ an arbitrary primitive idempotent corresponding to Young diagram ν is understood, the summation $\sum'$ goes over all diagrams $\nu\vdash(n+k)$ such that $\nu\setminus\lambda$ is a vertical strip, and in the last equality we used q-combinatorial formula (A.2) setting $\ell_{a_ia_j}=b_i-b_j$.

Substituting expression (2.22) for $Q(R)$ in eq.(2.5) we derive formula (2.3), which is a particular example of the Littlewood-Richardson rule.

Now we are ready to prove the general case. To this end, let us argue that elements $s_{(1^k)}(M)$, $k = 0, 1, \ldots$, form a $\mathbb{Z}$-basis of generators for the set of Schur functions. Indeed, with the help of eqs.(2.3) it is easy to see that

$$s_{(2^k 1^m)}(M) = s_{(1^{(k+m)})}(M) s_{(1^k)}(M) - s_{(1^{(k+m+1)})}(M) s_{(1^{(k-1)})}(M), \quad \forall\, k \geq 1, m \geq 0\,.$$

Then, using eqs.(2.3), elements $s_{(3^k,2^m,1^n)}(M)$ can be expressed as linear combinations of monomials of the type $s_{(2^l 1^p)}(M) s_{(1^r)}(M)$. Etc. Repeating this procedure finitely many times one can express any Schur function $s_\lambda(M)$ as a polynomial in generators $s_{(1^k)}(M)$, $k = 0, 1, \ldots$. The explicit expressions are given by famous Jacobi-Trudi identities (see [Mac], section 1.3).

At last, since the product of generators $s_{(1^k)}(M)$ is described by the specification (2.3) of the Littlewood-Richardson formula, the product of two arbitrary Schur functions $s_\lambda(M)$ and $s_\mu(M)$ is to be given by eq.(2.2). ■

2.2 Bilinear relations

In this subsection we derive a series of bilinear relations for the Schur symmetric functions $s_\lambda \in \Lambda$. By the homomorphic map (2.1) one can translate them to the characteristic subalgebra of the Hecke type quantum matrix algebra. These relations are used in section 3.1 to split the characteristic identity in the $GL(m|n)$ case into the product of two factors and, thereby, to separate "even" and "odd" parts of the spectra of quantum matrices.

Our derivation is based on the use of the Plücker relations and we start from their short reminding (for details see [Sturm]).

Consider a pair of $n \times n$ matrices $A = \|a_{ij}\|_1^n$ and $B = \|b_{ij}\|_1^n$. We denote the i-th row of the matrix A as a_{i*} and introduce notation

$$\det A \,:=\, [A]\,, \qquad A \,:=\, \left(\begin{array}{ccccc} a_{1*} & \ldots & a_{i*} & \ldots & a_{n*} \\ 1 & \ldots & i & \ldots & n \end{array}\right), \tag{2.23}$$

where the latter symbol contains a detailed information on the row content of A. Namely, it says that the row a_{i*} appears in the matrix A at the i-th place (counting downwards).

Let us fix a set of integer data $\{k, r_1, r_2, \ldots, r_k\}$ such that $1 \leq k \leq n$ and $1 \leq r_1 < \ldots < r_k \leq n$. Given these data the Plücker relation reads

$$[A][B] = \sum_{1 \leq s_1 < \ldots < s_k \leq n} \left[\begin{array}{ccccccccc} a_{1*} & \ldots & b_{s_1 *} & \ldots & b_{s_2 *} & \ldots & b_{s_k *} & \ldots & a_{n*} \\ 1 & \ldots & r_1 & \ldots & r_2 & \ldots & r_k & \ldots & n \end{array}\right] \times$$
$$\left[\begin{array}{ccccccccc} b_{1*} & \ldots & a_{r_1 *} & \ldots & a_{r_2 *} & \ldots & a_{r_k *} & \ldots & b_{n*} \\ 1 & \ldots & s_1 & \ldots & s_2 & \ldots & s_k & \ldots & n \end{array}\right], \tag{2.24}$$

where the sum is taken over all possible sets $\{k, s_1, \ldots, s_k\}$. We now apply the Plücker relations for the proof of

Proposition 3 *Let us fix four integers r, p, l and k, such that $1 \leq l \leq r$ and $1 \leq k \leq p$. Then in the ring Λ of symmetric functions the following bilinear relations are satisfied (for the notation see (1.12))*

$$s_{[r|p]_k^l}\, s_{[r|p]} = s_{[r-1|p-1]_{(k-1)}^{(l-1)}}\, s_{[r+1|p+1]} + s_{[r|p]_k}\, s_{[r|p]^l}\,. \tag{2.25}$$

Proof. For the Schur symmetric function s_λ corresponding to a partition $\lambda = (\lambda_1, \dots, \lambda_p)$, the Jacobi-Trudi relation reads (see [Mac], section 1.3, eq.(3.4))

$$s_\lambda = \det \| h_{\lambda_i - i + j} \|_{i,j=1}^{m}, \tag{2.26}$$

where $m \ge p$ and the components of the matrix in the right hand side are the complete symmetric functions (that is, the single row Schur symmetric functions) $h_i := s_{(i)}$. By convention, $h_i := 0$ if $i < 0$.

Substituting expressions (2.26) into the left hand side of relation (2.25) and using notation (2.23) we have

$$\begin{aligned} s_{[r|p]^l_k} s_{[r|p]} \; = \; & \left[\begin{array}{cccccccc} h_{p+1*} & h_{p*} & \dots & h_{p-l+2*} & h_{p-l*} & \dots & h_{p-r+1*} & h_{k-r*} \\ 1 & 2 & \dots & l & l+1 & \dots & r & r+1 \end{array} \right] \times \\ & \left[\begin{array}{cccccccc} h_{p*} & h_{p-1*} & \dots & h_{p-l+1*} & h_{p-l*} & \dots & h_{p-r+1*} & h_{-r*} \\ 1 & 2 & \dots & l & l+1 & \dots & r & r+1 \end{array} \right], \end{aligned} \tag{2.27}$$

where the symbol $h_{i*} := (h_i, h_{i+1}, h_{i+2}, \dots)$ is used for the rows of the matrices appearing in the Jacobi-Trudi formula (2.26).

Now, we transform the right hand side of eq.(2.27) using the Plücker relation for the set of data $\{k = 1, r_1 = r+1\}$. In this case most of the summands in formula (2.24) vanish, since they contain determinants of matrices with coinciding pairs of rows. The only two contributing terms correspond to $s_1 = l$ and $s_1 = r+1$. So, we get

$$\begin{aligned} s_{[r|p]^l_k} s_{[r|p]} \; = \; & \left[\begin{array}{ccccccc} h_{p+1*} & \dots & h_{p-l+2*} & h_{p-l*} & \dots & h_{p-r+1*} & h_{p-l+1*} \\ 1 & \dots & l & l+1 & \dots & r & r+1 \end{array} \right] \times \\ & \left[\begin{array}{ccccccc} h_{p*} & \dots & h_{k-r*} & h_{p-l*} & \dots & h_{p-r+1*} & h_{-r*} \\ 1 & \dots & l & l+1 & \dots & r & r+1 \end{array} \right] + \\ & \left[\begin{array}{ccccccc} h_{p+1*} & \dots & h_{p-l+2*} & h_{p-l*} & \dots & h_{p-r+1*} & h_{-r*} \\ 1 & \dots & l & l+1 & \dots & r & r+1 \end{array} \right] \times \\ & \left[\begin{array}{ccccccc} h_{p*} & \dots & h_{p-l+1*} & h_{p-l*} & \dots & h_{p-r+1*} & h_{k-r*} \\ 1 & \dots & l & l+1 & \dots & r & r+1 \end{array} \right] \end{aligned} \tag{2.28}$$

which, by the Jacobi-Trudi relations, is exactly the right hand side of the eq.(2.25) (to represent the first summand in the right hand side of (2.28) as a product of two Schur functions one has to move $(r+1)$-th row in its first factor up to the $(l+1)$-th place, and l-th row in its second factor down to the r-th place). ■

3 Various presentations of the Cayley-Hamilton identity

In this section we derive three alternative expressions for the characteristic identity (1.13).

In subsection 3.1 we use the results of section 2 to present the characteristic identity for the $GL(m|n)$ type QM algebra as a product of two factors of orders m and n. The factorization allows us to introduce separately the sets of "even" and "odd" eigenvalues for the quantum matrix M of generators of the algebra.

In the subsection 3.2 we derive two other forms of the Cayley-Hamilton identity. They are written in terms of symmetric and skew-symmetric powers of the quantum matrix M, respectively. The coefficients of these identities are elements of the characteristic subalgebra and we find their expressions in terms of the Schur functions $s_\lambda(M)$, and in terms of the eigenvalues of M. For the case of supermatrices these two expressions for the characteristic identity were first derived in [KT2, T].

3.1 Separation of "even" and "odd" spectral values

¿From the condition v) in the definition of the $GL(m|n)$ type QM algebra it follows immediately that for all Young diagrams λ containing the diagram $((n+1)^{m+1}) = [m+1|n+1]$ their corresponding Schur functions s_λ belong to the kernel of the homomorphism (2.1)

$$s_\lambda \mapsto s_\lambda(M) = 0, \quad \forall\, \lambda:\ ((n+1)^{m+1}) \subset \lambda\,. \tag{3.1}$$

Therefore, the image of bilinear relations (2.25) with $r = m$, $p = n$ in the characteristic subalgebra of the $GL(m|n)$ type QM algebra reduces to

$$s_{[m|n]^l_k}(M)\, s_{[m|n]}(M) = s_{[m|n]_k}(M)\, s_{[m|n]^l}(M)\,, \quad \forall\, k,l:\ 0\le k\le n,\ 0\le l\le m\,. \tag{3.2}$$

We shall use these relations to factorize the characteristic polynomial (1.13). To this end we multiply the identity (1.13) by the Schur function $s_{[m|n]}(M)$ from the right and apply eqs.(3.2). The resulting expression reads

$$\sum_{i=0}^{m+n} M^{\overline{m+n-i}} \sum_{k=\max(0,i-n)}^{\min(i,m)} (-q)^k s_{[m|n]^k}(M)\ q^{k-i} s_{[m|n]_{(i-k)}}(M)\ \equiv\ 0\,. \tag{3.3}$$

With the use of relations (1.11) it can be immediately turned into the quantum matrix product of two factors.

Theorem 4 (Cayley-Hamilton identity in a factorized form) *In the assumptions of theorem 1 the identity (1.13) implies*

$$\Bigl(\sum_{k=0}^{m}(-q)^k\, M^{\overline{m-k}} s_{[m|n]^k}(M)\Bigr) * \Bigl(\sum_{r=0}^{n} q^{-r}\, M^{\overline{n-r}} s_{[m|n]_r}(M)\Bigr) \equiv 0\,. \tag{3.4}$$

The identities (1.13) and (3.4) are equivalent iff the Schur function $s_{[m|n]}(M)$ is invertible (i.e., in case if all conditions i)–v) are satisfied).

The factorization suggests a natural parameterization for the characteristic subalgebra. Namely, assuming that the conditions i)–v) on the $GL(m|n)$ type QM algebra $\mathcal{M}(R,F)$ are satisfied we consider a homomorphic map from the characteristic subalgebra $\mathrm{Char}(R,F)$ into the algebra $\mathbb{C}[\mu,\nu]$ of polynomials in two sets of (mutually commuting) variables $\mu := \{\mu_i\}_{1\le i\le m}$ and $\nu := \{\nu_j\}_{1\le j\le n}$. The map $\mathrm{Char}(R,F) \to \mathbb{C}[\mu,\nu] : s_\lambda(M) \mapsto s_\lambda(\mu,\nu)$ called the *parameterization map* is given by relations[8]

$$\frac{s_{[m|n]^k}(M)}{s_{[m|n]}(M)} \quad\mapsto\quad \frac{s_{[m|n]^k}(\mu,\nu)}{s_{[m|n]}(\mu,\nu)} := \sum_{1\le i_1<\ldots<i_m\le m} q^{-k}\mu_{i_1}\ldots\mu_{i_k} = e_k(q^{-1}\mu)\,, \quad 1\le k\le m\,, \tag{3.5}$$

$$\frac{s_{[m|n]_r}(M)}{s_{[m|n]}(M)} \quad\mapsto\quad \frac{s_{[m|n]_r}(\mu,\nu)}{s_{[m|n]}(\mu,\nu)} := \sum_{1\le j_1<\ldots<j_r\le n} (-q)^r\nu_{j_1}\ldots\nu_{j_r} = e_r(-q\nu)\,, \quad 1\le r\le n\,. \tag{3.6}$$

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[8]Here we implicitly assume the algebraic independence of the elements $\frac{s_{[m|n]^k}(M)}{s_{[m|n]}(M)}$, $1\le k\le m$, and $\frac{s_{[m|n]_r}(M)}{s_{[m|n]}(M)}$, $1\le r\le n$.

Here $e_k(\cdot)$ denotes the specialization of the elementary symmetric function $e_k \in \Lambda$ to the elementary symmetric polynomial in finitely many variables — the arguments of $e_k(\cdot)$. The powers of the parameter q are introduced in order to get the simple form of the identity (3.7) below.

Note that for the above parameterization we need assuming an invertibility of the Schur function $s_{[m|n]}(M)$ (see condition v)-c)). As we shall see in section 4, relations (3.5), (3.6) define consistently the homomorphism of the characteristic subalgebra $\mathrm{Char}(R,F)$ to a subalgebra of the supersymmetric polynomials in variables $\{q^{-1}\mu_i\}$ and $\{-q\nu_j\}$ (see the definition in section 4) [9].

Now, it is straightforward to derive a completely factorized formula for the characteristic polynomial (1.13). Namely, the parameterization map defines naturally a left $\mathrm{Char}(R,F)$-module structure on the algebra $\mathbb{C}[\mu,\nu]$. We shall use this structure to construct *completion* of the space of quantum matrices:

$$\overline{\mathrm{Pow}}(R,F) := \mathrm{Pow}(R,F) \bigotimes_{\mathrm{Char}(R,F)} \mathbb{C}[\mu,\nu]\,.$$

The quantum matrix product for the completed space $\overline{\mathrm{Pow}}(R,F)$ is given by formula

$$(N \bigotimes_{\mathrm{Char}(R,F)} x) * (K \bigotimes_{\mathrm{Char}(R,F)} y) := (N*K) \bigotimes_{\mathrm{Char}(R,F)} (xy)\,, \quad \forall\, N,K \in \mathrm{Pow}(R,F)\,, \ \ \forall\, x,y \in \mathbb{C}[\mu,\nu]\,.$$

It is associative as well as commutative (see discussion below eq.(1.10)).

Finally, notice that the characteristic identities (1.13) and (3.4) are written in the algebra $\mathrm{Pow}(R,F)$. When passing to the completed algebra $\overline{\mathrm{Pow}}(R,F)$ we can apply substitutions (3.5) and (3.6) in the characteristic polynomial (3.4) and, thus, we turn it to a completely factorized form

$$(s_{[m|n]}(M)\, I)^{*2} * \prod_{i=1}^{m}(M-\mu_i I) * \prod_{j=1}^{n}(M-\nu_j I) \equiv 0\,. \tag{3.7}$$

Here all products are understood as the quantum matrix products.

The above totally factorized form of the Cayley-Hamilton theorem confirms interpretation of the indeterminates $\{\mu_i\}$ and $\{\nu_j\}$ as, respectively, *"even" and "odd" eigenvalues of the quantum supermatrix M.*

Let us stress that the parameterization formulae (3.5)–(3.7) are obtained here at a formal algebraic level. A different approach based on the representation theory of the algebras was adopted in papers [JG, GL] (see also references therein and ref.[Mudr]). The latter approach is well applicable for the family of RE algebras, in which case the characteristic subalgebra belongs to the center of the algebra (see, e.g., [I], section 3.2, proposition 5). However, it seems hardly possible to apply this approach for the QM algebras in general.

3.2 Cayley-Hamilton identities for skew-symmetric and symmetric matrix powers

We have already mentioned in the introduction that for the Hecke type QM algebras the corresponding $\mathrm{Char}(R,F)$-module $\mathrm{Pow}(R,F)$ is spanned linearly by the set $M^{\overline{k}}$, $k = 0,1,2,\dots$. The Cayley-Hamilton identity (1.13) then states that $\mathrm{Pow}(R,F)$ is *not a free span* of the quantum matrix powers of M. In this subsection we consider two other spanning sets for the space of

[9] The characteristic subalgebra augmented by the inverse Schur function $(s_{[m|n]}(M))^{-1}$ is parameterized by rational functions in μ_i and ν_j which are symmetric (separately) with respect to variables in the subsets μ and ν (see eqs.(3.5), (3.6) and proposition 13).

quantum matrices $\mathrm{Pow}(R,F)$ and derive equivalent forms of the Cayley-Hamilton identity in their terms.

Consider quantum matrices (c.f., with eq.(1.9))

$$M^{[k|1]} \;:=\; \mathrm{Tr}_{R(2\dots k)}\Big(M_{\overline{1}}\dots M_{\overline{k}}\,\rho_R(E^{[k|1]})\Big)\,, \tag{3.8}$$

$$M^{[1|k]} \;:=\; \mathrm{Tr}_{R(2\dots k)}\Big(M_{\overline{1}}\dots M_{\overline{k}}\,\rho_R(E^{[1|k]})\Big)\,, \quad (\text{recall } [r|p] := (p^r))\,. \tag{3.9}$$

Following to A.M. Lopshits (see [GGB], p.342, or [KT2, T]) we introduce series of *skew-symmetric and symmetric quantum matrix powers* of the quantum matrix M, respectively,

$$M^{\wedge 0} := I\,, \qquad M^{\wedge k} := (-1)^{k-1}k_q M^{[k|1]} + (-q)^k s_{[k|1]}(M)\, I, \quad k = 1,2,\dots, \tag{3.10}$$

and

$$M^{\mathcal{S}0} := I\,, \qquad M^{\mathcal{S}k} := k_q M^{[1|k]} + q^{-k} s_{[1|k]}(M)\, I, \quad k = 1,2,\dots. \tag{3.11}$$

In [IOP2] (see the Cayley-Hamilton-Newton theorem there) expressions for $M^{[k|1]}$ and $M^{[1|k]}$ in terms of the quantum matrix powers of M were derived. Therefrom we calculate

$$M^{\wedge k} \;=\; \sum_{r=0}^{k}(-q)^r M^{\overline{k-r}} s_{[r|1]}(M)\,, \qquad M^{\mathcal{S}k} \;=\; \sum_{r=0}^{k} q^{-r} M^{\overline{k-r}} s_{[1|r]}(M)\,. \tag{3.12}$$

These relations can be inverted with the use of the inverse Cayley-Hamilton-Newton theorem [IOP2] and the Wronski relations (see, e.g., [Mac], eq.(2.6'))[10]

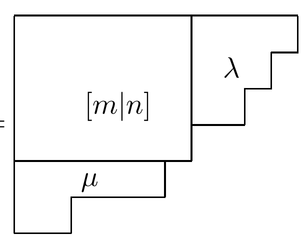
$$\sum_{r=0}^{k}(-1)^r s_{[r|1]}\, s_{[1|k-r]} \;=\; \delta(k)\,, \tag{3.13}$$

where $\delta(i) := 1$ if $i = 0$, and $\delta(i) := 0$ otherwise. The inverse relations read

$$M^{\overline{k}} \;=\; \sum_{r=0}^{k} q^r M^{\wedge(k-r)} s_{[1|r]}(M) \;=\; \sum_{r=0}^{k}(-q)^{-r} M^{\mathcal{S}(k-r)} s_{[r|1]}(M)\,. \tag{3.14}$$

Formulae (3.14) show that the space of quantum matrices $\mathrm{Pow}(R,F)$ is a $\mathrm{Char}(R,F)$-span of each one of the sets $\{M^{\wedge k}\}_{k\geq 0}$, $\{M^{\mathcal{S}k}\}_{k\geq 0}$. We shall use them also for rewriting the Cayley-Hamilton identity (1.13) in terms of the (skew-)symmetric matrix powers. To simplify formulation we introduce one more notation for the Young diagrams of a particular shape. It is easier to explain it on the picture

$\langle\mu|\lambda\rangle :=$



that is, the Young diagram $\langle\mu|\lambda\rangle$ is a composition of the rectangular diagram $[m|n]$ and the two diagrams λ and μ, such that the length of λ does not exceed m, and the length of μ^T is less or equal to n (we use the standard notation from [Mac]).

[10]For the elements of the characteristic subalgebra the Wronski relation was proved in [IOP2].

Theorem 5 (Cayley-Hamilton identity for the (skew-)symmetric matrix powers) *In the assumptions of theorem 1 the identity (1.13) can be written in the following equivalent forms*

$$\sum_{k=0}^{\min\{2n,m+n\}} M^{\wedge(m+n-k)} d_k(M) \equiv 0\,, \qquad or \qquad \sum_{k=0}^{\min\{2m,m+n\}} M^{\mathcal{S}(m+n-k)} f_k(M) \equiv 0\,, \quad (3.15)$$

where we denote

$$d_k(M) \;:=\; \sum\nolimits_{r=\max\{0,k-n\}}^{\left[\frac{k}{2}\right]} (k-2r+1)_q\, s_{\langle (k-r,r)|0\rangle}(M)\,, \quad (3.16)$$

$$f_k(M) \;:=\; \sum\nolimits_{r=\max\{0,k-m\}}^{\left[\frac{k}{2}\right]} (-1)^{k-2r}(k-2r+1)_q\, s_{\langle 0|(2^r,1^{k-2r})\rangle}(M)\,. \quad (3.17)$$

Here the symbol $\left[\frac{k}{2}\right]$ *stands for the integral part of the fraction* $\frac{k}{2}$.

Proof. The proof of the theorem is a straightforward calculation on the base of relations (3.14). We shall carry it out for the left identity in (3.15). Checking the right identity is a similar calculation.

Substitute the expressions (3.14) for the quantum matrix powers $M^{\overline{m+n-i}}$ in terms of the skew-symmetric powers into the Cayley-Hamilton identity (1.13). Evidently, the identity takes the form

$$\sum_{k=0}^{m+n} M^{\wedge(m+n-k)} d_k(M) \equiv 0\,, \quad (3.18)$$

where the coefficients $d_k(M) \in \mathrm{Char}(R,F)$ are to be specified. We shall verify the explicit expressions (3.16) for $d_k(M)$ and refine the limits of the summation over k.

First of all, collecting the contributions to $d_k(M)$ from the expressions for the matrix powers $M^{\overline{m+n-i}}$, $0 \le i \le k$, we have

$$d_k(M) = \sum_{i=0}^{k} q^{k-i} s_{(k-i)}(M) \sum_{j=\max\{0,i-n\}}^{\min\{i,m\}} (-1)^j\, q^{2j-i}\, s_{[m|n]^j_{i-j}}(M)\,.$$

Then, introducing a new summation variable $r = i - j$ and changing the order of summation over i and r we get

$$d_k(M) = \sum_{r=0}^{\min\{k,n\}} (-1)^r q^{k-2r} \sum_{i=r}^{\min\{k,r+m\}} (-1)^i\, s_{(k-i)}(M) s_{[m|n]^{i-r}_r}(M)\,. \quad (3.19)$$

Let us separately calculate the second sum in the expression above.

Lemma 6 *For any fixed pair of integers m and n, and for all integers r and k satisfying conditions $0 \le r \le n$, $r \le k \le m+n$, the following equalities*

$$\sum_{i=r}^{\min\{k,r+m\}} (-1)^i s_{(k-i)} s_{[m|n]^{i-r}_r} = \begin{cases} 0\,, & k \ge n+r+1\,, \\ (-1)^r \sum_{i=\max\{0,k-n\}}^{\min\{r,k-r\}} s_{\langle (k-i,i)|0\rangle}\,, & k \le n+r\,, \end{cases} \quad (3.20)$$

take place in the ring Λ of the symmetric functions.

Proof. Denote $\omega_{k,r}$ the expression in the left hand side of eq.(3.20).

Consider the case $k \le r+m$. Introducing a new summation variable $j = k-i$ and denoting $p := k-r,\ \ 0 \le p \le m$, we rewrite the sum $\omega_{k,r}$ as

$$(-1)^k \omega_{k,r} = \sum_{j=0}^{p} (-1)^j s_{(j)} s_{[m|n]_r^{p-j}} = (-1)^p s_{(p)} s_{[m|n]_r} + \sum_{j=0}^{p-1} (-1)^j s_{(j)} s_{[m|n]_r^{p-j}} .$$

Applying the Littlewood-Richardson rule to the products $s_{(j)} s_{[m|n]_r^{p-j}}$ we can gather terms in the latter expression into two separate sums

$$\begin{aligned}(-1)^k \omega_{k,r} = (-1)^p s_{(p)} s_{[m|n]_r} &+ \sum_{j=0}^{p-1} (-1)^j \sum_{t=0}^{\min\{j,n\}} \sum_{i=\max\{0,r+t-n\}}^{\min\{r,t\}} s_{\langle (r+t-i,\, i) | (j-t+1,\, 1^{p-j-1}) \rangle} \\ &+ \sum_{j=1}^{p-1} (-1)^j \sum_{t=0}^{\min\{j-1,n\}} \sum_{i=\max\{0,r+t-n\}}^{\min\{r,t\}} s_{\langle (r+t-i,\, i) | (j-t,\, 1^{p-j}) \rangle} .\end{aligned}$$

As can be easily checked, the two triple sums in the expression above cancel each other except for the term $j = p-1$ in the first sum. So, we obtain

$$(-1)^k \omega_{k,r} = (-1)^p s_{(p)} s_{[m|n]_r} + (-1)^{p-1} \sum_{t=0}^{\min\{p-1,n\}} \sum_{i=\max\{0,r+t-n\}}^{\min\{r,t\}} s_{\langle (r+t-i,i) | (p-t) \rangle} . \tag{3.21}$$

Consider now expansion of the product $s_{(p)} s_{[m|n]_r}$ into the sum of Schur symmetric functions

$$(-1)^p s_{(p)} s_{[m|n]_r} = (-1)^p \sum_{t=0}^{\min\{p,n\}} \sum_{i=\max\{0,\, t+r-n\}}^{\min\{r,t\}} s_{\langle (r+t-i,\, i) | (p-t) \rangle} . \tag{3.22}$$

Comparing the double sums in the right hand sides of eqs.(3.21) and (3.22) we observe that they are exactly opposite in the sign in case $k \ge r+n+1 \Leftrightarrow p \ge n+1$, and they differ by the term with $t = p$ in case $k \le n+r \Leftrightarrow p \le n$. Therefore, substitution of the expression (3.22) into eq.(3.21) results in formula (3.20).

The case $k \ge r+m+1$ is treated in complete analogy with the above consideration. ■

Return to the proof of the theorem. By the homomorphism (2.1) the statement of lemma 6 translates to the ring of Schur functions $s_\lambda(M)$. So, formula (3.19) for $d_k(M)$ can be equivalently written as

$$d_k(M) = 0, \quad \text{if } k > 2n; \qquad d_k(M) = \sum_{r=\max\{0,k-n\}}^{\min\{k,n\}} q^{k-2r} \sum_{i=\max\{0,k-n\}}^{\min\{r,k-r\}} s_{\langle (k-i,\, i)|0 \rangle}, \quad \text{for } 0 \le k \le 2n. \tag{3.23}$$

The latter expression can be further simplified. In case $0 \le k \le n$ we have

$$d_k(M) = \sum_{r=0}^{k} q^{k-2r} \sum_{i=0}^{\min\{r,k-r\}} s_{\langle (k-i,\, i)|0 \rangle} = \sum_{i=0}^{[k/2]} s_{\langle (k-i,\, i)|0 \rangle} \sum_{r=i}^{k-i} q^{k-2r} = \sum_{i=0}^{[k/2]} (k-2i+1)_q\, s_{\langle (k-i,\, i)|0 \rangle} , \tag{3.24}$$

where in the second equality we changed the order of summation. In case $n < k \le 2n$ the similar calculation gives

$$d_k(M) = \sum_{r=k-n}^{n} q^{k-2r} \sum_{i=k-n}^{\min\{r,k-r\}} s_{\langle (k-i,\, i)|0 \rangle} = \sum_{i=k-n}^{[k/2]} (k-2i+1)_q\, s_{\langle (k-i,\, i)|0 \rangle} . \tag{3.25}$$

Combining together the results (3.18), (3.23), (3.24) and (3.25) we get formulae (3.15), (3.16). ■

Assuming additionally the Schur function $s_{[m|n]}(M) = d_0(M) = f_0(M)$ to be invertible we will now express the ratios $d_k(M)/d_0(M)$ and $f_k(M)/f_0(M)$ in terms of the eigenvalues of the quantum supermatrix M.

Proposition 7 *Let $\mathcal{M}(R,F)$ be a QM algebra of the $GL(m|n)$ type, that is the algebra defined by the set of conditions i)–v) (see introduction). Then, under the parameterization map (3.5), (3.6) we have*

$$\frac{d_k(M)}{d_0(M)} \quad\mapsto\quad (-1)^k \sum_{r=\max\{0,k-n\}}^{\min\{k,n\}} q^{2r} e_r(\nu)\, e_{k-r}(\nu)\,, \tag{3.26}$$

$$\frac{f_k(M)}{f_0(M)} \quad\mapsto\quad (-1)^k \sum_{r=\max\{0,k-m\}}^{\min\{k,m\}} (-q)^{-2r} e_r(\mu)\, e_{k-r}(\mu)\,. \tag{3.27}$$

Proof. We shall prove the equality (3.26). The relation (3.27) can be checked in a similar way.

Multiplying eq.(3.19) by $d_0(M)$ we obtain

$$\begin{aligned} d_k(M)d_0(M) &= \sum_{l=0}^{\min\{k,n\}} (-1)^l q^{k-2l} \sum_{j=l}^{\min\{k,l+m\}} (-1)^j\, s_{(k-j)}(M)\, s_{[m|n]_l^{(j-l)}}(M)\, s_{[m|n]}(M) \\ &= \sum_{l=0}^{\min\{k,n\}} q^{k-2l} s_{[m|n]_l}(M) \sum_{j=0}^{\min\{k-l,m\}} (-1)^j\, s_{(k-l-j)}(M)\, s_{[m|n]^j}(M), \end{aligned}$$

where in passing to the second line we apply the bilinear relations (3.2) and shift the summation index $j \to j-l$. The last sum in the second line can be calculated with the use of relation (3.20) (take there $r=0$ and substitute $k \to k-l$). The result is

$$d_k(M)d_0(M) = \sum_{l=\max\{0,k-n\}}^{\min\{k,n\}} q^{k-2l} s_{[m|n]_l}(M)\, s_{[m|n]_{(k-l)}}(M). \tag{3.28}$$

The parameterization formula (3.26) follows immediately from the relations (3.6) and (3.28). ■

4 Spectral parameterization of the characteristic subalgebra

In this section we complete the parameterization of the characteristic subalgebra in terms of the eigenvalues of quantum supermatrix M. To this end, in the subsection 4.1 we derive parametric expressions for the generators $s_{(1^k)}(M) = s_{[k|1]}(M)$ and $s_{(k)}(M) = s_{[1|k]}(M)$ and prove that the characteristic subalgebra is parameterized by the supersymmetric polynomials. This, in principle, solves the parameterization problem.

In the last subsection 4.2 we derive parameterization formula (4.12) for the Schur function $s_{[m|n]}(M)$. The latter result allows translating the condition of invertibility of $s_{[m|n]}(M)$ into conditions on the spectral variables $\{\mu_i\}_{1\le i\le m}$ and $\{\nu_j\}_{1\le j\le n}$. We shall prove formula (4.12) using yet another series of bilinear relations in the ring Λ of symmetric functions (see lemma 12). Note that relation (4.12) is a particular case of the factorization formula known in the theory of the supersymmetric polynomials [BR, PrT].

4.1 Parameterization of the single column and the single row Schur functions

Proposition 8 *Let $\mathcal{M}(R,F)$ be the $GL(m|n)$ type QM algebra satisfying the conditions i)–v) (see introduction). Then, the parameterization map (3.5), (3.6) assigns the following expressions to the generators $\{s_{[k|1]}(M)\}_{k\geq 0}$ and $\{s_{[1|k]}(M)\}_{k\geq 0}$ of the characteristic subalgebra* $\mathrm{Char}(R,F)$

$$s_{[k|1]}(M) \;\;\mapsto\;\; s_{[k|1]}(\mu,\nu) \;=\; \sum_{r=0}^{k} e_r(q^{-1}\mu)\, h_{k-r}(-q\nu)\,, \tag{4.1}$$

$$s_{[1|k]}(M) \;\;\mapsto\;\; s_{[1|k]}(\mu,\nu) \;=\; \sum_{r=0}^{k} e_r(-q\nu)\, h_{k-r}(q^{-1}\mu)\,. \tag{4.2}$$

Here $e_r(q^{-1}\mu) := \sum_{1\leq i_1<\ldots<i_r\leq m} q^{-r}\mu_{i_1}\mu_{i_2}\ldots\mu_{i_r}$ *and* $h_r(-q\nu) := \sum_{1\leq i_1\leq\ldots\leq i_r\leq n} (-q)^r\nu_{i_1}\nu_{i_2}\ldots\nu_{i_r}$ *are the elementary symmetric and complete symmetric polynomials in m and n variables, respectively ([Mac], section 1.2).*

Proof. We apply induction on k. By the Littlewood-Richardson rule (2.2)

$$s_{[m|n]}(M)\, s_{(1)}(M) = s_{[m|n]^1}(M) + s_{[m|n]_1}(M).$$

Dividing both sides of this equality by $s_{[m|n]}(M)$ and using relations (3.5), (3.6) we get the parameterization formula for $s_{(1)}(M)$

$$s_{(1)}(\mu,\nu) = s_{[1|1]}(\mu,\nu) = e_1(q^{-1}\mu) + e_1(-q\nu)\,,$$

which can be equivalently written as

$$s_{[1|1]}(\mu,\nu) = e_1(q^{-1}\mu) + h_1(-q\nu)\,, \quad \text{or as} \quad s_{[1|1]}(\mu,\nu) = e_1(-q\nu) + h_1(q^{-1}\mu)\,.$$

These formulae are nothing but the eqs. (4.1) and (4.2) in case $k=1$.

Now, assuming the relations (4.1) and (4.2) are valid for all values of the index $1\leq k<p$ we shall prove them for $k=p$. For definiteness, we check the eq.(4.2). The eq.(4.2) is worked out similarly.

Let us write down the image of the relation (3.20) in the characteristic subalgebra, the case $r=0,\ k=p$:

$$\sum_{i=0}^{\min\{p,m\}} (-1)^i s_{(p-i)}(M)\, s_{[m|n]^i}(M) \;=\; \theta(n-p)\, s_{[m|n]_p}(M)\,. \tag{4.3}$$

Here $\theta(i) := 0$ if $i<0$, and $\theta(i) := 1$ otherwise. Substituting the parametric expressions (3.5) and (3.6) for $s_{[m|n]^i}(M)/s_{[m|n]}(M)$ and $s_{[m|n]_p}(M)/s_{[m|n]}(M)$ into (4.3) we find

$$s_{[1|p]}(\mu,\nu) = e_p(-q\nu) - \sum_{i=1}^{p}(-1)^i s_{[1|p-i]}(\mu,\nu)\, e_i(q^{-1}\mu). \tag{4.4}$$

Now, using the induction assumption we substitute expressions (4.2) for the elements $s_{[1|p-i]}(\mu,\nu),\ 1\leq i\leq p,$ into (4.4) and calculate

$$\begin{aligned} s_{[1|p]}(\mu,\nu) \;&=\; e_p(-q\nu) - \sum_{j=0}^{p-1} e_j(-q\nu) \sum_{i=1}^{p-j}\Big((-1)^i h_{p-j-i}(q^{-1}\mu)\, e_i(q^{-1}\mu)\Big) \\ &=\; e_p(-q\nu) + \sum_{j=0}^{p-1} e_j(-q\nu)\, h_{p-j}(q^{-1}\mu) = \sum_{j=0}^{p} e_j(-q\nu)\, h_{p-j}(q^{-1}\mu)\,, \end{aligned} \tag{4.5}$$

where in passing to the second line we used the Wronski relations (3.13) for the substitution

$$\sum_{i=1}^{p-j}(-1)^i h_{p-j-i}(q^{-1}\mu)\, e_i(q^{-1}\mu) \;=\; -h_{p-j}(q^{-1}\mu)\,.$$

Calculation (4.5) completes the inductive proof of the eq.(4.2). ■

Let us recall the definition of the supersymmetric polynomials (see, e.g., [Stem]).

Definition 9 *Let $x = \{x_i\}_{1\le i\le m}$ and $y = \{y_j\}_{1\le j\le n}$ be two sets of independent commutative variables. A polynomial $p \in \mathbb{C}[x,y]$ is said to be supersymmetric if*

a) p is invariant under permutations of $x_1,\dots,x_m$;

b) p is invariant under permutations of $y_1,\dots,y_n$;

c) upon substituting $x_1 = y_1 = t$ in p, the resulting polynomial does not depend on t.

An algebra of the supersymmetric polynomials is further denoted as $T[x,y]$.

Obviously, the polynomials $s_{[k|1]}(\mu,\nu)$ and $s_{[1|k]}(\mu,\nu)$ given by eqs.(4.1), (4.2) satisfy the conditions *a)* and *b)* of the above definition with respect to variables $x_i = q^{-1}\mu_i$ and $y_j = -q\nu_j$. Validity of the property *c)* for them results from the following statement.

Lemma 10 *Denote $\{\mu'\} := \{\mu\}\setminus\{\mu_1\} = \{\mu_i\}_{2\le i\le m}$, $\{\nu'\} := \{\nu\}\setminus\{\nu_1\} = \{\nu_i\}_{2\le j\le n}$. Then the polynomials $s_{[1|k]}(\mu,\nu)$ and $s_{[k|1]}(\mu,\nu)$ satisfy expansions*

$$s_{[1|k]}(\mu,\nu) \;=\; s_{[1|k]}(\mu',\nu') + (q^{-1}\mu_1 - q\nu_1)\sum_{r=0}^{k-1}\Big(q^{-1}\mu_1\Big)^{k-r-1} s_{[1|r]}(\mu',\nu')\,, \tag{4.6}$$

$$s_{[k|1]}(\mu,\nu) \;=\; s_{[k|1]}(\mu',\nu') + (q^{-1}\mu_1 - q\nu_1)\sum_{r=0}^{k-1}(-q\nu_1)^{k-r-1}\, s_{[r|1]}(\mu',\nu'). \tag{4.7}$$

Proof. For the elementary and complete symmetric functions one has

$$e_k(\mu) \;=\; e_k(\mu') + \mu_1\, e_{k-1}(\mu')\,, \qquad h_k(\mu) \;=\; \sum_{r=0}^{k}(\mu_1)^r\, h_{k-r}(\mu')\,.$$

Substituting these formulae into eqs.(4.1) and (4.2) it is easy to derive formulae (4.6), (4.7). ■

We have checked that the polynomials $s_{[k|1]}(\mu,\nu)$ and $s_{[1|k]}(\mu,\nu)$ are supersymmetric. Moreover, as was proved in [PrT] (see theorem (3.1) and proposition (2.3) there) the algebra of supersymmetric polynomials $T[q^{-1}\mu,-q\nu]$ can be generated by any one of the sets $\{s_{[1|k]}(\mu,\nu)\}_{k\ge0}$ or $\{s_{[k|1]}(\mu,\nu)\}_{k\ge0}$. Therefore, as a direct consequence of the proposition 8 we get

Corollary 11 *In the conditions of the proposition 8 an image of the characteristic subalgebra* Char(R,F) *under the parameterization map (3.5), (3.6) is the algebra $T[q^{-1}\mu,-q\nu]$ of the supersymmetric polynomials in the variables $\{q^{-1}\mu_i\}_{1\le i\le m}$ and $\{-q\nu_j\}_{1\le j\le n}$.*

4.2 Parameterization of the Schur function $s_{[m|n]}(M)$

In a derivation of the parameterization formula for $s_{[m|n]}(M)$ we will use A. Kirillov's bilinear relations on the Schur functions (see [Kir, KR, Kl])

$$s_{[m|n]}\, s_{[m|n]} \;=\; s_{[m+1|n]}\, s_{[m-1|n]} \;+\; s_{[m|n+1]}\, s_{[m|n-1]}\,, \qquad \forall\, m,n = 1,2,\dots\,. \tag{4.8}$$

To keep our presentation self-contained let us briefly describe how one can prove them using the Plücker relations and the Jacobi-Trudi formulae. Actually, one can derive a more extensive set of relations.

Lemma 12 *For any integers $a,\, b,\, m,\, n\, :\, 1 \le a \le m,\;\; 1 \le b \le n,$ the equalities*

$$\begin{aligned} s_{[a|b]}\, s_{[m|n]} \;&=\; \sum_{k=\max\{1,a+b-n\}}^{a} (-1)^{a-k} s_{[m|n]_{a+b-k}}\, s_{[a-1|b-1]^{k-1}} \\ &+ \sum_{k=\max\{1,a+b-m\}}^{b} (-1)^{b-k} s_{[m|n]^{a+b-k}}\, s_{[a-1|b-1]_{k-1}} \end{aligned} \tag{4.9}$$

are satisfied in the ring of symmetric functions Λ.

Formulae (4.8) correspond to the choice $a = m$, $b = n$ in eqs.(4.9).

Proof. Applying the Jacobi-Trudi relation (2.26) and using elementary properties of determinants we can write down determinantal presentations for the Schur functions $s_{[m|n]}$ and $s_{[a|b]}$

$$s_{[m|n]} \;=\; \left[\begin{array}{ccccc} h_{n*} & h_{n-1*} & \dots & h_{n-m+1*} & \delta_{m+1*} \\ 1 & 2 & \dots & m & m+1 \end{array}\right], \tag{4.10}$$

$$s_{[a|b]} \;=\; \left[\begin{array}{cccccccc} \delta_{1*} & \delta_{2*} & \dots & \delta_{m+1-a*} & h_{b-m+a-1*} & \dots & h_{b-m+1*} & h_{b-m*} \\ 1 & 2 & \dots & m+1-a & m+2-a & \dots & m & m+1 \end{array}\right]. \tag{4.11}$$

Here we use the matrix notation introduced in (2.23) and the symbols h_{i*} and δ_{i*} denote the following matrix rows

$$h_{i*} := (h_i, h_{i+1}, h_{i+2}, \dots)\,, \qquad \delta_{i*} := (0,\dots,0,\overset{\downarrow i\text{-th place}}{1},0,\dots)\,.$$

Relations (4.9) result from an application of the Plücker relations (2.24) for the set of data $\{k = 1, r_1 = m+1\}$ to the product of determinants (4.10) and (4.11). ■

Now we are ready to prove the main result of this subsection.

Proposition 13 *Let $\mathcal{M}(R,F)$ be the $GL(m|n)$ type QM algebra satisfying the conditions i)–v) (see introduction). Then, the image of the Schur function $s_{[m|n]}(M)$ under the parameterization map (3.5), (3.6) is given by formula*

$$s_{[m|n]}(M) \;\mapsto\; s_{[m|n]}(\mu,\nu) = \prod_{i=1}^{m}\prod_{j=1}^{n}\Big(q^{-1}\mu_i - q\nu_j\Big)\,. \tag{4.12}$$

Therefore, the invertibility of the Schur function $s_{[m|n]}(M)$ implies invertibility of all factors $(q^{-1}\mu_i - q\nu_j)$ in the product (4.12) for $1 \le i \le m$ and $1 \le j \le n$.

Proof. Let us first multiply the image of the relation (4.8) in the characteristic subalgebra $\mathrm{Char}(R,F)$ by $(s_{[m|n]}(M))^{-1}$ and then apply the parameterization map. By virtue of the relations (3.5), (3.6) the resulting formula reads

$$s_{[m|n]}(\mu,\nu) = e_n(-q\nu)\, s_{[m-1|n]}(\mu,\nu) \;+\; e_m(q^{-1}\mu)\, s_{[m|n-1]}(\mu,\nu). \tag{4.13}$$

Noticing that

$$e_m(q^{-1}\mu)|_{\mu_i=0} = 0\,, \quad \forall i=1,\dots,m, \qquad e_n(-q\nu)|_{\nu_j=0} = 0\,, \quad \forall j=1,\dots,n,$$

we obtain for the supersymmetric polynomial $s_{[m|n]}(\mu,\nu)$

$$s_{[m|n]}(\mu,\nu)|_{q^{-1}\mu_i=q\nu_j} = s_{[m|n]}(\mu,\nu)|_{\mu_i=\nu_j=0} = 0\,, \quad \forall i,j\,:\; 1\le i\le m,\; 1\le j\le n. \tag{4.14}$$

As immediately follows from the Jacobi-Trudi relation (2.26), the Schur function $s_{[m|n]}(\mu,\nu)$ is a homogeneous polynomial in the variables $\{q^{-1}\mu_i\}_{1\le i\le m}$ and $\{q\nu_j\}_{1\le j\le n}$ of the order $(m+n)$. Together with eq.(4.14) this implies

$$s_{[m|n]}(\mu,\nu) = \alpha \prod_{i=1}^{m}\prod_{j=1}^{n}\Bigl(q^{-1}\mu_i - q\nu_j\Bigr)\,, \tag{4.15}$$

where α is a numeric factor. To define α, observe the following consequence of (4.2)

$$s_{(k)}(\mu,\nu)|_{\mu_1=\ldots=\mu_m=0} = e_k(-q\nu).$$

Therefore

$$s_{[m|n]}(\mu,\nu)_{|\mu_1=\ldots=\mu_m=0} = \det\Bigl(e_{n-i+j}(-q\nu)\Bigr)_{i,j=1}^{m} = \Bigl(e_n(-q\nu)\Bigr)^m = \Bigl(\prod_{i=1}^{n}(-q\nu_i)\Bigr)^m.$$

Comparing this result with eq.(4.15) at the point $\mu_1=\ldots=\mu_m=0$, we find $\alpha=1$ thereby ending the proof. ∎

Appendix

Here we derive the q-combinatorial relations which are used in the proof of theorem 2.

For an arbitrary set of pairwise different nonvanishing integers b_i, $i=1,2,\dots,k$, we shall prove following relations

$$q^k \;-\; \prod_{i=1}^{k}\frac{(b_i+1)_q}{(b_i)_q} \;=\; -\sum_{j=1}^{k}\frac{q^{-b_j}}{(b_j)_q}\prod_{\substack{i=1\\ i\neq j}}^{k}\frac{(b_i-b_j+1)_q}{(b_i-b_j)_q}\,, \tag{A.1}$$

$$k_q \;=\; \sum_{j=1}^{k}\prod_{\substack{i=1\\ i\neq j}}^{k}\frac{(b_i-b_j+1)_q}{(b_i-b_j)_q}, \tag{A.2}$$

$$\prod_{i=1}^{k}\frac{(b_i+1)_q}{(b_i)_q} \;=\; \sum_{j=1}^{k}\frac{(b_j+k)_q}{k_q(b_j)_q}\prod_{\substack{i=1\\ i\neq j}}^{k}\frac{(b_i-b_j+1)_q}{(b_i-b_j)_q} \tag{A.3}$$

A proof is by induction on k. Checking the case $k=1$ in relations (A.1)–(A.3) is an easy exercise. Now, assuming relations (A.1) are valid for all $k \leq m$ let us transform the expression in the left hand side of eq.(A.1) for $k=m+1$

$$
\begin{aligned}
q^{m+1} - \prod_{i=1}^{m+1} \frac{(b_i+1)_q}{(b_i)_q} &= \left(q^{m+1} - q^m \frac{(b_{m+1}+1)_q}{(b_{m+1})_q} \right) + \left(q^m - \prod_{i=1}^{m} \frac{(b_i+1)_q}{(b_i)_q} \right) \frac{(b_{m+1}+1)_q}{(b_{m+1})_q} \\
&= -\frac{q^{m-b_{m+1}}}{(b_{m+1})_q} - \left(\sum_{j=1}^{m} \frac{q^{-b_j}}{(b_j)_q} \prod_{\substack{i=1\\ i\neq j}}^{m} \frac{(b_i-b_j+1)_q}{(b_i-b_j)_q} \right) \frac{(b_{m+1}+1)_q}{(b_{m+1})_q} . \qquad \text{(A.4)}
\end{aligned}
$$

Here we used formula (A.1), case $k=m$, for the transformation of the last term in the first line. For further transformation we use the formula

$$
\frac{(b_{m+1}+1)_q}{(b_{m+1})_q} = \frac{(b_{m+1}-b+1)_q}{(b_{m+1}-b)_q} - \frac{(b)_q}{(b_{m+1})_q(b-b_{m+1})_q} . \qquad \text{(A.5)}
$$

Substituting b_j, $j=1,2,\ldots m$, for b in eq.(A.5) we continue the calculation

$$
\text{(A.4)} = -\sum_{j=1}^{m} \frac{q^{-b_j}}{(b_j)_q} \prod_{\substack{i=1\\ i\neq j}}^{m+1} \frac{(b_i-b_j+1)_q}{(b_i-b_j)_q} - \frac{q^{-b_{m+1}}}{(b_{m+1})_q} \left(q^m + \sum_{j=1}^{m} \frac{q^{-b_j+b_{m+1}}}{(b_j-b_{m+1})_q} \prod_{\substack{i=1\\ i\neq j}}^{m} \frac{(b_i-b_j+1)_q}{(b_i-b_j)_q} \right),
$$

and then, applying relation (A.1) with the shifted set of integers $b_i \to (b_i - b_{m+1})$, $i=1,2,\ldots,m$, for the transformation of the last term we obtain

$$
\begin{aligned}
&= -\sum_{j=1}^{m} \frac{q^{-b_j}}{(b_j)_q} \prod_{\substack{i=1\\ i\neq j}}^{m+1} \frac{(b_i-b_j+1)_q}{(b_i-b_j)_q} - \frac{q^{-b_{m+1}}}{(b_{m+1})_q} \left(\prod_{i=1}^{m} \frac{(b_i-b_{m+1}+1)_q}{(b_i-b_{m+1})_q} \right) \\
&= -\sum_{j=1}^{m+1} \frac{q^{-b_j}}{(b_j)_q} \prod_{\substack{i=1\\ i\neq j}}^{m+1} \frac{(b_i-b_j+1)_q}{(b_i-b_j)_q}
\end{aligned}
$$

which proves formula (A.1) in the case $k=m+1$.

In order to prove eqs.(A.2), (A.3) we rewrite eq.(A.1), inverting the parameter $q \to q^{-1}$

$$
q^{-k} - \prod_{i=1}^{k} \frac{(b_i+1)_q}{(b_i)_q} = -\sum_{j=1}^{k} \frac{q^{b_j}}{(b_j)_q} \prod_{\substack{i=1\\ i\neq j}}^{k} \frac{(b_i-b_j+1)_q}{(b_i-b_j)_q} , \qquad \text{(A.6)}
$$

and form a linear combination $\left((\text{A.1}) \cdot q^x - (\text{A.6}) \cdot q^{-x}\right)/(q-q^{-1})$, where x takes on integer values. The resulting equality reads

$$
(k+x)_q - (x)_q \prod_{i=1}^{k} \frac{(b_i+1)_q}{(b_i)_q} = \sum_{j=1}^{k} \frac{(b_j-x)_q}{(b_j)_q} \prod_{\substack{i=1\\ i\neq j}}^{k} \frac{(b_i-b_j+1)_q}{(b_i-b_j)_q} . \qquad \text{(A.7)}
$$

The relations (A.2) and (A.3) are particular cases of the relation (A.7) for $x=0$ and $x=-k$, respectively.

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