THE SIGNATURE OF A FIBRE BUNDLE IS MULTIPLICATIVE MOD 4

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Abstract. We express the signature modulo 4 of a closed, oriented, $4k$-dimensional PL manifold as a linear combination of its Euler characteristic and the new absolute torsion invariant defined in Korzeniewski [11]. Let $F \to E \to B$ be a PL fibre bundle, where F, E and B are closed, connected, and compatibly oriented PL manifolds. We give a formula for the absolute torsion of the total space E in terms of the absolute torsion of the base and fibre, and then combine these two results to prove that the signature of E is congruent modulo 4 to the product of the signatures of F and B.

The *signature* $\operatorname{sign}(M) \in \mathbf{Z}$ of a closed, oriented n-manifold M^n is the index of its cup product form when $n \equiv 0 \bmod 4$, and zero otherwise. For an orientable differentiable fibre bundle, Chern, Hirzebruch and Serre [4] proved that $\operatorname{sign}(E) = \operatorname{sign}(F) \cdot \operatorname{sign}(B)$ provided that the fundamental group $\pi_1(B)$ acts trivially on the real cohomology $H^*(F;\mathbb{R})$ of the fibre. In general, the signature is *not* multiplicative for differentiable fibre bundles: the first examples were constructed by Kodaira [10], Atiyah [2], and Hirzebruch [7]. These examples occur in the lowest possible dimension where $\dim B = \dim F = 2$. The total space E of the bundle in these examples has dimension 4 and non-zero signature, but $\operatorname{sign}(B) = \operatorname{sign}(F) = 0$ by convention. Surface bundles over surfaces were studied in detail by W. Meyer [16] and H. Endo [5]. For such bundles, the signature of the total space was shown to be divisible by 4.

The non-multiplicative behaviour of the signature in fibre bundles is explained by the characteristic class formula for $\operatorname{sign}(E)$ due to Atiyah [2], which contains a contribution from the action of $\pi_1(B)$ on the cohomology of the fibre. On the other hand, W. Neumann [19] showed that the signature is multiplicative in many cases where the structural group of the bundle is discrete, and gave a Witt group interpretation of the obstruction to multiplicativity under this assumption.

Stephan Klaus and Peter Teichner [9] have conjectured that the signature is multiplicative modulo 8, provided that $\pi_1(B)$ acts trivially on the mod 2 cohomology $H^*(F;\mathbf{Z}/2)$ of the fibre. This suggests studying the conditions under which the signature is multiplicative modulo other powers of 2. The

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first case is easy since $\mathrm{sign}(E) \equiv \chi(E) \bmod 2$, and the Euler characteristic $\chi(E) = \chi(F) \cdot \chi(B)$ is multiplicative. The following is our main result.

Theorem A. *Let $F \xrightarrow{q} E \xrightarrow{p} B$ be a PL fibre bundle of closed, connected, compatibly oriented PL manifolds. Then*

$$\mathrm{sign}(E) \equiv \mathrm{sign}(F) \cdot \mathrm{sign}(B) \bmod 4 \ .$$

We will use the new absolute torsion invariant $\tau^{NEW}(E) \in K_1(\mathbf{Z}[\pi_1(E)])$ defined and investigated in [11]. The new invariant is defined more generally, and has better properties than the "round torsion" invariant introduced in [24]. We show in Theorem 3.3 that the value of $\mathrm{sign}(E) \bmod 4$ can be expressed in terms of $\tau^{NEW}(E)$ and $\chi(E)$. Our main result follows from this fact, and a general formula for $\tau^{NEW}(E)$ in terms of $\tau^{NEW}(F)$ and $\tau^{NEW}(B)$.

Theorem B. *Let $F \xrightarrow{q} E \xrightarrow{p} B$ be a PL fibre bundle of closed, connected, compatibly oriented PL manifolds. Then, if $n = \dim E$,*

$$\tau^{NEW}(E) = p^!(\tau^{NEW}(B)) + \chi(B) q_*(\tau^{NEW}(F)) \in \widehat{H}^n(\mathbf{Z}/2; K_1(\mathbf{Z}\pi_1(E))) \ .$$

In this statement the maps $p^!$ and q_* are the transfer and push-forward maps associated to the fibre bundle (see [12], [13], and [14] for the surgery transfer).

In Section 1 we describe some of the results of [11], including the definition and main properties of the absolute torsion invariant. In Section 2 we use the new invariant to correct the definition of the round *simple* symmetric L-groups of [6], so they fit in the long exact sequence

$$\cdots \to L^n_{rs}(R) \longrightarrow L^n_{rh}(R) \xrightarrow{\tau^{NEW}} \widehat{H}^n(\mathbf{Z}/2; K_1(R)) \longrightarrow L^{n-1}_{rs}(R) \to \dots$$

Section 3 establishes the relationship between the signature mod 4 and the new torsion invariant of a $4k$-dimensional Poincaré space (see Theorem 3.3). Sections 4 and 5 study the set $\mathcal{S}_\bullet(X)$ of *pointed torsion structures* (PTS) on manifolds. In Theorem 5.2 we give a formula expressing the absolute torsion invariant for a closed manifold X in terms of a pairing

$$\Phi \colon \mathcal{S}_\bullet(X) \times \mathcal{S}_\bullet(X) \to K_1(\mathbf{Z}[\pi_1(X, x_0)])$$

After this preparation, we turn to the geometric applications of absolute torsion for fibre bundles. In Section 6 we define the fibre transport and transfer map $p^!$ associated to a fibre bundle, and establish some useful properties. Sections 7 and 8 deal with pointed torsion structures on fibre bundles, and the Φ-pairing for the total space E. Section 9 uses the results of Section 8 to prove Theorem B, and then Section 10 uses Theorem B to prove Theorem A. Section 11 is a brief discussion of fibrations of Poincaré spaces.

The last part of the paper (Section 12) contains the algebra needed to describe the absolute torsion of signed filtered chain complexes, such as the chain complexes of the total spaces of fibre bundles. This may be of independent interest, and this part can be read independently of the rest of the paper. We give a purely algebraic treatment of the absolute torsion of filtered chain complexes: the main result is the Invariance Theorem 12.14 identifying the torsion of a contractible filtered chain complex with the torsion of the contractible chain complex (in the derived category) of filtration quotients.

Contents

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1. Signed K-theory

In this section we recall the algebraic K-groups $K_0(R)$, $K_1(R)$ of a ring R, and the algebraic theory of torsion of Ranicki [24], [25], [26] and Korzeniewski [11]. The absolute torsion invariant $\tau^{NEW}(f) \in K_1(R)$ of [11] is based on the definition of "round torsion" given in [26, 7.20(ii)], but the sign correction terms are different.

The *class group* $K_0(\mathbb{A})$ of an additive category $\mathbb{A}$ is the abelian group with one generator $[M]$ for each object M in $\mathbb{A}$, and relations

(i) $[M] = [M']$ if M is isomorphic to M',

(ii) $[M \oplus N] = [M] + [N]$ for objects M, N in $\mathbb{A}$.

The *isomorphism torsion group* $K_1^{iso}(\mathbb{A})$ is the abelian group with one generator $\tau^{iso}(f)$ for each isomorphism $f\colon M \to N$ in $\mathbb{A}$, and relations

(i) $\tau^{iso}(gf) = \tau^{iso}(f) + \tau^{iso}(g)$ for isomorphisms $f\colon M \to N$, $g\colon N \to P$.

(ii) $\tau^{iso}(\begin{pmatrix} f & e \\ 0 & f' \end{pmatrix}) = \tau^{iso}(f) + \tau^{iso}(f')$ for isomorphisms $f\colon M \to N$, $f'\colon M' \to N'$ and any morphism $e\colon M' \to N$.

The class group $K_0(R)$ and the isomorphism torsion group $K_1^{iso}(R)$ of a ring R are the class and torsion groups of the additive category $\mathbb{A}(R)$ of based f.g. (= finitely generated) free R-modules. To obtain values in $K_1(R)$, we compose with the split surjection $K_1^{iso}(R) \to K_1(R)$ given by [24, Prop. 5.1], and let $\tau^{iso}(f) \mapsto \tau(f) \in K_1(R)$. In particular, there are defined isomorphisms

$$\begin{aligned} K_0(\mathbf{Z}) &\xrightarrow{\cong} \mathbf{Z} \ ; \ [M] \mapsto \mathrm{rank}_{\mathbf{Z}} M \ , \\ K_1(\mathbf{Z}) &\xrightarrow{\cong} \mathbf{Z}/2 \ ; \ [M,f] \mapsto \tau(\det(f)) \ . \end{aligned}$$

The *reduced class group* $\widetilde{K}_0(R)$ and the *reduced torsion group* $\widetilde{K}_1(R)$ are defined by

$$\widetilde{K}_i(R) = \mathrm{coker}(K_i(\mathbf{Z}) \to K_i(R)) \ (i = 0, 1) \ .$$

We shall assume that R is such that the rank of f.g. free R-modules is well-defined, so that $K_0(R) = K_0(\mathbf{Z}) \oplus \widetilde{K}_0(R)$.

We shall consider only *finite* chain complexes over R, meaning finite dimensional, positive chain complexes of f.g. free R-modules. The *Euler characteristic* of a finite R-module chain complex C is the chain homotopy invariant

$$\chi(C) = \sum_{r=0}^{\infty} (-1)^r \,\mathrm{rank}_R C_r \in \mathbf{Z} \ .$$

Our assumptions on C mean that $C_r = 0$ if $r < 0$ or $r > N$ for some integer N, so this sum is defined. A finite chain complex C is *round* if $\chi(C) = 0 \in \mathbf{Z}$.

The *torsion* $\tau(C) \in K_1(R)$ of a contractible finite based R-module chain complex

$$C: \ 0 \to \cdots \to C_{r+1} \xrightarrow{d} C_r \xrightarrow{d} C_{r-1} \to \cdots \to 0$$

was originally defined by Whitehead [29] to be

$$\tau(C) = \tau(d + \Gamma) \in K_1(R) \ ,$$

with the isomorphism

$$d + \Gamma \colon C_{odd} = \sum_{r \text{ odd}} C_r \longrightarrow C_{even} = \sum_{r \text{ even}} C_r$$

defined using any contraction $\Gamma\colon 1 \simeq 0 \colon C \to C$. The value of $\tau(C)$ is independent of the choice of Γ.

The torsion of a chain equivalence $f\colon C \to D$ of finite based R-module chain complexes can be defined by

$$\tau(f) = \tau(\mathcal{C}(f)) \in K_1(R)$$

with $\mathcal{C}(f)$ the algebraic mapping cone

$$d_{\mathcal{C}(f)} = \begin{pmatrix} d_D & (-)^{r-1} f \\ 0 & d_C \end{pmatrix} : \mathcal{C}(f)_r = D_r \oplus C_{r-1} \to \mathcal{C}(f)_{r-1} = D_{r-1} \oplus C_{r-2} \ .$$

However, as noted in [24], this definition of torsion only has good sum and composition properties modulo the sign subgroup

$$\mathrm{im}(K_1(\mathbf{Z}) \to K_1(R)) = \{\tau(\pm 1)\} \subseteq K_1(R) .$$

The new absolute torsion uses the notion of a *signed* chain complex [11, Def. 5]. This is a pair (C, η_C), where C is as above and $\eta_C \in \mathrm{Im}(K_1(\mathbf{Z}) \to K_1(R))$. The image of $K_1(\mathbf{Z}) = \mathbf{Z}/2$ in $K_1(R)$ is given by $a \mapsto a \cdot \tau(-1)$, but the symbol

$$\tau(-1) := \ \tau(-1\colon R \to R) \in K_1(R)$$

will usually be suppressed to simplify the notation.

If (C, η_C) is a signed chain complex with C contractible, then its absolute torsion is defined as

$$\tau^{NEW}(C, \eta_C) = \tau(C) + \eta_C \ .$$

If (C, η_C) is a signed complex, then the suspension SC of C, where $(SC)_r = C_{r-1}$, has the sign $\eta_{SC} = -\eta_C$. The sum $(C \oplus D, \eta_{C\oplus D})$ of two signed complexes involves some more sign terms from $K_1(\mathbf{Z}) = \mathbf{Z}/2$. Let

$$\epsilon(M, N) := \mathrm{rank}_R(M) \cdot \mathrm{rank}_R(N) \bmod 2$$

for any free R-modules M and N, and define [24, p. 213]:

$$\beta(C, D) = \sum_{i>j} (\epsilon(C_{2i}, D_{2j}) - \epsilon(C_{2i+1}, D_{2j+1})) \in \mathrm{Im}(K_1(Z) \to K_1(R)) \ .$$

This is just the difference of the torsions of the permutation isomorphisms $(C \oplus D)_{even} \to C_{even} \oplus D_{even}$ and $(C \oplus D)_{odd} \to C_{odd} \oplus D_{odd}$. We define

$$\eta_{C\oplus D} = \eta_C + \eta_D - \beta(C, D) + \mathrm{rank}(C_{odd}) \cdot \chi(D) \ .$$

From now on, we will usually denote a signed chain complex (C, η_C) just by C, even though all the formulas will involve these signs. It turns out that the absolute torsion of a Poincaré complex C is independent of the choice of sign η_C, and this justifies our abbreviated notation (see [11, Prop. 26.6]).

The absolute torsion of a chain equivalence $f\colon C \to D$ of finite based signed complexes is defined to be

$$\tau^{NEW}(f) = \tau^{NEW}(\mathcal{C}(f)) \in K_1(R) \ ,$$

with $\eta_{\mathcal{C}(f)} = \eta_{D\oplus SC}$. The extra sign terms needed (in comparison with the invariant of [24]) can be seen from the formula

$$\tau^{NEW}(f) = \tau(C(f)) - \beta(D, SC) + \mathrm{rank}_R(D_{odd}) \cdot \chi(SC) + \eta_D - \eta_C \in K_1(R)$$

given in [11, Lemma 12].

For chain equivalences of signed complexes, the absolute torsion invariant is a chain homotopy invariant, which is additive under compositions and direct sums (by an appropriate modification of the proof of [24, Prop. 4.2, 4.4, 4.5]).

Proposition 1.1 ([11, Prop. 13])**.** *If $f\colon C \to D$ and $g\colon D \to E$ are chain equivalences of finite based signed chain complexes, then*

$$\tau^{NEW}(g \circ f) = \tau^{NEW}(f) + \tau^{NEW}(g) \in K_1(R) \ .$$

If $f\colon C \to D$ and $f'\colon C' \to D'$ are chain equivalences of finite based signed chain complexes, then

$$\tau^{NEW}(f \oplus f') = \tau^{NEW}(f) + \tau^{NEW}(f') \in K_1(R) \ .$$

The formula for the absolute torsion is a modification of the definition in [24, p. 223, 226], which gave an invariant with values in $K_1^{iso}(\mathbb{A})$ for chain complexes over an additive category $\mathbb{A}$. In that setting, the signs η_C are in the image of the skew-symmetric pairing

$$\epsilon\colon K_0(\mathbb{A}) \otimes K_0(\mathbb{A}) \to K_1^{iso}(\mathbb{A})$$

defined in [24, Prop. 2.2]: for any two objects M, N of $\mathbb{A}$,

$$\epsilon([M],[N]) = \tau^{iso}(M \oplus N \to N \oplus M) \in K_1^{iso}(\mathbb{A})$$

is the torsion of the interchange map. See [11, Lemma 7] for the properties of this pairing.

This extra generality will be useful in Section 12 where we discuss the absolute torsion of filtered chain complexes. A finite chain complex over $\mathbb{A}$ is an object in the derived category $\mathbb{D}(\mathbb{A})$, whose morphisms are the chain homotopy classes of chain maps.

Definition 1.2. The *signed derived category* $\mathbb{SD}(\mathbb{A})$ of an additive category $\mathbb{A}$ is the additive category with objects signed complexes (C, η_C) in $\mathbb{A}$, and morphisms the chain homotopy classes of chain maps. An isomorphism in $\mathbb{SD}(\mathbb{A})$ is a chain homotopy class of chain equivalences.

If $f\colon C \to D$ is a morphism in $\mathbb{SD}(\mathbb{A})$, then the mapping cone $\mathcal{C}(f)$ is an object in $\mathbb{SD}(\mathbb{A})$ under our sign conventions. The absolute torsion gives an invariant in $\tau^{NEW}(f) \in K_1^{iso}(\mathbb{SD}(\mathbb{A}))$ for an isomorphism $f\colon C \to D$ in $\mathbb{SD}(\mathbb{A})$. We refer to [11] for the definition and properties of the absolute torsion in this setting, but we note that the map

$$i_*\colon K_1^{iso}(\mathbb{SD}(\mathbb{A})) \to K_1^{iso}(\mathbb{A})$$

defined by $i_*\tau^{iso}(f) = \tau^{NEW}(f)$ is a split surjection. The map i_* has the naturality property

$$i_*\epsilon(C, D) = \epsilon(\chi(C), \chi(D)) \in K_1^{iso}(\mathbb{A})$$

for any objects (C, η_C), (D, η_D) in $\mathbb{SD}(\mathbb{A})$. We will also use the relation

$$i_*\epsilon(C, SD) = -i_*\epsilon(C, D) \in K_1^{iso}(\mathbb{A})$$

and the formula

$$\eta_{C(f)} = \eta_{D\oplus SC} = \eta_D - \eta_C - \beta(D, SC) + \epsilon(D_{odd}, \chi(SC)) \in K_1^{iso}(\mathbb{SD}(\mathbb{A}))$$

for the mapping cone sign in our sign calculations (see Section 12).

2. Round L-theory

In this section we recall the round L-theory of Hambleton, Ranicki and Taylor [6]. Note that the round torsion is *not* a round cobordism invariant, contrary to the assertion in [26, p. 190], but the new absolute torsion of [11] does have this important property. Moreover, all the results claimed in [26, 7.22(ii)] and [6, p. 135] do hold after replacing the round torsion with the new absolute torsion invariant.

A finite, oriented Poincaré duality space X of dimension n, with universal covering $\widetilde{X}$, has an associated symmetric chain complex (C, φ), as defined in [22]. Here $C := C(\widetilde{X})$ is a n-dimensional finite chain complex over $R = \mathbf{Z}[\pi_1(X)]$ and $\varphi_0 \colon C^{n-*} \to C_*$ is a chain equivalence. The definition of this symmetric structure φ uses the standard involution $g \mapsto g^{-1}$ on the group ring. More generally, for any ring R with involution α, and unit $\varepsilon = \pm 1$, Ranicki [22] defines ε-symmetric structures on algebraic chain complexes and the notion of a symmetric (algebraic) Poincaré complex. The cobordism group of n-dimensional symmetric Poincaré is denoted $L^n(R, \varepsilon)$ [22, §3]. The additive inverse is defined to be $-(C, \varphi) := (C, -\varphi)$. In the rest of the paper we will assume that $\varepsilon = +1$, and denote these groups just by $L^n(R)$.

If we restrict to round complexes in both objects and bordisms and $n > 0$, we get the round symmetric L-groups $L^n_r(R)$. The groups $L^0_r(R)$ is the Witt group of formal differences $(M, \psi) - (M', \psi')$ of non-singular symmetric forms over (R, α) with $\operatorname{rank}_R M = \operatorname{rank}_R M'$ [6, Prop. 2.1]. We can compare the round L-groups to the ordinary ones by a long exact sequence. The anti-automorphism on R induces a $\mathbf{Z}/2$-action on $K_0(R)$ which is the identity on the subgroup $K_0(\mathbf{Z}) = \mathbf{Z}$.

Proposition 2.1 ([6, Prop. 3.2])**.** *For any ring with involution (R, α), there is a long exact sequence*

$$\cdots \to \widehat{H}^{n+1}(\mathbf{Z}/2; K_0(\mathbf{Z})) \to L^n_r(R) \to L^n(R) \to \widehat{H}^n(\mathbf{Z}/2; K_0(\mathbf{Z})) \to \dots$$

where the map $L^n(R) \to \widehat{H}^n(\mathbf{Z}/2; K_0(\mathbf{Z}))$ is defined by $(C, \varphi) \mapsto \chi(C)$.

We will also need to compare these L-groups to the cobordism groups $L^n_{ev}(R)$ of n-dimensional symmetric Poincaré complexes (C, φ) with *even* Euler characteristic $\chi(C) \equiv 0 \bmod 2$ for objects and bordisms:

$$\begin{tikzcd}
\widehat{H}^{n+1}(\mathbf{Z}/2; \mathbf{Z}/2) \arrow[dr] \arrow[rr, bend left] & & \widehat{H}^n(\mathbf{Z}/2; \mathbf{Z}) \arrow[dr] \arrow[rr, bend left] & & L^{n-1}_r(R) \\
& L^n_{ev}(R) \arrow[ur] \arrow[dr] & & \widehat{H}^n(\mathbf{Z}/2; \mathbf{Z}) \arrow[ur] \arrow[dr] & \\
L^n_r(R) \arrow[ur] \arrow[rr, bend right] & & L^n(R) \arrow[ur] \arrow[rr, bend right] & & \widehat{H}^n(\mathbf{Z}/2; \mathbf{Z}/2)
\end{tikzcd}$$

The map $L^n_{ev}(R) \to \widehat{H}^n(\mathbf{Z}/2;\mathbf{Z})$ is given by $(C,\varphi) \mapsto \chi(C)/2$.

In order to define the *torsion* of a symmetric Poincaré chain complex over a ring with involution (R,α), we need to work with *signed* chain complexes, in which each free chain module C_r has a preferred basis, along with a single choice of sign η_C. We denote by $L^n_{rh}(R)$ the corresponding cobordism group of round, n-dimensional, finite, signed symmetric Poincaré chain complexes of finitely generated based free R-modules. By forgetting the preferred bases and signs we can identify $L^n_{rh}(R) = L^n_r(R)$. Similarly, we can identify the based and unbased versions of the even L-groups $L^n_{ev\,h}(R) = L^n_{ev}(R)$.

Now let (C,φ) denote an n-dimensional signed symmetric Poincaré chain complex, of finitely generated based free R-modules, over a ring with involution (R,α). The absolute torsion

$$\tau^{NEW}(C,\varphi) := \tau^{NEW}(\varphi_0\colon C^{n-*} \to C_*) \in K_1(R)$$

is now defined, as the absolute torsion of the chain equivalence φ_0 with respect to the given bases on C_* and the dual bases on C^{n-*}. Recall that the differential for C^{n-*} has the sign convention

$$d_{C^{n-*}} = (-1)^r d_C^*\colon C^{n-r} \to C^{n-r+1}\ .$$

The dual signed chain complex $(C^{n-*}, \eta_{C^{n-*}})$ is given the sign

$$\eta_{C^{n-*}} = \eta_C + \beta(C,C) + \alpha_n(C) \in \mathrm{Im}(K_1(\mathbf{Z}) \to K_1(R)) \tag{2.2}$$

where

$$\alpha_n(C) = \sum_{r\equiv n+2,n+3(\bmod 4)} \mathrm{rank}_R(C^r) \in \mathrm{Im}(K_1(\mathbf{Z}) \to K_1(R))\ .$$

With this convention, the absolute torsion $\tau^{NEW}(C,\varphi)$ is independent of the initial choice of sign η_C (see [11, Prop. 26]). We say that (C,φ) is *simple* if $\tau^{NEW}(C,\varphi) = 0$.

Lemma 2.3 ([11, Lemma 10])**.** *If $\varphi_0\colon C^{n-*} \to C$ is an isomorphism, then*

$$\tau^{NEW}(C,\varphi) = \sum_{r=0}^{n} (-1)^r \tau(\varphi_0\colon C^{n-r} \to C_r) + \beta(C,C) + \alpha_n(C) \in K_1(R)\ .$$

Proof. The absolute torsion of a chain isomorphism $f\colon C \to D$ of signed chain complexes is just

$$\tau^{NEW}(f) = \sum_{r=0}^{n} (-1)^r \tau(f_r\colon C_r \to D_r) - \eta_C + \eta_D$$

according to [11, Def. 6]. The sign terms for φ_0 appear above. $\square$

The absolute torsion has the symmetry property [11, Prop. 26.2] (compare [26, 7.20(ii)])

$$\tau^{NEW}(C,\varphi)^* = (-1)^n \tau^{NEW}(C,\varphi) + \frac{n(n+1)}{2}\chi(C)$$

where $*\colon K_1(R) \to K_1(R)$ denotes the involution on K-theory induced by "α-conjugate-transpose" of matrices. The sign term vanishes over rings R for which skew-hermitian forms necessarily have even rank (such as an integral group ring $R = \mathbf{Z}[\pi]$). Also

$$\tau^{NEW}(C,-\varphi) = \tau^{NEW}(C,\varphi) + \chi(C)$$

(see [11, Prop. 26.5]). Notice that the extra sign terms in both formulas vanish for round or even symmetric complexes. Moreover, the absolute torsion is additive under direct sums of symmetric n-complexes. It follows that $\tau^{NEW}(C,\varphi)$ defines an additive map into $\widehat{H}^n(\mathbf{Z}/2; K_1(R))$ for round or even symmetric complexes.

For our applications, we will need the cobordism groups of simple round or even Poincaré complexes, and their relation to the L-groups already mentioned. However, to give a well-defined homomorphism

$$L^n_{ev}(R) \to \widehat{H}^n(\mathbf{Z}/2; K_1(R))$$

we need the invariant associated to a null-bordant symmetric complex to be trivial.

Example 2.4. Let $R = \mathbf{Z}$ with trivial involution. A zero-dimensional symmetric Poincaré complex over $\mathbf{Z}$ is just unimodular, symmetric bilinear form (L,h) on a finitely-generated free abelian group. In this case,

$$\tau^{NEW}(L,h) = \tau(\det h) \in \mathbf{Z}/2$$

by Lemma 2.3. To compare this invariant to the determinant, we map $\mathbf{Z}/2 \to \mathbf{Z}/4$ by $a \mapsto 2a \bmod 4$. Then $2\tau(\det h) \equiv \det h - 1 \bmod 4$ and

$$2\tau^{NEW}(L,h) \equiv \det h - 1 \bmod 4\ .$$

Note that for any form (L,h) of even rank, the absolute torsion is just $\tau(\det h)$. In particular, the absolute torsion of the hyperbolic plane $\mathbf{H}$ is non-zero, so $\mathbf{H}$ represents a non-trivial element of $L^0_{ev}(\mathbf{Z})$.

Lemma 2.5 ([11, Prop. 26.4])**.** *If (C,φ) is homotopy equivalent to the boundary of an $(n+1)$-dimensional symmetric signed complex (D,Φ), then*

$$\tau^{NEW}(C,\varphi) = (-1)^{n+1}\tau^{NEW}(C\to\partial D)^* - \tau^{NEW}(C\to\partial D) + \frac{(n+1)(n+2)}{2}\chi(D)\ .$$

Once again, the extra sign term vanishes for round or even complexes.

Corollary 2.6. *The absolute torsion $\tau^{NEW}(C,\varphi)$ induces a well-defined homomorphism $L^n_{ev}(R) \to \widehat{H}^n(\mathbf{Z}/2; K_1(R))$.*

The cobordism group of simple round (or even) symmetric Poincaré complexes is denoted $L^n_{rs}(R)$ or $L^n_{ev\,s}(R)$, following [6, p. 135].

Proposition 2.7. *There is a commutative braid of exact sequences*

$$\begin{tikzcd}
\widehat{H}^{n+1}(\mathbf{Z}/2;\mathbf{Z}) \arrow[dr] \arrow[rr, bend left] & & L^n_{rh}(R) \arrow[dr] \arrow[rr, bend left] & & \widehat{H}^n(\mathbf{Z}/2;K_1(R)) \\
& L^n_{rs}(R) \arrow[ur] \arrow[dr] & & L^n_{ev\,h}(R) \arrow[ur] \arrow[dr] & \\
\widehat{H}^{n+1}(\mathbf{Z}/2;K_1(R)) \arrow[ur] \arrow[rr, bend right] & & L^n_{ev\,s}(R) \arrow[ur] \arrow[rr, bend right] & & \widehat{H}^n(\mathbf{Z}/2;\mathbf{Z})
\end{tikzcd}$$

where the maps $L^n_{rh}(R) \to \widehat{H}^n(\mathbf{Z}/2;K_1(R))$ *and* $L^n_{ev\,h}(R) \to \widehat{H}^n(\mathbf{Z}/2;K_1(R))$ *are induced by the absolute torsion* $(C,\varphi) \mapsto \tau^{NEW}(\varphi_0) \in K_1(R)$. *The maps to* $\widehat{H}^n(\mathbf{Z}/2;\mathbf{Z})$ *are given by* $(C,\varphi) \mapsto \chi(C)/2$.

Lemma 2.8. *Let R be a ring with involution, and (C,φ) be an even based symmetric Poincaré complex over R, with $\chi(C) \equiv 0 \bmod 4$ and $\{\tau^{NEW}(C,\varphi)\} = 0 \in \widehat{H}^n(\mathbf{Z}/2;K_1(R))$. Then (C,φ) is even Poincaré bordant to a round simple based symmetric Poincaré complex (C',φ').*

Proof. This follows from the comparison braid given in Proposition 2.7 relating $L^n_{rh}(R)$ and $L^n_{ev\,h}(R)$. ☐

Example 2.9. For the special case $R = \mathbf{Z}$ and $n = 0$, we can substitute the calculation $L^0_{rh}(\mathbf{Z}) = 2\mathbf{Z}$ from [6, 4.2]:

$$\begin{tikzcd}
0 \arrow[dr] \arrow[rr, bend left] & & 2\mathbf{Z} \arrow[dr] \arrow[rr, bend left] & & \mathbf{Z}/2 \\
& 4\mathbf{Z} \arrow[ur] \arrow[dr] & & L^0_{ev\,h}(\mathbf{Z}) \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] \arrow[rr, bend right] & & L^0_{ev\,s}(\mathbf{Z}) \arrow[ur] \arrow[rr, bend right] & & \mathbf{Z}/2
\end{tikzcd}$$

We obtain $L^0_{rs}(\mathbf{Z}) = 4\mathbf{Z}$. Since the hyperbolic plane $\mathbf{H}$ represents a non-zero element of $L^0_{ev\,h}(\mathbf{Z})$ with $\chi \equiv 2 \mod 4$ and $\det(\mathbf{H}) = -1$, we have $L^0_{ev\,h}(\mathbf{Z}) = 2\mathbf{Z} \oplus \mathbf{Z}/2$, generated by $\langle 1 \rangle \perp \langle 1 \rangle$ and $\mathbf{H}$, and $L^0_{ev\,s}(\mathbf{Z}) = 2\mathbf{Z}$ generated by $\langle 1 \rangle \perp \langle 1 \rangle$. A diagram chase gives $L^2_{ev\,h}(\mathbf{Z}) = \mathbf{Z}/2$ and the fact that $\{\tau^{NEW}(C,\varphi)\} = 0 \in \widehat{H}^0(\mathbf{Z}/2;K_1(\mathbf{Z}))$ if $\dim C \equiv 2 \bmod 4$.

We will need the following calculation in a later section.

Lemma 2.10. *Let C be a symmetric Poincaré complex over $R = \mathbf{Z}[\mathbf{Z}/2]$. The absolute torsion $\{\tau^{NEW}(C,\varphi)\} = 0 \in \widehat{H}^0(\mathbf{Z}/2; K_1(\mathbf{Z}[\mathbf{Z}/2]))$ if $\dim C \equiv 2 \bmod 4$.*

Proof. Since $\dim C \equiv 2 \bmod 4$, C is an even symmetric Poincaré complex, so $\tau^{NEW}(C,\varphi)$ defines an element of $\widehat{H}^0(\mathbf{Z}/2; K_1(R))$. Let $\mathbf{Z}/2 = \langle T\rangle$ denote a generator of the group of order two, and we have an inclusion $\mathbf{Z}[\mathbf{Z}/2] \subset \mathbf{Z}\oplus\mathbf{Z}$ of rings given by $T \mapsto \pm 1$. The induced map

$$\widehat{H}^0(\mathbf{Z}/2; K_1(\mathbf{Z}[\mathbf{Z}/2])) \to \widehat{H}^0(\mathbf{Z}/2; K_1(\mathbf{Z}\oplus\mathbf{Z}))$$

is injective, and the absolute torsion map $L^{4k+2}_{ev\,h}(\mathbf{Z}[\mathbf{Z}/2]) \to \widehat{H}^0(\mathbf{Z}/2; K_1(\mathbf{Z}[\mathbf{Z}/2]))$ composed with this injection factors through $L^2_{ev\,h}(\mathbf{Z}) \to \widehat{H}^0(\mathbf{Z}/2; K_1(\mathbf{Z}))$, which we have seen is the zero map. □

3. Absolute torsion and signatures

Let X be a finite, connected CW-complex, and fix a base point $x_0 \in X$. We choose orientations and lifts to the universal covering $\widetilde{X}$ for each cell in X, together with a sign η_X. The cellular chain complex $(C(\widetilde{X}), \eta_X)$ is now a finitely-generated, signed, based chain complex over $R = \mathbf{Z}[\pi_1(X, x_0)]$. If X is an oriented, finite, geometric Poincaré n-complex, then we define

$$\tau^{NEW}(X) = \tau^{NEW}(C(\widetilde{X}), \varphi_0) \in \widehat{H}^n(\mathbf{Z}/2; K_1(\mathbf{Z}[\pi_1(X, x_0)]))$$

where φ_0 is the duality map from the symmetric structure on $C(\widetilde{X})$ defined by Ranicki [22, p. 92]. The symmetric structure is a homotopy invariant of X.

Lemma 3.1. *Let X be an oriented, finite, Poincaré complex of dimension n. The absolute torsion $\tau^{NEW}(X) \in \widehat{H}^n(\mathbf{Z}/2; K_1(\mathbf{Z}[\pi_1(X)]))$ depends only on the homotopy type of X. In particular, the absolute torsion is independent of the choice of preferred base for $C(\widetilde{X})$.*

Proof. The absolute torsion $\tau^{NEW}(X)$ is independent of all the choices made by [11, Prop. 26]. □

The *symmetric signature* of X is the element $\sigma^*(X) \in L^n(\mathbf{Z}[\pi_1(X, x_0)])$ given by the bordism class of the symmetric structure $(C(\widetilde{X}), \varphi)$. If $\chi(X) \equiv 0 \bmod 2$, then we get an even symmetric signature $\sigma^*_{ev}(X) \in L^n_{ev}(\mathbf{Z}[\pi_1(X)])$. In this case, $\tau^{NEW}(X)$ is a bordism invariant: it is the image of $\sigma^*_{ev}(X)$ under the homomorphism $L^n_{ev}(R) \to \widehat{H}^n(\mathbf{Z}/2; K_1(R))$ from Corollary 2.6. Similarly, if $\chi(X) = 0$ the image of $\sigma^*_r(X) \in L^n_r(\mathbf{Z}[\pi_1(X)])$ is again the absolute torsion $\tau^{NEW}(X)$.

Example 3.2. The manifold $X = \mathbb{C}P^2 \# \mathbb{C}P^2 \# 2(S^1\times S^3)$ has a round symmetric signature, whose absolute torsion invariant in $\widehat{H}^0(\mathbf{Z}/2, K_1(\mathbf{Z}))$ is non-zero. Its symmetric signature represents a generator of $L^4_{rh}(\mathbf{Z}) = 2\mathbf{Z}$.

By composing with the augmentation map $\epsilon\colon \mathbf{Z}[\pi_1(X)] \to \mathbf{Z}$, setting $\epsilon(g) = 1$ for all $g \in \pi_1(X, x_0)$, we get a symmetric signature $\sigma^*(X) \in L^n(\mathbf{Z})$ for any oriented, finite Poincaré complex of dimension n (e.g. any closed, oriented n-manifold). We define the *reduced* absolute torsion, as the image

$$\bar{\tau}^{NEW}(X) := \epsilon_*(\tau^{NEW}(X)) \in K_1(\mathbf{Z})$$

of the absolute torsion. This invariant is computed from the chain complex of X, instead of $\widetilde{X}$. Our main result depends on a mod 4 relationship between the ordinary signature of a manifold and the absolute torsion of its symmetric signature over $\mathbf{Z}$. This is an algebraic fact.

Theorem 3.3. *Let (C, φ) be a finite, based, $4k$-dimensional algebraic Poincaré complex over $\mathbf{Z}$. Then*

$$\mathrm{sign}(C) \equiv 2\tau^{NEW}(C, \varphi) + (2k+1)\chi(C) \bmod 4 .$$

Proof. The first step is to show that the right-hand side is an algebraic cobordism invariant for Poincaré complexes over $\mathbf{Z}$. Suppose that $(C \oplus C' \to D, \delta\varphi, \varphi \oplus \varphi')$ is an algebraic cobordism. Since $\dim C = 4k$, we have

$$\tau^{NEW}(C, \varphi) + \tau^{NEW}(C', -\varphi') = (4k+1)(2k+1)\chi(D) \in K_1(\mathbf{Z})$$

by Lemma 2.5. However, $\chi(D) = \frac{1}{2}(\chi(C) + \chi(C')) \in \mathbf{Z}$ and $\tau^{NEW}(C', -\varphi') = \tau^{NEW}(C', \varphi') + \chi(C')$, so that

$$2\tau^{NEW}(C, \varphi) + 2\tau^{NEW}(C', \varphi') \equiv (2k+1)\chi(C) + (2k-1)\chi(C') \bmod 4 .$$

Therefore

$$2\tau^{NEW}(C, \varphi) + (2k+1)\chi(C) \equiv 2\tau^{NEW}(C', \varphi') + (2k+1)\chi(C') \bmod 4$$

as required (since $2\chi(C) \equiv 2\chi(C') \bmod 4$). Next, we use the fact that every $4k$-dimensional symmetric Poincaré complex over $\mathbf{Z}$ is algebraically cobordant to a complex (C, φ) which is concentrated in dimension $2k$ (see [22, Prop. 4.5]). In that case φ_0 is an isomorphism, so

$$\tau^{NEW}(C, \varphi) = \tau(\det \varphi_0) + \beta(C, C) + \alpha_{4k}(C)$$

by Lemma 2.3. But the β-term is zero in this case, and $\alpha_{4k}(C)$ contributes $\chi(C)$ when k is odd and zero otherwise. We may express this as $\alpha_{4k}(C) = k\chi(C) \in K_1(\mathbf{Z})$. Therefore

$$\tau^{NEW}(C, \varphi) = \tau(\det \varphi_0) + k\chi(C) \in K_1(\mathbf{Z})$$

and the right-hand side becomes

$$2\tau^{NEW}(C, \varphi) + (2k+1)\chi(C) \equiv 2\tau(\det \varphi_0) + \chi(C) \equiv \det \varphi_0 + \chi(C) - 1 \bmod 4 .$$

But there is a classical formula

$$\mathrm{sign}(\phi) \equiv \mathrm{rank}(\phi) + \det(\phi) - 1 \bmod 4$$

relating the signature and the determinant mod 4 (see [8, Theorem 3.5]), for any unimodular symmetric bilinear form ϕ over the integers. We can apply this to φ_0, and note that $\chi(C) = \operatorname{rank} \varphi_0$, since C is concentrated in dimension $2k$.

$\square$

Corollary 3.4. *If (C, φ) is a simple, round Poincaré complex over $\mathbf{Z}$, the signature* $\operatorname{sign}(C) \equiv 0 \bmod 4$.

We now consider the case where X is an even-dimensional manifold. Let π^2 denote the subset of π^{ab} given by the images of squares from $\pi = \pi_1(X, x_0)$.

Proposition 3.5. *Let X be a closed, oriented PL manifold of even dimension. Then $\tau^{NEW}(X) \in \operatorname{Im}\Big(\widehat{H}^0(\mathbf{Z}/2; K_1(\mathbf{Z})) \oplus \widehat{H}^0(\mathbf{Z}/2; \pi^2) \to \widehat{H}^0(\mathbf{Z}/2; K_1(\mathbf{Z}[\pi]))\Big)$.*

Proof. Note that $\widehat{H}^0(\mathbf{Z}/2; \pi^2) = \pi^2 \cap \{\bar{g} \in \pi^{ab} \,|\, \bar{g}^2 = 1\}$. Suppose that $\tau^{NEW}(X) = \tau(\pm g)$ for some $g \in \pi$ with image $\bar{g} \neq 1 \in \pi^{ab}$. Then the duality property $\tau^{NEW}(X) = \tau^{NEW}(X)^*$ implies that $\bar{g}^2 = 1$. Suppose that $\bar{g} \notin \pi^2$. Then there is a projection $j \colon \pi \to \mathbf{Z}/2$ such that $j(g) = T$ is the generator of the quotient group $\mathbf{Z}/2$. Let $p \colon X' \to X$ denote the 2-fold covering of X induced by j. If $\dim X \equiv 2 \bmod 4$, then

$$j_*\{\tau^{NEW}(X)\} = \{\tau(\pm T)\} \in \widehat{H}^0(\mathbf{Z}/2; K_1(\mathbf{Z}[\mathbf{Z}/2]))$$

which is non-zero, contrary to Lemma 2.10.

If $\dim X = 4k$, then the transfer $p^!(\tau^{NEW}(X)) = \tau(-1)$, since for finite coverings the fibre bundle transfer is the same as the classical transfer induced by restriction (see [12, p. 108]). However, by Theorem B it follows that $\tau^{NEW}(X') = p^!(\tau^{NEW}(X))$, so we have $\bar{\tau}^{NEW}(X') \neq 0 \in K_1(\mathbf{Z})$. It follows that $\operatorname{sign}(X') \equiv 2 + 2\chi(X) \bmod 4$, from the relation $\chi(X') = 2\chi(X)$ and Theorem 3.3. The corresponding formula for X gives

$$\operatorname{sign}(X) \equiv 2\bar{\tau}^{NEW}(X) + (2k+1)\chi(X) \bmod 4$$

and by the Hirzebruch Signature Theorem we have

$$\operatorname{sign}(X') = 2\operatorname{sign}(X) \equiv 2\chi(X) \bmod 4$$

This is a contradiction, so the covering $X' \to X$ does not exist and $\bar{g} \in \pi^2$. $\square$

4. Absolute torsion structures on polyhedra

In this section we will define the notion of a (pointed) torsion structure on a polyhedron. First we recall some standard definitions (following [3, §1]) A *polyhedron* is a topological space equipped with a maximal family of PL related locally finite triangulations. A *cell complex* K is a collection of cells PL embedded in a polyhedron P such that

(i) K is a locally finite covering of P,
(ii) if $\beta, \gamma \in K$ then $\partial\beta$ and $\beta \cap \gamma$ are unions of cells of K,
(iii) if β, γ are distinct cells of K, then $\operatorname{Int}\beta \cap \operatorname{Int}\gamma = \emptyset$.

Let $|K|$ denote the underlying polyhedron of a cell complex, and use β to denote a cell in K or the subcomplex it determines. A cell complex K' is a *subdivision* of K if $|K'| = |K|$ and every cell of K is a union of cells of K'. Any two cell complexes K', K'' with $|K'| = |K''|$ have a common subdivision K_0. A base-point for K is a preferred vertex. A pointed map is one which preserves given base-points. A *cellular map* $h\colon K \to L$ between cell complexes is a PL map $h\colon |K| \to |L|$ such that for any cell $\beta \in K$, the image $h(\beta)$ is contained in a cell of L. Given any continuous map $f\colon |K| \to |L|$, there is a subdivision $h\colon K_0 \to K$ and a cellular map $k\colon K_0 \to L$ such that $f \circ h \simeq k$. In other words, any map is homotopic to a cellular map after subdividing the domain. A *cellular homeomorphism* is a cellular map $h\colon K \to L$ such that $h\colon |K| \to |L|$ is a homeomorphism. From now on, we will consider only *finite* cell complexes (those consisting of only finitely many cells), or coverings of finite cell complexes.

If K is a cell complex, we will say that a covering space $\widetilde{P} \to P$ over $P = |K|$ is *subordinate* to K if every cell of K is contained in a evenly-covered neighbourhood of P. In that case, we let $\widetilde{K}$ be the cell complex on $\widetilde{P}$ induced by the covering $p\colon \widetilde{P} \to P$. If $k_0 \in K$ is a base point and π denotes the structural group of the covering (i.e. there is an identification $\pi = p^{-1}(k_0))$, then the cellular chain complex $C(\widetilde{K})$ is a chain complex of free R-modules, where $R = \mathbf{Z}[\pi]$. We need some additional data to get a chain complex of based modules.

Let (K, p) denote a pointed finite cell complex K, with $|K| = P$, and a covering space $p\colon \widetilde{P} \to P$ subordinate to K, with structural group π. A *geometric basis* for (K, p) consists of the following data:

(i) an ordering for the cells of K compatible with the boundary partial ordering.
(ii) an orientation on each cell $\beta \in K$,
(iii) a preferred lift for each cell $\beta \in K$ to $\widetilde{K}$,
(iv) a sign $\eta_K \in \mathrm{Im}(K_1(\mathbf{Z}) \to K_1(\mathbf{Z}[\pi]))$.

We call (K, p) a *based cell complex* if it is equipped with a geometric basis. Note that the chain complex $C(\widetilde{K})$ of a based cell complex is a signed, based, finite chain complex of finitely-generated free $\mathbf{Z}[\pi]$-modules.

Let (X, x_0) be a pointed space, and let $p\colon \widetilde{X} \to X$ be a covering space with structural group π. We fix a base-point $\tilde{x}_0 \in \widetilde{X}$, corresponding to the identity element of π, with $p(\tilde{x}_0) = x_0$.

A *pointed torsion structure* on (X, p) is a pair (K, c, k_0) such that

(i) K is a pointed finite cell complex, with base-point $k_0 \in K$,
(ii) $c\colon |K| \to X$ is a pointed homeomorphism, and
(iii) $(K, c^*(p))$ is a based cell complex, with the lift of k_0 mapping to $\tilde{x}_0$.

The notation $c^*(p)\colon |\widetilde{K}| \to |K|$ means the pull-back covering via c. Usually we will suppress mentioning the base-points and use the notation (K, c).

Definition 4.1. Two pointed torsion structures (K_1, c_1) and (K_2, c_2) on (X, p) are *related* by (K_0, h, k) if K_0 is a pointed cell complex and there exists a homotopy commutative diagram

$$\begin{tikzcd} K_1 \arrow[rr, "f"] & & K_2 \\ & K_0 \arrow[ul, "h"] \arrow[ur, "k"'] & \end{tikzcd}$$

where h is a pointed cellular homeomorphism, k is a pointed cellular map, $f = c_2^{-1} \circ c_1$, and $f \circ h \simeq k$.

Unless explicitly mentioned, we assume that all homotopies of pointed maps are base-point preserving. Since $f = c_2^{-1} \circ c_1$ is homotopic to a cellular map on some subdivision K_0 of K, any two pointed torsion structures are related as above. If (K_1, c_1) and (K_2, c_2) are related by (K_0, h, k), we let (K_0, p_0) denote the pull-back covering $(c_1 \circ h)^*(p) \cong (c_2 \circ k)^*(p)$.

Lemma 4.2. *Suppose that (K_1, c_1) and (K_2, c_2) are related by (K_0, h, k), and that (K_0, p_0) is a based cell complex. Then the quantity*

$$\tau^{NEW}(\ K_1 \rightsquigarrow K_2\ ) := \tau^{NEW}(C(\widetilde{K}_0) \xrightarrow{k_*} C(\widetilde{K}_2)) - \tau^{NEW}(C(\widetilde{K}_0) \xrightarrow{h_*} C(\widetilde{K}_1))$$

in $K_1(\mathbf{Z}[\pi])$ is independent of the choice of (K_0, h, k), and independent of the choice of geometric basis for (K_0, p_0).

Proof. If we have two choices (K_0, h, k) and (K_0', h', k'), there is a homotopy commutative diagram

$$\begin{tikzcd} & K_0' \arrow[dl, "h'"'] \arrow[dr, "k'"] & \\ K_1 \arrow[rr, "f"] & & K_2 \\ & K_0 \arrow[ul, "h"] \arrow[ur, "k"'] & \end{tikzcd}$$

and we apply the chain homotopy invariance of the absolute torsion, and the composition formula of Proposition 1.1, to the composite $C(h)^{-1} \circ C(h') \simeq C(k)^{-1} \circ C(k')$ of chain equivalences (and chain homotopy inverses). $\square$

Definition 4.3. Two pointed torsion structures (K_1, c_1), (K_2, c_2) on (X, p) are *equivalent* if they are related by a based cell complex (K_0, p_0) and

$$\tau^{NEW}(\ K_1 \rightsquigarrow K_2\ ) = 0 \in K_1(\mathbf{Z}[\pi])\ .$$

Lemma 4.4. *This definition gives an equivalence relation on the set of pointed torsion structures on (X, p).*

Proof. The given relation is reflexive and symmetric. If (K_1, c_1), (K_2, c_2) are related by (K_0, h, k), and (K_2, c_2), (K_3, c_3) are related by (K'_0, h', k'), we have a homotopy commutative diagram

$$\begin{tikzcd} K_1 \arrow[rr, "f"] & & K_2 \arrow[rr, "f'"] & & K_3 \\ & K_0 \arrow[ul, "h"] \arrow[ur, "k"'] & & K'_0 \arrow[ul, "h'"] \arrow[ur, "k'"'] & \\ & & K''_0 \arrow[ul, "h''"] \arrow[ur, "k''"'] & & \end{tikzcd}$$

where (K''_0, h'', k'') is the pull-back of $K_0 \xrightarrow{k} K_2 \xleftarrow{h'} K'_0$. This is a cell complex (the common subdivision of K_0 and K'_0), and $h''\colon K''_0 \to K_0$ is a cellular homeomorphism, so (K_1, c_1) and (K_3, c_3) are related by $(K''_0, h \circ h'', k' \circ k'')$. We choose any geometric basis for (K''_0, p''_0), where p''_0 is the pull-back of the covering $c_2^*(p)$ by $k \circ h'' \simeq k' \circ h''$. It follows that

$$\tau^{NEW}(\ K_1 \rightsquigarrow K_3\ ) = \tau^{NEW}(\ K_1 \rightsquigarrow K_2\ ) + \tau^{NEW}(\ K_2 \rightsquigarrow K_3\ )$$

by the composition formula for the absolute torsion. Therefore $(K_1, c_1) \sim (K_2, c_2)$ and $(K_2, c_2) \sim (K_3, c_3)$ implies $(K_1, c_1) \sim (K_3, c_3)$. □

Definition 4.5. Let $\mathcal{S}_\bullet(X, p)$ denote the set of equivalence classes of pointed torsion structures on (X, p) for any polyhedron X. We will use the notation $[K, c]$ (or just $[K]$ when the reference map is understood) for the equivalence class of a torsion structure (K, c). If the covering space p is understood, we will use the notation $\mathcal{S}_\bullet(X)$ for short.

The basic idea of the equivalence relation is to identify two pointed torsion structures whenever the homeomorphism $c_2^{-1} \circ c_1\colon K_1 \to K_2$ is homotopic to a *cellular* homeomorphism with zero absolute torsion. The more general formulation above will be useful in dealing with subdivisions or amalgamations of cell complexes. We remark that if $K \subset L$ is a subcomplex (containing the base-point), then a pointed torsion structure on L induces a pointed torsion structure on K by restriction of the data.

We now define a pairing

$$\Phi\colon \mathcal{S}_\bullet(X, p) \times \mathcal{S}_\bullet(X, p) \to K_1(\mathbf{Z}[\pi])$$

by the formula

$$\Phi([K], [L]) = \tau^{NEW}(\ K \rightsquigarrow L\ )$$

where (K, c_K) and (L, c_L) are pointed torsion structures on (X, p) representing the classes $[K]$ and $[L]$ respectively. This is well-defined, and may be computed by taking the absolute torsion of a cellular approximation to $f = c_L^{-1} \circ c_K$ on a subdivision of K. Here are some basic properties of the pairing.

Lemma 4.6. *For any pointed torsion structures* $[K]$, $[L]$ *and* $[N]$ *on* (X, p)*:*

(i) $\Phi([K],[N]) = \Phi([K],[L]) + \Phi([L],[N])$
(ii) $\Phi([K],[L]) = -\Phi([L],[K])$
(iii) $[K]$ *is equivalent to* $[L]$ *if and only if* $\Phi([K],[L]) = 0$.

Proof. The formulas follow directly from the definitions. The details are left to the reader. □

5. Absolute torsion structures on manifolds

In Theorem 5.2 we give a formula for the absolute torsion of a closed, oriented manifold in terms of the Φ-pairing. This formula will be used in the proof of Theorem B. Our starting point is the fact that any closed PL manifold X of dimension n has a "normal" triangulation (see [27, §68]). More precisely, any simplicial n-complex K homeomorphic to X has the following properties:

(i) Each k-simplex ($k < n$) of K is incident with at least one n-simplex (this is called a *pure* n-complex).
(ii) Each $(n-1)$-simplex of K is incident with exactly two n-simplices.
(iii) Any two n-simplices of K can be connected by a sequence of successively incident simplices of dimensions n and $n-1$.

By subdivision, we can find such a normal triangulation of arbitrarily small diameter. If X is oriented, we can choose a consistent orientation for each n-simplex of K. In addition, for any element of $\mathcal{S}_\bullet(X,p)$ we may find a representative $[K]$ such that the underlying triangulation is normal. There is an analogous cell complex version of this notion, which we call a *normal cell structure* on X.

To compute the absolute torsion of X, we will use the classical approach to Poincaré duality (see [28, Theorem 2.1], [27, § 69]). Let K^* denote the *dual cell* complex associated to K. This is defined in terms of the triangulation, starting with the first barycentric subdivision K' of K. Let $\Delta_r(K)$ denote the set of r-simplices of K. The vertices of K' are the barycentres $\hat{\sigma}^r$ of the simplices $\sigma^r \in \Delta_r(K)$, and its simplices have the form $[\hat{\sigma}^{r_0}, \hat{\sigma}^{r_1}, \dots, \hat{\sigma}^{r_s}]$, where σ^{r_j} is a face of $\sigma^{r_{j+1}}$ for each j. Let $\Delta_r(K')$ denote the set of r-simplices of K'. The simplex σ^r of K is the union of the simplices in $\Delta_r(K')$ which terminate with $\hat{\sigma}^r$. The cells $\mathcal{D}\sigma^r$ of K^* correspond bijectively to the simplices of K, and $\mathcal{D}\sigma^r$ is the union of the simplices of K' which begin with $\hat{\sigma}^r$. Then $\dim \mathcal{D}\sigma^r = n - r$ and σ^r intersects $\mathcal{D}\sigma^r$ transversely only at $\hat{\sigma}^r$. We denote the set of $(n-r)$-cells of K^* by $\Delta_{n-r}(K^*)$.

For a suitable chain approximation to the diagonal in $K \times K^*$, cap product with the fundamental cycle takes the cochain on K dual to σ^r to the chain $\mathcal{D}\sigma^r$ on K^*.

Definition 5.1. Let (K, c) be a torsion structure on X. We define the *dual torsion structure* $(\mathcal{D}K, \mathcal{D}c)$ to be the torsion structure on X with underlying cell-complex $\mathcal{D}K$ described above and with geometric basis as follows:

(i) The ordering of the cells on $\mathcal{D}K$ is induced from the ordering of the cells on K: for cells $\mathcal{D}\sigma^r$ and $\mathcal{D}\tau^r$ of the same dimension, $\mathcal{D}\sigma^r$ comes before $\mathcal{D}\tau^r$ if and only if σ come before τ.

(ii) The lift of each cell of $\mathcal{D}K$ is induced by the lift of the corresponding dual cell.

(iii) The cells of $\mathcal{D}K$ are oriented such that cap product with the fundamental cycle induces the based identity map

$$\mathcal{D}\varphi_0 \colon C(K)^{n-r} \to C(\mathcal{D}K)_r$$

With the above choice of geometric basis the chain complex of $\mathcal{D}K$ is $C(\mathcal{D}K) = C(K)^{n-*}$. We choose the sign $\eta_{\mathcal{D}K}$ to be $\eta_{\mathcal{D}K} = \eta_{C(K)^{n-*}}$. Hence $C(\mathcal{D}K) = C(K)^{n-*}$ as signed complexes (identified by $\mathcal{D}\varphi_0$).

Theorem 5.2. *Let X be a closed, oriented manifold of dimension n. Then*

$$\tau^{NEW}(X) = \Phi([\mathcal{D}K], [K]) \in \widehat{H}^n(\mathbf{Z}/2; K_1(\mathbf{Z}[\pi_1(X, x_0)]))$$

Proof. We first choose a geometric basis for the first barycentric subdivision K' of K. Since X is an oriented manifold we may apply the symmetric construction of Ranicki [22], [23] to find a symmetric Poincaré complex $(C(K), \varphi)$; in particular, the map $\varphi_0 \colon C(K)^{n-*} \to C(K)$ is a chain level representative of the Poincaré duality map on X given by cup-product with the fundamental class. We have a diagram of signed chain complexes which commutes up to chain homotopy:

$$\begin{tikzcd}
C^{n-*}(K) \arrow[r, equal, "\mathcal{D}\varphi_0"] \arrow[d, "\varphi_0"'] & C(\mathcal{D}K) \arrow[d] \\
C(K) \arrow[r] & C(K')
\end{tikzcd}$$

Hence

$$\begin{aligned}
\tau^{NEW}(X) &= \tau^{NEW}(\varphi_0) \\
&= \tau^{NEW}(C(\mathcal{D}K) \to C(K')) - \tau^{NEW}(C(K) \to C(K')) \\
&= \Phi([\mathcal{D}K], [K']) - \Phi([K], [K']) \\
&= \Phi([\mathcal{D}K], [K]) \ .
\end{aligned}$$

□

6. Fibre transport and transfer

In this section we will describe a suitable algebraic setting for the fibre transport in a fibre bundle, and establish some useful properties for use in Section 8. Let $\mathbb{A}$ and $\mathbb{B}$ be (small) additive categories with involution, and recall that $\mathbb{SPD}_n(\mathbb{B})$ denotes the full sub-category of $\mathbb{SD}(\mathbb{B})$ consisting of signed n-dimensional chain complexes C in $\mathbb{B}$, such that C^{n-*} is chain equivalent to C, and $\chi(C) = 0$ if n is odd. This sub-category inherits an involution via $C \mapsto C^{n-*}$ (see Definition 12.18).

Definition 6.1. A *transfer functor* is a set of additive functors $J^D\colon \mathbb{A} \to \mathbb{SPD}_n(\mathbb{B})$, indexed by a set ${}_{\mathbb{A}}Mod_{\mathbb{B}}$ of objects D in $\mathbb{SPD}_n(\mathbb{B})$ called $\mathbb{A}$-$\mathbb{B}$ *bimodules*, such that

(i) If $D \in {}_{\mathbb{A}}Mod_{\mathbb{B}}$ and $D \cong D'$, then $D' \in {}_{\mathbb{A}}Mod_{\mathbb{B}}$.

(ii) For each isomorphism $g\colon D \to D'$ in $\mathbb{SPD}_n(\mathbb{B})$ there is a natural transformation $g_*\colon J^D \to J^{D'}$, with $(id)_* = id$ and $(h \circ g)_* = h_* \circ g_*$ for any isomorphism $h\colon D' \to D''$.

(iii) For each object $M \in \mathbb{A}$, $J^D(M)^* \cong J^{D^{n-*}}(M^*)$. □

If (C, η_C) is a signed complex in $\mathbb{A}$, then $J^D(C)$ is a signed chain complex in $\mathbb{SPD}_n(\mathbb{B})$, with $\eta_{J^D(C)} = J^D(\eta_C)$. This formula gives an allowable sign because

$$J^D(\epsilon(M,N)) = \epsilon(J^D(M), J^D(N))$$

since J^D is additive and $\epsilon(M,N) = \tau^{iso}(M \oplus N \to N \oplus M)$ for any objects M, N of $\mathbb{A}$. The composition

$$trf_D = i_* \circ J^D_*\colon K_1^{iso}(\mathbb{A}) \to K_1^{iso}(\mathbb{B})$$

is called the *transfer* induced by J^D.

The main examples for our applications arise from the fibre transport.

Example 6.2 (Fibre Transport)**.** Let R and S be rings with involution, and let D be an object in $\mathbb{SPD}_n(\mathbb{A}(S))$. Suppose that

$$p\colon R \to [D,D]^{op}$$

is a homomorphism of rings with involution, where $[D,D]^{op}$ denotes the set of chain homotopy classes of self-chain maps $D \to D$ over S. Then we let ${}_{\mathbb{A}(R)}Mod_{\mathbb{A}(S)}$ be the collection of all objects $D' \cong D$. Define

$$J^D\colon \mathbb{A}(R) \to \mathbb{SPD}_n(\mathbb{A}(S))$$

on objects by setting $J^D(R) = (D, \eta_D)$, and extending additively. As a chain complex, $J^D(M) = M \otimes_R D$ for each object M in $\mathbb{A}(R)$, and the sign $\eta_{J^D(M)}$ is determined by the usual formula for direct sums. If $f\colon M \to M'$ is a morphism in $\mathbb{A}(R)$, then we get an induced morphism

$$f \otimes 1\colon M \otimes_R D \to M' \otimes_R D,$$

where p is used to define a left R-module structure on D. If $g\colon D \to D'$ is an isomorphism in $\mathbb{SPD}_n(\mathbb{A}(S))$, the natural transformation $g_*\colon J^D(C) \to J^{D'}(C)$ is defined by

$$1 \otimes g\colon C \otimes_R D \to C \otimes_R D' \ .$$

The duality property (iii) follows from the identification $M = R^m$ as a based, free R-module. The involution $M \mapsto M^*$ maps the given base to the dual base. Under these identifications

$$M^* \otimes_R D^{n-*} = \oplus_{i=1}^m D^{n-*} = (\oplus_{i=1}^m D)^* = (M \otimes_R D)^*$$

In this example, the transfer induced by J^D is usually denoted $p^!\colon K_1(R) \to K_1(S)$ (see[12]).

As a special case (when p is trivial) we get a transfer functor

$$J^D\colon \mathbb{A}(R) \to \mathbb{SPD}_n(\mathbb{A}(R \otimes S))$$

for any two rings with involution (tensor product over $\mathbf{Z}$).

Example 6.3 (Tensor products)**.** Let C and D be signed chain complexes over $\mathbb{A}(R)$ and $\mathbb{A}(S)$ respectively, and let $F_*(C \otimes D)$ denote the filtered complex structure defined in Example 12.4, Section 12. Then the *signed tensor product* is the signed filtered complex whose underlying chain complex is $C \otimes D$ and whose associated graded complex is $J^D(C, \eta_C) = (\mathbb{G}_*(C \otimes D), J^D(\eta_C))$. The 'internal' sign $\eta_{G_r(C\otimes D)}$ is the sign of the chain complex $J^D(C_r)$ over S, as defined above. In this example, the transfer $p^! = trf_D$ is just multiplication by $\chi(D)$. □

The functorial properties of J^D give two useful formulas.

Lemma 6.4. *Let $J^D\colon \mathbb{A} \to \mathbb{SD}_n(\mathbb{B})$ be a transfer functor.*

(i) *For each chain equivalence $f\colon C \to C'$ in $\mathbb{A}$,*

$$i_*\tau^{NEW}(J^D(f)) = i_*J^D_*(\tau^{NEW}(f)) = trf_D(\tau^{NEW}(f)) \in K_1^{iso}(\mathbb{B}) \ .$$

(ii) *If J^D is induced by a fibre transport $R \xrightarrow{p} [D, D]^{op}$, then for each chain equivalence $g\colon D \to D'$ in $\mathbb{A}(S)$,*

$$i_*\tau^{NEW}(J^D(C) \xrightarrow{g_*} J^{D'}(C)) = \chi(C) \cdot \tau^{NEW}(g) \in K_1^{iso}(\mathbb{A}(S)) \ .$$

Proof. Part (i) follows directly from the definitions. For Part (ii) observe that $g_*\colon J^D(C) \to J^{D'}(C)$ is an isomorphism of chain complexes in $\mathbb{SD}(\mathbb{A}(S))$ and hence

$$\begin{aligned} & i_*\tau^{NEW}(J^D(C) \xrightarrow{g_*} J^{D'}(C)) \\ &\quad = i_*\left(\textstyle\sum(-)^r\tau^{iso}(1 \otimes g\colon C_r \otimes_R D \to C_r \otimes_R D')\right) \\ &\quad = \textstyle\sum(-)^r\tau^{NEW}(\bigoplus_{\mathrm{rank}_R(C_r)}(g\colon D \to D')) \\ &\quad = \chi(C)\tau^{NEW}(g\colon D \to D') \in K_1^{iso}(\mathbb{A}(S)) \end{aligned}$$

□

We now prove the product formula for the absolute torsion of symmetric Poincaré structures announced in [11].

Proposition 6.5. *Let (C, φ^C) and (D, φ^D) be symmetric Poincaré complexes over rings with involution R and S respectively. Then*

$$\tau^{NEW}(C \otimes D, \varphi^C \otimes \varphi^D) = \chi(C)\tau^{NEW}(D, \varphi^D) + \chi(D)\tau^{NEW}(C, \varphi^C)$$

evaluated in $K_1(R \otimes S)$.

Proof. Let $\dim C = k$ and $\dim D = n$. By definition

$$\tau^{NEW}(C \otimes D, \varphi^C \otimes \varphi^D) = \tau^{NEW}(\varphi_0 \otimes \phi_0 \colon (C \otimes D)^{n+k-*} \to (C \otimes D)) \ .$$

The map $\varphi_0^C \otimes \varphi_0^D$ is given by the composition:

$$(C \otimes D)^{n+k-*} \xrightarrow{\theta_{C \otimes D}} C^{k-*} \otimes D^{n-*} \xrightarrow{\varphi_0 \otimes 1} C \otimes D^{n-*} \xrightarrow{1 \otimes \varphi_0} C \otimes D$$

The result now follows from Lemma 6.4 and Lemma 12.28. □

7. Absolute torsion structures on fibre bundles

The notion of pointed torsion structures will now be extended to PL or smooth fibre bundles $F^n \xrightarrow{q} E^{n+k} \xrightarrow{p} B^k$ (see [1, p. 181] for the definition of a PL fibre bundle). We assume that F, E and B are compatibly oriented PL or smooth, closed manifolds (considered as polyhedra via the canonical PL structure compatible with their smooth structure). In both cases there exists a triangulation of the fibre bundle compatible with a given triangulation on the base (see [21] for the smooth case).

Our goal is to define a notion of a *pointed fibre bundle torsion structure* (PFBTS) which will have as data a PTS on both the base and fibre, and which will determine a PTS on the total space E.

Fix a base-point $b_0 \in B$ and a base-point $e_0 \in E$ with $p(e_0) = b_0$. We will always assume that B is connected. In the definitions below, the geometric bases for torsion structures on B or E will use their *universal* covering spaces (suppressed in the notation, together with the chosen lifts of the base-points). In particular, if $c \colon K \to B$ is a pointed torsion structure (PTS), and $\gamma \subset \partial\beta$ for some cell $\beta \in K$, then we have a component of the boundary chain map

$$\partial_{\beta,\gamma} \colon C(\widetilde{\beta}) \to SC(\widetilde{\gamma})$$

which can be identified with the module homomorphism

$$d_{\beta,\gamma} \colon \mathbf{Z}[\pi_1(B, b_0)] \to \mathbf{Z}[\pi_1(B, b_0)]$$

given by multiplication with some element $d_{\beta,\gamma} \in \{\pm\pi^{ab}\}$. This is a module isomorphism. We will use the images

$$p^!(\tau(d_{\beta,\gamma})) \in K_1(\mathbf{Z}[\pi_1(E, e_0)])$$

under the fibre bundle transfer $p^!$ in the discussion below.

Let $K \to B$ and $L \to E$ be cell structures on the base and total spaces such that the induced map $L \to K$ is cellular. We call this a *cell structure* on the fibre bundle. Then we may regard $C(\widetilde{L})$ as a filtered complex by defining:

$$F_r C(\widetilde{L}) = C(p^{-1}(K^{\langle \leqslant r \rangle}))$$

where $K^{\langle \leqslant r \rangle}$ is the subcomplex of K consisting of the cells of dimension $\leqslant r$. If $\dim F = n$ and $\dim B = k$, then we obtain an admissible $(n+k)$-dimensional k-filtered chain complex as defined in Section 12. The associated complex of $F_* C(\widetilde{L})$ has the direct sum decomposition:

$$G_r C(\widetilde{L}) = S^{-r} \bigoplus_{\beta \in K^{(r)}} C(p^{-1}(\beta))$$

indexed over the cells in K of dimension r. We will use the notation $C(\widetilde{F_\beta}) := S^{-r} C(p^{-1}(\beta))$ for $\beta \in K^{(r)}$, so we have the identification

$$G_r C(\widetilde{L}) = \bigoplus_{\beta \in K^{(r)}} C(\widetilde{F_\beta})$$

Note that the cell structure on E gives a cell structure on $F_0 := p^{-1}(b_0)$. Let $[F]$ denote a pointed torsion structure for F_0, with respect to the *pull-back* of the universal covering of E by $q\colon F \to E$. Fibre transport defines a ring morphism

$$\mathbf{Z}[\pi_1 B] \to [C(\widetilde{F_0}), C(\widetilde{F_0})]^{op}$$

so we have a transfer functor

$$J^{C(\widetilde{F_0})} \colon \mathbb{A}(\mathbf{Z}[\pi_1 B]) \to \mathbb{SPD}_n(\mathbb{A}(\mathbf{Z}[\pi_1 E]))$$

To shorten the notation, we will let $J^F := J^{C(\widetilde{F_0})}$ when the PTS on F is clear from the context.

We can apply this additive functor to the identification

$$C_r(\widetilde{K}) = \bigoplus_{\beta \in K^{(r)}} C(\widetilde{\beta})$$

to obtain

$$J^{C(\widetilde{F_0})}(C_r(\widetilde{K})) = \bigoplus_{\beta \in K^{(r)}} J^{C(\widetilde{F_0})}(C(\widetilde{\beta})) = \bigoplus_{\beta \in K^{(r)}} C(\widetilde{F_0})$$

One of our main goals to construct a chain equivalence

$$\mathcal{E} \colon \mathbb{G}_*(C(\widetilde{L})) \to J^{C(\widetilde{F_0})} C(\widetilde{K})$$

by constructing chain equivalences

$$\mathcal{E}^\beta \colon C(\widetilde{F_\beta}) \to C(\widetilde{F_0})$$

for each $\beta \in K$, and defining $\mathcal{E}$ as the direct sum of the $\mathcal{E}^\beta$ over all such β. This can be achieved if we can construct such maps which are compatible with boundary inclusions $\gamma < \beta$. We will first construct the maps $\mathcal{E}^\beta$ for a special type of PTS on the base B which comes from a normal unfolding (defined below). This construction will then be extended to any PTS on the base B.

A pointed torsion structure $c_K\colon K \to B$ is *small* with respect to the bundle $E \xrightarrow{p} B$, if $c(\beta)$ is contained in a locally trivial bundle neighbourhood for each $\beta \in K$.

Definition 7.1. A *geometric basis* for a pointed fibre bundle (E, p, e_0) consists of the following data:

(i) a pointed torsion structure (K, c_K, k_0) on B, which is small with respect to the bundle (E, p).
(ii) a cell structure $L \to K$ on $E \to B$, over $c_K\colon K \to B$.
(iii) a geometric basis for $F_\beta \subset L$, for each $\beta \in K$, with respect to the pull-back of the universal covering of E. □

Notice that a geometric basis for (E, p) and the fixed base-points give a PTS for the fibre F, represented by $(F_0, c_L|_{F_0})$. We denote this pointed torsion structure by $[F] \in \mathcal{S}_\bullet(F)$.

Definition 7.2. A *pointed fibre bundle torsion structure* (PFBTS) on (E, p) consists of a geometric basis $L \to K$ for (E, p, e_0)), such that:

(i) for each $\beta \in K$, a chain equivalence $\mathcal{E}^\beta\colon C(\widetilde{F_\beta}) \to C(\widetilde{F_0})$ where the map $\mathcal{E}^{b_0}\colon C(\widetilde{F_0}) \to C(\widetilde{F_0})$ is the identity.

(ii) for each $\beta \in K$,
$$\tau^{NEW}(\mathcal{E}^\beta\colon C(\widetilde{F_\beta}) \to C(\widetilde{F_0})) = 0 \in K_1(\mathbf{Z}[\pi_1(E, e_0)])$$
(iii) the sum
$$\mathcal{E} = \bigoplus_{\beta \in K} \mathcal{E}^\beta\colon \mathbb{G}_*(C(\widetilde{L})) \to J^{C(\widetilde{F_0})} C(\widetilde{K})$$
is a chain isomorphism in $\mathbb{SD}(\mathbb{A}(\mathbf{Z}[\pi_1(E, e_0)]))$. □

We will denote a PFBTS for (E, p) by $(L \to K, \mathcal{E})$, and the induced PTS on the fibre by $[F]$. We will say that the PFBTS *realizes* the given structures on base and fibre.

A PFBTS on $E \to B$ determines a PTS on E as follows. The choice of geometric basis on each fibre F_β determines lifts and orientations of each of the cell in L. The associated complex $\mathbb{G}_*(C(\widetilde{L}))$ may be made into a signed complex by setting
$$G_r(C(\widetilde{L})) = \bigoplus_{\beta \in K^{\langle r \rangle}} C(\widetilde{F_\beta})$$
as signed complexes, and setting $\eta_{\mathbb{G}_* C(\widetilde{L})} = J_*^F(\eta_K)$. We then give $C(\widetilde{L})$ the filtered sign determined by the signs for $\mathbb{G}_*(C(\widetilde{L}))$ and define $\eta_L = \eta_{F_* C(\widetilde{L})}$.

Lemma 7.3. $i_*\tau^{NEW}(\mathcal{E}_*\colon \mathbb{G}_*(C(\widetilde{L})) \to J^{C(\widetilde{F_0})}C(\widetilde{K})) = 0 \in K_1(\mathbf{Z}[\pi_1(E, e_0)])$.

Proof. This follows directly from the sign conventions chosen, and the formula

$$\tau^{NEW}(\mathcal{E}_*) = \sum_r (-1)^r \tau^{iso}(\mathcal{E}_r\colon G_r(C(\widetilde{L})) \to J^F C_r(\widetilde{K})) + \eta_{J^F C(\widetilde{K})} - \eta_{\mathbb{G}_*(C(\widetilde{L}))}$$

for the absolute torsion of a chain isomorphism. □

8. Absolute torsion of fibre bundles

We will show in Theorem 8.5 that every smooth (or PL) fibre bundle $F \to E \to B$ as above admits a pointed fibre bundle pointed torsion structure. First we define the *normal unfolding* of a normal triangulation of a closed manifold. The idea is to cut X open along certain $(n-1)$-simplices of the triangulation K to obtain a contractible complex projecting onto X.

Definition 8.1. Let $K = \bigcup\{\Delta^n_\alpha \,|\, \alpha \in J_K\}$, where the Δ^n_α are the distinct n-simplices in the triangulation of X. Let $j_\alpha\colon \Delta^n_\alpha \to K$ denote the inclusion maps of the n-simplices. A *normal unfolding* of K is a pure simplicial n-complex $\widehat{K}$ equipped with a simplicial map $f\colon \widehat{K} \to K$, such that

(i) $\widehat{K}$ is a union of n-simplices, with index set $J_{\widehat{K}}$, and each $(n-1)$-simplex of $\widehat{K}$ is incident with one or two n-simplices.
(ii) There is a bijection $\theta\colon J_{\widehat{K}} \to J_K$ such that $f|\Delta^n_\alpha = j_{\theta(\alpha)}$ for all $\alpha \in J_{\widehat{K}}$.
(iii) $\widehat{K}$ is contractible.

We again have an analogous notion for normal cell structures.

Lemma 8.2. *Let X be a closed smooth manifold of dimension n. Each normal cell structure K on X admits a normal unfolding $(\widehat{K}, f)$.*

Proof. We give the proof for the case of triangulations. Consider the set of barycentres $V = \{v_\alpha \,|\, \alpha \in J_K\}$ of the n-simplices of K. We can join the barycentres of each pair $\{\Delta^n_\alpha, \Delta^n_\beta\}$ of incident n-simplices with an edge $e_{\alpha\beta}$ through their common $(n-1)$ face to obtain a connected graph (the dual 1-skeleton of K). Let $T = (V, E)$ denote a maximal tree in this graph, and note that T has vertex set consisting of all the barycentres. Let $\widehat{K}$ be the quotient space of the disjoint union of the n-simplices in K, where we identify two n-simplices Δ^n_α, Δ^n_β in $\widehat{K}$ along an $(n-1)$ face if and only if $e_{\alpha\beta} \in E$. By construction, $\widehat{K}$ is a thickening of the tree T, so $\widehat{K}$ is contractible. The other properties of a normal unfolding are clear. □

A normal unfolding $\widehat{K} \to K$ can be used to make consistent choices of lifts to the universal covering for simplices in K. Any cell $\beta \in K$ is contained in a unique n-cell of minimal ordering. We can specify a unique lift $\hat{\beta} \in \widehat{K}$ by requiring $\hat{\beta}$ to be contained in the lift $\hat{\Delta}$ of Δ.

Definition 8.3. A pointed torsion structure $K \to B$ is *normal* if for some normal unfolding $\widehat{K} \to K$, the chosen lifts of adjacent cells $\beta, \gamma \in K$ are adjacent in the universal covering $\widetilde{K}$ whenever their lifts $\hat{\beta}$ and $\hat{\gamma}$ are adjacent in $\widehat{K}$.

Remark 8.4. If $f\colon \widehat{K} \to K$ is a normal unfolding, the map f induces an equivalence relation on the r-simplices of $\widehat{K}$. Two r-simplices are equivalent if they are identified by f to the same simplex in K. Let $\partial\widehat{K}$ denote the subset of K consisting of all the simplices whose equivalence class contains more than one element. This is a sub-complex of $\widehat{K}$ called its *boundary.*

Let v_0 denote the barycentre of one of the n-simplices (chosen as a base point, so $f(v_0) = k_0$), and let $v_i \in \tau_i$ denote the barycentre of each of the n-simplices Δ_i^n with a face $\tau_i \subset \partial\widehat{K}$. Let $x_i \in \tau_i$ denote the barycentre of each such face, and $e_i = [v_i, x_i] \subset \Delta_i^n$ the linear path joining v_i to x_i. Let σ_i denote the unique path in the maximal tree joining v_0 with the barycentre $v_i \in \Delta_i^n$, followed by the path e_i to x_i. If τ_i and τ_j, $i < j$, are a pair of $(n-1)$-faces identified under $f\colon \widehat{K} \to K$, then $f(e_i) \cup f(e_j) = e_{ij}$ is an edge in the dual 1-skeleton of K. The fundamental group $\pi_1(K, k_0)$ is generated by loops of the form $u_{ij} := f(\sigma_i) \cup e_{ij} \cup f(\sigma_j)$. $\square$

Here is our main result about the existence of pointed fibre bundle torsion structures.

Theorem 8.5 (**Realization**). *Let $E \to B$ be a fibre bundle of compatibly oriented closed PL manifolds. Given a PTS on $K \to B$, a cell structure $L \to K$ on the fibre bundle, and a geometric basis on F, there exists a PFBTS $(L \to K, \mathcal{E})$ realizing this data.*

The main steps in the proof are contained in the following special case.

Proposition 8.6. *There exists a PFBTS $(L \to K, \mathcal{E})$ on $E \to B$ realizing a given normal pointed torsion structure $K \to B$ on B, a given cell structure $L \to K$ on the fibre bundle, and a given geometric basis for F.*

Proof. We are required to construct chain maps $\mathcal{E}^\beta\colon C(\widetilde{F_\beta}) \to C(\widetilde{F_0})$ such that the direct sum $\mathcal{E}\colon \mathbb{G}_*(C(\widetilde{L})) \to J^F C(\widetilde{K})$ is a chain equivalence. Once these maps are constructed, we can choose a geometric basis on each F_β such that the maps $\mathcal{E}^\beta\colon C(\widetilde{F_\beta}) \to C(\widetilde{F_0})$ have trivial absolute torsion (this is condition (ii) for a PFBTS). The condition that $\mathcal{E}$ is a chain map is equivalent to the statement that for a pair of simplices $\gamma < \beta$ where γ is an $(r-1)$-dimensional cell in the boundary of an r-dimensional cell β, the diagram

$$\begin{tikzcd} C(\widetilde{F_\beta}) \arrow[r] \arrow[d] & J^F C(\beta) \arrow[d] \\ C(\widetilde{F_\gamma}) \arrow[r] & J^F C(\gamma) \end{tikzcd} \tag{8.7}$$

commutes. We will first construct the maps $\mathcal{E}$ and then show that this diagram commutes for all $\gamma < \beta$.

Let $f\colon \widehat{K} \to K$ be the normal unfolding of K, and $g\colon \widehat{L} \to L$ the pull-back of $L \to K$ over $\widehat{K}$. The chain complex $\mathbb{G}_*(C(\widehat{L}))$ splits over the cells $\hat{\beta} \in \widehat{K}$ as:

$$\mathbb{G}_*(C(\widehat{L})) = \bigoplus_{\hat{\beta}\in\widehat{K}} C(\widetilde{F_{\hat{\beta}}})$$

We assume that the base-point k_0 is uniquely covered ($f^{-1}(k_0) = v_0$). The chain complex $C(\widehat{K})$ is a based $\mathbf{Z}$-module chain complex since we have an orientation for each cell. Since $\widehat{K}$ is contractible the pull-back of the universal cover of B over the map $\widehat{K} \to K \to B$ is trivial and hence we have a chain map

$$f_*\colon C(\widehat{K}) \otimes_{\mathbf{Z}} \mathbf{Z}[\pi_1 B] \to C(\widetilde{K})$$

of based signed chain complexes over $\mathbf{Z}[\pi_1 B]$. On the total space we have a filtered chain map $g_*\colon C(\widehat{L}) \to C(\widetilde{L})$ whose associated map

$$\mathbb{G}_*(g_*)\colon \mathbb{G}_*(C(\widehat{L})) \to \mathbb{G}_*(C(\widetilde{L}))$$

splits over each cell β of K, and lifts $\hat{\beta}$ of β giving an isomorphism of chain complexes:

$$g_{\hat{\beta}}\colon C(\widetilde{F_{\hat{\beta}}}) \to C(\widetilde{F_\beta})$$

Note that

$$J^F(C(\widehat{K}) \otimes_{\mathbf{Z}} \mathbf{Z}[\pi_1 B]) = C(\widehat{K}) \otimes_{\mathbf{Z}} C(\widetilde{F_0}) \ .$$

Since $\widehat{K}$ is contractible and the triangulation $K \to B$ is small with respect to the bundle $E \to B$, there is a trivializing homeomorphism

$$\Psi_K\colon |\widehat{L}| \to |\widehat{K}| \times F$$

of the pull-back bundle $\hat{p}\colon \widehat{L} \to \widehat{K}$ which gives rise to a chain homotopy commutative diagram of chain maps:

$$\begin{tikzcd}
C(\hat{p}^{-1}(\hat{\beta})) \arrow[r] \arrow[d] & C(\widetilde{\beta}) \otimes_{\mathbf{Z}} C(\widetilde{F_0}) = C(\widetilde{F_0}) \arrow[d] \\
C(\widehat{L}) \arrow[r] & C(\widehat{K}) \otimes_{\mathbf{Z}} C(\widetilde{F_0}) = J^F(C(\widehat{K}) \otimes_{\mathbf{Z}} \mathbf{Z}[\pi_1 B])
\end{tikzcd}$$

for each $\hat{\beta} \in \widehat{K}$. Let r be the dimension of β. By taking the associated complexes and maps in this diagram, we construct maps $\mathcal{E}^{\hat{\beta}}$:

$$\begin{tikzcd}
C(F_\beta) \arrow[r, "\mathcal{E}^{\hat{\beta}}"] \arrow[d] & C(\widetilde{F_0}) \arrow[d] \\
G_r(C(\widehat{L})) \arrow[r] & J^F(C_r(\widehat{K}) \otimes_{\mathbf{Z}} \mathbf{Z}[\pi_1 B])
\end{tikzcd}$$

The sum $\widehat{\mathcal{E}}\colon \mathbb{G}_*(C(\widehat{L})) \to J^{C(\widetilde{F_0})}(C(\widehat{K}) \otimes_{\mathbf{Z}} \mathbf{Z}[\pi_1 B])$ is a chain map and each $\mathcal{E}^{\hat{\beta}}$ is a chain equivalence.

Each cell β in K lies in an n-simplex of minimal ordering so we have a particular lift $\hat{\beta}$ of β. We define the chain equivalence $\mathcal{E}_\beta$ to be

$$\mathcal{E}^\beta = \mathcal{E}^{\hat{\beta}} \circ (g_{\hat{\beta}})^{-1}\colon C(\widetilde{F_\beta}) \to C(\widetilde{F_{\hat{\beta}}}) \to C(\widetilde{F_0})$$

where $g_{\hat{\beta}}$ is the restriction of $g\colon \widehat{L} \to L$ to $\hat{\beta}$. It remains to show that $\mathcal{E}_*$ is a chain map, in other words that the diagram (8.7) commutes for all pairs β, γ. This is true if the lifts of the cells β and γ are adjacent in $\widehat{K}$, since we have a commutative diagram:

$$\begin{tikzcd}
C(\widetilde{F_\beta}) \arrow[r, "(g_{\hat{\beta}})^{-1}"] \arrow[d] & C(\widetilde{F_{\hat{\beta}}}) \arrow[r, "\mathcal{E}^{\hat{\beta}}"] \arrow[d] & C(\widetilde{F}) = J^F C(\beta) \arrow[d] \\
C(\widetilde{F_\gamma}) \arrow[r, "(g_{\hat{\gamma}})^{-1}"] & C(\widetilde{F_{\hat{\gamma}}}) \arrow[r, "\mathcal{E}^{\hat{\gamma}}"] & C(\widetilde{F}) = J^F C(\gamma)
\end{tikzcd}$$

Suppose now that the lift of γ is not adjacent to that of β. Then the component of the differential in $C(\widetilde{K})$ given by $C(\beta) \to C(\gamma)$ is $d_{\beta,\gamma} = \pm t$ for some non-trivial group element $t \in \pi_1 B$, when considered as a map of based complexes. We will denote by $\pm$ the sign in front of t. Let $\hat{\gamma}'$ be the lift of γ which is adjacent to the chosen lift of $\hat{\beta}$ of β. The two faces $\hat{\gamma}$, $\hat{\gamma}'$ are identified under $\widehat{K} \to K$. Let $f_{\hat{\gamma}}$, $f_{\hat{\gamma}'}$ be paths in $\widehat{K}$ joining the barycentres $x_{\hat{\gamma}}$, $x_{\hat{\gamma}'}$ with the base point v_0. Then the image of $f_{\hat{\gamma}} \cup f_{\hat{\gamma}'}$ in K is a loop representing t. Hence fibre transport around t is described by the following composition:

$$F \to F \times x_{\hat{\gamma}} \xrightarrow{\Psi|_{x_{\hat{\gamma}}}} \hat{p}^{-1}(x_{\hat{\gamma}}) \to p^{-1}(x_\gamma) \to \hat{p}^{-1}(x_{\hat{\gamma}'}) \xrightarrow{(\Psi|_{x_{\hat{\gamma}'}})^{-1}} F \times x_{\hat{\gamma}'} \to F$$

where x_γ is the barycentre of γ in K. Therefore the composition of homotopy equivalences:

$$F \to F \times |\hat{\gamma}| \xrightarrow{\Psi_K|_{\hat{\gamma}}} F_{\hat{\gamma}} \xrightarrow{g|_\gamma} F_\gamma \xrightarrow{(g|_{\gamma'})^{-1}} F_{\hat{\gamma}'} \xrightarrow{(\Psi_K|_{\hat{\gamma}'})^{-1}} F \times |\hat{\gamma}'| \to F$$

is described up to homotopy by fibre transport around t. Hence the composition

$$C(\widetilde{F}) \xrightarrow{(\mathcal{E}^{\hat{\gamma}})^{-1}} C(\widetilde{F_{\hat{\gamma}}}) \xrightarrow{g_{\hat{\gamma}}} C(\widetilde{F_\gamma}) \xrightarrow{(g_{\hat{\gamma}'})^{-1}} C(\widetilde{F_{\hat{\gamma}'}}) \xrightarrow{\mathcal{E}^{\hat{\gamma}'}} C(\widetilde{F})$$

is given (up to chain homotopy) by $J^F(t)\colon C(\widetilde{F}) \to C(\widetilde{F})$. From this we deduce that

$$\mathcal{E}^\gamma = J^F(t) \circ (\mathcal{E}^{\hat{\gamma}'} \circ (g_{\hat{\gamma}'})^{-1})$$

Since $\hat{\gamma}'$ is adjacent to $\hat{\beta}$ we know that

$$\begin{tikzcd} C(\widetilde{F_\beta}) \arrow[r, "\mathcal{E}^\beta"] \arrow[d] & C(\widetilde{F}) \arrow[d, "J^F(\pm)"] \\ C(\widetilde{F_\gamma}) \arrow[r, "\mathcal{E}^{\hat{\gamma}'} \circ (g_{\hat{\gamma}'})^{-1}"] & C(\widetilde{F}) \end{tikzcd}$$

commutes. It now follows that

$$\begin{tikzcd} C(\widetilde{F_\beta}) \arrow[r, "\mathcal{E}^\beta"] \arrow[d] & C(\widetilde{F}) \arrow[d, "J^F(\pm t)"] \\ C(\widetilde{F_\gamma}) \arrow[r, "\mathcal{E}^{\gamma}"] & C(\widetilde{F}) \end{tikzcd}$$

commutes as required. □

The proof of Theorem 8.5. Let $K' \to B$ be a normal PTS on B with the same underlying cell-structure as K. Then we can find a PFBTS $(L' \to K', \mathcal{E}')$ with L' having the same underlying cell structure as L and fibre structure F by the above proposition. For each cell $\beta \in K$ we have a corresponding cell $\beta' \in K'$ and an identification of $C(F_\beta)$ with $C(F_{\beta'})$. We use this identification to choose a geometric basis for F_0 such that $C(F_{k_0}) \to C(F_{k_0'})$ is the based identity map. We now define the maps $\mathcal{E}^\beta$ by

$$\mathcal{E}^\beta = J^F(C(\beta') \to C(\beta)) \circ (\mathcal{E}')^{\beta'} \colon C(\widetilde{F_\beta}) \to C(\widetilde{F_0})$$

using the identification of $C(F_\beta)$ with $C(F_{\beta'})$. We again choose a geometric basis on each F_β such that the maps $\mathcal{E}^\beta \colon C(\widetilde{F_\beta}) \to C(\widetilde{F_0})$ have trivial absolute torsion. We claim that $(L \to K, \mathcal{E})$ is a PFBTS. The only condition remaining to check is that $\mathcal{E}$ is a chain map. However, by construction, $\mathcal{E}$ is the composition

$$\mathbb{G}_*(C(\widetilde{L})) \to \mathbb{G}_*(C(\widetilde{L'})) \xrightarrow{\mathcal{E}'} J^F C(\widetilde{K'}) \xrightarrow{J^F(C(\widetilde{K'}) \to C(\widetilde{K}))} J^F C(\widetilde{K})$$

of chain maps and hence is itself a chain map. □

9. The Proof of Theorem B

In order to prove Theorem B we will need to show how the absolute torsions computed from a pointed torsion structure on a fibre bundle vary when we change the PTS on the base or on the fibre. Let $\mathcal{S}_\bullet(F \xrightarrow{q} E)$ denote the set of pointed torsion structures on $F_0 = p^{-1}(b_0)$ with respect to the pull-back of the universal covering of E. We summarize the results as follows:

Proposition 9.1. *Given $[B] \in \mathcal{S}_\bullet(B)$ and $[F] \in \mathcal{S}_\bullet(F \xrightarrow{q} E)$, there exists a fibre bundle torsion structure $(L \to K, \mathcal{E})$ on $F \to E \to B$, such that $[K] = [B]$*

and the class of the fibre structure is $[F]$. Moreover any two such fibre bundle torsion structures represent the same element in $\mathcal{S}_\bullet(E)$.

Let $\mathfrak{E}_{[B][F]} \in \mathcal{S}_\bullet(E)$ denote the induced torsion structure on the total space.

Proposition 9.2. *Let $[B_1], [B_2] \in S_\bullet(B)$ be any two torsion structures for the base, and let $[F_1], [F_2] \in \mathcal{S}_\bullet(F \xrightarrow{q} E))$ be any two fibre structures. Then*

$$\Phi(\mathfrak{E}_{[B_1][F_1]}, \mathfrak{E}_{[B_2][F_2]}) = p^!\Phi([B_1], [B_2]) + \chi(B)\Phi([F_1], [F_2])$$

The *dual torsion structure* $\mathcal{D}\mathfrak{E}_{[B][F]}$ of $\mathfrak{E}_{[B][F]}$ was defined in Definition 5.1.

Proposition 9.3. *Let $\mathcal{D}\mathfrak{E}_{[B][F]}$ denote the dual torsion structure of $\mathfrak{E}_{[B][F]}$ in $\mathcal{S}_\bullet(E)$. Then*

$$\Phi(\mathcal{D}\mathfrak{E}_{[B][F]}, \mathfrak{E}_{[\mathcal{D}B][\mathcal{D}F]}) = 0$$

These properties will be established in a sequence of lemmas.

Lemma 9.4. *Given PTSs on K and F the maps $\mathcal{E}^\beta \colon C(\widetilde{F_\beta}) \to C(\widetilde{F_0})$ are uniquely determined by the properties:*

(i) *The map $\mathcal{E}^{b_0} \colon C(\widetilde{F_0}) \to C(\widetilde{F_0})$ is the identity.*
(ii) *The direct sum $\mathcal{E} = \bigoplus_\beta \mathcal{E}^\beta$ is a chain equivalence.*

Proof. Suppose that $\{\mathcal{E}_1^\beta\}$ and $\{\mathcal{E}_2^\beta\}$ are two sets of chain maps satisfying these conditions. Since B is connected, any cell $\beta \in K$ can be connected to the base-point by a sequence of adjacent pairs of cells. If $\gamma \subset \partial\beta$, we have a commutative diagram

$$\begin{tikzcd} C(\widetilde{F_\beta}) \arrow[r, "\mathcal{E}_1^\beta"] \arrow[d] & C(\widetilde{F_0}) \arrow[d, equal] \\ C(\widetilde{F_\gamma}) \arrow[r, "\mathcal{E}_1^\gamma"] & C(\widetilde{F_0}) \end{tikzcd}$$

and similarly for $\mathcal{E}_2$. We are using the identification $J^F C(\beta) = C(\widetilde{F_0})$, valid for every cell. But the horizontal maps are isomorphisms in $\mathbb{SPD}_n(\mathbf{Z}[\pi_1(E, e_0)])$, so $\mathcal{E}_1^\beta = \mathcal{E}_2^\beta$ if and only if $\mathcal{E}_1^\gamma = \mathcal{E}_2^\gamma$. However, these two maps agree at the base-point b_0, so they agree over every cell and $\mathcal{E}_1 = \mathcal{E}_2$. $\square$

Lemma 9.5. *Suppose that $(L_1 \to K, \mathcal{E}_1)$ and $(L_2 \to K, \mathcal{E}_2)$ are pointed fibre bundle torsion structures on $E \to B$ with fibre structures F_1 and F_2 respectively. If $h \colon L_1 \to L_2$ is a subdivision over K, then there is a commutative diagram*

$$\begin{tikzcd} \mathbb{G}_*(C(\widetilde{L}_1)) \arrow[r, "\mathcal{E}_1"] \arrow[d, "\mathbb{G}_*(h)"'] & J^{F_1}(C(\widetilde{K})) \arrow[d, "h_*"] \\ \mathbb{G}_*(C(\widetilde{L}_2)) \arrow[r, "\mathcal{E}_2"'] & J^{F_2}(C(\widetilde{K})) \end{tikzcd}$$

in $\mathbb{SPD}_n(\mathbf{Z}[\pi_1(E, e_0)])$.

Proof. Since any subdivision over K is a filtered map, we can pass to the associated graded complexes. The composition

$$\mathcal{E}_1' \colon \mathbb{G}_*(C(\widetilde{L_1})) \xrightarrow{\mathbb{G}_*(h)} \mathbb{G}_*(C(\widetilde{L_2})) \xrightarrow{\mathcal{E}_2} J^{F_2}C(\widetilde{K}) \xrightarrow{(h_*)^{-1}} J^{F_1}C(\widetilde{K})$$

splits over each simplex $\beta \in K$ to the composition:

$$\mathcal{E}_1'^{\beta} \colon C(\widetilde{F_{1\beta}}) \xrightarrow{\mathbb{G}_*(h_*|_\beta)} C(\widetilde{F_{2\beta}}) \xrightarrow{\mathcal{E}_2^{\beta}} J^{F_2}C(\beta) \xrightarrow{(h_*)^{-1}} J^{F_1}C(\beta)$$

In particular the map $\mathcal{E}_1'^{b_0}$ coincides with $\mathcal{E}_1{}^{b_0}$, so the above remark implies that $\mathcal{E}_1' = \mathcal{E}_1$. Hence the diagram commutes. $\square$

Corollary 9.6. *Suppose that $(L_1 \to K, \mathcal{E}_1)$ and $(L_2 \to K, \mathcal{E}_2)$ are pointed fibre bundle torsion structures on a fibre bundle $F \to E \to B$. The induced pointed torsion structures $[L_1]$ and $[L_2]$ on E satisfy*

$$\Phi([L_1], [L_2]) = \chi(B)\Phi([F_1], [F_2])$$

Proof. If L_2 is a subdivision of L_1, then the commutativity of the diagram implies that

$$\tau^{NEW}(\mathbb{G}_*(C(\widetilde{L_1})) \to \mathbb{G}_*(C(\widetilde{L_2}))) = \tau^{NEW}(J^{F_1}(C(\widetilde{K})) \to J^{F_2}(C(\widetilde{K})))$$

since the maps $\mathcal{E}_1$ and $\mathcal{E}_2$ have trivial absolute torsion. By Lemma 6.4

$$\begin{aligned} i_*\tau^{NEW}(\mathbb{G}_*(C(\widetilde{L_1})) \to \mathbb{G}_*(C(\widetilde{L_2}))) &= \chi(B)\tau^{NEW}(C(\widetilde{F_1}) \to C(\widetilde{F_2})) \\ &= \chi(B)\Phi([F_1], [F_2]) \end{aligned}$$

From Theorem 12.17 on the torsion of filtered maps we see that:

$$\begin{aligned} \Phi([L_1], [L_2]) &= \tau^{NEW}(C(\widetilde{L_1}) \to C(\widetilde{L_2})) \\ &= i_*\tau^{NEW}(\mathbb{G}_*(C(\widetilde{L_1})) \to \mathbb{G}_*(C(\widetilde{L_2}))) \\ &= \chi(B)\Phi([F_1], [F_2]) \end{aligned}$$

as required. We now consider the general case where L_2 is not necessarily a subdivision of L_1; in this case we may find a filtered cellular map $L_3 \to L_1$ which is a common subdivision (as cell-complexes) of L_1 and L_2. By Theorem 8.5 we may find a PFBTS whose underlying triangulation is L_3, we will also denote this by L_3. It now follows that

$$\begin{aligned} \Phi([L_1], [L_2]) &= \Phi([L_2], [L_3]) - \Phi([L_1], [L_3]) \\ &= \chi(B)\Phi([F_2], [F_3]) - \chi(B)\Phi([F_1], [F_3]) \\ &= \chi(B)\Phi([F_1], [F_2]) \end{aligned}$$

as required. $\square$

Lemma 9.7. *Suppose that $(L_1 \to K_1, \mathcal{E}_1)$ and $(L_2 \to K_2, \mathcal{E}_2)$ are pointed fibre bundle torsion structures on $E \to B$. Let $h \colon L_1 \to L_2$, $f \colon K_1 \to K_2$ be cellular homeomorphisms, with $p_{L_2} \circ h = f \circ p_{L_1}$ and h inducing the identity* *on $F = p^{-1}(b_0)$. If both PFBTS have the same fibre structure under this identification, then there is a commutative diagram*

$$\begin{tikzcd}
\mathbb{G}_*(C(\widetilde{L}_1)) \arrow[r, "\mathcal{E}_1"] \arrow[d, "\mathbb{G}_*(h)"'] & J^F(C(\widetilde{K}_1)) \arrow[d, "J^F(f)"] \\
\mathbb{G}_*(C(\widetilde{L}_2)) \arrow[r, "\mathcal{E}_2"'] & J^F(C(\widetilde{K}_2))
\end{tikzcd}$$

in $\mathbb{SD}_n(\mathbf{Z}[\pi_1(E, e_0)])$.

Proof. We first consider the case where K_1 and K_2 differ only by the choice of PTS. Then the composition

$$\mathbb{G}_*(C(\widetilde{L_1})) \xrightarrow{\mathbb{G}_*(h)} \mathbb{G}_*(C(\widetilde{L_2})) \xrightarrow{\mathcal{E}_2} J^F C(\widetilde{K_2}) \xrightarrow{J^F(f_*^{-1})} J^F C(\widetilde{K_1})$$

splits over the cells of $K_1 = K_2$ to give maps ${\mathcal{E}'}^{\beta}_1 \colon \mathbb{G}_*(C(\widetilde{F_\beta})) \to J^F C(\widetilde{K_1})$. However Remark 9.4 shows that these maps must coincide with the $\mathcal{E}^{\beta}_1$, so the diagram commutes in this case.

It is now sufficient to consider the case where the map $K_1 \to K_2$ is an elementary subdivision of one cell β and where $L_1 \to L_2$ is a cellular isomorphism away from $p^{-1}(\beta)$. We may also assume the geometric basis for K_2 is the same as that for K_1 away from β and that the geometric basis for each $C(\widetilde{F_{1\gamma}})$ coincides with that for $C(\widetilde{F_{2\gamma}})$ for $\gamma \neq \beta$. We may also assume that the maps $\mathcal{E}^{\gamma}_1$ and $\mathcal{E}^{\gamma}_2$ are equal for all $\gamma \neq \beta$ so it is sufficient to show that the diagram commutes on the $C(\widetilde{F_\beta})$ factor. Let $\{\beta_i\}_i$ be the components of the subdivision of β in K_2 which have the same dimension as β. Then we must show that the diagram

$$\begin{tikzcd}
C(\widetilde{F_\beta}) \arrow[r, "\mathcal{E}^{\beta}"] \arrow[d, "\mathbb{G}_*(h|_\beta)"'] & J^F C(\beta) \arrow[d, "J^F(f|_\beta)"] \\
\bigoplus_i C(\widetilde{F_{\beta_i}}) \arrow[r, "\oplus \mathcal{E}^{\beta_i}"] & \bigoplus_i J^F C(\beta_i)
\end{tikzcd}$$

commutes. The RHS of the diagram splits over the components $C(\widetilde{F_{\sigma_i}})$ so it is sufficient to show that

$$\begin{tikzcd}
C(\widetilde{F_\beta}) \arrow[r, "\mathcal{E}^{\beta}"] \arrow[d, "\mathbb{G}_*(h|_{\beta_i})"'] & J^F C(\beta) \arrow[d, "J^F(f|_{\beta_i})"] \\
C(\widetilde{F_{\beta_i}}) \arrow[r, "\mathcal{E}^{\beta_i}"] & J^F C(\beta_i)
\end{tikzcd}$$

commutes for each i. Let τ be a boundary component of both β_i and β (such a component can always be found for an elementary subdivision). Then we have a diagram of chain equivalences:

$$\begin{tikzcd}
C(\widetilde{F_\tau}) \arrow[r] \arrow[d, "\mathcal{E}^\tau"] & C(\widetilde{F_\beta}) \arrow[r, "\mathbb{G}_*(h|_{\beta_i})"] \arrow[d, "\mathcal{E}^\beta"] & C(\widetilde{F_{\beta_i}}) \arrow[r] \arrow[d, "\mathcal{E}^{\beta_i}"] & C(\widetilde{F_\tau}) \arrow[d, "\mathcal{E}^\tau"] \\
J^F C(\tau) \arrow[r] & J^F C(\beta) \arrow[r, "J^F(f|_{\beta_i})"] & J^F C(\beta_i) \arrow[r] & J^F C(\tau)
\end{tikzcd}$$

The outer square commutes since the compositions along the top and bottom are the identity maps. The left and right squares commute since both $\mathcal{E}_1$ and $\mathcal{E}_2$ are chain maps. Hence the middle square commutes as required. □

Corollary 9.8. *Let K_1 and K_2 be cell-decompositions of B; let $L_1 \to K_1$ and $L_2 \to K_2$ be PFBTSs on $E \to B$ with fibre class $[F]$. Then the induced PTSs on E, $[L_1]$ and $[L_2]$ satisfy:*

$$\Phi([L_1],[L_2]) = p^!\Phi([K_1],[K_2])$$

Proof. If L_2 and K_2 are subdivisions of L_1 and K_1 respectively, then commutativity of the diagram of the above lemma and Lemma 6.4 give:

$$\begin{aligned}
i_*\tau^{NEW}(\mathbb{G}_*(C(\widetilde{L_1})) \to \mathbb{G}_*(C(\widetilde{L_2}))) &= \tau^{NEW}(J^F C(\widetilde{K_1}) \to J^F C(\widetilde{K_2})) \\
&= p^!\tau^{NEW}(C(\widetilde{K_1}) \to C(\widetilde{K_2}))
\end{aligned}$$

Using Theorem 12.17 on the torsion of filtered maps we see that:

$$\begin{aligned}
\Phi([L_1],[L_2]) &= \tau^{NEW}(C(\widetilde{L_1}) \to C(\widetilde{L_2}) \\
&= \tau^{NEW}(\mathbb{G}_*(C(\widetilde{L_1})) \to \mathbb{G}_*(C(\widetilde{L_2}))) \\
&= p^!\tau^{NEW}(C(\widetilde{K_1}) \to C(\widetilde{K_2}))
\end{aligned}$$

In the general case we choose a common subdivision $(L_3 \to K_3, \mathcal{E})$ with fibre structure F. Then:

$$\begin{aligned}
\Phi([L_1],[L_2]) &= \Phi([L_1],[L_3]) + \Phi([L_3],[L_2]) \\
&= p^!\Phi([K_1],[K_3]) + p^!\Phi([K_3],[K_2]) \\
&= p^!\Phi([K_1],[K_2])
\end{aligned}$$

□

The proof of Proposition 9.1. We choose a torsion structure $K \to B$ representing $[B]$. Then by Theorem 8.5 we may find a fibre bundle torsion structure $(L \to K, \mathcal{E})$ with the given fibre class. Let $(L' \to K', \mathcal{E}')$ be another choice of fibre bundle torsion structure. Then by Proposition 9.8 we have

$$\Phi([L],[L']) = p^!([K],[K']) = p^!([B],[B]) = 0$$

Therefore $[L]$ and $[L']$ are equivalent. □

The proof of Proposition 9.2. Choose representatives $(L_1 \to K_1, \mathcal{E}_1)$, $(L_2 \to K_2, \mathcal{E}_2)$ and $(L_3 \to K_2, \mathcal{E}_3)$ for $\mathfrak{E}_{[B_1][F_1]}$,$\mathfrak{E}_{[B_2][F_1]}$ and $\mathfrak{E}_{[B_2][F_2]}$ respectively. Then

$$\begin{aligned}\Phi(\mathfrak{E}_{[B_1][F_1]}, \mathfrak{E}_{[B_2][F_2]}) &= \Phi((L_1 \to K_1, \mathcal{E}_1), (L_2 \to K_2, \mathcal{E}_2)) \\ &\quad +\Phi((L_2 \to K_2, \mathcal{E}_2), (L_3 \to K_2, \mathcal{E}_3)) \\ &= p^! \Phi([B_1], [B_2]) + \chi(B)\Phi([F_1], [F_2])\end{aligned}$$

by Corollaries 9.6 and 9.8. □

The proof of Proposition 9.3. Let $(L \to K, \mathcal{E})$ be a PFBTS representing $\mathfrak{E}_{[B][F]}$. Then $\mathcal{D}L \to \mathcal{D}K$ is a cellular map and we have a dual PTS for $C(\mathcal{D}K)$. We first construct a PFBTS $(\mathcal{D}L \to \mathcal{D}K, \mathcal{D}\mathcal{E})$ representing $\mathfrak{E}_{[\mathcal{D}B][\mathcal{D}F]}$. We choose the geometric basis on each fibre of $\mathcal{D}L$ such that the composition

$$C(\widetilde{\mathcal{D}L}) \to C(\widetilde{L})^{n+k-*} \xrightarrow{\theta_{C(\widetilde{L})}} F_*^{dual} C(\widetilde{L}) \tag{9.9}$$

is the based identity map of signed filtered complexes. We now have an identification

$$\mathbb{G}_*(C(\widetilde{\mathcal{D}L})) \to \mathbb{G}_*(C(\widetilde{L}))^{k-*}$$

by taking the associated complex of the above composition, and applying Lemma 12.23 to identify the associated complex of the filtered dual $F_*^{dual}C(\widetilde{L})$ with the k-dual $\mathbb{G}_*(C(\widetilde{L}))^{k-*}$ of the associated complex. Note that

$$(\mathbb{G}_*(C(\widetilde{L})))_r^{k-*} = \sum_{\beta \in K^{<k-r>}} C(\widetilde{F_\beta})^{k-*} = \sum_{\mathcal{D}\beta \in \mathcal{D}K^{<r>}} C(\widetilde{F_{\mathcal{D}\beta}})$$

We have a chain equivalence

$$(\mathcal{E}^{k-*})^{-1} \colon \mathbb{G}_*(C(\mathcal{D})) \to (J^{C(\widetilde{F_0})} C(\widetilde{K}))^{k-*} = J^{C(\widetilde{\mathcal{D}L_{\mathcal{D}k_0}})} C(\widetilde{K})$$

We now choose a base-point $\mathcal{D}k_0$ in $\mathcal{D}K$, denote its dual k-cell in K by σ_0. We have a chain equivalence

$$\mathcal{E}_{\sigma_0}^{k-*} \colon C(\widetilde{\mathcal{D}F_{\mathcal{D}k_0}}) \to C(\widetilde{\mathcal{D}F_0})$$

and hence a natural transformation

$$h_* \colon J^{C(\widetilde{\mathcal{D}L_{\mathcal{D}k_0}})} \to J^{C(\widetilde{\mathcal{D}L_0})}$$

We define the chain map $\mathcal{D}\mathcal{E}$ by $\mathcal{D}\mathcal{E} = h_*(\mathcal{E}^{k-*})^{-1}$ with the components given by $\mathcal{D}\mathcal{E}_{\mathcal{D}\beta} = \mathcal{E}_{\sigma_0}^{k-*}(\mathcal{E}_\beta^{k-*})^{-1}$. It remains to show that

$$\Phi(\mathcal{D}(L \to K, \mathcal{E}), (\mathcal{D}L \to \mathcal{D}K, \mathcal{D}\mathcal{E}) = 0$$

However,

$$\begin{aligned}&\Phi(\mathcal{D}(L \to K, \mathcal{E}), (\mathcal{D}L \to \mathcal{D}K, \mathcal{D}\mathcal{E})) \\ &\qquad = \tau^{NEW}(C(\widetilde{\mathcal{D}L}) \to C(\widetilde{L})^{n+k-*}) \\ &\qquad = -\tau^{NEW}(\theta_{C(\widetilde{L})} \colon (C(\widetilde{L}))^{n+k-*} \to F_*^{dual} C(\widetilde{L})) \\ &\qquad = 0\end{aligned}$$

using the fact that (9.9) is the based identity map and Proposition 12.26. ☐

The proof of Theorem B. We can now complete the proof of Theorem B. Let $[B]$ and $[F]$ be torsion structures on B and F respectively. We may assume that $[F] \in \mathcal{S}_\bullet(F \xrightarrow{q} E)$ is induced by a PTS on the universal covering of F, and therefore $\Phi([\mathcal{D}F],[F]) = q_*(\tau^{NEW}(F))$ by Theorem 5.2. Then

$$\begin{aligned}
\tau^{NEW}(E) &= \Phi(\mathcal{D}\mathfrak{E}_{[B][F]}, \mathfrak{E}_{[B][F]}) \\
&= \Phi(\mathfrak{E}_{[\mathcal{D}B][\mathcal{D}F]}, \mathfrak{E}_{[B][F]}) \\
&= \Phi(\mathfrak{E}_{[\mathcal{D}B][F]}, \mathfrak{E}_{[B][F]}) + \Phi(\mathfrak{E}_{[\mathcal{D}B][\mathcal{D}F]}, \mathfrak{E}_{[\mathcal{D}B][F]}) \\
&= p^!\Phi([\mathcal{D}B][B]) + \chi(B)\Phi([\mathcal{D}F][F]) \\
&= p^!(\tau^{NEW}(B)) + \chi(B)q_*(\tau^{NEW}(F)) \\
&\qquad \in \widehat{H}^{n+k}(\mathbf{Z}/2; K_1(\mathbf{Z}[\pi_1 E]))
\end{aligned}$$

as required. ☐

10. The Proof of Theorem A

Our main result on the multiplicativity of signatures will be deduced from Theorem 3.3 and the following application of Theorem B, in which we compute the reduced absolute torsion E. If $\dim B$ is odd, the vanishing of $\mathrm{sign}(E)$ follows immediately (as remarked in [2]), so we may assume that $\dim B$ is even.

Theorem 10.1. *Let $F \xrightarrow{q} E \xrightarrow{p} B$ be a PL bundle of compatibly oriented closed manifolds. If $\dim B$ is even, then the reduced absolute torsion*

$$\bar{\tau}^{NEW}(E) = \chi(F) \cdot \bar{\tau}^{NEW}(B)) + \chi(B) \cdot \bar{\tau}^{NEW}(F)) \ .$$

Proof. The augmentation map induces a direct sum splitting

$$K_1(\mathbf{Z}[\pi_1(E, e_0)]) \cong K_1(\mathbf{Z}) \oplus \widetilde{K}_1(\mathbf{Z}[\pi_1(E, e_0)])$$

and ϵ_* restricted to the subgroup $K_1(\mathbf{Z})$ is the identity map. We will apply the augmentation map ϵ_* to the formula for $\tau^{NEW}(E)$ given in Theorem B. Note that $\epsilon_* \circ q_* = \epsilon_*$ so the second term

$$\chi(B) \cdot \epsilon_*(q_*\tau^{NEW}(F)) = \chi(B) \cdot \bar{\tau}^{NEW}(F))$$

as required.

To evaluate the first term, we need to compute

$$\epsilon_*(p^!(\tau^{NEW}(B))) = \epsilon_*(p_*p^!(\tau^{NEW}(B))) \ .$$

By Proposition 3.5 we know that $\tau^{NEW}(B) = \tau(\pm h^2)$ for some $h \in \pi_1(B, b_0)$. But the map $\phi\colon \pi_1(B, b_0) \to K_1(\mathbf{Z})$ defined by $\phi(g) = \epsilon_*(p^!(\tau(g)))$ is a group homomorphism. Since $K_1(\mathbf{Z}) = \mathbf{Z}/2$, ϕ vanishes on squares of group elements in $\pi_1(B, b_0)$. Therefore

$$\epsilon_*(p^!(\tau^{NEW}(B))) = \epsilon_*(p^!(\tau(\pm 1))) \ .$$

However, $\tau(\pm 1)$ lies in the image of the map $i_*\colon K_1(\mathbf{Z}) \to K_1(\mathbf{Z}[\pi_1(B, b_0)])$ induced by inclusion. By pulling back the bundle $E \xrightarrow{p} B$ over the base point $b_0 \in B$, and applying the naturality formula for pullbacks (see [12, Cor. 5.3]), we obtain the relation

$$p^!(\tau^{NEW}(B)) = i_*(p_0^*(\tau(\pm 1))) = i_*(p_0^*(\tau(\pm 1)))$$

by comparison with the trivial bundle $F \times \{b_0\} \xrightarrow{p_0} \{b_0\}$. But for a trivial bundle $p_* \circ p^!$ is just multiplication by $\chi(F)$, by an easy special case of [13, Thm. 7.1]. Therefore

$$\epsilon_*(p^!(\tau^{NEW}(B))) = \epsilon_*(p_*(p^!(\tau^{NEW}(B)))) == \chi(F) \cdot \bar{\tau}^{NEW}(B)$$

as required, since $\epsilon_* \circ i_*$ is the identity map. □

Corollary 10.2. *If $\dim B$ is even, then $\bar{\tau}^{NEW}(E) = \bar{\tau}^{NEW}(F \times B)$.*

Proof. We apply the product formula, Proposition 6.5, and the formula just proved. □

This result says that the reduced absolute torsion of E is the same as that of the total space $F \times B$ for the trivial bundle, provided $\dim B$ is even, We can now conclude that the signature $\operatorname{sign}(E)$ agrees with $\operatorname{sign}(F \times B) = \operatorname{sign}(F) \cdot \operatorname{sign}(B)$ modulo 4.

(i) If $\dim B = 4j$, $\dim F = 4l$, and $\dim E = 4(j+l) = 4k$

$$\begin{aligned} \operatorname{sign}(E) &= 2\bar{\tau}^{NEW}(E) + (2k+1)\chi(E) \\ &= 2(\bar{\tau}^{NEW}(B)\chi(F) + \chi(B)\bar{\tau}^{NEW}(F)) + (2k+1)\chi(B)\chi(F) \\ \operatorname{sign}(B)\operatorname{sign}(F) &= \left[2\bar{\tau}^{NEW}(b) + (2j+1)\chi(B)\right]\left[2\bar{\tau}^{NEW}(F) + (2l+1)\chi(F)\right] \end{aligned}$$

and these agree modulo 4.

(ii) If $\dim B = 4j+2$, $\dim F = 4l-2$, and $\dim E = 4(j+l) = 4k$

$$\operatorname{sign}(E) = 2(\bar{\tau}^{NEW}(B)\chi(F) + \chi(B)\bar{\tau}^{NEW}(F)) + (2k+1)\chi(B)\chi(F)$$

and $\chi(F) \equiv \chi(B) \equiv 0 \bmod 2$, so $\operatorname{sign}(E) \equiv 0 \bmod 4$.

(iii) If $\dim B$ is even, but $\dim F$ is odd, then both sides are zero.

This completes the proof of Theorem A. □

11. Fibrations of PD spaces

The conjecture of [9] is stated in a more general situation: for fibrations of Poincaré duality spaces. We don't know yet if the signature is multiplicative mod 4 for such fibrations, and hope to return to this topic in a future paper. We will just check that there are no counter-examples arising from finite coverings. A formula for the Whitehead torsions in a fibration of Poincaré duality spaces was given in [20].

Recall that C. T. C. Wall constructed examples of finite coverings of oriented Poincaré complexes X, with $\pi_1(X) = \mathbf{Z}/p$, p prime, and the property that $\mathrm{sign}(\widetilde{X}) \neq p \cdot \mathrm{sign}(X)$. On the other hand, the algebraic theory of surgery gives the following congruence.

Lemma 11.1. *Let $(f, b)\colon Y \to X$ be a degree one normal map of oriented, finite Poincaré duality spaces. Then* $\mathrm{sign}(Y) - \mathrm{sign}(X) \equiv 0 \bmod 8$.

Proof. By [23, p. 229] any such degree one normal map has a quadratic signature $\sigma_*(f, b)$, with the property that

$$\mathrm{sign}(\sigma_*(f, b)) = \mathrm{sign}(\sigma^*(Y)) - \mathrm{sign}(\sigma^*(X))$$

and $\mathrm{sign}(X) = \mathrm{sign}(\sigma^*(X))$ for any Poincaré complex. On the other hand, $\mathrm{sign}(\sigma_*(f, b)) \equiv 0 \bmod 8$ since the signature of an even, unimodular symmetric bilinear form over $\mathbf{Z}$ is divisible by 8 (see [8, 3.11]). □

Corollary 11.2. *Let $X' \to X$ be a finite covering of degree d of oriented, finite Poincaré complexes. Then* $\mathrm{sign}(X') - d \cdot \mathrm{sign}(X) \equiv 0 \bmod 8$.

Proof. For any finite covering $X' \to X$ of oriented, finite Poincaré duality spaces, we can construct a degree one normal map

$$(f, b)\colon X' \sqcup -(d-1)X \to X \ .$$

Then Lemma 11.1 implies that

$$\mathrm{sign}(X') - d \cdot \mathrm{sign}(X) = \mathrm{sign}(X' \sqcup -(d-1)X) - \mathrm{sign}(X) \equiv 0 \bmod 8 \ .$$

□

12. Filtered chain complexes

In this section, we give a self-contained treatment of the absolute torsion of signed filtered chain complexes. These results are the algebraic foundation for the torsion calculations in this paper, but this section can be read independently of the previous sections. The main results are the Invariance Theorems 12.14, 12.17, generalizing results previously obtained for the reduced torsion by Milnor [17], Maumary [15], and Munkholm [18]. These results express the absolute torsion $\tau^{NEW}(f) \in K_1(\mathbb{A})$ of a filtered chain equivalence $f\colon C \to D$ of signed filtered complexes in terms of the filtration quotients.

§12A. Filtered complexes. We first need some notation and definitions.

Definition 12.1. Let $\mathbb{A}$ be an additive category.

(i) A *k-filtered object* F_*M in $\mathbb{A}$ is an object M in $\mathbb{A}$ together with a direct sum decomposition

$$M = M_0 \oplus M_1 \oplus \cdots \oplus M_k$$

which we regard as a length k filtration

$$F_{-1}M = 0 \subseteq F_0M \subseteq F_1M \subseteq \cdots \subseteq F_kM = M$$

with

$$F_jM = M_0 \oplus M_1 \oplus \cdots \oplus M_j \ (0 \leqslant j \leqslant k)\ .$$

(ii) A *filtered morphism* $f\colon F_*M \to F_*N$ of k-filtered objects in $\mathbb{A}$ is a morphism in $\mathbb{A}$ of the type

$$f = \begin{pmatrix} f_0 & f_1 & f_2 & \dots & f_k \\ 0 & f_0 & f_1 & \dots & f_{k-1} \\ 0 & 0 & f_0 & \dots & f_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f_0 \end{pmatrix} \ :\ M = \bigoplus_{s=0}^{k} M_s \to N = \bigoplus_{s=0}^{k} N_s$$

so that

$$f(F_jM) \subseteq F_jN \ (0 \leqslant j \leqslant k)\ .$$

The (u,v)-component of this upper triangular matrix is a morphism $f_{v-u}\colon M_v \to N_u$, $0 \leqslant u \leqslant v \leqslant k$, where $f_j\colon M_* \to N_{*-j}$, $0 \leqslant j \leqslant k$, are graded morphisms in $\mathbb{A}$.

(iii) A *k-filtered complex* F_*C in $\mathbb{A}$ is a finite chain complex C in $\mathbb{A}$ with k-filtered objects $C_r = \bigoplus_{s=0}^{k} C_{r,s}$ such that the differentials $d\colon F_*C_r \to F_*C_{r-1}$ are filtered morphisms. The matrix components of d are maps $d_j\colon C_{r,s} \to C_{r-1,s-j}$.

(iv) A *filtered chain map* $f\colon F_*C \to F_*D$ is a chain map $f\colon C \to D$ such that $f_r\colon F_*C_r \to F_*D_r$ is a filtered morphism in each degree. The component maps have the form $f_j\colon C_{r,s} \to D_{r,s-j}$.

(v) A *filtered chain homotopy* $g\colon f \simeq f'\colon F_*C \to F_*D$ between filtered chain maps f, f' is a collection $\{g\colon C_r \to D_{r+1} \,|\, r \in \mathbf{Z}\}$ of morphisms in $\mathbb{A}$ such that

$$f - f' = dg + gd\colon C_r \to D_r$$

and

$$g = \begin{pmatrix} g_0 & g_1 & g_2 & \dots & g_k \\ g_{-1} & g_0 & g_1 & \dots & g_{k-1} \\ 0 & g_{-1} & g_0 & \dots & g_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & g_0 \end{pmatrix}$$

has component maps

$$g_j\colon C_{r,s} \to D_{r+1,s-j} \ (-1 \leqslant j \leqslant k)\ .$$

(vi) A *filtered contraction* of a k-filtered complex F_*C in $\mathbb{A}$ is a filtered chain homotopy

$$\Gamma\colon 1 \ \simeq\ 0\colon F_*C \to F_*C\ .$$

(vii) The *filtered mapping cone* of a filtered chain map $f\colon C \to D$ of k-filtered complexes is the $(k+1)$-filtered complex $F_*\mathcal{C}^{fil}(f)$ with

$$\mathcal{C}^{fil}(f)_{r,s} = D_{r,s} \oplus C_{r-1,s-1} \ (0 \leqslant s \leqslant k+1)$$

and the differential $d^{\mathcal{C}^{fil}(f)}\colon \mathcal{C}^{fil}(f)_r \to \mathcal{C}^{fil}(f)_{r-1}$ is in upper triangular block form with 2×2 block entries $(0 \leqslant j \leqslant k+1)$:

$$d_j^{\mathcal{C}^{fil}(f)} = \begin{pmatrix} d_j^D & (-)^{r-1} f_{j-1} \\ 0 & d_j^C \end{pmatrix} : D_{r,s} \oplus C_{r-1,s-1} \to D_{r-1,s-j} \oplus C_{r-2,s-j-1}$$

□

The rearrangement map

$$\mathcal{C}^{fil}(f)_r = \sum_{s=0}^{k} (D_{r,s} \oplus C_{r-1,s-1}) \xrightarrow{\cong} \sum_{s=0}^{k} D_{r,s} \oplus \sum_{s=1}^{k} C_{r-1,s-1} = \mathcal{C}(f)_r$$

define an isomorphism of unfiltered chain complexes $\rho\colon \mathcal{C}^{fil}(f) \xrightarrow{\cong} \mathcal{C}(f)$.

Theorem 12.2. *A filtered chain map $f\colon F_*C \to F_*D$ of k-filtered chain complexes is a filtered chain equivalence if and only if the filtered mapping cone $F_*\mathcal{C}^{fil}(f)$ is filtered contractible.*

Proof. By definition, f is a filtered chain equivalence if and only if there exist a filtered chain map $g\colon F_*D \to F_*C$ and filtered chain homotopies

$$\Gamma\colon 1 \simeq gf\colon F_*C \to F_*C \ , \ \Delta\colon 1 \simeq fg\colon F_*D \to F_*D \ .$$

A filtered contraction $e\colon 1 \simeq 0\colon F_*\mathcal{C}^{fil}(f) \to F_*\mathcal{C}^{fil}(f)$ is of the type

$$e = \begin{pmatrix} \Delta & h \\ (-)^r g & \Gamma \end{pmatrix} : \mathcal{C}^{fil}(f)_r \cong D_r \oplus C_{r-1} \to \mathcal{C}^{fil}(f)_{r+1} \cong D_{r+1} \oplus C_r$$

with $g\colon F_*D \to F_*C$ a filtered chain map and $\Gamma\colon 1 \simeq gf$, $\Delta\colon 1 \simeq fg$ filtered chain homotopies, so that f is a filtered chain equivalence.

Conversely, suppose that f is a filtered chain equivalence. The morphisms defined by

$$e' = \begin{pmatrix} \Delta & 0 \\ (-)^r g & \Gamma \end{pmatrix} : \mathcal{C}^{fil}(f)_r \cong D_r \oplus C_{r-1} \to \mathcal{C}^{fil}(f)_{r+1} \cong D_{r+1} \oplus C_r$$

are such that the morphisms

$$de' + e'd = \begin{pmatrix} 1 & (-)^r(f\Gamma - \Delta f) \\ 0 & 1 \end{pmatrix} : \mathcal{C}^{fil}(f)_r \cong D_r \oplus C_{r-1} \to \mathcal{C}^{fil}(f)_r \cong D_r \oplus C_{r-1}$$

are automorphisms. The morphisms

$$e = (de' + e'd)^{-1} e'\colon \mathcal{C}^{fil}(f)_r \cong D_r \oplus C_{r-1} \to \mathcal{C}^{fil}(f)_{r+1} \cong D_{r+1} \oplus C_r$$

define a filtered contraction $e\colon 1 \simeq 0\colon F_*\mathcal{C}^{fil}(f) \to F_*\mathcal{C}^{fil}(f)$. □

§12B. The associated complex. We now describe how to pass from a filtered complex to its associated graded complex. Notice that the differentials $d\colon C_r \to C_{r-1}$ of a k-filtered complex F_*C the identities

$$d^2 = 0\colon C_r \to C_{r-2}$$

lead to a number of recursion formulas among the component maps $d_j\colon C_{r,s} \to C_{r-1,s-j}$ by carrying out the multiplication d^2 as a product of upper triangular matrices. In particular, we have the formulas

$$\begin{aligned}&(d_0)^2 = 0\colon C_{r,s} \to C_{r-2,s}\ ,\\ &d_0d_1 + d_1d_0 = 0\colon C_{r,s} \to C_{r-2,s-1}\ ,\\ &(d_1)^2 + d_0d_2 + d_2d_0 = 0\colon C_{r,s} \to C_{r-2,s-2}\ .\end{aligned}$$

Definition 12.3. The *associated complex* of a k-filtered complex F_*C in $\mathbb{A}$ is the k-dimensional chain complex in the derived category $\mathbb{D}(\mathbb{A})$

$$\mathbb{G}_*(C):\quad G_k(C) \to \cdots \to G_{r+1}(C) \xrightarrow{d_*} G_r(C) \xrightarrow{d_*} \cdots \to G_0(C)$$

with differential

$$d_* = (-)^s d_1\colon G_r(C)_s = C_{r+s,r} \to G_{r-1}(C)_s = C_{r+s-1,r-1}\ .$$

The individual terms $G_r(C)$ are objects in $\mathbb{D}(\mathbb{A})$, with 'internal' differential

$$d_{G_r(C)} = d_0\colon G_r(C)_s = C_{r+s,r} \to G_r(C)_{s-1} = C_{r+s-1,r}\ .$$

□

Example 12.4 (Tensor Products)**.** Let C and D be chain complexes in $\mathbb{A}(R)$ and $\mathbb{A}(S)$ respectively, where R and S are rings. If $\dim C = k$, then the tensor product complex $C \otimes D$ (over $\mathbf{Z}$) admits a k-filtered structure with $(C\otimes D)_{r,s} = C_s \otimes D_{r-s}$ for $0 \leqslant s \leqslant k$. The filtered differential is defined by

$$\begin{aligned}d_0 &= 1\otimes d^D\colon C_s\otimes D_{r-s} \to C_s \otimes D_{r-s-1}\\ d_1 &= (-)^{r-s}d^C\otimes 1\colon C_{s-1}\otimes D_{r-s} \to C_{s-1}\otimes D_{r-s}\end{aligned}$$

and $d_j = 0$ for $j \geqslant 2$. The associated complex has $G_r(C\otimes D)_s = C_r \otimes D_s$ and differential $d_* = d^C \otimes 1\colon C_r \otimes D_s \to C_{r-1}\otimes D_s$. □

When we work with the associated complex, it is useful to translate the maps $d_j\colon C_{r,s} \to C_{r-1,s-j}$ into the new notation, so that

$$d_j\colon G_r(C)_s \to G_{r-j}(C)_{s+j-1}\ .$$

The relation $d_0d_1 + d_1d_0 = 0$ then implies that $d_*\colon G_r(C) \to G_{r-1}(C)$ is a chain map. The relation $(d_1)^2 + d_0d_2 + d_2d_0 = 0$ shows that $(d_1)^2$ is chain homotopic to zero, and hence $(d_*)^2 = 0$ in the derived category.

Proposition 12.5.

(i) *A filtered chain map $f\colon F_*C \to F_*D$ induces a chain map in $\mathbb{D}(\mathbb{A})$*

$$\mathbb{G}_*(f)\colon \mathbb{G}_*(C) \to \mathbb{G}_*(D)\ .$$

(ii) *A filtered chain homotopy* $g\colon f \simeq f'\colon F_*C \to F_*D$ *induces a chain homotopy in* $\mathbb{D}(\mathbb{A})$

$$\mathbb{G}_*(g)\colon \mathbb{G}_*(f) \simeq \mathbb{G}_*(f')\colon \mathbb{G}_*(C) \to \mathbb{G}_*(D)\ .$$

(iii) *A filtered chain equivalence* $f\colon F_*C \to F_*D$ *induces a chain equivalence* $\mathbb{G}_*(f)\colon \mathbb{G}_*(C) \to \mathbb{G}_*(D)$ *in* $\mathbb{D}(\mathbb{A})$.

(iv) *A filtered chain contraction* $\Gamma\colon 1 \simeq 0\colon F_*C \to F_*D$ *induces a chain equivalence* $\mathbb{G}_*(\Gamma)\colon 1 \simeq 0\colon \mathbb{G}_*(C) \to \mathbb{G}_*(D)$ *in* $\mathbb{D}(\mathbb{A})$.

Proof. (i) For a filtered chain map $f\colon F_*C \to F_*D$ the identities

$$df = fd\colon C_r \to D_r$$

expressed in upper triangular matrix form lead to relations involving the component maps f_i and d_j. In particular we obtain $d_0f_0 = f_0d_0\colon C_{r,s} \to D_{r-1,s}$ and the map $G_r(f) = f_0\colon G_r(C) \to G_r(D)$ is a chain map. The relation $d_1f_0 - f_0d_1 = f_1d_0 - d_0f_1$ shows that $(-)^sf_1\colon G_r(C)_s \to G_{r-1}(D)_{s+1}$ gives a chain homotopy $d_*f_0 - f_0d_* \simeq 0$, and so f_0 gives a chain map $\mathbb{G}_*(f)\colon \mathbb{G}_*(C) \to \mathbb{G}_*(D)$ in $\mathbb{D}(\mathbb{A})$.

(ii) For a filtered chain homotopy $g\colon f \simeq f'\colon F_*C \to F_*D$ the identities

$$f - f' = dg + gd\colon C_r \to D_r$$

expressed in upper triangular matrix form again lead to various relations. In particular

$$\begin{aligned} &d_0g_{-1} + g_{-1}d_0 = 0\colon C_{r,s} \to D_{r,s+1}\ , \\ &f_0 - f_0' = d_0g_0 + g_0d_0\ + d_1g_{-1} + g_{-1}d_1\colon C_{r,s} \to D_{r,s}\ . \end{aligned}$$

In the $\mathbb{G}_*$-notation, we have maps

$$g_j\colon G_r(C)_s \to G_{r-j}(D)_{s+j+1}\ .$$

Since $g_0\colon G_r(C)_s \to G_r(D)_{s+1}$ gives a chain null-homotopy $d_0g_0 + g_0d_0 \simeq 0\colon G_r(C) \to G_r(D)$, the map

$$\mathbb{G}_*(g) = (-)^sg_{-1}\colon G_r(C)_s \to G_{r+1}(D)_s$$

gives a null-homotopy $f_0 - f_0' \simeq 0$ in the derived category $\mathbb{D}(\mathbb{A})$. Parts (iii) and (iv) follow immediately from parts (i) and (ii). □

Theorem 12.6. (i) *A* k*-filtered complex* F_*C *in* $\mathbb{A}$ *is filtered contractible if and only if the associated complex* $\mathbb{G}_*(C)$ *is contractible in* $\mathbb{D}(\mathbb{A})$.
(ii) *A filtered chain map* $f\colon F_*C \to F_*D$ *of* k*-filtered chain complexes is a filtered chain equivalence if and only if the associated chain map* $\mathbb{G}_*(f)\colon \mathbb{G}_*(C) \to \mathbb{G}_*(D)$ *is a chain equivalence in* $\mathbb{D}(\mathbb{A})$.

Proof. (i) A filtered contraction $\Gamma\colon 1 \simeq 0\colon F_*C \to F_*C$ induces a contraction $\mathbb{G}_*(\Gamma)\colon 1 \simeq 0\colon \mathbb{G}_*(C) \to \mathbb{G}_*(C)$ by Proposition 12.5 (iv).

Conversely, suppose given a contraction of $\mathbb{G}_*(C)$ in $\mathbb{D}(\mathbb{A})$

$$e\colon 1 \ \simeq\ 0\colon \mathbb{G}_*(C) \to \mathbb{G}_*(C)\ ,$$

as represented by chain maps $e\colon (G_r(C), d_0) \to (G_{r+1}(C), d_0)$, for which there exist chain homotopies

$$h\colon 1 \ \simeq\ d_*e + ed_*\colon G_r(C) \to G_r(C)$$

with

$$(-)^s(d_1e + ed_1) + d_0h + hd_0 = 1\colon G_r(C)_s = C_{r+s,r} \to G_r(C)_s = C_{r+s,r}\ .$$

In terms of the filtration on C, the maps $h\colon G_r(C)_s \to G_r(C)_{s+1}$ and $e\colon G_r(C)_s \to G_{r+1}(C)_s$ give maps $h_s\colon C_{r,s} \to C_{r+1,s}$ and $e_{r,s}\colon C_{r,s} \to C_{r+1,s+1}$. The relation given by $h\colon 1 \simeq d_*e + ed_*$ gives

$$(-)^{r+s}(d_1e + ed_1) + d_0h + hd_0 = 1\colon C_{r,s} \to C_{r,s}$$

Let $\hat{e}_{r,s} = (-)^{r-1}e$, and define morphisms

$$\beta = \begin{pmatrix} h & 0 & 0 & \dots & 0 \\ \hat{e} & h & 0 & \dots & 0 \\ 0 & \hat{e} & h & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \dots & h \end{pmatrix} : C_r \to C_{r+1}$$

The morphisms $\alpha\colon C_r \to C_r$ defined by

$$\alpha = d\beta + \beta d\colon C_r \to C_r$$

are in upper triangular form, with the identity on the main diagonal. For the components in the $(s+1, s)$ positions, note that we get $(-)^{r-1}d_0e + (-)^r ed_0 = 0$ since e a chain map. In the diagonal (s, s) positions we get the sum of composites

$$C_{r,s} \xrightarrow{(-)^s d_1} C_{r-1,s-1} \xrightarrow{(-)^r e} C_{r,s} \qquad C_{r,s} \xrightarrow{(-)^{r-1} e} C_{r+1,s+1} \xrightarrow{(-)^{s+1} d_1} C_{r,s}$$

$$C_{r,s} \xrightarrow{d_0} C_{r-1,s} \xrightarrow{h} C_{r,s} \qquad C_{r,s} \xrightarrow{h} C_{r+1,s} \xrightarrow{d_0} C_{r,s}$$

which equals $(-)^{r+s}(d_1e + ed_1) + d_0h + hd_0 = 1$ by the relation above.

Each $\alpha\colon C_r \to C_r$ is an automorphism, with

$$\begin{aligned} d\alpha = d\beta d = \alpha d\colon C_r \to C_{r-1}\ , \\ \alpha^{-1}d = d\alpha^{-1}\colon C_r \to C_{r-1}\ . \end{aligned}$$

The morphisms

$$\Gamma = \beta\alpha^{-1}\colon C_r \to C_{r+1}$$

are such that

$$d\Gamma + \Gamma d = (d\beta + \beta d)\alpha^{-1} = 1 \colon C_r \to C_r \ ,$$

and define a filtered contraction of F_*C

$$\Gamma \colon 1 \ \simeq \ 0 \colon C \to C \ .$$

(ii) By Theorem 12.2 f is a filtered chain equivalence if and only if $F_*\mathcal{C}^{fil}(f)$ is filtered contractible. By (i) $F_*\mathcal{C}^{fil}(f)$ is filtered contractible if and only if $\mathbb{G}_*(\mathcal{C}^{fil}(f)) = \mathcal{C}(G_*(f))$ is chain contractible. □

§12C. Splitting and Folding. In an additive category $\mathbb{A}$ we don't necessarily have kernels and cokernels, but we can define split exact sequences. A *direct sum system* $A \underset{\Gamma}{\overset{f}{\rightleftarrows}} B \underset{\Delta}{\overset{g}{\rightleftarrows}} C$ is a collection of morphisms in $\mathbb{A}$ such that

$$(\Delta \ f) \colon C \oplus A \to B \ , \ \begin{pmatrix} g \\ \Gamma \end{pmatrix} \colon B \to C \oplus A$$

are inverse isomorphisms in $\mathbb{A}$. Then we say that a morphism $f \colon A \to B$ in $\mathbb{A}$ is a *split injection* (respectively $g \colon B \to C$ is a *split surjection*) if the morphism extends to a direct sum system. Then

$$0 \to A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \to 0$$

is a *short exact sequence* in $\mathbb{A}$ if f and g can be extended to a direct sum system. A short exact sequence of chain complexes in $\mathbb{A}$ is one which is short exact in each degree.

There is a useful criterion for the existence of a split injection in the derived category $\mathbb{D}(\mathbb{A})$, expressed in terms of any representative chain map f for a given morphism.

Proposition 12.7. *A morphism $f \colon C \to D$ in $\mathbb{D}(\mathbb{A})$ is a split injection if and only if there exists a chain map $\Gamma \colon D \to C$ and a chain homotopy $h \colon \Gamma f \simeq 1 \colon C \to C$. If (Γ, h) exists, then there is a direct sum system*

$$C \underset{\Gamma}{\overset{f}{\rightleftarrows}} D \underset{\Delta}{\overset{g}{\rightleftarrows}} \mathcal{C}(f)$$

in $\mathbb{D}(\mathbb{A})$, with $g = \binom{1}{0} \colon D_r \to \mathcal{C}(f)_r = D_r \oplus C_{r-1}$ and

$$\Delta = (1 - f\Gamma \ \ (-)^{r+1} fh) \colon \mathcal{C}(f)_r = D_r \oplus C_{r-1} \to D_r \ .$$

Proof. Assuming there exist such Γ, h we define a direct sum system in $\mathbb{D}(\mathbb{A})$ by the given formulas. The chain maps

$$(\Delta \ f) \colon \mathcal{C}(f) \oplus C \to D \ , \ \begin{pmatrix} g \\ \Gamma \end{pmatrix} \colon D \to \mathcal{C}(f) \oplus C$$

are inverse chain equivalences, and the morphisms

$$e = \begin{pmatrix} 0 & 0 & 0 \\ (-)^r\Gamma & -h & (-)^{r+1} \\ h\Gamma & (-)^{r+1}h^2 & h \end{pmatrix} : D_r \oplus C_{r-1} \oplus C_r \to D_{r+1} \oplus C_r \oplus C_{r+1}$$

define a chain homotopy

$$e\colon \begin{pmatrix} g \\ \Gamma \end{pmatrix}(\Delta\ f) \ \simeq\ 1\colon \mathcal{C}(f) \oplus C \to \mathcal{C}(f) \oplus C\ .$$

The converse is clear. □

Corollary 12.8. *Let $f\colon C \to D$ be a split injection in $\mathbb{SD}(\mathbb{A})$. Then*

$$i_*\tau^{NEW}(\begin{pmatrix} g \\ \Gamma \end{pmatrix}) = i_*\tau^{NEW}(\Delta\ f) = 0 \in K_1^{iso}(\mathbb{A})\ .$$

Proof. This follows immediately from the last result and the formula in [11, Prop. 13.6], since $\tau^{NEW}(f) = \tau^{NEW}(\mathcal{C}(f))$. □

We will need a variant of the folding construction used by Whitehead [29] to define the torsion of a contractible complex. Let

$$C\ :\ C_k \xrightarrow{\ d\ } C_{k-1} \xrightarrow{\ d\ } C_{k-2} \to \cdots \to C_0$$

be a k-dimensional chain complex in $\mathbb{A}$, such that the boundary map $d\colon C_k \to C_{k-1}$ extends to a direct sum system

$$C_k \underset{\Gamma}{\overset{d}{\rightleftarrows}} C_{k-1} \underset{\Delta}{\overset{g}{\rightleftarrows}} C'_{k-1}$$

We use this direct sum system to define the $(k-1)$-dimensional chain complex

$$C'\ :\quad C'_{k-1} \xrightarrow{d\Delta} C_{k-2} \xrightarrow{\ d\ } C_{k-3} \longrightarrow \cdots \longrightarrow C_0$$

in $\mathbb{A}$, called the *abelian k-folding* of C. An *elementary* chain complex in $\mathbb{A}$ is a contractible complex with non-zero chain groups only in two adjacent degrees.

Proposition 12.9. *Let C' be the abelian k-folding of C, and let E be the elementary chain complexes with $C_k \xrightarrow{1} C_k$ in adjacent degrees $(k, k-1)$. Then there exist an isomorphism $\binom{i}{j}\colon C\ \cong\ C' \oplus E$ with isomorphism torsion*

$$\tau^{iso}(\begin{pmatrix} i \\ j \end{pmatrix}) = (-)^{k-1}\tau^{iso}(\begin{pmatrix} g \\ \Gamma \end{pmatrix}\colon C_{k-1} \to C'_{k-1} \oplus C_k) \in K_1^{iso}(\mathbb{A})\ .$$

Proof. The chain isomorphism $\binom{i}{j}\colon C \to C' \oplus E$ is defined by the diagram

$$\begin{tikzcd}
C_k \arrow[r, "d"] \arrow[d, equal] & C_{k-1} \arrow[r, "d"] \arrow[d, "\binom{g}{\Gamma}"] & C_{k-2} \arrow[r, "d"] \arrow[d, equal] & \cdots \arrow[r] & C_0 \arrow[d, equal] \\
C_k \arrow[r, "\binom{0}{1}"] & C'_{k-1} \oplus C_k \arrow[r, "(d\Delta\ 0)"] & C_{k-2} \arrow[r, "d"] & \cdots \arrow[r] & C_0
\end{tikzcd}$$

and the relations $gd = 0$, $\Gamma d = 1$. □

§12D. The Invariance Theorem for filtered contractible complexes. We will first establish the sign conventions for filtered complexes.

Definition 12.10. A *k-filtered signed complex* (F_*C, η_{F_*C}) in $\mathbb{A}$ is a k-filtered complex C together with signs

$$\eta_{\mathbb{G}_*(C)} \in \mathrm{im}(\epsilon\colon K_0(\mathbb{SD}(\mathbb{A})) \otimes K_0(\mathbb{SD}(\mathbb{A})) \to K_1^{iso}(\mathbb{SD}(\mathbb{A})))\ ,$$
$$\eta_{G_j(C)} \in \mathrm{im}(\epsilon\colon K_0(\mathbb{A}) \otimes K_0(\mathbb{A}) \to K_1^{iso}(\mathbb{A}))\ (0 \leqslant j \leqslant k)\ .$$

The sign of F_*C and the sign of C (as an unfiltered complex) are set to be

$$\begin{aligned} \eta_{F_*C} &= \eta_C = i_*\eta_{\mathbb{G}_*(C)} + \eta_{G_0C \oplus S(G_1C \oplus S(G_2C \oplus \cdots \oplus SG_kC)\dots)} \\ &\in \mathrm{im}(\epsilon\colon K_0(\mathbb{A}) \otimes K_0(\mathbb{A}) \to K_1^{iso}(\mathbb{A}))\ . \end{aligned}$$

The *associated chain complex* $(\mathbb{G}_*(C), \eta_{\mathbb{G}_*(C)})$ is a k-dimensional chain complex in $\mathbb{SD}(\mathbb{A})$. We will usually denote this *signed* complex in $\mathbb{D}(\mathbb{A})$ just by $\mathbb{G}_*(C)$. □

The sign conventions have been chosen so that for a filtered contractible k-filtered signed complex (F_*C, η_{F_*C}) in $\mathbb{A}$

$$\tau^{NEW}(C, \eta_C) = \tau^{NEW}(F_*C, \eta_{F_*C}) \in K_1^{iso}(\mathbb{A})\ .$$

The underlying (unfiltered) complex of a filtered complex has a useful *iterated mapping cone* description. Given a k-filtered chain complex F_*C in $\mathbb{A}$, we define chain complexes $T_{\ell,r}(C) = S^{-r}(F_\ell C/F_{r-1}C)$ with chain groups

$$T_{\ell,r}(C)_s = (F_\ell C/F_{r-1}C)_{r+s} = C_{r+s,r} \oplus C_{r+s,r+1} \oplus \cdots \oplus C_{r+s,\ell}$$

for $1 \leqslant r \leqslant \ell \leqslant k$. The differential on $T_{\ell,r}(C)$ is the one induced by $d\colon C \to C$, and the sign

$$\eta_{T_{\ell,r}(C)} = \eta_{G_rC \oplus S(G_{r+1}C \oplus S(G_{r+2}C \oplus \cdots \oplus SG_\ell C)\dots)}$$

Note that the definition $G_r(C)_s = C_{r+s,r}$ gives the formula

$$T_{\ell,r}(C)_s = G_r(C)_s \oplus G_{r+1}(C)_{s-1} \oplus \cdots \oplus G_\ell(C)_{r+s-\ell}\ .$$

We use this expression to define maps

$$\partial_{\ell,r}\colon T_{\ell,r}(C)_s \to G_{r-1}(C)_s \quad (1 \leqslant j \leqslant k)$$

by the row matrix

$$\partial_{\ell,r} = (-)^s(d_1\ d_2\ \dots\ d_{\ell-r+1})$$

where $d_j\colon G_r(C)_s \to G_{r-j}(C)_{s+j-1}$ are the entries in the matrix expression for d. If $\ell = k$ we will set $\partial_r := \partial_{k,r}$ to simplify the notation. We remark that $G_r(C) = T_{r,r}(C)$ and $\partial_{r,r} = d_*\colon G_r(C) \to G_{r-1}(C)$.

Proposition 12.11. *Let (F_*C, η_{F_*C}) be a k-filtered signed complex in $\mathbb{A}$.*

(i) *The maps $\partial_{\ell,r}\colon T_{\ell,r}(C) \to G_{r-1}(C)$ are chain maps.*

(ii) $T_{\ell,r-1}(C) = \mathcal{C}(T_{\ell,r}(C) \xrightarrow{\partial_{\ell,r}} G_{r-1}(C))$, *for* $1 \leqslant r \leqslant \ell \leqslant k$, *as signed complexes.*

(iii) *There is a short exact sequence*

$$T_{\ell_1,r}(C) \to T_{\ell_2,r}(C) \to S^{\ell_1-r+1}T_{\ell_2,\ell_1+1}(C)$$

in $\mathbb{SD}(\mathbb{A})$, *for* $1 \leqslant r \leqslant \ell_1 < \ell_2 \leqslant k$.

(iv) *The signed complex* $(C, \eta_C - i_*\eta_{\mathbb{G}_*(C)}) = T_{k,0}(C)$ *as an object in* $\mathbb{SD}(\mathbb{A})$.

Proof. The claim that $\partial_{\ell,r}$ is a chain map follows immediately from the relation $d^2 = 0$. Part (ii) follows directly from the definition, and part (iii) can be checked inductively. Note that, according to our conventions, the sign term

$$\eta_C - i_*\eta_{\mathbb{G}_*(C)} = \eta_{G_0C\oplus S(G_1C\oplus S(G_2C\oplus\cdots\oplus SG_kC)\dots)}$$

is just the sign of the iterated mapping cone structure on $T_{k,0}(C)$. □

Definition 12.12. Let (F_*C, η_{F_*C}) be a k-filtered signed complex in $\mathbb{A}$. The $(k-1)$-*amalgamation* of (F_*C, η_{F_*C}) is the $(k-1)$-filtered signed complex $(F_*C', \eta_{F_*C'})$ in $\mathbb{A}$ with filtration summands

$$C'_{r,s} = C_{r,s},\ \text{if } 0 \leqslant s < k-1 \text{ and } C'_{r,k-1} = C_{r,k-1} \oplus C_{r,k}$$

in each degree r. The differentials $d'_j = d_j$ except in filtration degree $k-1$, where

$$d'_j = \begin{pmatrix} d_j & d_{j+1} \\ 0 & d_j \end{pmatrix} : C_{r,k-1} \oplus C_{r,k} \to C_{r-1,k-1-j} \oplus C_{r-1,k-j}$$

and

$$G_r(C') = \begin{cases} G_r(C) & \text{if } 0 \leqslant r \leqslant k-2 \\ \mathcal{C}(G_k(C) \xrightarrow{\partial_k} G_{k-1}(C)) & \text{if } r = k-1\ . \end{cases}$$

where $\partial_k := \partial_{k,k}$. The signs are given by the formulas

$$\eta_{\mathbb{G}_*(C')} = \eta_{\mathbb{G}_*(C)}\ ,$$
$$\eta_{G_j(C')} = \begin{cases} \eta_{G_j(C)} & \text{if } 0 \leqslant j \leqslant k-2 \\ \eta_{G_{k-1}(C)\oplus SG_k(C)} & \text{if } j = k-1\ . \end{cases}$$

□

Lemma 12.13. $G_{k-1}(C') = \mathcal{C}(G_k(C) \xrightarrow{\partial_k} G_{k-1}(C))$ *as objects in* $\mathbb{SD}(\mathbb{A})$.

Proof. The associated $(k-1)$-dimensional chain complex $\mathbb{G}_*(C')$ in $\mathbb{SD}(\mathbb{A})$ has the differential $d'_*\colon G_r(C') \to G_{r-1}(C')$ given by

$$d_{\mathbb{G}_*(C')} = \begin{cases} d_*\colon G_j(C) \to G_{j-1}(C) & \text{if } 0 \leqslant j \leqslant k-2 \\ \partial_{k-1}\colon \mathcal{C}(G_k(C) \xrightarrow{\partial_k} G_{k-1}(C)) \to G_{k-2}(C) & \text{if } j = k-1\ . \end{cases}$$

Since $G_{k-1}(C') = G_{k-1}(C) \oplus SG_k(C)$, the signs on $G_{k-1}(C')$ and $\mathcal{C}(\partial_k)$ agree. □

Note that $\eta_{G_0C' \oplus S(G_1C' \oplus S(G_2C' \oplus \cdots \oplus SG_{k-1}C')\ldots)}$ is the sign of the new iterated mapping cone structure on C, and so $\eta_{F_*C'} = \eta_{F_*C} = \eta_C$ as required,

Theorem 12.14 (Invariance). *For a filtered contractible k-filtered signed complex (F_*C, η_{F_*C}) in $\mathbb{A}$*

$$\tau^{NEW}(C, \eta_C) = i_*\tau^{NEW}(\mathbb{G}_*(C), \eta_{\mathbb{G}_*(C)}) \in K_1^{iso}(\mathbb{A}) \ .$$

Proof. The proof is by induction on k. The result is true for $k = 0$, since in that case $(\mathbb{G}_*(C), \eta_{\mathbb{G}*(C)})$ is concentrated in degree 0, and $\eta_C = i_*\eta_{\mathbb{G}*(C)} + \eta_{G_0(C)}$. Therefore

$$i_*\tau^{NEW}(\mathbb{G}_*(C), \eta_{\mathbb{G}_*(C)}) = \tau^{NEW}(G_0(C), \eta_{G_0(C)}) + i_*\eta_{\mathbb{G}*(C)} = \tau^{NEW}(C, \eta_C) \ .$$

So assume that $k \geqslant 1$, and that the result is true for $(k-1)$. Let $(F_*C', \eta_{F_*C'})$ be the $(k-1)$-amalgamation of C. ¿From the construction of F_*C', it is clear that

$$\tau^{NEW}(F_*C, \eta_{F_*C}) = \tau^{NEW}(F_*C', \eta_{F_*C'}) \in K_1^{iso}(\mathbb{A}) \ ,$$

and by the inductive hypothesis

$$\tau^{NEW}(F_*C', \eta_{F_*C'}) = i_*\tau^{NEW}(\mathbb{G}_*(C'), \eta_{\mathbb{G}_*(C')}) \in K_1^{iso}(\mathbb{A}) \ .$$

By Proposition 12.5 (iv) a filtered contraction $\Gamma\colon 1 \simeq 0\colon C \to C$ of (F_*C, η_{F_*C}) determines a contraction

$$\mathbb{G}_*(\Gamma)\colon 1 \ \simeq \ 0\colon \mathbb{G}_*(C) \to \mathbb{G}_*(C)$$

of the associated k-dimensional chain complex $\mathbb{G}_*(C)$ in the signed derived category $\mathbb{SD}(\mathbb{A})$ with

$$G_r(\Gamma) = (-)^s\Gamma_{-1}\colon G_r(C)_s = C_{r+s,r} \to G_{r+1}(C)_s = C_{r+s+1,r+1} \ .$$

In particular, the chain maps

$$\partial_k = d_*\colon G_k(C) \to G_{k-1}(C) \ , \ \Gamma_{-1}\colon G_{k-1}(C) \to G_k(C)$$

are related by a chain homotopy

$$\Gamma_0\colon \Gamma_{-1}\partial_k \ \simeq \ 1\colon G_k(C) \to G_k(C) \ .$$

Now Proposition 12.7 gives a direct sum system in $\mathbb{SD}(\mathbb{A})$

$$G_k(C) \underset{\Gamma_{-1}}{\overset{\partial_k}{\rightleftarrows}} G_{k-1}(C) \underset{\Delta}{\overset{g}{\rightleftarrows}} \mathcal{C}(\partial_k) \ .$$

This defines a k-splitting of $\mathbb{G}_*(C)$, and by Lemma 12.13 the associated amalgamated complex $\mathbb{G}_*(C')$ is the corresponding abelian folding. By Proposition 12.9 there is defined an isomorphism of chain complexes in $\mathbb{SD}(\mathbb{A})$

$$\binom{i}{j}\colon \mathbb{G}_*(C) \ \cong \ \mathbb{G}_*(C') \oplus \mathbb{E}$$

where $\mathbb{E}$ is the elementary complex in $\mathbb{SD}(\mathbb{A})$ given by $G_k(C) \xrightarrow{1} G_k(C)$ in degrees $(k, k-1)$. Since $\eta_{\mathbb{G}_*(C)} = \eta_{\mathbb{G}_*(C')\oplus\mathbb{E}}$, the formula in Proposition 12.9 for $\tau^{iso}\binom{i}{j}$ gives

$$i_*\tau^{NEW}\binom{i}{j} = (-)^{k-1}\tau^{NEW}\left(\begin{pmatrix} g \\ \Gamma_{-1}\end{pmatrix} : G_{k-1}(C) \to \mathcal{C}(\partial_k) \oplus G_k(C)\right) \in K_1^{iso}(\mathbb{SD}(\mathbb{A})) .$$

By Corollary 12.8 this torsion has image

$$i_*\tau^{NEW}\binom{i}{j} = 0 \in K_1^{iso}(\mathbb{A}) ,$$

so that

$$\begin{aligned} i_*\tau^{NEW}(\mathbb{G}_*(C)) &= i_*\tau^{NEW}(\mathbb{G}_*(C') \oplus \mathbb{E}) \\ &= \tau^{NEW}(\mathbb{G}_*(C')) \in K_1^{iso}(\mathbb{A}) \end{aligned}$$

which gives the inductive step

$$\begin{aligned} \tau^{NEW}(F_*C, \eta_{F_*C}) &= \tau^{NEW}(F_*C', \eta_{F_*C'}) \\ &= i_*\tau^{NEW}(\mathbb{G}_*(C'), \eta_{\mathbb{G}_*(C')}) \\ &= i_*\tau^{NEW}(\mathbb{G}_*(C), \eta_{\mathbb{G}_*(C)}) \in K_1^{iso}(\mathbb{A}) . \end{aligned}$$

□

§12E. The Invariance Theorem for filtered chain equivalences. We have already defined the filtered mapping cone $F_*\mathcal{C}^{fil}(f)$ of a filtered chain map (see Definition 12.1 (vii)), with associated complex $\mathbb{G}_*(\mathcal{C}^{fil}(f))$ given by

$$G_r(\mathcal{C}^{fil}(f))_s = G_r(D)_s \oplus G_{r-1}(C)_s ,$$

but now we need the signed version.

Definition 12.15. The *filtered signed mapping cone* $(F_*\mathcal{C}^{fil}(f), \eta_{F_*\mathcal{C}^{fil}(f)})$ of a filtered chain map is the $(k+1)$-filtered signed complex defined by the filtered mapping cone $F_*\mathcal{C}^{fil}(f)$ with sign terms

$$\begin{aligned} \eta_{\mathbb{G}_*(\mathcal{C}^{fil}(f))} &= \eta_{\mathbb{G}_*(D)\oplus S\mathbb{G}_*(C)} \\ &= -\beta(\mathbb{G}_*(D), S\mathbb{G}_*(C)) + \epsilon(\mathbb{G}_*(D)_{odd}, \chi(S\mathbb{G}_*(C)) + \eta_{\mathbb{G}_*(D)} - \eta_{\mathbb{G}_*(C)} \\ &\in \mathrm{im}(\epsilon\colon K_0(\mathbb{SD}(\mathbb{A})) \otimes K_0(\mathbb{SD}(\mathbb{A})) \to K_1^{iso}(\mathbb{SD}(\mathbb{A}))) , \\ \eta_{G_r(\mathcal{C}^{fil}(f))} &= \eta_{G_r(D)\oplus G_{r-1}(C)} \\ &\in \mathrm{im}(\epsilon\colon K_0(\mathbb{A}) \otimes K_0(\mathbb{A}) \to K_1^{iso}(\mathbb{A}))\ (0 \leqslant r \leqslant k+1) . \end{aligned}$$

□

With these sign conventions, the rearrangement map $\rho\colon \mathcal{C}^{fil}(f) \to \mathcal{C}(f)$ is a *simple* isomorphism.

Lemma 12.16. $\tau^{NEW}\left((\mathcal{C}^{fil}(f), \eta_{\mathcal{C}^{fil}(f)}) \xrightarrow{\rho} (\mathcal{C}(f), \eta_{\mathcal{C}(f)})\right) = 0 \in K_1^{iso}(\mathbb{A})$.

Proof. The proof is by induction on k. The result is true for $k = 0$, since in that case $F_*\mathcal{C}^{fil}(f) = \mathcal{C}(f)$, with $\mathcal{C}^{fil}(f)_{r,0} = D_r$ and $\mathcal{C}^{fil}(f)_{r,1} = C_{r-1}$, and $\eta_{F_*\mathcal{C}^{fil}(f)} = \eta_{\mathcal{C}(f)}$.

Assume that $k \geqslant 1$, and that the result is true for $(k-1)$. Let

$$f'\colon (F_*C', \eta_{F_*C'}) \to (F_*D', \eta_{F_*D'})$$

be the chain equivalence of the $(k-1)$-amalgamations induced by f, with

$$\tau^{NEW}((\mathcal{C}^{fil}(f'), \eta_{\mathcal{C}^{fil}(f')}) \to (\mathcal{C}(f), \eta_{\mathcal{C}(f)})) = 0 \in K_1^{iso}(\mathbb{A})$$

by the inductive hypothesis. By Proposition 12.11 the unfiltered complex $\mathcal{C}(f) = \mathcal{C}(f')$ has two iterated mapping cone descriptions: from the signed filtered complexes $F_*E = \mathcal{C}^{fil}(f)$ and $F_*E' = \mathcal{C}^{fil}(f')$ respectively, we have signed complexes $T_{k,0}(E)$ and $T_{k,0}(E')$, and a commutative diagram

$$\begin{tikzcd}
T_{k-2,0}(E) \arrow[r] \arrow[d, equal] & T_{k+1,0}(E) \arrow[r] \arrow[d] & S^{k-1}T_{k+1,k-1}(E) \arrow[d] \\
T_{k-2,0}(E') \arrow[r] & T_{k+1,0}(E') \arrow[r] & S^{k-1}T_{k+1,k-1}(E')
\end{tikzcd}$$

But $\eta_E = \eta_{E'}$, so the rearrangement map $\rho\colon E \to E'$ has same absolute torsion as the following composite (distribute S, then interchange)

$$\begin{array}{c}
S^{k-1}\big(G_{k-2}(C) \oplus S\big(G_k(D) \oplus G_{k-1}(C) \oplus SG_k(C)\big)\big) \\
\big\downarrow {\scriptstyle 1\oplus S} \\
S^{k-1}\big(G_{k-2}(C) \oplus SG_k(D) \oplus S\big(G_{k-1}(C) \oplus SG_k(C)\big)\big) \\
\big\downarrow {\scriptstyle flip\oplus 1} \\
S^{k-1}\big(SG_k(D) \oplus G_{k-2}(C) \oplus S\big(G_{k-1}(C) \oplus SG_k(C)\big)\big)
\end{array}$$

since $T_{k-2,0}(E) = T_{k-2}(E')$. Therefore

$$\tau^{NEW}(E \to E') = (-)^{k-1} i_*\epsilon(G_k(D), G_k(C) \oplus SG_{k-1}(C) \oplus G_{k-2}(C)) \in K_1^{iso}(\mathbb{A})$$

from the formulas in [11, Lemma 7]. By Proposition 12.11,

$$\tau^{NEW}(\mathcal{C}^{fil}(f) \to \mathcal{C}(f')) = \tau^{NEW}(E \to E') + i_*\eta_{\mathbb{G}_*(E')} - i_*\eta_{\mathbb{G}_*(E)}$$

where the sign term from the filtered mapping cones is

$$\begin{aligned}
\eta_{\mathbb{G}_*(E)} - \eta_{\mathbb{G}_*(E')} \;&= -\beta(\mathbb{G}_*(D), S\mathbb{G}_*(C)) + \epsilon(\mathbb{G}_{odd}(D), \chi(S\mathbb{G}_*(C))) \\
&\quad + \beta(\mathbb{G}_*(D'), S\mathbb{G}_*(C')) - \epsilon(\mathbb{G}_{odd}(D'), \chi(S\mathbb{G}_*(C')))
\end{aligned}$$

We first notice that $\mathbb{G}_{odd}(D) - \mathbb{G}_{odd}(D') = G_k(D)$ and

$$i_*\chi(S\mathbb{G}_*(C)) = i_*\chi(S\mathbb{G}_*(C'))$$

since $G_{k-1}(C') = G_{k-1}(C) \oplus SG_k(C)$. Therefore

$$\begin{aligned} & i_*\epsilon(\mathbb{G}_{odd}(D), \chi(S\mathbb{G}_*(C))) - i_*\epsilon(\mathbb{G}_{odd}(D'), \chi(S\mathbb{G}_*(C'))) \\ & \qquad = i_*\epsilon(G_k(D), \chi(S\mathbb{G}_*(C))) = -i_*\epsilon(G_k(D), \chi(\mathbb{G}_*(C))) \ . \end{aligned}$$

For the β-terms, we observe that $\beta(\mathbb{G}_*(D), S\mathbb{G}_*(C)) = \beta(\mathbb{G}_*(D), S\mathbb{G}_*(C'))$, and compute

$$\begin{aligned} & i_*\beta(\mathbb{G}_*(D), S\mathbb{G}_*(C')) - i_*\beta(\mathbb{G}_*(D'), S\mathbb{G}_*(C')) \\ & \qquad = (-)^k i_* \left[\epsilon(G_k(D), \sum_{j\geq 1} G_{k-2j-1}(C)) - \epsilon(G_k(D), \sum_{j\geq 2} G_{k-2j}(C)) \right] \ . \end{aligned}$$

It follows that

$$i_*\eta_{\mathbb{G}_*(E)} - i_*\eta_{\mathbb{G}_*(E')} = (-)^{k-1} i_*\epsilon(G_k(D), G_k(C) \oplus SG_{k-1}(C) \oplus G_{k-2}(C))$$

so the sign terms cancel and $\tau^{NEW}(\mathcal{C}^{fil}(f) \to \mathcal{C}(f')) = 0$ as required. □

We proceed now to the statement of the Invariance Theorem. It is immediate from the definitions that the associated complex of the filtered mapping cone $(F_*\mathcal{C}^{fil}(f), \eta_{F_*\mathcal{C}^{fil}(f)})$ of a chain map $f\colon (F_*C, \eta_{F_*C}) \to (F_*D, \eta_{F_*D})$ of k-filtered signed complexes in $\mathbb{A}$ is the mapping cone of the associated chain map $\mathbb{G}_*(f)\colon (\mathbb{G}_*(C), \eta_{\mathbb{G}_*(C)}) \to (\mathbb{G}_*(D), \eta_{\mathbb{G}_*(D)})$ in the signed derived category $\mathbb{SD}(\mathbb{A})$, so we have

$$(\mathbb{G}_*(\mathcal{C}^{fil}(f)), \eta_{\mathbb{G}_*(\mathcal{C}^{fil}(f))}) = (\mathcal{C}(\mathbb{G}_*(f)), \eta_{\mathbb{G}_*(D)\oplus S\mathbb{G}_*(C)}) \ .$$

By Theorem 12.6, the filtered chain map f is a filtered chain equivalence if and only if $\mathbb{G}_*(f)$ is a chain equivalence, or $\mathcal{C}(\mathbb{G}_*(f))$ is contractible.

Theorem 12.17 (**Invariance**). *The torsion of a filtered chain equivalence $f\colon (F_*C, \eta_{F_*C}) \to (F_*D, \eta_{F_*D})$ of k-filtered signed complexes is*

$$\tau^{NEW}(f) = i_*\tau^{NEW}(\mathbb{G}_*(f)) \in K_1^{iso}(\mathbb{A}) \ .$$

Proof. By definition

$$\tau^{NEW}(\mathbb{G}_*(f)) = \tau^{NEW}(\mathcal{C}(\mathbb{G}_*(f))) \in K_1^{iso}(\mathbb{SD}(\mathbb{A}))$$

so we have the formulas

$$\begin{aligned} i_*\tau^{NEW}(\mathbb{G}_*(f)) &= i_*\tau^{NEW}(\mathcal{C}(\mathbb{G}_*(f))) \\ &= i_*\tau^{NEW}(\mathbb{G}_*(\mathcal{C}^{fil}(f))) \\ &= \tau^{NEW}(\mathcal{C}^{fil}(f), \eta_{\mathcal{C}^{fil}(f)}) \\ &= \tau^{NEW}(\mathcal{C}(f), \eta_{\mathcal{C}(f)}) \\ &= \tau^{NEW}(f) \in K_1^{iso}(\mathbb{A}) \ . \end{aligned}$$

□

§12F. **The filtered dual complex.** Let $(\mathbb{A}, *)$ be an additive category with involution (see [26]). We now define an involution on a sub-category of the additive category $\mathbb{SD}(\mathbb{A})$ (following [11, §3]).

Definition 12.18. Given a signed chain complex C in $\mathbb{A}$, the *n-dual signed chain complex* is $(C^{n-*}, \eta_{C^{n-*}})$, where

$$d_{C^{n-*}} = (-1)^r d_C^* \colon C^{n-r} = (C_{n-r})^* \to C^{n-r+1}$$

and the sign

$$\eta_{C^{n-*}} = (-)^n(\eta_C)^* + \beta(C, C) + \alpha_n(C) \in K_1^{iso}(\mathbb{A})$$

where

$$\alpha_n(C) = \sum_{r \equiv n+2,\, n+3\, (\mathrm{mod} 4)} \epsilon(C^r, C^r) \in K_1^{iso}(\mathbb{A})\ .$$

For any integer $n \geq 0$, let $\mathbb{SPD}_n(\mathbb{A})$ denote the full sub-category of $\mathbb{SD}(\mathbb{A})$ whose objects are signed chain complexes (C, η_C) in $\mathbb{A}$ with $\dim C = n$, such that C^{n-*} is chain equivalent to C and $\chi(C) = 0$ for n odd. □

Remark 12.19. The condition that $\chi(C) = 0$ when n is odd is necessary to ensure that $C \mapsto C^{n-*}$ is an additive functor on $\mathbb{SPD}_n(\mathbb{A})$. This condition and the requirement that C^{n-*} be chain equivalent to C are both satisfied (by duality) for symmetric Poincaré n-complexes of R-modules, and these are the main objects of interest in the rest of the paper.

Example 12.20. Let $\mathbb{A} = \mathbb{A}(R)$ denote the additive category of finitely generated based free modules over a ring R with involution. Then $\mathbb{A}(R)$ has an involution $* \colon \mathbb{A}(R) \to \mathbb{A}(R)$ (see [26]), and we get an involution on $\mathbb{SPD}_n(R) := \mathbb{SPD}_n(\mathbb{A}(R))$ given by

$$* \colon C \mapsto C^{n-*}$$

$$* \colon (f \colon C \to D) \mapsto (f^{n-*} \colon D^{n-*} \to C^{n-*})$$

(see [11]) □

We will now define the *filtered dual* $F_*^{dual}C$ of a k-filtered complex F_*C (under some filtration assumptions to ensure that each $G_r(C)$ is an n-dimensional chain complex). It will turn out that the associated complex $\mathbb{G}_*(F^{dual}C)$ is the k-dual of $\mathbb{G}_*(C)$. Our main example is the k-filtered chain complex $F_*C(\widetilde{E})$ of the total space of a fibre bundle $F \to E \to B$, where $\dim B = k$ and $\dim F = n$ (see Section 5).

Definition 12.21. Let F_*C be a k-filtered $(n+k)$-dimensional chain complex in $\mathbb{A}(R)$.

(i) We say that F_*C is *n-admissible* if
  (a) $C_{r,s} = 0$ unless $0 \leqslant s \leqslant k$ and $0 \leqslant r - s \leqslant n$.
  (b) $(G_rC)^{n-*}$ is chain equivalent to G_rC.

(ii) We define the *filtered dual* $F_*^{dual}C$ of F_*C to be the $(n+k)$-filtered complex with modules $(F_*^{dual}C)_{r,s} = C^*_{n+k-r,k-s}$ for $0 \leqslant s \leqslant k$ and differentials $d^{dual} \colon (F_*^{dual}C)_r \to (F_*^{dual}C)_{r-1}$ the filtered morphism given by the upper triangular matrix with components

$$d_j^{dual} = (-)^{r+s+j(n+r)} d_j^* \colon C^*_{n+k-r,k-s} \to C^*_{n+k-r+1,k-s+j}\ .$$

The associated complex of $F_*^{dual}C$ will be denoted $\mathbb{G}_*(F^{dual}C)$.

(iii) We define the *signed filtered dual* of F_*C as follows: Since F_*C is n-admissible $\mathbb{G}_*C$ may be considered to lie in $\mathbb{SPD}_n(R)$. This is a category with involution so the dual sign $\eta_{(\mathbb{G}_*C)^{k-*}} \in K_1^{iso}(\mathbb{SPD}_n(\mathbb{A}(R))$ is defined. We have an obvious functor $\mathbb{SPD}_n(R) \to \mathbb{SD}(R)$ so we may consider $\eta_{(\mathbb{G}_*C)^{k-*}}$ to lie in $K_1^{iso}(\mathbb{SD}(R))$. We define the signed filtered dual of F_*C to be the signed filtered complex whose underlying filtered chain complex is $F_*^{dual}C$ and with signs:

$$\eta_{\mathbb{G}_*(F^{dual}C)} = \eta_{(\mathbb{G}_*C)^{k-*}} \in K_1^{iso}(\mathbb{SD}(R))$$

$$\eta_{G_r(F^{dual}C)} = \eta_{G_{k-r}(C)^{n-*}} \in K_1^{iso}(R)$$

□

Remark 12.22. It follows from the condition $(G_rC)^{n-*}$ is chain equivalent to G_rC that $\chi(F_*C)$ is zero if n is odd.

Lemma 12.23. *The associated complex $\mathbb{G}_*(F^{dual}C)$ of the filtered dual of F_*C is the k-dual of $\mathbb{G}_*(C)$ in $\mathbb{SPD}_n(\mathbb{A})$.*

Proof. We have $G_r(F^{dual}C)_s = (F_*^{dual}C)_{r+s,r} = C^*_{n+k-r-s,k-r}$ by definition. The internal differential $d_0 \colon G_r(F^{dual}C)_s \to G_r(F^{dual}C)_{s-1}$ on this term is given by $(-)^s d_0^*$. On the other hand, $G_{k-r}(C)^{n-s} = C^*_{k-r+n-s,k-r}$, so the dual complex has the same chain groups as the associated complex. The differential on $G_{k-r}(C)^{n-s}$ is again $(-)^s d_0^*$, since $\dim G_{k-r}(C) = n$. The signs on $\mathbb{G}_*(F^{dual}C)$ were defined to agree with those of the k-dual of $\mathbb{G}_*(C)$, and in particular $G_r(F^{dual}C) = (G_{k-r}C)^{n-*}$ as signed chain complexes, for $0 \leqslant r \leqslant k$. □

We define the map

$$\theta_{F_*C} \colon C^{n+k-*} \to F_*^{dual}C \tag{12.24}$$

as the direct sum of the following sign maps on the (r,s)-summands of the underlying modules:

$$(-)^{s(n+r+1)} \colon C^*_{n+k-r,k-s} \to C^*_{n+k-r,k-s}$$

Lemma 12.25. *θ_{F_*C} is a chain equivalence of (unfiltered) complexes.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
C^*_{n+k-r,k-s} \arrow[r, "(-)^{s(n+r+1)}"] \arrow[d, "(-)^r d_j^*"'] & C^*_{n+k-r,k-s} \arrow[d, "(-)^{r+s+j(n+r)} d_j^*"] \\
C^*_{n+k-r+1,k-s+j} \arrow[r, "(-)^{s(n+r)+j(n+r)}"] & C^*_{n+k-r+1,k-s+j}
\end{tikzcd}$$

where the vertical maps are the components of the differential on C^{n+k-*} and $F_*^{dual}C$ respectively. □

Proposition 12.26. *Let F_*C be an admissible, signed, k-filtered chain complex over $\mathbb{A}(R)$, with $\dim C = n+k$. Then $\tau^{NEW}(\theta_{F_*C}\colon C^{n+k-*} \to F_*^{dual}C) = 0$.*

Proof. The proof will be by induction on $m(C)$, defined as the smallest integer m such that $F_mC = C$. Suppose first that $m(C) = 0$. Then $C_r = C_{r,0}$ and $G_r(C) = 0$ unless $r = 0$. We still allow non-zero signs $\eta_{G_r(C)} \in K_1(\mathbf{Z})$ if $r > 0$. The map $\theta_{F_*C}\colon C^{n+k-r} \to C^{n+k-r}$ has absolute torsion

$$\begin{aligned}
\tau^{NEW}(\theta) &= \sum_{r=0}^{n+k} \tau^{iso}((-)^{k(n+r+1)}) + \eta_{F_*^{dual}C} - \eta_{C^{n+k-*}} \\
&= k\cdot\chi(C_{odd}) + \tfrac{k}{2}(k+1)\chi(C) + \alpha_{n+k}(C) + \alpha_n(C) \\
&= \begin{cases} 0 \\ \chi(C_{odd}) \\ 0 \\ \chi(C_{odd}) \end{cases} + \begin{cases} 0 \\ \chi(C) \\ \chi(C) \\ 0 \end{cases} + \begin{cases} 0 \\ \chi(C_{even}) \\ \chi(C) \\ \chi(C_{odd}) \end{cases} \quad \text{for } k \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \mod 4 \\
&= 0 \in K_1(\mathbf{Z})\ .
\end{aligned}$$

We now move on to the inductive step. Let $m = M(C) > 0$ for F_*C and let F_*C' be the m-amalgamation of F_*C, considered as a k-filtered complex, with

$$C'_{r,s} = \begin{cases} C_{r,s} & s < m-1 \\ C_{r,m-1} \oplus C_{r,m} & s = m-1 \\ 0 & s > m-1 \end{cases}$$

with signs $\eta_{\mathbb{G}_*(C')} = \eta_{\mathbb{G}_*(C)}$ and

$$\eta_{G_r(C')} = \begin{cases} \eta_{G_r(C)} & r \neq m-1 \\ \eta_{G_{m-1}(C) \oplus SG_m(C)} & r = m-1 \end{cases}$$

Note that $M(C') = m-1$, and that $C = C'$ as unfiltered signed chain complexes. Then we have a commutative diagram:

$$\begin{tikzcd}
C^{n+k-*} \arrow[rr, "\theta_{F_*C}"] \arrow[dr, "\theta_{F_*C'}"'] & & F_*^{dual}C \\
& F_*^{dual}C' \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

with the map $\sigma\colon F_*^{dual}C' \to F_*^{dual}C$ induced by the maps on the underlying modules: $\sigma = (-)^{k+1+r}\colon C^*_{r,m} \to C^*_{r,}$, and otherwise $\sigma = 1$. By our inductive assumption, $\tau^{NEW}(\theta_{F_*C'}) = 0$, so it remains to compute $\tau^{NEW}(\sigma)$.

We use the k-filtered complexes F_C and F_*C' to express C and C' as an iterated mapping cones. Then

$$\tau^{NEW}(\sigma) = \tau^{iso}(\sigma) + i_*\eta_{\mathbb{G}(F^{dual}C)} - i_*\eta_{\mathbb{G}(F^{dual}C')} \in K_1(\mathbf{Z}) = \mathbf{Z}/2$$

where $\tau^{iso}(\sigma)$ is the sum of the term

$$\tau^{NEW}((-)^{k+m+n+s+1}\colon G_m(C)^{n-s} \to S(SG_m(C)^{n-s})) = (1+k+m+n)\cdot\chi(G_m(C))$$

by [11, Lemma 18], plus the torsions of rearrangements of the direct sum

$$S^{k-m}(G_m(C)^{n-*} \oplus S(G_{m-1}(C)^{n-*} \oplus S(G_{m-2}(C)^{n-*} \oplus \dots)\dots))$$

Note that the chain groups $G_m(C)^{n-s} = S(SG_m(C)^{n-s})$, although these chain complexes have different signs on their differentials. Let $X = SG_m(C)$, $Y = G_{m-1}(C)$ and $Z = S(G_{m-2}(C)^{n-*} \oplus \dots)\dots)$. Then we rearrange by

$$\begin{aligned} SX^{n-*} \oplus S(Y^{n-*} \oplus Z) &\xrightarrow{a} S(X^{n-*} \oplus Y^{n-*} \oplus Z) \\ \xrightarrow{b} S((X \oplus Y)^{n-*} \oplus Z) &\xrightarrow{c} S((Y \oplus X)^{n-*} \oplus Z) \end{aligned}$$

The formulas of [11, Lemma 7] give

$$\tau^{NEW}(a) + \tau^{NEW}(b) + \tau^{NEW}(c) = \epsilon(G_m(C), G_{m-2}(C) \oplus \cdots \oplus G_0(C)) \ .$$

Now the sign term $i_*\eta_{\mathbb{G}(F^{dual}C)} - i_*\eta_{\mathbb{G}(F^{dual}C')}$ is the sum of two terms

$$\alpha_k(\mathbb{G}_*(C)) - \alpha_k(\mathbb{G}_*(C')) = (1+k+m+n)\cdot\chi(G_m(C))$$

and

$$\beta(\mathbb{G}_*(C), \mathbb{G}_*(C)) - \beta(\mathbb{G}_*(C'), \mathbb{G}_*(C')) = \epsilon(G_m(C), G_{m-2}(C) \oplus \cdots \oplus G_0(C))$$

so $\tau^{NEW}(\sigma) = 0$. Hence $\tau^{NEW}(\theta_{F_*C}) = \tau^{NEW}(\theta_{F_*C'}) = 0$ as required. ☐

Example 12.27 (Tensor Products)**.** The following special case is used in deriving the product formula for the absolute torsion of symmetric Poincaré spaces (see Proposition 6.5).

Lemma 12.28. *Let C and D be chain complexes of dimension over $\mathbb{A}(R)$ and ($\mathbb{A}(S)$) respectively, where R and S are rings with involution. Let* $\dim C = k$ *and* $\dim D = n$*. Then we have a chain equivalence*

$$\theta_{C\otimes D}\colon (C \otimes D)^{n+k-*} \to C^{k-*} \otimes D^{n-*}$$

given by

$$\theta_{C\otimes D} = (-)^{(k+n+r)(s+n)}\colon C^{k-r} \otimes D^{n-s} \to C^{k-r} \otimes D^{n-s}$$

Moreover

$$\tau^{NEW}(\theta_{C\otimes D}) = 0 \in K_1(R \otimes S)$$

Proof. The associated complex of $F_*(C^{k-*} \otimes D^{n-*})$ is given by

$$\mathbb{G}_*(C^{k-*} \otimes D^{n-*}) : \ \ldots \to C^{k-r} \otimes D^{n-*} \xrightarrow{(-)^r d_C^* \otimes 1} C^{k-r+1} \otimes D^{n-*}$$

thus $\mathbb{G}_*(C^{k-*} \otimes D^{n-*}) = \mathbb{G}_*(C \otimes D)^{k-*}$ considered as the k-dual of a chain complex in $\mathbb{SPD}_n(\mathbb{A}(R \otimes S))$. Hence $F_*(C^{k-*} \otimes D^{n-*}) = F_*^{dual}(C \otimes D)$; moreover

$$\theta_{C\otimes D} = \theta_{F_*(C\otimes D)} \colon (C \otimes D)^{n+k-*} \to F_*(C^{k-*} \otimes D^{n-*}) = F_*^{dual}(C \otimes D)$$

The result now follows from Proposition 12.26. □

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