IRREDUCIBILITY AND p-ADIC MONODROMIES ON THE SIEGEL MODULI SPACES

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ABSTRACT. We generalize the surjectivity result of the p-adic monodromy for the ordinary locus of a Siegel moduli space by Faltings and Chai (independently by Ekedahl) to that for any p-rank stratum. We discuss irreducibility and connectedness of some p-rank strata of the moduli spaces with parahoric level structure. Finer results are obtained on the Siegel 3-fold with Iwahoric level structure.

1. INTRODUCTION

The present paper is a continuation of the author's work [20]. In loc. cit. we have determined the number of irreducible components of a mod p Siegel module space with Iwahoric level structure. The main ingredients are a result of Ngô and Genestier [14] that the ordinary locus is dense in the moduli space, and the surjectivity of a p-adic monodromy due to Faltings and Chai [7], also due to Ekedahl [5]. The goal of this paper is to investigate the same problem for the the non-ordinary locus and smaller strata.

Let p be a rational prime number. Let $N \geq 3$ be a prime-to-p positive integer. We choose a primitive N-th root of unity ζ_N in $\overline{\mathbb{Q}} \subset \mathbb{C}$ and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let $\mathcal{A}_{g,1,N}$ denote the moduli space over $\mathbb{Z}_{(p)}[\zeta_N]$ of g-dimensional principally polarized abelian varieties with a full symplectic level-N structure with respect to ζ_N. The moduli scheme $\mathcal{A}_{g,1,N}$ has irreducible geometric fibers. Let $\mathcal{A}$ be the reduction $\mathcal{A}_{g,1,N} \otimes \overline{\mathbb{F}}_p$ modulo p. For each integer $0 \leq f \leq g$, let $\mathcal{A}^f \subset \mathcal{A}$ be the locally closed reduced subscheme that classifies the objects (A, λ, η) whose p-rank is f. The p-rank of an abelian variety A is the dimension of $A[p](\bar{k})$ over $\mathbb{F}_p$. It is known due to Koblitz [11] that each stratum $\mathcal{A}^f$ is equi-dimensional of co-dimension f and the closure of the stratum $\mathcal{A}^f$ contains $\mathcal{A}^{f-1}$ for all f. This result is generalized to the moduli spaces of arbitrary polarized abelian varieties by Norman and Oort [15]

Let $(\mathcal{X}, \lambda, \eta) \to \mathcal{A}^f$ be the universal family. The maximal etale quotient $\mathcal{X}[p^\infty]^{\mathrm{et}}$ of the p-divisible group $\mathcal{X}[p^\infty]$ gives rise to a p-adic monodromy

$$\rho^f : \pi_1(\mathcal{A}^f, \bar{x}) \to \mathrm{GL}_f(\mathbb{Z}_p),$$

where $\bar{x}$ is a geometric point of $\mathcal{A}^f$.

In this paper, we prove

Theorem 1.1. *The homomorphism ρ^f is surjective.*

The case where $f = g$ is a well-known result proved by Faltings and Chai [7] and independently by Ekedahl [5]. Theorem 1.1 answers a question raised in Tilouine [19, Remark below Theorem 2, p.792].

Date: January 25, 2007. The research is partially supported by NSC 96-2115-M-001-001.

A direct consequence of Theorem 1.1 is that the associated Igusa tower over each stratum $\mathcal{A}^f$ is irreducible except when $f=0$ and $g\le 2$ (this is the case where the stratum is supersingular); see Section 3. We apply Theorem 1.1 to the almost ordinary locus of the moduli spaces with parahoric level structure and determine the number of irreducible components; see Section 4. In the special case of Iwahoric level structure, we have the following result:

Let $\mathcal{A}_{g,\Gamma_0(p),N}$ denote the moduli space over $\mathbb{Z}_{(p)}[\zeta_N]$ which parametrizes equivalence classes of objects $(A,\lambda,\eta,H_\bullet)_S$, where S is a $\mathbb{Z}_{(p)}[\zeta_N]$-scheme, (A,λ,η) is in $\mathcal{A}_{g,1,N}(S)$, and $H_\bullet$ is a flag of finite flat subgroup schemes of $A[p]$

$$H_\bullet : \quad 0\subset H_1\subset H_2\subset\cdots\subset H_g\subset A[p]$$

such that each H_i is of rank p^i and H_g is isotropic for the Weil pairing e_λ induced by λ. Let $\mathcal{A}_{\Gamma_0(p)}:=\mathcal{A}_{g,\Gamma_0(p),N}\otimes\overline{\mathbb{F}}_p$ be the reduction modulo p and let $\mathcal{A}^{g-1}_{\Gamma_0(p)}:=\mathcal{A}_{\Gamma_0(p)}\times_{\mathcal{A}}\mathcal{A}^{g-1}$, the almost ordinary locus of $\mathcal{A}_{\Gamma_0(p)}$. We prove (Corollary 4.3)

Theorem 1.2. *For $g\ge 2$, the almost ordinary locus $\mathcal{A}^{g-1}_{\Gamma_0(p)}$ has $g2^{g-1}$ irreducible components.*

One might expect that the almost ordinary locus $\mathcal{A}^{g-1}_{\Gamma_0(p)}$ is dense in the non-ordinary locus $\mathcal{A}^{\mathrm{non-ord}}_{\Gamma_0(p)}$. If this is true, then it would imply the same for the moduli spaces with any parahoric level structure as well (see an argument in [20] for the ordinary case), and then we could determine the number of irreducible components of these non-ordinary loci. However, it is false in general. We examine an example in Section 6.

In Section 5 we show how to use Theorem 1.1 to determine the numbers of connected components of the p-rank strata.

In Sections 7 and 8, we give a geometric characterization of Kottwitz-Rapoport strata for the case $g=2$. The characterization requires the knowledge on the supersingular locus. Therefore, the description of the supersingular locus is included. On the other hand, the characterization also gives extra information on the supersingular locus by the induced Kottwitz-Rapoport stratification. This enables us to determine the number of irreducible components of each Kottwitz-Rapoport stratum.

2. Proof of Theorem 1.1

We may assume that $1\le f<g$ because the case $f=g$ is done in [7] and there is nothing to show for $f=0$. Choose a point $x_0=\underline{A}_0$ in $\mathcal{A}_{g-f,1,N}\otimes\overline{\mathbb{F}}_p$ whose p-rank is zero. Consider the morphism

$$\alpha:\mathcal{A}_f^{\mathrm{ord}}\to\mathcal{A}^f,\quad \underline{A}\mapsto\underline{A}_0\times\underline{A},$$

where $\mathcal{A}_f^{\mathrm{ord}}$ is the ordinary locus of the reduction $\mathcal{A}_{f,1,N}\otimes\overline{\mathbb{F}}_p$ mod p. Choose a geometric point $\bar{x}_1$ of $\mathcal{A}_f^{\mathrm{ord}}$. We have the commutative diagram for the p-adic monodromies

$$\begin{tikzcd} \pi_1(\mathcal{A}_f^{\mathrm{ord}},\bar{x}_1) \arrow[r, "\alpha_*"] \arrow[dr, "\rho_f^{\mathrm{ord}}"'] & \pi_1(\mathcal{A}^f,\bar{x}_0\times\bar{x}_1) \arrow[d, "\rho^f"] \\ & \mathrm{GL}_f(\mathbb{Z}_p). \end{tikzcd}$$

Since $\rho_f^{\rm ord}$ is surjective [7], ρ^f is also surjective. This completes the proof of Theorem 1.1 .

3. Irreducibility of the Igusa towers

3.1. Let f be an integer with $0 \le f \le g$. For each integer $m \ge 0$, let $\mathcal{I}_m^f$ be the cover of $\mathcal{A}^f$ over $\overline{\mathbb{F}}_p$ which parametrizes equivalence classes of objects $(A, \lambda, \eta, \xi)_S$ where S is an $\overline{\mathbb{F}}_p$-scheme, $(A, \lambda, \eta)_S$ is in $\mathcal{A}^f(S)$ and ξ is an isomorphism form $\mu_{p^m,S}$ to the multiplicative part $A[p^m]^{\rm mul}$ of $A[p^m]$ over S. Let

$$\mathcal{I}^f := \{\mathcal{I}_m^f\}_{m\ge 0}$$

be the Igusa tower over the stratum $\mathcal{A}^f$.

Proposition 3.1. *The Isuga tower $\mathcal{I}^f$ is irreducible except when $f = 0$ and $g \le 2$.*

Proof. The cover $\mathcal{I}_m^f$ is etale over $\mathcal{A}^f$ and it represents the etale sheaf

$$\underline{Isom}(\mu_{p^m}^{\oplus f}, \mathcal{X}[p^m]^{\rm mul}),$$

where $(\mathcal{X}, \lambda, \eta) \to \mathcal{A}^f$ is the universal family. Therefore, the cover gives rise to the p-adic monodromy $\rho_m^f : \pi_1(\mathcal{A}^f, \bar{x}) \to \mathrm{GL}_f(\mathbb{Z}/p^m\mathbb{Z})$. By Theorem 1.1, the homomorphism ρ_m^f is surjective for all $m \ge 0$. Therefore, each $\mathcal{I}_m^f$ is irreducible if the base $\mathcal{A}^f$ is irreducible. From the possible symmetric Newton polygons, we know that

(a) the stratum $\mathcal{A}^f$ is supersingular (means that every maximal point of $\mathcal{A}^f$ is supersingular) if and only if $f = 0$ and $g \le 2$, and
(b) the stratum $\mathcal{A}^f$ contains a unique maximal Newton polygon stratum as an open dense subset.

Then the proposition follows from the following theorem. ∎

Theorem 3.2 (Oort)**.** *Every non-supersingular Newton polygon stratum of $\mathcal{A}$ is irreducible.*

Proof. See the proof in Oort [17]. ∎

When $f = 0$, each member $\mathcal{I}_m^f$ of $\mathcal{I}^f$ is $\mathcal{A}^0$; this is the trivial case. For the non-trivial cases $f \ge 1$, the Igusa towers are all irreducible.

3.2. We consider a variant of the Igusa towers. Let f be an integer with $1 \le f \le g$. For each integer $m \ge 0$, let $H_m^f := (\mathbb{Z}/p^m\mathbb{Z} \times \mu_{p^m})^{\oplus f}$ and let $\varphi_m^f : H_m^f \times H_m^f \to \mu_{p^m}$ be the alternating pairing defined by

$$\varphi_m^f((m_i, \zeta_i), (n_i, \eta_i)) = \prod_{i=1}^{f} \eta_i^{m_i} \zeta_i^{-n_i}, \quad \forall\, m_i, n_i \in \mathbb{Z}/p^m\mathbb{Z},\ \zeta_i, \eta_i \in \mu_{p^m}.$$

Let $\mathcal{J}_m^f$ be the cover of $\mathcal{A}^f$ over $\overline{\mathbb{F}}_p$ which parametrizes equivalence classes of objects $(A, \lambda, \eta, \xi)_S$ where S is an $\overline{\mathbb{F}}_p$-scheme, $(A, \lambda, \eta)_S$ is in $\mathcal{A}^f(S)$ and

$$\xi : (H_m^f)_S \to A[p^m]$$

is a monomorphism (both homomorphism and closed immersion) over S such that

$$\varphi_m^f(x, y) = e_\lambda(\xi(x), \xi(y)), \quad \forall\, x, y \in H_m^f,$$

where e_λ is the Weil pairing induced by λ. Let

$$\mathcal{J}^f := \{\mathcal{J}^f_m\}_{m\geq 0}$$

be a tower over the stratum $\mathcal{A}^f$.

Theorem 3.3. *The tower $\mathcal{J}^f$ is irreducible, that is, each member $\mathcal{J}^f_m$ is irreducible.*

Proof. Let $(\mathcal{X},\lambda,\iota)\to\mathcal{A}^f$ be the universal family. Consider the canonical filtration

$$0\subset \mathcal{X}[p^m]^{\mathrm{mul}}\subset \mathcal{X}[p^m]^0\subset \mathcal{X}[p^m],$$

where $A[p^m]^0$ is the neutral connected component of $A[p^m]$. So we have two canonical short exact sequences

$$0\to \mathcal{X}[p^m]^{\mathrm{mul}}\to \mathcal{X}[p^m]^0\to \mathcal{X}[p^m]^{\mathrm{loc,loc}}\to 0,$$
$$0\to \mathcal{X}[p^m]^0\to \mathcal{X}[p^m]\to \mathcal{X}[p^m]^{\mathrm{et}}\to 0.$$

Since these short exact sequences split over a perfect affine base in characteristic p, we can find a finite radical surjective morphism $\pi:\mathcal{A}'\to A^f$ such that the base change $\mathcal{X}[p^m]\times_{\mathcal{A}^f}\mathcal{A}'$ admits the canonical decomposition

$$\mathcal{X}[p^m]_{\mathcal{A}'}=\left(\mathcal{X}[p^m]^{\mathrm{mul}}_{\mathcal{A}'}\oplus \mathcal{X}[p^m]^{\mathrm{et}}_{\mathrm{sub}}\right)\oplus \mathcal{X}[p^m]^{\mathrm{loc,loc}}_{\mathrm{sub}}, \tag{3.1}$$

where $\mathcal{X}[p^m]^{\mathrm{mul}}_{\mathcal{A}'}$ is the base change $\mathcal{X}[p^m]^{\mathrm{mul}}\times_{\mathcal{A}^f}\mathcal{A}'$, the middle part $\mathcal{X}[p^m]^{\mathrm{et}}_{\mathrm{sub}}$ is the maximal etale subgroup scheme of $\mathcal{X}[p^m]_{\mathcal{A}'}$ and $\mathcal{X}[p^m]^{\mathrm{loc,loc}}_{\mathrm{sub}}$ is the maximal local-local subgroup scheme of $\mathcal{X}[p^m]_{\mathcal{A}'}$. Furthermore, we may choose $\mathcal{A}'$ to be irreducible. To see this, let $\mathcal{A}_0$ be a scheme over $\mathbb{F}_q$ such that $\mathcal{A}^f\simeq\mathcal{A}_0\otimes_{\mathbb{F}_q}\overline{\mathbb{F}}_p$. Let $\{U\}$ be a finite open covering of affine subschemes of $\mathcal{A}_0$. Choose a positive integer n large enough such that $\mathcal{X}[p^m]$ admits the canonical decomposition as (3.1) over $U^{(q^{-n})}$ for each U, where $F^n:U^{(q^{-n})}\to U$ is the iterated relative Frobenius morphism over $\mathbb{F}_q$. The subgroup schemes

$$\left\{(\mathcal{X}[p^m]_{U^{(q^{-n})}})^{\mathrm{mul}}_{\mathrm{sub}}\right\}_{U^{(q^{-n})}},\quad (\text{resp. } \left\{(\mathcal{X}[p^m]_{U^{(q^{-n})}})^{\mathrm{loc,loc}}_{\mathrm{sub}}\right\}_{U^{(q^{-n})}})$$

glue to a subgroup scheme $\mathcal{X}[p^m]^{\mathrm{mul}}_{\mathrm{sub}}$ (resp. $\mathcal{X}[p^m]^{\mathrm{loc,loc}}_{\mathrm{sub}}$) over $\mathcal{A}_0^{(q^{-n})}$. Clearly $\mathcal{A}_0^{(q^{-n})}$ is irreducible. We may take $\mathcal{A}':=\mathcal{A}_0^{(q^{-n})}\otimes_{\mathbb{F}_q}\overline{\mathbb{F}}_p$ and let $\pi=F^n$, then $\mathcal{X}[p^m]_{\mathcal{A}'}$ admits the canonical decomposition.

Let $\mathcal{J}'_m$ be the etale cover of $\mathcal{A}'$ that represents the etale sheaf

$$\mathcal{P}_m:=\underline{Isom}\left((H^f_m,\varphi^f_m)_{\mathcal{A}'},(\mathcal{X}[p^m]^{\mathrm{mul}}_{\mathcal{A}'}\oplus\mathcal{X}[p^m]^{\mathrm{et}}_{\mathrm{sub}},e_\lambda)\right).$$

Since any section ξ of $\mathcal{P}_m$ is determined by its restriction ξ on $(\mathbb{Z}/p^m)^{\oplus f}$, the restriction map $\xi\mapsto\xi|_{(\mathbb{Z}/p^m)^{\oplus f}}$ gives an isomorphism

$$\mathcal{P}_m\simeq\underline{Isom}((\mathbb{Z}/p^m)^{\oplus f}_{\mathcal{A}'},\mathcal{X}[p^m]^{\mathrm{et}}_{\mathrm{sub}}).$$

Therefore, $\mathcal{J}'_m$ corresponds to the p-adic monodromy $\rho'_m:\pi_1(\mathcal{A}',\bar x')\to\mathrm{GL}_f(\mathbb{Z}/p^m\mathbb{Z})$ and we have the commutative diagram

$$\begin{tikzcd}
\pi_1(\mathcal{A}',\bar x') \arrow[r, "\pi_*"] \arrow[dr, "\rho'_m"'] & \pi_1(\mathcal{A}^f,\bar x) \arrow[d, "\rho^f_m"] \\
 & \mathrm{GL}_f(\mathbb{Z}/p^m\mathbb{Z}).
\end{tikzcd}$$

Since π_* is an isomorphism and ρ_m^f is surjective, ρ'_m is surjective. Therefore, $\mathcal{J}'_m$ is irreducible as $\mathcal{A}'$ is so. Let $(\mathcal{X}', \lambda', \eta')$ be the base change of $(\mathcal{X}, \lambda, \eta)$ over $\mathcal{A}'$ and let $\xi' \in \mathcal{P}_m(\mathcal{J}'_m)$ be the universal section. Then the family $(\mathcal{X}', \lambda', \eta', \xi') \to \mathcal{J}'_m$ gives rise to a morphism $\alpha : \mathcal{J}'_m \to \mathcal{J}_m^f$. Clearly this map is surjective, hence $\mathcal{J}_m^f$ is irreducible. ∎

Remark 3.4. The irreducibility of $\mathcal{J}_1^g$ is studied in [20] (that was denoted $\mathcal{A}_{\Gamma(p)}^{\rm ord}$ there). The lines 1-2 of p. 2593 in loc. cit. are incorrect. The moduli scheme $\mathcal{A}_{\Gamma(p)}^{\rm ord}$ is not etale over $\mathcal{A}_{g,1,N}^{\rm ord}$ because the extension

$$0 \to \mathcal{X}[p]^0 \to \mathcal{X}[p] \to \mathcal{X}[p]^{\rm et} \to 0$$

does not split over any finite etale base change. However, this does not effect the conclusion on irreducibility of $\mathcal{A}_{\Gamma(p)}^{\rm ord}$; we just need a modified argument as in the proof above.

4. The almost ordinary locus of the moduli spaces with parahoric level structure

4.1. We keep the notation as before. Let $\underline{k} = (k_1, \dots, k_r)$ be a tuple of positive integers $k_i \ge 1$ with $\sum_{i=1}^r k_i \le g$. Set $h(i) := \sum_{j=1}^i k_j$ for $1 \le i \le r$ and $h(0) = 0$. Let $\mathcal{A}_{g,\underline{k},N}$ denote the moduli space over $\mathbb{Z}_{(p)}[\zeta_N]$ that parametrizes equivalence classes of objects $(A, \lambda, \eta, H_\bullet)_S$, where S is a $\mathbb{Z}_{(p)}[\zeta_N]$-scheme, (A, λ, η) is in $\mathcal{A}_{g,1,N}(S)$, and $H_\bullet$ is a flag of finite subgroup schemes of $A[p]$

$$H_\bullet \quad 0 = H_{h(0)} \subset H_{h(1)} \subset \cdots \subset H_{h(r)} \subset A[p]$$

such that $H_{h(i)}$ is locally free of rank $p^{h(i)}$ and $H_{h(r)}$ is isotropic for the Weil pairing e_λ induced by the polarization λ. When $r = g$, the moduli scheme $\mathcal{A}_{g,\underline{k},N}$ is $\mathcal{A}_{g,\Gamma_0(p),N}$ defined in Section 1.

Let $\mathcal{A}_{\underline{k}} := \mathcal{A}_{g,\underline{k},N} \otimes \overline{\mathbb{F}}_p$ be the reduction modulo p. For $0 \le f \le g$, let $\mathcal{A}_{\underline{k}}^f := \mathcal{A}_{\underline{k}} \times_{\mathcal{A}} \mathcal{A}^f$, the p-rank f stratum of the moduli space $\mathcal{A}_{\underline{k}}$. For an $\overline{\mathbb{F}}_p$-scheme S, the S-valued set $\mathcal{A}_{\underline{k}}^f(S)$ consists of objects $(A, \lambda, \eta, H_\bullet)_S$ in $\mathcal{A}_{\underline{k}}(S)$ such that the canonical morphism $S \to \mathcal{A}$ given by the family $(A, \lambda, \eta)_S$ factors through the subscheme $\mathcal{A}^f$. Note that from the definition one can not determine whether $\mathcal{A}_{\underline{k}}^f$ is reduced. We will compute the number of irreducible components of $\mathcal{A}_{\underline{k}}^f$ in the case where $f = g - 1$, the almost ordinary locus.

We first seek discrete invariants for geometric points on $\mathcal{A}_{\underline{k}}^{g-1}$. Let k be an algebraically closed field of characteristic p. Fix a supersingular elliptic curve E_0 over $\overline{\mathbb{F}}_p$. Let $(A, \lambda, \eta, H_\bullet)$ be a point of $\mathcal{A}_{\underline{k}}(k)$. We have

$$A[p] \simeq (\mathbb{Z}/p\mathbb{Z} \times \mu_p)^{g-1} \times E_0[p].$$

There are two cases:

(a) $H_{h(r)}$ does not have local-local part. This occurs only when $h(r) < g$. For each $1 \le i \le r$, the finite group scheme $H_h(i)/H_{h(i-1)}$ has the form $\mu_p^{\tau(i)} \times (\mathbb{Z}/p\mathbb{Z})^{k_i - \tau(i)}$ for a non-negative integer $0 \le \tau(i) \le k_i$.

(b) $H_{h(r)}$ has non-trivial local-local part. There is an integer $1 \le j \le r$ such that $H_{h(j)}$ has non-trivial local-local part and $H_{h(j-1)}$ has no local-local part. For each $1 \le i \le r$, the finite group scheme $H_{h(i)}/H_{h(i-1)}$ has the form

$$\begin{cases} \mu_p^{\tau(i)} \times (\mathbb{Z}/p\mathbb{Z})^{k_i-\tau(i)} \text{ for some integer } 0 \le \tau(i) \le k_i & \text{if } i \ne j; \\ \mu_p^{\tau(i)} \times (\mathbb{Z}/p\mathbb{Z})^{k_i-1-\tau(i)} \times \alpha_p \text{ for some integer } 0 \le \tau(i) \le k_i - 1 & \text{if } i = j. \end{cases}$$

4.2. For each $m \ge 0$, define $I(m) := [0, m] \cap \mathbb{Z}$. Set

$$I^0_{\underline{k}} := \begin{cases} \emptyset & \text{if } h(r) = g; \\ \{0\} \times \prod_{i=1}^r I(k_i) & \text{if } h(r) < g. \end{cases}$$

For each $1 \le j \le r$, set

$$I^j_{\underline{k}} := \{\underline{\tau} = (j, \tau(1), \dots, \tau(r)); \quad \tau(i) \in I(k_i) \text{ for } i \ne j \text{ and } \tau(j) \in I(k_j - 1)\}.$$

For $\underline{\tau} = (0, \tau(1), \dots, \tau(r)) \in I^0_{\underline{k}}$, we say a geometric point $\underline{A}$ in $\mathcal{A}^{g-1}_{\underline{k}}$ is of type $\underline{\tau}$ if

- $H_{h(r)}$ has no local-local part, and
- the multiplicative part of $H_{h(i)}/H_{h(i-1)}$ is of rank $p^{\tau(i)}$ for all $1 \le i \le r$.

For $\underline{\tau} = (j, \tau(1), \dots, \tau(r)) \in I^j_{\underline{k}}$, where $1 \le j \le r$, we say a geometric point $\underline{A}$ in $\mathcal{A}^{g-1}_{\underline{k}}$ is of type $\underline{\tau}$ if

- $H_{h(j)}$ has local-local part and $H_{h(j-1)}$ has no local-local part, and
- the multiplicative part of $H_{h(i)}/H_{h(i-1)}$ is of rank $p^{\tau(i)}$ for all $1 \le i \le r$.

Then we have an assignment $\underline{A} \mapsto \underline{\tau}(\underline{A})$, which gives a surjective map

$$\tau : \mathcal{A}^{g-1}_{\underline{k}} \to \coprod_{0 \le j \le r} I^j_{\underline{k}}.$$

It is easy to see that this map is locally constant for the Zariski topology (use the argument in the proof of Theorem 3.3). For a fixed type $\underline{\tau}$, let $\mathcal{A}^{g-1}_{\underline{k},\underline{\tau}}$ be the union of the connected components of $\mathcal{A}^{g-1}_{\underline{k}}$ whose objects are of type $\underline{\tau}$. We write $\mathcal{A}^{g-1}_{\underline{k}}$ into a disjoint union of open subschemes

$$\mathcal{A}^{g-1}_{\underline{k}} = \coprod_{0 \le j \le r} \coprod_{\underline{\tau} \in I^j_{\underline{k}}} \mathcal{A}^{g-1}_{\underline{k},\underline{\tau}}.$$

Theorem 4.1. *For each integer $0 \le j \le r$ and each type $\underline{\tau} \in I^j_{\underline{k}}$, there is a finite surjective morphism $f : \mathcal{J}^{g-1}_1 \to \mathcal{A}^{g-1}_{\underline{k},\underline{\tau}}$. Consequently, each stratum $\mathcal{A}^{g-1}_{\underline{k},\underline{\tau}}$ is irreducible for $g \ge 2$.*

Proof. Let $(\mathcal{X}, \lambda, \eta, \xi) \to \mathcal{J}^{g-1}_1$ be the universal family. The image $\xi(H^{g-1}_1)$ is the etale-multiplicative part $\mathcal{X}[p]^{\mathrm{em}}$ of $\mathcal{X}[p]$, and the orthogonal complement of $\xi(H^{g-1}_1)$ for the Weil pairing e_λ is the local-local part $\mathcal{X}[p]^{\mathrm{loc,loc}}$ of $\mathcal{X}[p]$. Namely, we have

$$\mathcal{X}[p] = \mathcal{X}[p]^{\mathrm{em}} \times \mathcal{X}[p]^{\mathrm{loc,loc}}, \quad \xi : (\mu_p \times \mathbb{Z}/p\mathbb{Z})^{g-1} \xrightarrow{\sim} \mathcal{X}[p]^{\mathrm{em}}.$$

Let C be the kernel of the relative Frobenius morphism

$$F_{\mathcal{X}/\mathcal{J}^{g-1}_1} : \mathcal{X}[p]^{\mathrm{loc,loc}} \to (\mathcal{X}[p]^{\mathrm{loc,loc}})^{(p)}.$$

(a) If $\tau \in I^0_{\underline{k}}$, let $K_i := \mu_p^{\tau(i)} \times (\mathbb{Z}/p\mathbb{Z})^{k_i - \tau(i)}$ for $1 \le i \le r$. For $1 \le m \le r$, set $H_{h(m)} := \xi(\prod_{i=1}^m K_i)$. Then we define a family $(\mathcal{X}, \lambda, \eta, H_\bullet) \to \mathcal{J}^{g-1}_1$ and this family induces a natural morphism $f : \mathcal{J}^{g-1}_1 \to \mathcal{A}_{\underline{k}}$.

(b) If $\tau \in I^j_{\underline{k}}$ for some $1 \le j \le r$, let

$$K_i := \begin{cases} \mu_p^{\tau(i)} \times (\mathbb{Z}/p\mathbb{Z})^{k_i-\tau(i)} & \text{if } i \neq j; \\ \mu_p^{\tau(i)} \times (\mathbb{Z}/p\mathbb{Z})^{k_i-1-\tau(i)} & \text{if } i = j. \end{cases}$$

For $1 \le m \le r$, set

$$H_{h(m)} := \begin{cases} \xi(\prod_{i=1}^m K_i) & \text{if } m < j; \\ \xi(\prod_{i=1}^m K_i) \times C & \text{if } m \ge j. \end{cases}$$

Then we define a family $(\mathcal{X}, \lambda, \eta, H_\bullet) \to \mathcal{J}_1^{g-1}$ and this family induces a natural morphism $f : \mathcal{J}_1^{g-1} \to \mathcal{A}_{\underline{k}}$.

It is clear that f factors through the almost ordinary locus $\mathcal{A}^{g-1}_{\underline{k}}$. Moreover, the image lands in the open subscheme $\mathcal{A}^{g-1}_{\underline{k},\tau}$ by the construction. So we get the morphism $f : \mathcal{J}_1^{g-1} \to \mathcal{A}^{g-1}_{\underline{k},\tau}$. One checks easily that f is surjective. Since the composition $\mathcal{J}_1^{g-1} \to \mathcal{A}^{g-1}_{\underline{k},\tau} \to \mathcal{A}^{g-1}$ is finite, f is finite. This completes the proof. ∎

Note that in the proof we use the universal family to define the morphism f instead of defining $f(x)$ pointwisely. The reason is that $\mathcal{J}_1^{g-1}$ or $\mathcal{A}^{g-1}_{\underline{k}}$ (defined by the fiber product) could be non-reduced.

Corollary 4.2. *For $g \ge 2$, the almost ordinary stratum $\mathcal{A}^{g-1}_{\underline{k}}$ has*

$$\sum_{j=0}^{r} |I^j_{\underline{k}}| = (k_1+1)\dots(k_r+1)\left[\epsilon + \frac{k_1}{k_1+1} + \cdots + \frac{k_r}{k_r+1}\right]$$

irreducible components, where $\epsilon = 1$ if $h(r) < g$ and $\epsilon = 0$ if $h(r) = g$.

For the Iwahoric case, $r = g$ and $k_i = 1$ for all i, so Corollary 4.2 gives

Corollary 4.3. *For $g \ge 2$, the almost ordinary locus $\mathcal{A}^{g-1}_{\Gamma_0(p)}$ has $g2^{g-1}$ irreducible components.*

5. Connected components of p-rank strata

In the previous section we study irreducible components of the almost ordinary locus. In this section we consider lower p-rank strata. We know that when $g \ge 2$ and $0 \le f \le g-2$, the natural morphism $\mathcal{A}^f_{\underline{k}} \to \mathcal{A}^f$ is not finite in general. This limits the method of using p-adic monodromy to the irreducibility problem in the present case. The obstacle results from the fibration "moving α_p-subgroups". If one contracts the fibration, then one obtains a finite morphism for which the p-adic monodromy results can be applied. Proceeding this approach, we obtain information on connected components instead.

Keep the notion in the previous sections. Assume that $g \ge 2$.

5.1. Fix a tuple $\underline{k}=(k_1,\dots,k_r)$ of positive integers with $\sum_{i=1}^r k_i\le g$ and an integer f with $0\le f\le g$. Again, we first seek discrete invariants for geometric points in $\mathcal{A}^f_{\underline{k}}$. Let $(A,\lambda,\eta,H_\bullet)$ be a point in $\mathcal{A}^f_{\underline{k}}(k)$. We have a decomposition

$$H_{h(i)}=\left(H^{\rm et}_{h(i)}\oplus H^{\rm mul}_{h(i)}\right)\oplus H^{\rm loc,loc}_{h(i)}$$

into etale-multiplicative part and local-local part. Suppose that $H^{\rm et}_{h(i)}\oplus H^{\rm mul}_{h(i)}$ has rank $p^{a(i)}$ and $H^{\rm loc,loc}_{h(i)}$ has rank $p^{b(i)}$. Put $a(0)=0$ and $m(i):=a(i)-a(i-1)$ for each $1\le i\le r$. It is easy to see that

$$0\le m(i)\le k_i,\quad \text{and}\quad f-(g-h(r))\le\sum_{i=1}^r m(i)\le f, \tag{5.1}$$

where $h(i)=\sum_{j=1}^i k_j$ as before. Let $G_i:=H^{\rm et}_{h(i)}\oplus H^{\rm mul}_{h(i)}$. Then the successive quotient G_i/G_{i-1} has rank $p^{m(i)}$. Let

$$\tau(i):=\log_p \operatorname{rank}(G_i/G_{i-1})^{\rm mul},\quad \forall\, 1\le i\le r.$$

We have $0\le\tau(i)\le m(i)$ for all i. We call the pair of r-tuples

$$(\underline{m},\underline{\tau})=[(m(1),\dots,m(r)),(\tau(1),\dots\tau(r))]$$

the *graded etale-multiplicative type* associated to the object $(A,\lambda,\eta,H_\bullet)$, abbreviated as *gem* type.

Conversely, fix a tuple of integers $(m(1),\dots,m(r))$ satisfying (5.1). Let (A,λ,η) be an object in $\mathcal{A}^f(k)$ and $G_\bullet$ a flag of finite flat subgroup schemes

$$0=G_0\subset G_1\subset\cdots\subset G_r\subset A[p]$$

such that (1) the group scheme G_r is isotropic with respect to the Weil pairing e_λ, and (2) each G_i has no local-local part and the quotient G_i/G_{i-1} has rank $p^{m(i)}$. Then one can lift to an object $(A,\lambda,\eta,H_\bullet)\in\mathcal{A}^f_{\underline{k}}(k)$ such that $H^{\rm et}_{h(i)}\oplus H^{\rm mul}_{h(i)}=G_i$ for all i.

5.2. Define

$$\begin{split}\Sigma_0(\underline{k},f):=\{\,\underline{m}=(m(1),\dots,m(r))\in\mathbb{Z}^r\mid\\ 0\le m(i)\le k_i,\forall\, i,\ \text{and}\ f-(g-h(r))\le\sum_i^r m(i)\le f\,\},\end{split}\tag{5.2}$$

$$\begin{split}\Sigma(\underline{k},f):=\{\,(\underline{m},\underline{\tau})=((m(1),\dots,m(r)),(\tau(1),\dots,\tau(r)))\in\mathbb{Z}^r\times\mathbb{Z}^r\mid\\ \underline{m}\in\Sigma_0(\underline{k},f),\ \text{and}\ 0\le\tau(i)\le m(i),\ \forall\, i\,\}.\end{split}\tag{5.3}$$

For any element $\underline{m}\in\Sigma_0(\underline{k},f)$, we define a scheme $T(\underline{m})$ over $\overline{\mathbb{F}}_p$ as follows. For any locally Noetherian $\overline{\mathbb{F}}_p$-scheme S, the S-valued set $T(\underline{m})(S)$ classifies equivalence classes of objects $(A,\lambda,\eta,G_\bullet)_S$, where

- $(A,\lambda,\eta)_S$ is in $\mathcal{A}^f(S)$, and
- $G_\bullet$ is a flag of finite flat subgroup schemes

$$G_\bullet:\quad 0=G_0\subset G_1\subset\cdots\subset G_r\subset A[p]$$

satisfying the conditions (1) and (2) above.

For any element $(\underline{m},\underline{\tau}) \in \Sigma(\underline{k},f)$, let $T(\underline{m},\underline{\tau})$ be the (open) subscheme of $T(\underline{m})$ consists of objects $(A,\lambda,\eta,G_\bullet)_S$ such that the multiplicative part of the quotient G_i/G_{i-1} has rank $p^{\tau(i)}$ for all $1 \le i \le r$. Put $T^f_{\underline{k}} := \coprod_{\underline{m}\in\Sigma_0(\underline{k},f)} T(\underline{m})$. Clearly, we have

$$T^f_{\underline{k}} = \coprod_{(\underline{m},\underline{\tau})\in\Sigma(\underline{k},f)} T(\underline{m},\underline{\tau}).$$

Let $(\mathcal{X},\lambda,\eta,\widetilde{H}_\bullet) \to \mathcal{A}^f_{\underline{k}}$ be the universal family over $\mathcal{A}^f_{\underline{k}}$. Then there is a finite dominant homeomorphic morphism $\pi : \mathcal{A}' \to \mathcal{A}^f_{\underline{k}}$ such that the base change $\mathcal{X}[p]_{\mathcal{A}'}$ admits the canonical decomposition

$$\mathcal{X}[p]_{\mathcal{A}'} = (\mathcal{X}[p]^{\mathrm{mul}} \oplus \mathcal{X}[p]^{\mathrm{et}}) \oplus \mathcal{X}[p]^{\mathrm{loc,loc}} \tag{5.4}$$

into etale-multiplicative part and local-local part; see the proof of Theorem 3.3. Accordingly, we have the same decomposition

$$\widetilde{H}_{h(i),\mathcal{A}'} = (\widetilde{H}^{\mathrm{mul}}_{h(i)} \oplus \widetilde{H}^{\mathrm{et}}_{h(i)}) \oplus \widetilde{H}^{\mathrm{loc,loc}}_{h(i)} \tag{5.5}$$

for all i.

Since these subgroup schemes are locally free, their ranks are constant on each connected component of $\mathcal{A}'$. Let $\mathcal{A}'(\underline{m},\underline{\tau}) \subset \mathcal{A}'$ be the union of the connected components whose objects are of gem type $(\underline{m},\underline{\tau})$. Let $\mathcal{A}(\underline{m},\underline{\tau}) \subset \mathcal{A}^f_{\underline{k}}$ be the open subscheme $\pi(\mathcal{A}'(\underline{m},\underline{\tau}))$. Again, we have

$$\mathcal{A}^f_{\underline{k}} = \coprod_{(\underline{m},\underline{\tau})} \mathcal{A}(\underline{m},\underline{\tau}), \quad \mathcal{A}' = \coprod_{(\underline{m},\underline{\tau})} \mathcal{A}'(\underline{m},\underline{\tau}), \tag{5.6}$$

where $(\underline{m},\underline{\tau})$ runs through all elements in $\Sigma(\underline{k},f)$.

Consider the universal family $(\mathcal{X},\lambda,\eta,\widetilde{H}_\bullet)_{\mathcal{A}'(\underline{m},\underline{\tau})}$ restricted on the open subscheme $\mathcal{A}'(\underline{m},\underline{\tau})$. Put

$$\widetilde{G}_i := \widetilde{H}^{\mathrm{mul}}_{h(i)} \oplus \widetilde{H}^{\mathrm{et}}_{h(i)},$$

for all i. We get a family $(\mathcal{X},\lambda,\eta,\widetilde{G}_\bullet)_{\mathcal{A}'(\underline{m},\underline{\tau})}$. This gives rise to a natural morphism

$$c(\underline{m},\underline{\tau}) : \mathcal{A}'(\underline{m},\underline{\tau}) \to T(\underline{m},\underline{\tau}), \tag{5.7}$$

which is proper and surjective (Subsection 5.1).

If $f=0$, then the set $\Sigma(\underline{k},f)$ consists of only one element $(\underline{0},\underline{0})$. In this case, we have

$$\mathcal{A}' = \mathcal{A}^0_{\underline{k}}, \quad T^0_{\underline{k}} = \mathcal{A}^0, \quad \text{and} \quad c(\underline{0},\underline{0}) : \mathcal{A}^0_{\underline{k}} \to \mathcal{A}^0. \tag{5.8}$$

Proposition 5.1.
(1) *The stratum $\mathcal{A}^0$ is connected and it is irreducible if $g \ge 3$.*
(2) *Suppose that $f \ge 1$. Then there is a finite surjective morphism $\mathcal{J}^f_1 \to T(\underline{m},\underline{\tau})$. Consequently, each scheme $T(\underline{m},\underline{\tau})$ is irreducible.*

Proof. (1) When $g=2$, this is a special case of Theorem 7.3 in Oort [16]. When $g \ge 3$, this is obtained in the proof of Proposition 3.1 using Theorem 3.2, a result of Oort.

(2) The construction is similar as that as in Theorem 4.1. Therefore, we do not repeat it. ∎

5.3. Let $\underline{n}=(n(1),\dots n(r'))$ be a tuple of positive integers with $\sum_{i=1}^{r'} n(i)\le g-f$. We may identify the moduli space $\mathcal{A}_{\underline{n}}$ with the moduli space that parametrizes equivalence classes of chains of isogenies

$$\underline{A}_\bullet:\quad \underline{A}_0 \xrightarrow{\alpha_1} \underline{A}_1 \longrightarrow \dots \longrightarrow \underline{A}_{r'-1} \xrightarrow{\alpha_{r'}} \underline{A}_{r'},$$

where

- each $\underline{A}_i=(A_i,\lambda_i,\eta_i)$ is a polarized abelian scheme with a symplectic level-N structure,
- $\underline{A}_0$ is an object in $\mathcal{A}$, and
- each α_i is an isogeny of degree $p^{n(i)}$ that preserves the level structures and the polarizations except when $i=1$, and in this case one has $\alpha_1^*\lambda_1=p\lambda_0$.

Define $W_{\underline{n}}\subset\mathcal{A}_{\underline{n}}$ to be the reduced subscheme consisting of objects $\underline{A}_\bullet$ such that the kernel $\ker\alpha_i$ is of local-local type for all i.

Let $(\underline{m},\underline{\tau})$ be an element in $\Sigma(\underline{k},f)$. Let $a(i):=\sum_{j=1}^{i} m(j)$ and $b(i):=h(i)-a(i)$. Put $a(0)=b(0)=0$. Write the set $\{b(i); 0\le i\le r\}$ as

$$\{b'(0),b'(1),\dots b'(r')\}\quad\text{with } 0=b'(0)<b'(1)<\dots<b'(r').$$

We have $r'\le r$, $r'\le b'(r')$ and $h(r)-f\le b'(r')\le g-f$. Set $n(i):=b'(i)-b'(i-1)$ for $1\le i\le r'$. So we define a tuple $\underline{n}$ of integers from the pair $(\underline{m},\underline{\tau})$, and have a scheme $W_{\underline{n}}$.

Let $x=(A_x,\lambda_x,\eta_x,G_\bullet)$ be a point in $T(\underline{m},\underline{\tau})(k)$. The fiber $c(\underline{m},\underline{\tau})^{-1}(x)$ consists of flags of finite flat subgroup schemes

$$0=K_{b(0)}\subset K_{b(1)}\subset\dots\subset K_{b(r)}\subset A_x[p]$$

such that $K_{b(r)}$ is isotropic for the Weil pairing e_{λ_x} and each $K_{b(i)}$ is local-local of rank $p^{b(i)}$. This is the same as flags of finite flat subgroup schemes

$$0=K_{b'(0)}\subset K_{b'(1)}\subset\dots\subset K_{b'(r')}\subset A_x[p]$$

with the same properties. This proves $c(\underline{m},\underline{\tau})^{-1}(x)=W_{\underline{n}}(x)$, where

$$W_{\underline{n}}(x):=\{\underline{A}_\bullet\in W_{\underline{n}}\,;\,\underline{A}_0=(A_x,\lambda_x,\eta_x)\,\}.$$

It is easy to see that the reduced scheme $W_{\underline{n}}(x)$ is connected. Indeed, for $1\le d\le g-f$, let $\underline{1}^d:=(1,\dots,1)$ with length d. We see that the fiber of the natural morphism $W_{\underline{1}^i}(x)\to W_{\underline{1}^{i-1}}(x)$ is a projective space, as it is the family of α_p-subgroups in $\ker\lambda_{i-1}$. This shows that the scheme $W_{\underline{1}^{g-f}}(x)$ is connected. Since the forgetful morphism $W_{\underline{1}^{g-f}}(x)\to W_{\underline{n}}(x)$ is surjective, the scheme $W_{\underline{n}}(x)$ is connected. In conclusion, we have proved

Proposition 5.2. *Any fiber of the morphism $c(\underline{m},\underline{\tau})$ is connected.*

Theorem 5.3. *Every open subscheme $\mathcal{A}(\underline{m},\underline{\tau})\subset\mathcal{A}^f_{\underline{k}}$ is connected. Consequently, the p-rank f stratum $\mathcal{A}^f_{\underline{k}}$ has $|\Sigma(\underline{k},f)|$ connected components.*

PROOF. It follows from Propositions 5.1 and 5.2 that $\mathcal{A}'(\underline{m},\underline{\tau})$ is connected, and thus that $\mathcal{A}(\underline{m},\underline{\tau})$ is connected. ∎

Remark 5.4. The scheme $T^f_{\underline{k}}$ is closely related to the Stein factorization of the natural morphism $\mathcal{A}^f_{\underline{k}}\to\mathcal{A}^f$. Indeed, let T (resp. T') be the Stein factorization of $\mathcal{A}^f_{\underline{k}}\to\mathcal{A}^f$ (resp. of $\mathcal{A}'\to\mathcal{A}^f$). Then there are natural finite morphisms $\pi_1: T'\to T$

and $\pi_2 : T' \to T^f_{\underline{k}}$. It is not hard to see that these morphisms are homeomorphic. In some sense, $T^f_{\underline{k}}$ provides a "modular interpretation" of the scheme T.

6. The Siegel 3-fold with Iwahoric level structure

In this section we describe the Kottwitz-Rapoport stratification on the Siegel 3-fold $\mathcal{A}_{2,\Gamma_0(p)}$ with Iwahoric level structure. Our references are de Jong [3], Kottwitz and Rapoport [12], T. Haines [8], and Ngô and Genestier [14]. The geometric part of Tilouine [19] is also helpful to us. Then we conclude the following results as consequences:

(a) The almost ordinary locus $\mathcal{A}^1_{2,\Gamma_0(p)}$ is not dense in the non-ordinary locus $\mathcal{A}^{\mathrm{non-ord}}_{2,\Gamma_0(p)}$.

(b) The supersingular locus $\mathcal{S}_{2,\Gamma_0(p)}$ of $\mathcal{A}_{2,\Gamma_0(p)}$ is not equi-dimensional. It consists of both one-dimensional components and two-dimensional components.

6.1. **Local models.** Let $\mathcal{O}$ be a complete discrete valuation ring, K its fraction field, π an uniformizer of $\mathcal{O}$, and $\kappa := \mathcal{O}/\pi\mathcal{O}$ the residue field. We require that $\mathrm{char}\kappa = p > 0$. Set $V := K^{2n}$ and let $e_1, \dots, e_{2n}$ be the standard basis. Denote by $\psi : V \times V \to K$ the non-degenerate alternating form whose non-zero pairings are

$$\psi(e_i, e_{2n+1-i}) = 1, \quad 1 \le i \le n,$$
$$\psi(e_i, e_{2n+1-i}) = -1, \quad i \ge n+1,$$

The representing matrix for ψ is

$$\begin{pmatrix} 0 & \widetilde{I} \\ -\widetilde{I} & 0 \end{pmatrix}, \quad \widetilde{I} = \text{anti-diag}(1,\dots,1).$$

Let GSp_{2n} be the reductive algebraic group of symplectic similitudes with respect to ψ. Let

$$\pi L_0 = L_{-2n} \subset L_{-2n+1} \subset \cdots \subset L_{-1} \subset L_0 = \mathcal{O}^{2n}$$

be a chain of $\mathcal{O}$-lattices in V where the lattice L_{-i} is generated by $e_1, \dots, e_{2n-i}$, $\pi e_{2n-i+1}, \dots, \pi e_{2n}$. The $\mathcal{O}$-submodule in V generated by $x_1, \dots, x_k \in V$ is denoted by $< x_1, \dots, x_k >$. Thus $L_{i-2n} =< e_1, \dots, e_i, \pi e_{i+1}, \dots, \pi e_{2n} >$.

For $0 \le i \le 2n$, let $\Lambda_{i-2n} = \mathcal{O}^{2n}$ and define $\beta_{i-2n} : \Lambda_{i-2n} \to \Lambda_{i-2n+1}$ for $i < 2n$ by

$$\beta_{i-2n}(e_{i+1}) = \pi e_{i+1}, \quad \text{and} \quad \beta_{i-2n}(e_j) = e_j \quad \text{for } j \ne i+1.$$

We have

$$\Lambda_{-2n} \xrightarrow{\beta_{-2n}} \Lambda_{-2n+1} \longrightarrow \dots \longrightarrow \Lambda_{-1} \xrightarrow{\beta_{-1}} \Lambda_0,$$

and there is a unique isomorphism $a_{-i} : \Lambda_{-i} \to L_{-i}$ with $a_0 = \mathrm{id}$ such that the diagram

$$\begin{array}{ccc} \Lambda_{-i} & \xrightarrow{\beta_{-i}} & \Lambda_{-i+1} \\ \Big\downarrow{\scriptstyle a_{-i}} & & \Big\downarrow{\scriptstyle a_{-i+1}} \\ L_{-i} & \xrightarrow{\text{incl}} & L_{-i+1} \end{array}$$

commutes. Let $\widetilde{\beta}_{-i} : \Lambda_{-i} \to \Lambda_0$ denote the composition of the morphisms

$$\Lambda_{-i} \xrightarrow{\beta_{-i}} \Lambda_{-i+1} \longrightarrow \dots \longrightarrow \Lambda_{-1} \xrightarrow{\beta_{-1}} \Lambda_0.$$

Let $\psi_0=\psi$ be on the form on $\Lambda_0=L_0$. There is a perfect non-degenerate alternating form ψ_{-n} on Λ_{-n} such that

$$\psi_0(\widetilde{\beta}_{-n}(x),\widetilde{\beta}_{-n}(y))=\pi\psi_n(x,y),\quad \forall\, x,y\in\Lambda_{-n}.$$

Let $\mathbf{M}^{\rm loc}$ denote the projective $\mathcal{O}$-scheme that represents the functor which sends an $\mathcal{O}$-scheme S to the set of the collections of locally free $\mathcal{O}_S$-submodules $\mathcal{F}_{-i}\subset\Lambda_{-i}\otimes\mathcal{O}_S$ of rank n for $0\le i\le n$ such that

(i) $\mathcal{F}_0$ and $\mathcal{F}_{-n}$ are isotropic with respect to ψ_0 and ψ_{-n}, respectively.
(ii) $\mathcal{F}_{-i}$ locally is a direct summand of $\Lambda_{-i}\otimes\mathcal{O}_S$ for all i,
(iii) $\beta_{-i}(\mathcal{F}_{-i})\subset\mathcal{F}_{-i+1}$ for all i.

By an automorphism on $\Lambda_\bullet\otimes\mathcal{O}_S$, where S is an $\mathcal{O}$-scheme, we mean a collection of automorphisms g_{-i} on $\Lambda_{-i}\otimes\mathcal{O}_S$ such that g_{-i} commutes with the morphisms β_{-i} for all i and g_0 and g_{-n} preserve the forms ψ_0 and ψ_{-n}, respectively, up to invertible scalars. We denote by $\mathrm{Aut}(\Lambda_\bullet\otimes\mathcal{O}_S,\psi_0,\psi_{-n})$ the group of automorphisms on $\Lambda_\bullet\otimes\mathcal{O}_S$.

Let $\mathcal{G}$ be the group scheme over $\mathcal{O}$ that represents the functor

$$S\mapsto \mathrm{Aut}(\Lambda_\bullet\otimes\mathcal{O}_S,\psi_0,\psi_{-n}).$$

We know that $\mathcal{G}$ is an affine smooth group scheme over $\mathcal{O}$ whose generic fibre $\mathcal{G}_K$ is equal to GSp_{2n}. Furthermore, there is a left action of $\mathcal{G}$ on $\mathbf{M}^{\rm loc}$.

6.2. **The Kottwitz-Rapoport stratification on $\mathbf{M}^{\rm loc}_\kappa$.** Let $\mathcal{F}l$ be the space of chains of $\mathcal{O}$-lattices in V

$$\pi\mathcal{L}_0=\mathcal{L}_{-2n}\subset\cdots\subset\mathcal{L}_{-1}\subset\mathcal{L}_0$$

such that

(i) $\mathcal{L}_i/\mathcal{L}_{i-1}\simeq\kappa$ for each i,
(ii) there is a non-degenerate alternating pairing ψ' on $\mathcal{L}_0$ with values in $\mathcal{O}$ such that $\pi^m\psi'=\psi$ for some $m\in\mathbb{Z}$, and
(iii) set $\overline{\mathcal{L}}_{-i}:=\mathcal{L}_{-i}/\mathcal{L}_{-2n}$, we require that the orthogonal complement $\overline{\mathcal{L}}_{-i}^{\perp}$ with respect to ψ' is equal to $\overline{\mathcal{L}}_{i-2n}$ for all i.

Note that a lattice chain $(\mathcal{L}_\bullet)$ in $\mathcal{F}l$ is determined by it members $\mathcal{L}_{-i}$ for $0\le i\le n$.

We regard V as a space of column vectors. For $g\in\mathrm{GSp}_{2n}(K)$, the map

$$g\mapsto(\mathcal{L}_i)=(gL_i)$$

gives a bijection $\mathrm{GSp}_{2n}(K)/I\simeq\mathcal{F}l$, where I is the stabilizer of the standard lattice chain.

Now we restrict to the equi-characteristic case $\mathcal{O}=\kappa[[t]]$ and $\pi=t$. The space $\mathcal{F}l$ has a natural ind-scheme structure over κ and is called the *affine flag variety* associated to GSp_{2n} over κ. For any field extension κ' of κ, we have a natural bijection

$$\mathrm{GSp}_{2n}\left(\kappa'((t))\right)/I(\kappa')\simeq\mathcal{F}l(\kappa'),$$

where $I(\kappa')$ is the stabilizer of the standard lattice chain $(L_\bullet\otimes\kappa'[[t]])$ base change over $\kappa'[[t]]$.

Let $\mathcal{Y}$ be the closed subscheme of $\mathcal{F}l$ consisting of the lattice chains $\mathcal{L}_\bullet$ such that

$$tL_{-i}\subset\mathcal{L}_{-i}\subset L_{-i},\quad 0\le i\le n,$$

such that $L_0/\mathcal{L}_0 \simeq \kappa^n$.

The group I acts on the ind-scheme $\mathcal{F}l$ by the left translation; it leaves the subscheme $\mathfrak{Y}$ invariant. Using the Bruhat-Iwahori decomposition

$$\mathrm{GSp}_{2n}(\kappa((t))) = \coprod_{x\in\widetilde{W}} IxI,$$

the I-orbits are indexed by the extended affine Weyl group $\widetilde{W}$ of GSp_{2n}:

$$\mathcal{F}l = \coprod_{x\in\widetilde{W}} \mathcal{F}l_x.$$

The extended affine Weyl group $\widetilde{W}$ is the semi-direct product $X_*(T) \rtimes W$ of the Weyl group W of GSp_{2n} and the cocharacter groups $X_*(T)$, where T is the group of diagonal matrices in GSp_{2n}. The cocharacter group $X_*(T)$ is

$$\{(u_1, \dots, u_{2n}) \in \mathbb{Z}^{2n} \mid u_1 + u_{2n} = \cdots = u_n + u_{n+1}, \}.$$

The Weyl group W is a subgroup of the symmetric group $S_{2n} = W(\mathrm{GL}_{2n})$ consisting of elements that commute with the permutation

$$\theta = (1, 2n)(2, 2n-1)\dots(n, n+1).$$

We identity S_{2n} with the group of permutation matrices in GL_{2n} as follows:

(6.1)

$$\forall\, \sigma \in S_{2n},\ \text{the permutation } \sigma \text{ corresponding to } w_\sigma \in \mathrm{GL}_{2n}, \text{ where } w_\sigma(e_i) = e_{\sigma(i)}, \forall\, i.$$

We may regard the group $\widetilde{W}$ as a subgroup of the group $\mathbf{A}(\mathbb{R}^{2n})$ of affine transformations on the space $\mathbb{R}^{2n}$ of column vectors. For $\nu \in X_*(T)$, we write t_ν for the image of t under ν in $\mathrm{GSp}_{2n}(\kappa((t)))$, also for the translation by ν in $\mathbf{A}(\mathbb{R}^{2n})$. The element $x = t_\nu w$ then is identified with the function $x(v) = w \cdot v + \nu$ for $v \in \mathbb{R}^{2n}$. If $x = (x_i) \in \mathbb{R}^{2n}$, we write $|x| = \sum_i x_i$.

There is a partial order on the extended affine Weyl group $\widetilde{W}$ called the Bruhat order. According to the definition (see [12, 1.8], for example), one defines the Bruhat order on the extended affine Weyl group $\widetilde{W}_{\mathrm{der}}$ of Sp_{2n} first, then the partial order on $\widetilde{W}$ is inherited from $\widetilde{W}_{\mathrm{der}}$ as follows:

$$x \le y \text{ in } \widetilde{W} \iff [x] = [y] \text{ in } \widetilde{W}_{\mathrm{der}}\backslash\widetilde{W} \text{ and } 1 \le yx^{-1} \text{ in } \widetilde{W}_{\mathrm{der}}.$$

The choice of the Bruhat order on $\widetilde{W}_{\mathrm{der}}$ depends on the choice of the Borel subgroup of Sp_{2n}. We choose the Borel subgroup B to be the subgroup of upper triangular matrices in Sp_{2n}. This also agrees with the choice (following Haines [8]) of the lattice chain $L_{-2n} \subset \cdots \subset L_0$.

Let $v_{-2n}, \dots, v_0 \in \mathbb{Z}^{2n}$ be the alcove corresponding the lattices $L_{-2n}, \dots, L_0$ (see [12, Subsection 3.2, 4.2]). We have $v_{-i} = (0^{2n-i}, 1^i)$. Let $\mu = (1^n, 0^n) \in X_*(T)$, a dominant coweight. Following Kottwitz and Rapoport [12], we define the sets of *μ-permissible* and *μ-admissible* elements:

$$\mathrm{Perm}(\mu) := \{x \in \widetilde{W} \mid (0, \dots, 0) \le x(v_{-i}) - v_{-i} \le (1, \dots, 1) \text{ for all } i \text{ and } |x(0)| = n\}$$

$$\mathrm{Adm}(\mu) := \{x \in \widetilde{W} \mid \text{there is an element } w \in W \text{ such that } x \le t_{w(\mu)}\}$$

Proposition 6.1. *Notation as above.*

1. *The stratum $\mathcal{F}l_x$ is contained in $\mathfrak{Y}$ if and only if $x \in \mathrm{Perm}(\mu)$.*
2. $\mathrm{Adm}(\mu) = \mathrm{Perm}(\mu)$.

PROOF. (1) Let $x \in \widetilde{W} \subset \mathrm{GL}_{2n}(\kappa((t)))$. The lattice xL_{-i} corresponds to the element $x(v_{-i})$ in $\mathbb{Z}^{2n}$. Then the condition $tL_{-i} \subset xL_{-i} \subset L_{-i}$ is easily seen to be $(0,\dots,0) \le x(v_{-i}) - v_{-i} \le (1,\dots,1)$. We also have $\dim_\kappa L_0/xL_0 = |x(0)|$. Therefore, the statement follows.

(2) This is Theorem 4.5 (3) of Kottwitz and Rapoport [12]. ∎

Remark 6.2. The embedding $\sigma \mapsto w_\sigma$ from S_{2n} to GL_{2n} in (6.1) does not send the Weyl group W into GSp_{2n}. In fact for each $\sigma \in W$, there is a unique element $\epsilon_\sigma = \mathrm{diag}(1,\dots,1,\epsilon_{\sigma,n+1},\dots,\epsilon_{\sigma,2n})$ with $\epsilon_{\sigma,i} \in \{\pm 1\}$ such that $w'_\sigma = w_\sigma \epsilon_\sigma \in \mathrm{Sp}_{2n}$. However, since $w'_\sigma t_\nu (w'_\sigma)^{-1} = w_\sigma t_\nu w_\sigma^{-1}$ and $t_\nu w'_\sigma L_{-i} = t_\nu w_\sigma L_{-i}$, it won't effect any results if we choose the presentation of $\widetilde{W}$ in GL_{2n} either by $(\nu,\sigma) \mapsto t_\nu w_\sigma$ or by $(\nu,\sigma) \mapsto t_\nu w'_\sigma$. We make the first choice as it is easier not to deal with the signs. Another reason for this choice is that the lattice point in $\mathbb{Z}^{2n}$ corresponding to $t_\nu w'_\sigma L_{-i} = t_\nu w_\sigma L_{-i}$ is $w_\sigma \cdot v_{-i} + \nu$ not $w'_\sigma \cdot v_{-i} + \nu$.

To avoid confusing the standard lattice chain that defines the local model $\mathbf{M}^{\mathrm{loc}}$ and the lattice chain for the affine flag variety $\mathcal{F}l$, we use different notation to distinguish them. Let $\Lambda'_{-i} = \kappa[[t]]^{2n}$ for $0 \le i \le 2n$, and $L'_{i-2n} =< e_1, \dots, e_i, te_{i+1}, \dots te_{2n} >$. Define β'_{-i}, a'_{-i}, ψ'_0, ψ'_{-n}, $\mathcal{G}'$ as in Subsection 6.1. In particular, $\mathcal{G}'$ is a smooth affine group scheme over $\kappa[[t]]$ and one has

$$\mathcal{G}'(S) = \mathrm{Aut}(\Lambda'_\bullet \otimes \mathcal{O}_S, \psi'_0, \psi'_{-n})$$

for any $\kappa[[t]]$-scheme S. One also sees that the generic fibre of $\mathcal{G}'$ is GSp_{2n}, $\mathcal{G}'(\kappa[[t]])$ is equal to I, and that the special fibre $\mathcal{G}'_\kappa$ is canonically isomorphic to $\mathcal{G}_\kappa$.

Using the isomorphism $a'_{-i} : \Lambda'_{-i} \simeq L'_{-i}$, we regard the lattice $\mathcal{L}_{-i}$, where $(\mathcal{L}_{-i})$ is a member in $\mathcal{Y}$, as a $\kappa[[t]]$-submodule of Λ'_{-i} containing $t\Lambda'_{-i}$. Then we have an isomorphism $b : \mathcal{Y} \simeq \mathbf{M}^{\mathrm{loc}}_\kappa$, which maps any κ'-point of $\mathcal{Y}$ to $\mathbf{M}^{\mathrm{loc}}_\kappa(\kappa')$ by

$$(\mathcal{L}_{-i}) \mapsto (\mathcal{F}_{-i}), \quad \mathcal{F}_{-i} := \mathcal{L}_{-i}/t\Lambda'_{-i} \subset \Lambda_{-i} \otimes \kappa'$$

where κ' is any field extension of κ.

The action of I on the scheme $\mathcal{Y}$ factors through the quotient $\mathcal{G}'(\kappa)$. We also know that the isomorphism b is $\mathcal{G}_\kappa = \mathcal{G}'_\kappa$-equivariant. Therefore, the stratification

$$\mathcal{Y} = \coprod_{x \in \mathrm{Adm}(\mu)} \mathcal{F}l_x$$

induces a stratification, called the Kottwitz-Rapoport stratification, on $\mathbf{M}^{\mathrm{loc}}_\kappa$

$$\mathbf{M}^{\mathrm{loc}}_\kappa = \coprod_{x \in \mathrm{Adm}(\mu)} \mathbf{M}^{\mathrm{loc}}_x$$

so that the $\mathcal{G}_\kappa$-orbit $\mathbf{M}^{\mathrm{loc}}_x$ corresponds to the I-orbit $\mathcal{F}l_x$.

6.3. **Local model diagrams.** Let $\mathcal{A}'_{g,\Gamma_0(p),N}$ denote the moduli space over $\mathbb{Z}_{(p)}[\zeta_N]$ that parametrizes equivalence classes of objects

$$\underline{A}_\bullet : \quad \underline{A}_0 \xrightarrow{\alpha_1} \underline{A}_1 \longrightarrow \dots \longrightarrow \underline{A}_{g-1} \xrightarrow{\alpha_g} \underline{A}_g,$$

where

- each $\underline{A}_i = (A_i, \lambda_i, \eta_i)$ is a polarized abelian scheme with a symplectic level-N structure,
- $\underline{A}_0$ and $\underline{A}_g$ are objects in $\mathcal{A}_{g,1,N}$, and

- each α_i is an isogeny of degree p that preserves the level structures and the polarizations except when $i = 1$, and in this case one has $\alpha_1^*\lambda_1 = p\lambda_0$.

There is a natural isomorphism from $\mathcal{A}_{g,\Gamma_0(p),N}$ to $\mathcal{A}'_{g,\Gamma_0(p),N}$ [3, Propoistion 1.7]. We will identify $\mathcal{A}_{g,\Gamma_0(p),N}$ with $\mathcal{A}'_{g,\Gamma_0(p),N}$ via this natural isomorphism.

Put $n = g$ and $\mathcal{O} = \mathbb{Z}_p$ in Subsection 6.1. We get a projective $\mathbb{Z}_p$-scheme $\mathbf{M}^{\mathrm{loc}}$. Let S be a $\mathbb{Z}_p[\zeta_N]$-scheme and $\underline{A}_\bullet$ is an object in $\mathcal{A}_{g,\Gamma_0(p),N}(S)$. A *trivialization* γ from the de Rham cohomologies $H^1_{\mathrm{DR}}(A_\bullet/S)$ to $\Lambda_\bullet \otimes \mathcal{O}_S$ is a collection of isomorphisms $\gamma_i : H^1_{\mathrm{DR}}(A_i/S) \to \Lambda_{-i} \otimes \mathcal{O}_S$ of $\mathcal{O}_S$-modules such that

- the diagram
$$\begin{array}{ccc} H^1_{\mathrm{DR}}(A_i/S) & \xrightarrow{\alpha_i^*} & H^1_{\mathrm{DR}}(A_{i-1}/S) \\ \downarrow{\scriptstyle\gamma_i} & & \downarrow{\scriptstyle\gamma_{i-1}} \\ \Lambda_{-i} \otimes \mathcal{O}_S & \xrightarrow{\beta_{-i}} & \Lambda_{-i+1} \otimes \mathcal{O}_S \end{array}$$
commutes for $1 \le i \le g$,
- if $e_{\lambda_0}, e_{\lambda_g}$ are the non-degenerate symplectic pairings induced by the principal polarizations λ_0, λ_g, $\gamma_i^*\psi_i$ is a scalar multiple of e_{λ_i} by some element in $\mathcal{O}_S^\times$ for $i = 0, g$.

With the terminology as above, let $\widetilde{\mathcal{A}}_{g,\Gamma_0(p),N}$ denote the moduli space over $\mathbb{Z}_p[\zeta_N]$ that parametrizes equivalence classes of objects $(\underline{A}_\bullet, \gamma)_S$, where

- $\underline{A}_\bullet$ is an object in $\mathcal{A}_{g,\Gamma_0(p),N}(S)$, and
- γ is a trivialization from $H^1_{\mathrm{DR}}(A_\bullet/S)$ to $\Lambda_\bullet \otimes \mathcal{O}_S$.

The moduli scheme $\widetilde{\mathcal{A}}_{g,\Gamma_0(p),N}$ has two natural projections φ^{mod} and φ^{loc}. The morphism
$$\varphi^{\mathrm{mod}} : \widetilde{\mathcal{A}}_{g,\Gamma_0(p),N} \to \mathcal{A}_{g,\Gamma_0(p),N} \otimes \mathbb{Z}_p[\zeta_N]$$
forgets the trivialization. The morphism
$$\varphi^{\mathrm{loc}} : \widetilde{\mathcal{A}}_{g,\Gamma_0(p),N} \to \mathbf{M}^{\mathrm{loc}} \otimes \mathbb{Z}_p[\zeta_N]$$
sends an object $(\underline{A}_\bullet, \gamma)$ to $(\gamma(\omega_\bullet))$, where $\omega_\bullet = (\omega_i)$ is a system of $\mathcal{O}_S$-submodules in the Hodge filtration
$$0 \to \omega_i \to H^1_{\mathrm{DR}}(A_i/S) \to R^1 f_*(\mathcal{O}_{A_i}) \to 0,$$
and $f : A_i \to S$ is the structure morphism. Thus, we have the diagram:
$$\begin{array}{ccccc} & & \widetilde{\mathcal{A}}_{g,\Gamma_0(p),N} & & \\ & \swarrow{\scriptstyle\varphi^{\mathrm{mod}}} & & \searrow{\scriptstyle\varphi^{\mathrm{loc}}} & \\ \mathcal{A}_{g,\Gamma_0(p),N} \otimes \mathbb{Z}_p[\zeta_N] & & & & \mathbf{M}^{\mathrm{loc}} \otimes \mathbb{Z}_p[\zeta_N]. \end{array}$$

The moduli scheme $\widetilde{\mathcal{A}}_{g,\Gamma_0(p),N}$ also has a left action by the group scheme $\mathcal{G}$. By the works of Rapoport-Zink [18], de Jong [3], and Genestier (cf. Remark below Theorem 1.3 of [14]), we know

(a) φ^{mod} is a left $\mathcal{G}$-torsor, and hence it is affine and smooth.
(b) φ^{loc} is smooth, surjective, $\mathcal{G}$-equivariant, and of relative dimension same as φ^{mod}.

Let

$$\mathcal{A}_{\Gamma_0(p)} := \mathcal{A}_{g,\Gamma_0(p),N} \otimes \overline{\mathbb{F}}_p, \quad \widetilde{\mathcal{A}}_{\Gamma_0(p)} := \widetilde{\mathcal{A}}_{g,\Gamma_0(p),N} \otimes \overline{\mathbb{F}}_p, \quad \mathbf{M}^{\rm loc}_{\overline{\mathbb{F}}_p} := \mathbf{M}^{\rm loc} \otimes \overline{\mathbb{F}}_p$$

be the reduction modulo p, respectively. Let $\widetilde{\mathcal{A}}_{\Gamma_0(p),x}$ be the pre-image of a KR-stratum $\mathbf{M}^{\rm loc}_x$. By (b), $\widetilde{\mathcal{A}}_{\Gamma_0(p),x}$ is stable under the $\mathcal{G}_{\overline{\mathbb{F}}_p}$-action. Since $\varphi^{\rm mod}$ is a $\mathcal{G}_{\overline{\mathbb{F}}_p}$-torsor, the stratification

$$\widetilde{\mathcal{A}}_{\Gamma_0(p)} = \coprod_{x\in \mathrm{Adm}(\mu)} \widetilde{\mathcal{A}}_{\Gamma_0(p),x}$$

descends to a stratification, called the Kottwitz-Rapoport stratification, on $\mathcal{A}_{\Gamma_0(p)}$:

$$\mathcal{A}_{\Gamma_0(p)} = \coprod_{x\in \mathrm{Adm}(\mu)} \mathcal{A}_{\Gamma_0(p),x}.$$

Each stratum $\mathcal{A}_{\Gamma_0(p),x}$ is smooth of dimension same as $\dim \mathbf{M}^{\rm loc}_x$, which is the length $\ell(x)$ of x.

6.4. **g=2.** We describe the set $\mathrm{Adm}(\mu)$ of μ-admissible elements and the Bruhat order on this set, in the special case where $g=2$. The closure $\bar{\mathbf{a}}$ of the base alcove $\mathbf{a}$ is the set of points $u\in\mathbb{R}^4$ such that $u_1+u_4=u_2+u_3$ and

$$1+u_1 \ge u_4 \ge u_3 \ge u_2.$$

This is obtained from [12, 12.2] by applying the involution θ since our choice of the standard alcove $\{v_{-i}\}$ differ from $\{w_i\}$ in [12, 4.2] by the involution θ. The simple reflections corresponding to the faces

$$u_3=u_4, \quad u_2=u_3, \quad 1+u_1=u_4,$$

are $s_1=(1\,2)(3\,4)$, $s_2=(2\,3)$ and

$$s_0=((-1,0,0,1),(14)) : (u_1,u_2,u_3,u_4)\mapsto (u_4-1,u_2,u_3,u_1+1).$$

One checks that $\tau := ((0,0,1,1),(13)(24)) \in \widetilde{W}$ is the element in $\mathrm{Adm}(\mu)$ that fixes $\bar{\mathbf{a}}$. It is not hard to compute the set $\mathrm{Perm}(\mu)$ from the definition. From this and the fact $\mathrm{Adm}(\mu)=\mathrm{Perm}(\mu)$, we have

$$\mathrm{Adm}(\mu) := \begin{Bmatrix} \tau,\, s_1\tau,\, s_0\tau,\, s_2\tau,\, s_0s_1\tau,\, s_0s_2\tau,\, s_1s_2\tau,\, s_2s_1\tau,\, s_1s_0\tau, \\ s_0s_1s_0\tau,\, s_1s_0s_2\tau,\, s_2s_1s_2\tau,\, s_0s_2s_1\tau \end{Bmatrix}.$$

We compute and express these elements x as (ν,σ):

$$\begin{aligned} \tau &= [(0,0,1,1),(13)(24)], & s_1\tau &= [(0,0,1,1),(14)(23)], \\ s_0\tau &= [(0,0,1,1),(1342)], & s_2\tau &= [(0,1,0,1),(1243)], \\ s_0s_1\tau &= [(0,0,1,1),(23)], & s_0s_2\tau &= [(0,1,0,1),(12)(34)], \\ s_1s_2\tau &= [(1,0,1,0),(23)], & s_2s_1\tau &= [(0,1,0,1),(14)], \\ s_1s_0\tau &= [(0,0,1,1),(14)], & s_0s_1s_0\tau &= [(0,0,1,1),(1)], \\ s_1s_0s_2\tau &= [(1,0,1,0),(1)], & s_2s_1s_2\tau &= [(1,1,0,0),(1)], \\ s_0s_2s_1\tau &= [(0,1,0,1),(1)]. \end{aligned} \tag{6.2}$$

For a later use, we also express these elements as $t_\nu w_\sigma$ in $\mathrm{GL}_4(\kappa((t)))$:

$$\tau = \begin{pmatrix} & & 1 & \\ & & & 1 \\ t & & & \\ & t & & \end{pmatrix}, \quad s_1\tau = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & t & & \\ t & & & \end{pmatrix},$$

$$s_0\tau = \begin{pmatrix} & 1 & & \\ & & & 1 \\ t & & & \\ & & t & \end{pmatrix}, \quad s_2\tau = \begin{pmatrix} & & 1 & \\ t & & & \\ & & & 1 \\ & t & & \end{pmatrix},$$

$$s_0s_1\tau = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & t & & \\ & & & t \end{pmatrix}, \quad s_0s_2\tau = \begin{pmatrix} & 1 & & \\ t & & & \\ & & & 1 \\ & & t & \end{pmatrix}, \quad s_1s_2\tau = \begin{pmatrix} t & & & \\ & & 1 & \\ & t & & \\ & & & 1 \end{pmatrix},$$

(6.3)

$$s_2s_1\tau = \begin{pmatrix} & & & 1 \\ & t & & \\ & & 1 & \\ t & & & \end{pmatrix}, \quad s_1s_0\tau = \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & t & \\ t & & & \end{pmatrix}, \quad s_0s_1s_0\tau = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & t & \\ & & & t \end{pmatrix},$$

$$s_1s_0s_2\tau = \begin{pmatrix} t & & & \\ & 1 & & \\ & & t & \\ & & & 1 \end{pmatrix}, \quad s_2s_1s_2\tau = \begin{pmatrix} t & & & \\ & t & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad s_0s_2s_1\tau = \begin{pmatrix} 1 & & & \\ & t & & \\ & & 1 & \\ & & & t \end{pmatrix}.$$

By a result of Ngô-Genestier [14, Theorem 4.1], the points on each KR-stratum $\mathcal{A}_{\Gamma_0(p),x}$ have constant p-rank, which we denote by p-rank(x). Moreover, if $x = (\nu, \sigma)$, then p-rank(x) is given by the formula

$$p\text{-rank}(x) = \frac{1}{2}\,\#\mathrm{Fix}(\sigma), \quad \text{where } \mathrm{Fix}(\sigma) := \{\, i;\ \sigma(i) = i \,\}. \tag{6.4}$$

Put $\mathrm{Adm}^i(\mu) := \{x \in \mathrm{Adm}(\mu);\, p\text{-rank}(x) = i\,\}$. One easily computes the p-ranks of elements in $\mathrm{Adm}(\mu)$ using (6.4) and gets

$$\begin{aligned} \mathrm{Adm}^2(\mu) &= \{s_0s_1s_0\tau, s_1s_0s_2\tau, s_2s_1s_2\tau, s_0s_2s_1\tau\}, \\ \mathrm{Adm}^1(\mu) &= \{s_0s_1\tau, , s_1s_2\tau, s_2s_1\tau, s_1s_0\tau\}, \\ \mathrm{Adm}^0(\mu) &= \{\tau, s_1\tau, s_0\tau, s_2\tau, s_0s_2\tau\}. \end{aligned} \tag{6.5}$$

The Bruhat order on $\mathrm{Adm}(\mu)$ can be described by the following diagram

$$\begin{tikzcd}
 & s_0\tau \arrow[r] \arrow[dr] \arrow[ddr] & s_0s_1\tau \arrow[r] \arrow[dr] & s_0s_1s_0\tau \\
 & & s_1s_0\tau \arrow[ur] \arrow[ddr] & s_0s_2s_1\tau \\
\tau \arrow[uur] \arrow[r] \arrow[ddr] & s_1\tau \arrow[ur] \arrow[dr] \arrow[ddr] & s_0s_2\tau \arrow[ur] \arrow[dr] & \\
 & & s_1s_2\tau \arrow[r] \arrow[dr] & s_1s_0s_2\tau \\
 & s_2\tau \arrow[uur] \arrow[ur] \arrow[r] & s_2s_1\tau \arrow[uuur] \arrow[r] & s_2s_1s_2\tau .
\end{tikzcd}$$

Here $x \to y$ means $\ell(y) = \ell(x) + 1$ and $y = sx$ for some reflection s associated an affine root, and $x \le y$ if and only if there exists a chain

$$x = x_1 \to x_2 \cdots \to x_k = y,$$

(cf. [12, 1.1,1.2]). This diagram is obtained from reading the picture in Haines [8, Figure 2 in Section 4] of admissible alcoves for GSp_4

From (6.5) we see that the stratum $\mathcal{A}_{\Gamma_0(p),s_0s_2\tau}$ is supersingular and has dimension two. From the above diagram we see that the stratum $\mathcal{A}_{\Gamma_0(p),s_1\tau}$ is not contained in the closure of $\mathcal{A}_{\Gamma_0(p),s_0s_2\tau}$. Therefore, we conclude

Proposition 6.3.

(1) *The almost ordinary locus* $\mathcal{A}^1_{2,\Gamma_0(p)}$ *is not dense in the non-ordinary locus* $\mathcal{A}^{\mathrm{non-ord}}_{2,\Gamma_0(p)}$.

(2) *The supersingular locus* $\mathcal{S}_{2,\Gamma_0(p)}$ *of* $\mathcal{A}_{2,\Gamma_0(p)}$ *consists of both one-dimensional irreducible components and two-dimensional irreducible components.*

Remark 6.4. (1) It follows from Proposition 6.3 (2) that for $g \ge 2$ and $0 \le f \ge g-2$, the natural morphism $\mathcal{A}^f_{\Gamma_0(p)} \to \mathcal{A}^f$ is not finite.

(2) Tilouine [19, p.790] examines the intersections of four components of the special fiber of the local model. He also concludes the same results above.

7. Geometric characterization ($g = 2$)

Let $a = (\underline{A}_0 \xrightarrow{\alpha} \underline{A}_1 \xrightarrow{\alpha} \underline{A}_2)$ be a point in $\mathcal{A}_{2,\Gamma_0(p)}(k)$, where k is an algebraically closed field of characteristic p. Then a lies in a Kottwitz-Rapoport stratum $\mathcal{A}_{\Gamma_0(p),KR(a)}$ for a unique element $KR(a)$ in $\mathrm{Adm}(\mu)$. We would like to describe $KR(a)$ from the geometric properties of the point a.

Put $M_i := H^1_{\mathrm{DR}}(A_i/k)$ and $\omega_i := \omega_{A_i}$. We have

$$M_2 \xrightarrow{\alpha} M_1 \xrightarrow{\alpha} M_0.$$

Set $G_0 := \ker(\alpha : A_0 \to A_1)$ and $G_1 := \ker(\alpha : A_1 \to A_2)$; they are finite flat group scheme of rank p, which is isomorphic to $\mathbb{Z}/p, \mu_p$, or α_p.. From the Dieudonné theory we know that

$$\omega_{G_i} = \omega_i/\alpha(\omega_{i+1}), \quad \text{and} \quad \mathrm{Lie}(G_i^t) = M_i/(\omega_i + \alpha(M_{i+1})),$$

where G_i^t is the Cartier dual of G_i. Following de Jong [3], we define

$$\sigma_i(a) := \dim \omega_i/\alpha(\omega_{i+1}), \quad \tau_i(a) := \dim M_i/(\omega_i + \alpha(M_{i+1})).$$

Clearly we have the following characterization of G_i:

Table 1.

$(\sigma_i(a), \tau_i(a))$	$(0,1)$	$(1,0)$	$(1,1)$
G_i	$\mathbb{Z}/p\mathbb{Z}$	μ_p	α_p

When the p-rank of a is ≥ 1, the chain of the p-divisible groups of a is determined by the invariants $(\sigma_i(a), \tau_i(a))$ up to isomorphism. In particular, the element $KR(a)$ is determined by the invariants $(\sigma_i(a), \tau_i(a))$. To describe the correspondence, it suffices to compute these invariants for the distinguished point x in the stratum $\mathbf{M}^{\mathrm{loc}}_x$.

Recall how to associate a member in $\mathbf{M}^{\mathrm{loc}}_x$ to an element $x = t_\nu w_\sigma$ in $\mathrm{Adm}(\mu)$. We first apply x to the standard lattice chain and get a lattice chain $(xL'_{-i})_{0\le i\le 2}$. It follows from the permissibility that $tL'_{-i} \subset xL'_{-i} \subset L'_{-i}$. . Then there is a lattice $\mathcal{L}_{-i}$ in Λ'_{-i} so that its image under the isomorphism $\Lambda'_{-i} \simeq L'_{-i}$ is xL'_{-i}. This way we associate an element $(\mathcal{L}_{-i}/t\Lambda'_{-i})$ in $\mathbf{M}^{\mathrm{loc}}_x$ (via the isomorphism $b : \mathcal{Y} \simeq \mathbf{M}^{\mathrm{loc}}_{\overline{\mathbb{F}}_p}$). We use this element to compute the invariants (σ_i, τ_i).

Below we write $[L'_0] = (e_1, e_2, e_3, e_4)^t, [L'_{-1}] = (e_1, e_2, e_3, te_4)^t$, and $[L'_{-2}] = (e_1, e_2, te_3, te_4)^t$ and write $\overline{\mathcal{L}}_{-i} = \mathcal{L}_{-i}/t\Lambda'_{-i}$ and $\overline{\Lambda}'_{-i} = \Lambda'_{-i}/\mathcal{L}_{-i}$. Recall that $\beta's$ are the maps between the lattices Λ'_{-i} which correspond to the maps α on M_i under a trivialization map γ.

(1) When $x = s_0 s_1 s_0 \tau = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & t & \\ & & & t \end{pmatrix}$, we compute

$$x[L'_0] = \begin{pmatrix} e_1 \\ e_2 \\ te_3 \\ te_4 \end{pmatrix}, \quad x[L'_{-1}] = \begin{pmatrix} e_1 \\ e_2 \\ te_3 \\ t^2 e_4 \end{pmatrix}, \quad x[L'_{-2}] = \begin{pmatrix} e_1 \\ e_2 \\ t^2 e_3 \\ t^2 e_4 \end{pmatrix}.$$

It follows that $\mathcal{L}_{-2} = \mathcal{L}_{-1} = \mathcal{L}_0 =< e_1, e_2, te_3, te_4 >$, $\overline{\mathcal{L}}_{-2} = \overline{\mathcal{L}}_{-1} = \overline{\mathcal{L}}_0 =< e_1, e_2 >$, and $\overline{\Lambda}'_{-2} = \overline{\Lambda}'_{-1} = \overline{\Lambda}'_0 =< e_3, e_4 >$. It follows that $\beta(\overline{\mathcal{L}}_{-2}) = \beta(\overline{\mathcal{L}}_{-1}) =< e_1, e_2 >$, $\beta(\overline{\Lambda}'_{-2}) =< e_4 >$, and $\beta(\overline{\Lambda}'_{-1}) =< e_3 >$. This gives $(\sigma_0, \tau_0) = (0, 1)$ and $(\sigma_1, \tau_1) = (0, 1)$.

Note that if x is diagonal, then $\mathcal{L}_{-2} = \mathcal{L}_{-1} = \mathcal{L}_0$. Therefore, it is enough to compute $x[L'_0]$, which is done this way in (2)–(4).

(2) When $x = s_0 s_2 s_1 \tau = \begin{pmatrix} 1 & & & \\ & t & & \\ & & 1 & \\ & & & t \end{pmatrix}$, we compute $x[L'_0] = \begin{pmatrix} e_1 \\ te_2 \\ e_3 \\ te_4 \end{pmatrix}$ and obtain

$$\mathcal{L}_0 =< e_1, te_2, e_3, te_4 >, \quad \overline{\mathcal{L}}_0 =< e_1, e_3 >, \quad \overline{\Lambda}'_0 =< e_2, e_4 > .$$

It follows that $\beta(\overline{\mathcal{L}}_{-1}) =< e_1, e_3 >$, $\beta(\overline{\mathcal{L}}_{-2}) =< e_1 >$, $\beta(\overline{\Lambda}'_{-1}) =< e_2 >$, and $\beta(\overline{\Lambda}'_{-2}) =< e_2, e_4 >$. This gives $(\sigma_0, \tau_0) = (0, 1)$ and $(\sigma_1, \tau_1) = (1, 0)$.

(3) When $x = s_1 s_0 s_2 \tau = \begin{pmatrix} t & & & \\ & 1 & & \\ & & t & \\ & & & 1 \end{pmatrix}$, we compute $x[L'_0] = \begin{pmatrix} te_1 \\ e_2 \\ te_3 \\ e_4 \end{pmatrix}$ and obtain

$$\mathcal{L}_0 =< te_1, e_2, te_3, e_4 >, \quad \overline{\mathcal{L}}_0 =< e_2, e_4 >, \quad \overline{\Lambda}'_0 =< e_1, e_3 > .$$

It follows that $\beta(\overline{\mathcal{L}}_{-1}) =< e_2 >$, $\beta(\overline{\mathcal{L}}_{-2}) =< e_2, e_4 >$, $\beta(\overline{\Lambda}'_{-1}) =< e_1, e_3 >$, and $\beta(\overline{\Lambda}'_{-2}) =< e_1 >$. This gives $(\sigma_0, \tau_0) = (1, 0)$ and $(\sigma_1, \tau_1) = (0, 1)$.

(4) When $x = s_2 s_1 s_2 \tau = \begin{pmatrix} t & & & \\ & t & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, we compute $x[L'_0] = \begin{pmatrix} te_1 \\ te_2 \\ e_3 \\ e_4 \end{pmatrix}$ and obtain

$$\mathcal{L}_0 =< te_1, te_2, e_3, e_4 >, \quad \overline{\mathcal{L}}_0 =< e_3, e_4 >, \ \overline{\Lambda}'_0 =< e_1, e_2 > .$$

It follows that $\beta(\overline{\mathcal{L}}_{-1}) =< e_3 >$, $\beta(\overline{\mathcal{L}}_{-2}) =< e_4 >$, $\beta(\overline{\Lambda}'_{-1}) =< e_1, e_2 >$, and $\beta(\overline{\Lambda}'_{-2}) =< e_1, e_2 >$. This gives $(\sigma_0, \tau_0) = (1, 0)$ and $(\sigma_1, \tau_1) = (1, 0)$.

(5) When $x = s_0 s_1 \tau = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & t & & \\ & & & t \end{pmatrix}$, we compute $x([L'_0], [L'_{-1}], [L'_{-2}]) = \begin{pmatrix} e_1 & e_1 & e_1 \\ te_3 & te_3 & te_3 \\ e_2 & e_2 & te_2 \\ te_4 & t^2 e_4 & t^2 e_4 \end{pmatrix}$ and obtain

$$\mathcal{L}_0 =< e_1, e_2, te_3, te_4 >, \quad \overline{\mathcal{L}}_0 =< e_1, e_2 >, \quad \overline{\Lambda}'_0 =< e_3, e_4 >,$$

$$\mathcal{L}_{-1} =< e_1, e_2, te_3, te_4 >, \quad \overline{\mathcal{L}}_{-1} =< e_1, e_2 >, \quad \overline{\Lambda}'_{-1} =< e_3, e_4 >,$$

$$\mathcal{L}_{-2} =< e_1, te_2, e_3, te_4 >, \quad \overline{\mathcal{L}}_{-2} =< e_1, e_3 >, \quad \overline{\Lambda}'_{-2} =< e_2, e_4 > .$$

It follows that $\beta(\overline{\mathcal{L}}_{-1}) =< e_1, e_2 >$, $\beta(\overline{\mathcal{L}}_{-2}) =< e_1 >$, $\beta(\overline{\Lambda}'_{-1}) =< e_3 >$, and $\beta(\overline{\Lambda}'_{-2}) =< e_4 >$. This gives $(\sigma_0, \tau_0) = (0, 1)$ and $(\sigma_1, \tau_1) = (1, 1)$.

(6) When $x = s_1 s_2 \tau = \begin{pmatrix} t & & & \\ & & 1 & \\ & t & & \\ & & & 1 \end{pmatrix}$, we compute $x([L'_0], [L'_{-1}], [L'_{-2}]) = \begin{pmatrix} te_1 & te_1 & te_1 \\ te_3 & te_3 & te_3 \\ e_2 & e_2 & te_2 \\ e_4 & te_4 & te_4 \end{pmatrix}$ and obtain

$$\mathcal{L}_0 =< te_1, e_2, te_3, e_4 >, \quad \overline{\mathcal{L}}_0 =< e_2, e_4 >, \quad \overline{\Lambda}'_0 =< e_1, e_3 >,$$

$$\mathcal{L}_{-1} =< te_1, e_2, te_3, e_4 >, \quad \overline{\mathcal{L}}_{-1} =< e_2, e_4 >, \quad \overline{\Lambda}'_{-1} =< e_1, e_3 >,$$

$$\mathcal{L}_{-2} =< te_1, te_2, e_3, e_4 >, \quad \overline{\mathcal{L}}_{-2} =< e_3, e_4 >, \quad \overline{\Lambda}'_{-2} =< e_1, e_2 > .$$

It follows that $\beta(\overline{\mathcal{L}}_{-1}) =< e_2 >$, $\beta(\overline{\mathcal{L}}_{-2}) =< e_4 >$, $\beta(\overline{\Lambda}'_{-1}) =< e_1, e_3 >$, and $\beta(\overline{\Lambda}'_{-2}) =< e_1 >$. This gives $(\sigma_0, \tau_0) = (1, 0)$ and $(\sigma_1, \tau_1) = (1, 1)$.

(7) When $x = s_2 s_1 \tau = \begin{pmatrix} & & & 1 \\ & t & & \\ & & 1 & \\ t & & & \end{pmatrix}$, we compute $x([L'_0], [L'_{-1}], [L'_{-2}]) = \begin{pmatrix} te_4 & te_4 & te_4 \\ te_2 & te_2 & te_2 \\ e_3 & e_3 & te_3 \\ e_1 & te_1 & te_1 \end{pmatrix}$ and obtain

$$\mathcal{L}_0 =< e_1, te_2, e_3, te_4 >, \quad \overline{\mathcal{L}}_0 =< e_1, e_3 >, \quad \overline{\Lambda}'_0 =< e_2, e_4 >,$$

$$\mathcal{L}_{-1} =< te_1, te_2, e_3, e_4 >, \quad \overline{\mathcal{L}}_{-1} =< e_3, e_4 >, \quad \overline{\Lambda}'_{-1} =< e_1, e_2 >,$$

$$\mathcal{L}_{-2} =< te_1, te_2, e_3, e_4 >, \quad \overline{\mathcal{L}}_{-2} =< e_3, e_4 >, \quad \overline{\Lambda}'_{-2} =< e_1, e_2 > .$$

It follows that $\beta(\overline{\mathcal{L}}_{-1}) =< e_3 >$, $\beta(\overline{\mathcal{L}}_{-2}) =< e_4 >$, $\beta(\overline{\Lambda}'_{-1}) =< e_2 >$, and $\beta(\overline{\Lambda}'_{-2}) =< e_1, e_2 >$. This gives $(\sigma_0, \tau_0) = (1, 1)$ and $(\sigma_1, \tau_1) = (1, 0)$.

(8) When $x = s_1 s_0 \tau = \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & t & \\ t & & & \end{pmatrix}$, we compute $x([L'_0],[L'_{-1}],[L'_{-2}]) = \begin{pmatrix} te_4 & te_4 & te_4 \\ e_2 & e_2 & e_2 \\ te_3 & te_3 & t^2e_3 \\ e_1 & te_1 & te_1 \end{pmatrix}$ and obtain

$$\mathcal{L}_0 =< e_1, e_2, te_3, te_4 >, \quad \overline{\mathcal{L}}_0 =< e_1, e_2 >, \quad \overline{\Lambda}'_0 =< e_3, e_4 >,$$

$$\mathcal{L}_{-1} =< te_1, e_2, te_3, e_4 >, \quad \overline{\mathcal{L}}_{-1} =< e_2, e_4 >, \quad \overline{\Lambda}'_{-1} =< e_1, e_3 >,$$

$$\mathcal{L}_{-2} =< te_1, e_2, te_3, e_4 >, \quad \overline{\mathcal{L}}_{-2} =< e_2, e_4 >, \quad \overline{\Lambda}'_{-2} =< e_1, e_3 > .$$

It follows that $\beta(\overline{\mathcal{L}}_{-1}) =< e_2 >$, $\beta(\overline{\mathcal{L}}_{-2}) =< e_2, e_4 >$, $\beta(\overline{\Lambda}'_{-1}) =< e_3 >$, and $\beta(\overline{\Lambda}'_{-2}) =< e_1 >$. This gives $(\sigma_0,\tau_0) = (1,1)$ and $(\sigma_1,\tau_1) = (0,1)$.

We conclude the characterization of $KR(a)$ when p-rank of a is ≥ 1 in the following table:

Table 2.

p-rank(a)	2	2	2	2	1	1	1	1
$(\sigma_0(a),\tau_0(a))$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(1,1)$
$(\sigma_1(a),\tau_1(a))$	$(0,1)$	$(1,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(1,1)$	$(1,0)$	$(0,1)$
$KR(a)$	$s_0s_1s_0\tau$	$s_0s_2s_1\tau$	$s_1s_0s_2\tau$	$s_2s_1s_2\tau$	$s_0s_1\tau$	$s_1s_2\tau$	$s_2s_1\tau$	$s_1s_0\tau$

When a is supersingular (i.e. any member A_i is a supersingular abelian variety), the element $KR(a)$ is not determined by the invariants $(\sigma_i(a),\tau_i(a))$. In fact, they are all $(1,1)$, as the schemes G_i are isomorphic to α_p. We treat this case separately.

8. Geometric characterization ($g = 2$): the supersingular case

8.1. We continue with a geometric point a in $\mathcal{A}_{2,\Gamma_0(p)}(k)$, and suppose that a has p-rank 0. We know that $(\sigma_i(a),\tau_i(a)) = (1,1)$ for $i = 0,1$. We define an invariant $(\sigma_{02}(a),\tau_{02}(a))$ by

$$\sigma_{02}(a) := \dim \omega_0/\alpha^2(\omega_2), \quad \tau_{02}(a) := \dim M_0/(\omega_0 + \alpha^2(M_2).$$

As in the previous section, we associated a distinguished point in $\mathbf{M}_x^{\mathrm{loc}}$ to each element $x = t_\nu w_\sigma$ in $\mathrm{Adm}(\mu)$. We use this point to calculate the invariant $(\sigma_{02}(a),\tau_{02}(a))$. Below $[L'_0],[L'_{-1}],[L'_{-2}],\overline{\mathcal{L}}_{-i}$, $\overline{\Lambda}'_{-i}$ are as in the previous section.

(1) When $x = s_0 s_2 \tau = \begin{pmatrix} & 1 & & \\ t & & & \\ & & & 1 \\ & & t & \end{pmatrix}$, we compute $x([L_0'],[L_{-1}'],[L_{-2}']) = \begin{pmatrix} te_2 & te_2 & te_2 \\ e_1 & e_1 & e_1 \\ te_4 & te_4 & t^2 e_4 \\ e_3 & te_3 & te_3 \end{pmatrix}$ and obtain

$$\mathcal{L}_0 =< e_1, te_2, e_3, te_4 >, \quad \overline{\mathcal{L}}_0 =< e_1, e_3 >, \quad \overline{\Lambda}'_0 =< e_2, e_4 >,$$
$$\mathcal{L}_{-1} =< e_1, te_2, te_3, e_4 >, \quad \overline{\mathcal{L}}_{-1} =< e_1, e_4 >, \quad \overline{\Lambda}'_{-1} =< e_2, e_3 >,$$

$$\mathcal{L}_{-2} =< e_1, te_2, e_3, te_4 >, \quad \overline{\mathcal{L}}_{-2} =< e_1, e_3 >, \quad \overline{\Lambda}'_{-2} =< e_2, e_4 > .$$

It follows that $\beta^2(\overline{\mathcal{L}}_{-2}) =< e_1 >$ and $\beta^2(\overline{\mathcal{L}}_{-2}) =< e_2 >$. This gives $(\sigma_{02}, \tau_{02}) = (1, 1)$.

(2) When $x = s_0 \tau = \begin{pmatrix} & 1 & & \\ & & & 1 \\ t & & & \\ & & t & \end{pmatrix}$, we compute $x([L_0'],[L_{-1}'],[L_{-2}']) = \begin{pmatrix} te_3 & te_3 & te_3 \\ e_1 & e_1 & e_1 \\ te_4 & te_4 & t^2 e_4 \\ e_2 & te_2 & te_2 \end{pmatrix}$ and obtain

$$\mathcal{L}_0 =< e_1, e_2, te_3, te_4 >, \quad \overline{\mathcal{L}}_0 =< e_1, e_2 >, \quad \overline{\Lambda}'_0 =< e_3, e_4 >,$$
$$\mathcal{L}_{-1} =< e_1, te_2, te_3, e_4 >, \quad \overline{\mathcal{L}}_{-1} =< e_1, e_4 >, \quad \overline{\Lambda}'_{-1} =< e_2, e_3 >,$$
$$\mathcal{L}_{-2} =< e_1, te_2, e_3, te_4 >, \quad \overline{\mathcal{L}}_{-2} =< e_1, e_3 >, \quad \overline{\Lambda}'_{-2} =< e_2, e_4 > .$$

It follows that $\beta^2(\overline{\mathcal{L}}_{-2}) =< e_1 >$ and $\beta^2(\overline{\mathcal{L}}_{-2}) = 0$. This gives $(\sigma_{02}, \tau_{02}) = (1, 2)$.

(3) When $x = s_1 \tau = \begin{pmatrix} & 1 & & \\ & & & 1 \\ t & & & \\ & & t & \end{pmatrix}$, we compute $x([L_0'],[L_{-1}'],[L_{-2}']) = \begin{pmatrix} te_4 & te_4 & te_4 \\ te_3 & te_3 & te_3 \\ e_2 & e_2 & te_2 \\ e_1 & te_1 & te_1 \end{pmatrix}$ and obtain

$$\mathcal{L}_0 =< e_1, e_2, te_3, te_4 >, \quad \overline{\mathcal{L}}_0 =< e_1, e_2 >, \quad \overline{\Lambda}'_0 =< e_3, e_4 >,$$
$$\mathcal{L}_{-1} =< te_1, e_2, te_3, e_4 >, \quad \overline{\mathcal{L}}_{-1} =< e_2, e_4 >, \quad \overline{\Lambda}'_{-1} =< e_1, e_3 >,$$
$$\mathcal{L}_{-2} =< te_1, te_2, e_3, e_4 >, \quad \overline{\mathcal{L}}_{-2} =< e_3, e_4 >, \quad \overline{\Lambda}'_{-2} =< e_1, e_2 > .$$

It follows that $\beta^2(\overline{\mathcal{L}}_{-2}) = 0$ and $\beta^2(\overline{\mathcal{L}}_{-2}) = 0$. This gives $(\sigma_{02}, \tau_{02}) = (2, 2)$.

(4) When $x = s_2 \tau = \begin{pmatrix} & & 1 & \\ t & & & \\ & & & 1 \\ & t & & \end{pmatrix}$, we compute $x([L_0'],[L_{-1}'],[L_{-2}']) = \begin{pmatrix} te_2 & te_2 & te_2 \\ te_4 & te_4 & te_4 \\ e_1 & e_1 & te_1 \\ e_3 & te_3 & te_3 \end{pmatrix}$ and obtain

$$\mathcal{L}_0 =< e_1, te_2, e_3, te_4 >, \quad \overline{\mathcal{L}}_0 =< e_1, e_3 >, \quad \overline{\Lambda}'_0 =< e_2, e_4 >,$$
$$\mathcal{L}_{-1} =< e_1, te_2, te_3, e_4 >, \quad \overline{\mathcal{L}}_{-1} =< e_1, e_4 >, \quad \overline{\Lambda}'_{-1} =< e_2, e_3 >,$$
$$\mathcal{L}_{-2} =< te_1, te_2, e_3, e_4 >, \quad \overline{\mathcal{L}}_{-2} =< e_3, e_4 >, \quad \overline{\Lambda}'_{-2} =< e_1, e_2 > .$$

It follows that $\beta^2(\overline{\mathcal{L}}_{-2}) = 0$ and $\beta^2(\overline{\mathcal{L}}_{-2}) =< e_2 >$. This gives $(\sigma_{02}, \tau_{02}) = (2,1)$.

(5) When $x = \tau = \begin{pmatrix} & & 1 & \\ & & & 1 \\ t & & & \\ & t & & \end{pmatrix}$, we compute $x([L'_0],[L'_{-1}],[L'_{-2}]) = \begin{pmatrix} te_3 & te_3 & te_3 \\ te_4 & te_4 & te_4 \\ e_1 & e_1 & te_1 \\ e_2 & te_2 & te_2 \end{pmatrix}$ and obtain

$$\mathcal{L}_0 =< e_1, e_2, te_3, te_4 >, \quad \overline{\mathcal{L}}_0 =< e_1, e_2 >, \quad \overline{\Lambda}'_0 =< e_3, e_4 >,$$

$$\mathcal{L}_{-1} =< e_1, te_2, te_3, e_4 >, \quad \overline{\mathcal{L}}_{-1} =< e_1, e_4 >, \quad \overline{\Lambda}'_{-1} =< e_2, e_3 >,$$

$$\mathcal{L}_{-2} =< te_1, te_2, e_3, e_4 >, \quad \overline{\mathcal{L}}_{-2} =< e_3, e_4 >, \quad \overline{\Lambda}'_{-2} =< e_1, e_2 > .$$

It follows that $\beta^2(\overline{\mathcal{L}}_{-2}) = 0$ and $\beta^2(\overline{\mathcal{L}}_{-2}) = 0$. This gives $(\sigma_{02}, \tau_{02}) = (2,2)$.

We conclude the result of our computation for characterizing $KR(a)$ when p-rank of a is 0 in the following table:

Table 3.

p-rank(a)	0	0	0	0
$(\sigma_0(a), \tau_0(a))$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$
$(\sigma_1(a), \tau_1(a))$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$
$(\sigma_{02}(a), \tau_{02}(a))$	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
$KR(a)$	$s_0 s_2 \tau$	$s_0 \tau$	$s_2 \tau$	$s_1 \tau,\ \tau$

To distinguish the types $s_1\tau$ and τ, we need to know a global description of the supersingular locus.

8.2. **Description of the supersingular locus.** Let $\mathcal{A}_{2,p,N}$ denote the moduli space of polarized abelian surfaces of degree p^2 with a level-N structure with respect to ζ_N. Let $\Lambda_{2,1,N} \subset \mathcal{A}_{2,1,N} \otimes \overline{\mathbb{F}}_p$ denote the subset of superspecial (geometric) points. Let $\Lambda \subset \mathcal{A}_{2,p,N} \otimes \overline{\mathbb{F}}_p$ be the subset of (geometric) points (A, λ, η) such that $\ker \lambda \simeq \alpha_p \times \alpha_p$. Any member $\underline{A}$ of Λ is superspecial, as A contains $\ker \lambda = \alpha_p \times \alpha_p$. The sets $\Lambda_{2,1,N}$ and Λ are finite, and every member of them is defined over $\overline{\mathbb{F}}_p$.

Recall that $\mathcal{A}_{2,\Gamma_0(p)}$ denotes the reduction modulo p of the Siegel 3-fold with Iwahoric level structure, which parametrizes equivalence classes of objects $(\underline{A}_0 \xrightarrow{\alpha} \underline{A}_1 \xrightarrow{\alpha} \underline{A}_2)$ in characteristic p with conditions as before (Subsection 6.3). Let $\mathcal{S}_{2,\Gamma_0(p)} \subset \mathcal{A}_{2,\Gamma_0(p)}$ denote the supersingular locus, the reduced closed subscheme consisting of supersingular points. Clearly, we have (6.5)

$$\mathcal{S}_{2,\Gamma_0(p)} = \coprod_{x \in \mathrm{Adm}^2(\mu)} \mathcal{A}_{\Gamma_0(p),x} \quad (g = 2). \tag{8.1}$$

For each $\xi = (A_\xi, \lambda_\xi, \eta_\xi) \in \Lambda$, let $W_\xi \subset \mathcal{S}_{2,\Gamma_0(p)}$ be the reduced closed subscheme consisting of points $(\underline{A}_0 \to \underline{A}_1 \to \underline{A}_2)$ such that $\underline{A}_1 \simeq \xi$. For each $\gamma = (A_\gamma, \lambda_\gamma, \eta_\gamma) \in \Lambda_{2,1,N}$, let $U_\gamma \subset \mathcal{S}_{2,\Gamma_0(p)}$ be the locally closed reduced subscheme consisting of points $(\underline{A}_0 \to \underline{A}_1 \to \underline{A}_2)$ such that $\underline{A}_0 \simeq \gamma$ and $\underline{A}_1 \notin \Lambda$. Let $\mathcal{S}_\gamma$ be the Zariski closure of U_γ in $\mathcal{S}_{2,\Gamma_0(p)}$. Clearly, $\mathcal{S}_{\gamma_1} \cap \mathcal{S}_{\gamma_2} = \emptyset$ if $\gamma_1 \neq \gamma_2$ and $W_{\xi_1} \cap W_{\xi_2} = \emptyset$ if $\xi_1 \neq \xi_2$.

Theorem 8.1. *Notation as above.*

(1) *One has*

$$\mathcal{S}_{2,\Gamma_0(p)} = \left(\coprod_{\xi\in\Lambda} W_\xi\right) \cup \left(\coprod_{\gamma\in\Lambda_{2,1,N}} S_\gamma\right)$$

as the union of irreducible components. Consequently, the supersingular locus has $|\Lambda| + |\Lambda_{2,1,N}|$ *irreducible components.*

(2) *For each* $\xi \in \Lambda$, *the subscheme* W_ξ *is isomorphic to* $\mathbf{P}^1 \times \mathbf{P}^1$ *over* $\overline{\mathbb{F}}_p$. *For each* $\gamma \in \Lambda_{2,1,N}$, *the subscheme* S_γ *is isomorphic to* $\mathbf{P}^1$ *over* $\overline{\mathbb{F}}_p$. *Furthermore,* W_ξ *and* S_γ *intersects transversally at most one point. The singularity occurs at the intersection of components* S_γ *with components* W_ξ.

(3) *One has* $|\mathcal{S}^{\mathrm{sing}}_{2,\Gamma_0(p)}| = |\Lambda_{2,1.N}|(p+1)$ *and*

$$\begin{aligned} |\Lambda| &= |\operatorname{Sp}_4(\mathbb{Z}/N\mathbb{Z})| \frac{(-1)\zeta(-1)\zeta(-3)}{4}(p^2-1), \\ |\Lambda_{2,1,N}| &= |\operatorname{Sp}_4(\mathbb{Z}/N\mathbb{Z})| \frac{(-1)\zeta(-1)\zeta(-3)}{4}(p-1)(p^2+1), \end{aligned} \tag{8.2}$$

where $\zeta(s)$ *is the Riemann zeta function.*

The proof will be given in Subsection 8.6.

8.3. We use the classical contravariant Dieudonné theory. We refer the reader to Demazure [4] for a basic account of this theory. For a perfect field k of characteristic p, write $W := W(k)$ for the ring of Witt vectors over k, and $B(k)$ for the fraction field of $W(k)$. Let σ be the Frobenius map on $B(k)$. A quasi-polarization on a Dieudonné module M over k here is a non-degenerate (meaning of non-zero discriminant) alternating pairing

$$\langle\,,\rangle : M \times M \to B(k),$$

such that $\langle Fx, y\rangle = \langle x, Vy\rangle^\sigma$ for $x, y \in M$ and $\langle M^t, M^t\rangle \subset W$. Here the dual M^t of M is regarded as a Dieudonné submodule in $M \otimes B(k)$ using the pairing. A quasi-polarization is called *separable* if $M^t = M$. Any polarized abelian variety (A,λ) over k naturally gives rise to a quasi-polarized Dieudonné module. The induced quasi-polarization is separable if and only if $(p, \deg\lambda) = 1$.

Assume that k is an algebraically closed field field of characteristic p.

Lemma 8.2.

(1) *Let* M *be a separably quasi-polarized superspecial Dieudonné module over* k *of rank* 4. *Then there exists a basis* f_1, f_2, f_3, f_4 *for* M *over* $W := W(k)$ *such that*

$$Ff_1 = f_3, Ff_3 = pf_1, \quad Ff_2 = f_4, Ff_4 = pf_2$$

and the non-zero pairings are

$$\langle f_1, f_3\rangle = -\langle f_3, f_1\rangle = \beta_1, \quad \langle f_2, f_4\rangle = -\langle f_4, f_2\rangle = \beta_1,$$

where $\beta_1 \in W(\mathbb{F}_{p^2})^\times$ *with* $\beta_1^\sigma = -\beta_1$.

(2) *Let* ξ *be a point in* Λ, *and let* M_ξ *be the Dieudonné module of* ξ. *Then there is a* W*-basis* e_1, e_2, e_3, e_4 *for* M_ξ *such that*

$$Fe_1 = e_3, \quad Fe_2 = e_4, \quad Fe_3 = pe_1, \quad Fe_4 = pe_2,$$

and the non-zero pairings are

$$\langle e_1, e_2\rangle = -\langle e_2, e_1\rangle = \frac{1}{p}, \quad \langle e_3, e_4\rangle = -\langle e_4, e_3\rangle = 1.$$

PROOF. This is Lemma 4.2 of [21]. Statement (1) is a special case of Proposition 6.1 of [13], and statement (2) is deduced from that proposition. ∎

Lemma 8.3.

(1) *Let $(M_0, \langle\, ,\rangle_0)$ be a separably quasi-polarized supersingular Dieudonné module of rank 4 and suppose $a(M_0) = 1$. Let $M_1 := (F, V)M_0$ and N be the unique Dieudonné module containing M_0 with $N/M_0 = k$. Let $\langle\, ,\rangle_1 := \frac{1}{p}\langle\, ,\rangle_0$ be the quasi-polarization for M_1. Then one has $a(N) = a(M_1) = 2$, $VN = M_1$, and $M_1/M_1^t \simeq k \oplus k$ as Dieudonné modules.*

(2) *Let $(M_1, \langle\, ,\rangle_1)$ be a quasi-polarized supersingular Dieudonné module of rank 4. Suppose that M_1/M_1^t is of length 2, that is, the quasi-polarization has degree p^2.*

- (i) *If $a(M_1) = 1$, then letting $M_2 := (F, V)M_1$, one has that $a(M_2) = 2$ and $\langle\, ,\rangle_1$ is a separable quasi-polarization on M_2.*
- (ii) *Suppose $(M_1, \langle\, ,\rangle_1)$ decomposes as the product of two quasi-polarized Dieudonné submodules of rank 2. Then there are a unique Dieudonné submodule M_2 of M_1 with $M_1/M_2 = k$ and a unique Dieudonné module M_0 containing M_1 with $M_0/M_1 = k$ so that $\langle\, ,\rangle_1$ (resp. $p\langle\, ,\rangle_1$) is a separable quasi-polarization on M_2 (resp. M_0).*
- (iii) *Suppose $M_1/M_1^t \simeq k \oplus k$ as Dieudonné modules. Let $M_2 \subset M_1$ be any Dieudonné submodule with $M_1/M_2 = k$, and $M_0 \supset M_1$ be any Dieudonné overmodule with $M_0/M_1 = k$. Then $\langle\, ,\rangle_1$ (resp. $p\langle\, ,\rangle_1$) is a separable quasi-polarization on M_2 (resp. M_0).*

This is well-known; the proof is elementary and omitted.

8.4. Let (A_0, λ_0) be a superspecial principally polarized abelian surface and $(M_0, \langle\, ,\rangle_0)$ be the associated Dieudonné module. Let $\varphi : (A_0, \lambda_0) \to (A, \lambda)$ be an isogeny of degree p with $\varphi^*\lambda = p\,\lambda_0$. Write $(M, \langle\, ,\rangle)$ for the Dieudonné module of (A, λ). Choose a basis f_1, f_2, f_3, f_4 for M_0 as in Lemma 8.2. Put $M_2 := (F, V)M_0 = VM_0$. We have the inclusions

$$M_2 \subset M \subset M_0.$$

Modulo M_2, a module M corresponds a one-dimensional subspace M/M_2 in M_0/M_2. As $M_0/M_2 = k < f_1, f_2 >$, the subspace M/M_2 has the form

$$k < af_1 + bf_2 >, \quad [a : b] \in \mathbf{P}^1(k).$$

Let $\overline{M}_0 := M_0/pM_0$, and let

$$\langle\, ,\rangle_0 : \overline{M}_0 \times \overline{M}_0 \to k.$$

be the induced perfect pairing.

Lemma 8.4. *Notation as above, the following conditions are equivalent*

- (a) $\ker\lambda \simeq \alpha_p \times \alpha_p$.
- (b) $\langle\overline{M}, F\overline{M}\rangle_0 = 0$, *where* $\overline{M} := M/pM_0$.
- (c) $\langle\overline{M}, V\overline{M}\rangle_0 = 0$.
- (d) *The corresponding point* $[a : b]$ *satisfies* $a^{p+1} + b^{p+1} = 0$

PROOF. One has $\overline{M} = k < f_1', f_3, f_4 >$ with $f_1' = af_1 + bf_2$. It is easy to see that

$$\langle \overline{M}, F\overline{M} \rangle_0 = 0 \iff \langle f_1', F f_1' \rangle_0 = 0 \iff a^{p+1} + b^{p+1} = 0,$$

and

$$\langle \overline{M}, V\overline{M} \rangle_0 = 0 \iff \langle f_1', V f_1' \rangle_0 = 0 \iff a^{p+1} + b^{p+1} = 0.$$

This shows that the conditions (b), (c) and (d) are equivalent.

Since $\varphi^*\lambda = p\lambda_0$, we have $\langle\, ,\rangle = \frac{1}{p}\langle\, ,\rangle_0$. The Dieudonné module $M(\ker\lambda)$ of the subgroup $\ker\lambda$ is equal to M/M^t. Hence the condition (a) is equivalent to that F and V vanish on $M(\ker\lambda) = M/M^t$. On the other hand, the subspace $\overline{M^t} := M^t/pM_0$ is equal to $\overline{M}^{\perp}$ with respect to $\langle\, ,\rangle_0$. It follows that (a) is equivalent to the conditions (b) and (c). ∎

It follows from Lemma 8.4 that there are $p+1$ isogenies φ so that $\ker\lambda \simeq \alpha_p \times \alpha_p$. Conversely, fix a polarized superspecial abelian surface (A, λ) such that $\ker\lambda \simeq \alpha_p \times \alpha_p$. Then there are p^2+1 degree-p isogenies $\varphi : (A_0, \lambda_0) \to (A, \lambda)$ such that A_0 is superspecial and $\varphi^*\lambda = p\,\lambda_0$. Indeed, each isogeny φ always has the property $\varphi^*\lambda = p\,\lambda_0$ for a principal polarization λ_0 (Lemma 8.3 (iii)), and there are $|\mathbf{P}^1(\mathbb{F}_{p^2})|$ isogenies with A_0 superspecial.

8.5. Let $\mathcal{A}_P$ be the moduli space of isogenies $\alpha : \underline{A}_0 \to \underline{A}_1$ of degree p, where $\underline{A}_0$ is an object in $\mathcal{A}_{2,1,N}$ and $\underline{A}_1$ is an object in $\mathcal{A}_{2,p,N}$ such that $\alpha^*\lambda_1 = p\lambda_0$ and $\alpha_*\eta_0 = \eta_1$. Let $\mathcal{S}_P \subset \mathcal{A}_P \otimes \overline{\mathbb{F}}_p$ be the supersingular locus, the reduced closed subscheme consisting of supersingular points. For each $\xi = (A_\xi, \lambda_\xi, \eta_\xi) \in \Lambda$, let $V_\xi \subset \mathcal{S}_P$ be the closed subvariety consisting of the isogenies $\alpha : \underline{A}_0 \to \underline{A}_1$ such that $\underline{A}_1 = \xi$. For each $\gamma = (A_\gamma, \lambda_\gamma, \eta_\gamma) \in \Lambda_{2,1,N}$, let $S'_\gamma \subset \mathcal{S}_P$ be the closed subvariety consisting of the isogenies $\alpha : \underline{A}_0 \to \underline{A}_1$ such that $\underline{A}_0 = \gamma$.

It is known that the varieties V_ξ and S'_γ are isomorphic to $\mathbf{P}^1$ over $\overline{\mathbb{F}}_p$ (cf. [10]). We also know ([21, Proposition 4.5]) that

$$\mathcal{S}_P = \left(\coprod_{\xi\in\Lambda} V_\xi \right) \cup \left(\coprod_{\gamma\in\Lambda_{2,1,N}} S'_\gamma \right)$$

as the union of irreducible components.

Let $\mathrm{pr} : \mathcal{S}_{2,\Gamma_0(p)} \to \mathcal{S}_P$ be the natural projection.

8.6. **Proof of Theorem 8.1.** (1) It is easy to see that

$$\mathcal{S}_{2,\Gamma_0(p)} = \left(\coprod_{\xi\in\Lambda} W_\xi \right) \coprod \left(\coprod_{\gamma\in\Lambda_{2,1,N}} U_\gamma \right).$$

The statement follows from this.

(2) Clearly we have

$$W_\xi \simeq V_\xi \times V'_\xi \simeq \mathbf{P}^1 \times \mathbf{P}^1 \quad (\text{over } \overline{\mathbb{F}}_p),$$

where V'_ξ is the variety parameterizing isogenies $\alpha : \xi \to \underline{A}_2$ of degree p with $\underline{A}_2$ in $\mathcal{A}_{2,1,N} \otimes \overline{\mathbb{F}}_p$ satisfying $\alpha^*\lambda_2 = \lambda_\xi$ and $\alpha_*\eta_\xi = \eta_2$. This completes the first assertion.

Let $a = (\underline{A}_0 \stackrel{\alpha}{\to} \underline{A}_1 \stackrel{\alpha}{\to} \underline{A}_2)$ be a point in $U_\gamma(k)$. Since $\underline{A}_2$ is determined by $\underline{A}_1$ (cf. Lemma 8.3 (i) (ii)), the projection pr induces an isomorphism

$$\mathrm{pr} : U_\gamma \stackrel{\sim}{\to} \mathrm{pr}(U_\gamma) \subset S'_\gamma.$$

As U_γ is dense in S_γ and S'_γ is proper, $\mathrm{pr}(S_\gamma) \subset S'_\gamma$. Since pr is proper and S'_γ is a smooth curve, the section $s : \mathrm{pr}(U_\gamma) \to U_\gamma$ extends uniquely to a section $s : S'_\gamma \to S_\gamma$. This shows $\mathrm{pr} : S_\gamma \simeq S'_\gamma$, and hence $S_\gamma \simeq \mathbf{P}^1$ over $\overline{\mathbb{F}}_p$.

A component S_γ meets a component W_ξ if and only if S'_γ and $\mathrm{pr}(W_\xi)$ meet. Since S'_γ and V_ξ meet transversally at most one point [21, Proposition 4.5], the components S_γ and W_ξ meet transversally at most one point. Since any irreducible component of $\mathcal{S}_{2,\Gamma_0(p)}$ is smooth, the singularity occurs only at the intersection of components S_γ and W_ξ.

(3) We know that S'_γ contains $p+1$ points with $\underline{A}_1 \in \Lambda$ (Subsection 8.4). Each component S_γ meets $p+1$ components of the form W_ξ, and hence has $p+1$ singular points. This proves the first part.

The result (8.2) is due to Katsura and Oort [10, Theorem 5.1, Theorem 5.3] in a slightly different form. For another proof (using a mass formula due to Ekedahl [6] and some others), see [21, Corollary 3.3, Corollary 4.6]. ∎

8.7. It follows from the description of the supersingular locus that

(i) The closure of the stratum $\mathcal{A}_{\Gamma_0(p),s_0s_2\tau}$ is $\coprod_{\xi\in\Lambda} W_\xi$.

(ii) The stratum $\mathcal{A}_{\Gamma_0(p),s_1\tau}$ is $\coprod_{\gamma\in\Lambda_{2,1,N}} U_\gamma$, as this is the complement of the closure $\overline{\mathcal{A}}_{\Gamma_0(p),s_0s_2\tau}$ in the supersingular locus $\mathcal{S}_{2,\Gamma_0(p)}$.

(iii) The minimal stratum $\mathcal{A}_{\Gamma_0(p),\tau}$ is

$$\left(\coprod_{\xi\in\Lambda} W_\xi\right) \cap \left(\coprod_{\gamma\in\Lambda_{2,1,N}} S_\gamma\right) = \mathcal{S}^{\mathrm{sing}}_{2,\Gamma_0(p)}.$$

We compute the locus $\overline{\mathcal{A}}_{\Gamma_0(p),s_0\tau}$, the closure of the stratum $\mathcal{A}_{\Gamma_0(p),s_0\tau}$. This is the disjoint union of the subvarieties in the component W_ξ, for $\xi \in \Lambda$, defined by the closed condition $\tau_{02} = 2$ (Table 3.).

Let $\underline{A}_1 = \xi \in \Lambda$ and $(M_1, \langle\,,\rangle_1)$ be the Dieudonné module of $\underline{A}_1$. Choose a basis e_1, e_2, e_3, e_4 for M_1 as in Lemma 8.2. Let $M_2 \subset M_1 \subset M_0$ be a chain of Dieudonné modules with $M_0/M_1 \simeq M_1/M_2 \simeq k$. As $M_1/VM_1 = k < e_1, e_2 >$, the subspace M_2/VM_1 has the form

$$k < ae_1 + be_2 >, \quad [a:b] \in \mathbf{P}^1(k).$$

As $V^{-1}M_1/M_1 = k < \frac{1}{p}e_3, \frac{1}{p}e_4 >$, the subspace M_0/M_1 has the form

$$k < c\frac{1}{p}e_3 + d\frac{1}{p}e_4 >, \quad [c:d] \in \mathbf{P}^1(k).$$

Use this as coordinates for W_ξ, we get and fix an isomorphism $\Phi : W_\xi \simeq \mathbf{P}^1 \times \mathbf{P}^1$. Let A, B, C, D be lifts in W of a, b, c, d, respectively. We have

$$M_2 =< Ae_1 + Be_2, pe_1, pe_2, e_3, e_4 >, \quad \text{and} \quad M_0 =< e_1, e_2, e_3, e_4, C\frac{1}{p}e_3 + D\frac{1}{p}e_4 > .$$

The condition $\tau_{02} = 2$ says that in $\overline{M}_0 := M_0/pM_0$, the subspaces $\overline{VM_0}$ and $\overline{M}_2$ generates a two-dimensional subspace. As both have dimension two, the condition means $VM_0 = M_2$. One has

$$VM_0 =< C^{\sigma^{-1}}e_1 + D^{\sigma^{-1}}e_2, pe_1, pe_2, e_3, e_4 > .$$

As both submodules contain VM_1, modulo VM_1, we get

$$< c^{p^{-1}}e_1 + d^{p^{-1}}e_4 >=< ae_1 + be_2 > .$$

This gives the equation $a^p d - b^p c = 0$. We have shown that

$$\Phi(\overline{\mathcal{A}}_{\Gamma_0(p),s_0\tau} \cap W_\xi) = \{ ([a:b],[a^p:b^p]) \in \mathbf{P}^1 \times \mathbf{P}^1 \, ; \, [a:b] \in \mathbf{P}^1 \} \simeq \mathbf{P}^1.$$

This is the graph of the relative Frobenius morphism $F_{\mathbf{P}^1/\mathbb{F}_p} : \mathbf{P}^1 \to \mathbf{P}^1$. We carry out the similar computation and get

$$\Phi(\overline{\mathcal{A}}_{\Gamma_0(p),s_2\tau} \cap W_\xi) = \{ ([c^p:d^p],[c:d]) \in \mathbf{P}^1 \times \mathbf{P}^1 \, ; \, [c:d] \in \mathbf{P}^1 \} \simeq \mathbf{P}^1.$$

This is the transpose of the graph of the relative Frobenius morphism.

We summarize the results as follows.

Proposition 8.5. *We have*

$$\overline{\mathcal{A}}_{\Gamma_0(p),s_0s_2\tau} = \coprod_{\xi\in\Lambda} W_\xi, \quad \overline{\mathcal{A}}_{\Gamma_0(p),s_1\tau} = \coprod_{\gamma\in\Lambda_{2,1,N}} S_\gamma,$$

$$\overline{\mathcal{A}}_{\Gamma_0(p),s_0\tau} \simeq \coprod_{\xi\in\Lambda} \mathbf{P}^1, \quad \overline{\mathcal{A}}_{\Gamma_0(p),s_2\tau} \simeq \coprod_{\xi\in\Lambda} \mathbf{P}^1, \quad \mathcal{A}_{\Gamma_0(p),s_2\tau} = \mathcal{S}^{\mathrm{sing}}_{2,\Gamma_0(p)}.$$

Consequently, we have

(i) *The stratum $\mathcal{A}_{\Gamma_0(p),s_0s_2\tau}$ has $|\Lambda|$ irreducible components.*
(ii) *The stratum $\mathcal{A}_{\Gamma_0(p),s_1\tau}$ has $|\Lambda_{2,1,N}|$ irreducible components.*
(iii) *The stratum $\mathcal{A}_{\Gamma_0(p),s_0\tau}$ has $|\Lambda|$ irreducible components.*
(iv) *The stratum $\mathcal{A}_{\Gamma_0(p),s_2\tau}$ has $|\Lambda|$ irreducible components.*
(v) *The stratum $\mathcal{A}_{\Gamma_0(p),\tau}$ consists of $|\Lambda_{2,1,N}|(p+1)$ points.*

Proposition 8.5, Theorem 4.1, and Proposition 2.1 of [20] answer the question on irreducible components of each Kottwitz-Rapoport stratum in the moduli space $\mathcal{A}_{2,\Gamma_0(p)}$.

We end this paper with the following criterion to distinguish the types $s_1\tau$ and τ. This finishes our geometric characterization of Kottwitz-Rapoport strata for $g = 2$.

Lemma 8.6. *Let $a = (\underline{A}_0 \xrightarrow{\alpha} \underline{A}_1 \xrightarrow{\alpha} \underline{A}_2)$ be a point in $\overline{\mathcal{A}}_{\Gamma_0(p),s_1\tau}(k)$, and let $\overline{M}_2 \xrightarrow{\alpha} \overline{M}_1 \xrightarrow{\alpha} \overline{M}_0$ be the chain of the associated de Rham cohomologies. Let $\omega_i := \omega_{A_i} \subset \overline{M}_i$ be the Hodge subspace. Then the point a lies in $\mathcal{A}_{\Gamma_0(p),\tau}$ if and only if the condition $\langle \alpha(\overline{M}_1), \alpha(\omega_1) \rangle_0 = 0$ holds.*

Proof. It follows from Theorem 8.1 that a lies in $\mathcal{A}_{\Gamma_0(p),\tau}$ if and only if the object $\underline{A}_1$ lies in Λ (Subsection 8.2). The statement then follows from Lemma 8.4, as one has $\omega_1 = V\overline{M}_1$. ∎

Acknowledgments. The author would like to express his appreciation to C.-L. Chai, M.-T. Chuan and J. Tilouine for helpful discussions. Part of the manuscript is prepared during the author's stay at MPIM in Bonn. He wishes to thank the Institute for kind hospitality and excellent working environment.

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