On products in algebraic K-theory

Dominique Arlettaz, Grzegorz Banaszak and Wojciech Gajda

Abstract. This paper investigates the product structure in algebraic K-theory. The first objective is to understand the relationships between products and the kernel of the Hurewicz homomorphism relating the algebraic K-theory of any ring to the integral homology of its linear groups. The second part of the paper is devoted to the ring of integers $\mathbb{Z}$. Using recent results of V. Voevodsky and the 2-adic cyclotomic elements, we determine completely the products in $K_*(\mathbb{Z})$ tensored with the ring of 2-adic integers.

Introduction

The purpose of this paper is to study the product homomorphism

$$\star : K_i(R) \otimes K_k(\mathbb{Z}) \longrightarrow K_{i+k}(R)$$

in the algebraic K-theory of any ring R with identity, for positive integers i and k. Our first goal is to exhibit very strong connections between the image of that product and the kernel of the non-stable Hurewicz homomorphisms relating the K-groups of R to the integral homology groups of its linear groups

$$h_i : K_i(R) = \pi_i BGL(R)^+ \longrightarrow H_i BGL(R)^+ \cong H_i GL(R) \quad \text{for} \quad i \geq 1,$$

respectively $h_i : K_i(R) \to H_i E(R)$ for $i \geq 2$ and $h_i : K_i(R) \longrightarrow H_i St(R)$ for $i \geq 3$, where $GL(R)$ is the infinite general linear group (considered as a discrete group) over R, $E(R)$ its subroup generated by elementary matrices, and $St(R)$ the infinite Steinberg group over R. A universal approximation of the exponent of the kernel and some information on the cokernel of these Hurewicz homomorphisms have been obtained in [2], [3] and Section 5 of [4]. Our argument is based on the understanding, from various viewpoints, of the stable Hurewicz homomorphism between the algebraic K-theory and the homology of the K-theory spectrum. We establish in particular the following result (see Theorem 3.2):

> *For any ring R and any integer $i \geq 2$, the image of $\star : K_i(R) \otimes K_1(\mathbb{Z}) \to K_{i+1}(R)$ is contained in the kernel of the Hurewicz homomorphisms $h_{i+1} : K_{i+1}(R) \to H_{i+1}GL(R)$ and $h_{i+1} : K_{i+1}(R) \to H_{i+1}E(R)$; the same holds for $h_{i+1} : K_{i+1}(R) \to H_{i+1}St(R)$ if $i \geq 3$.*

In low dimensions, we actually prove exactness results for any ring R (see Theorems 4.1 and 4.3).

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(a) *There is an exact sequence*

$$K_4(R) \xrightarrow{h_4} H_4E(R) \longrightarrow \Gamma(K_2(R)) \longrightarrow K_3(R) \xrightarrow{h_3} H_3E(R) \longrightarrow 0\,,$$

where $\Gamma(-)$ *is the quadratic functor defined on abelian groups by J.H.C. Whitehead in Section 5 of* [43]*; moreover,* $\ker h_3$ *is isomorphic to* $K_2(R) \star K_1(\mathbb{Z})$.

(b) *There is an exact sequence*

$$K_5(R) \xrightarrow{h_5} H_5St(R) \longrightarrow K_3(R) \otimes K_1(\mathbb{Z}) \xrightarrow{\star} K_4(R) \xrightarrow{h_4} H_4St(R) \longrightarrow 0$$

and the kernel of h_5 *fits into a short exact sequence*

$$0 \longrightarrow K_4(R) \star K_1(\mathbb{Z}) \longrightarrow \ker h_5 \longrightarrow Q \longrightarrow 0\,,$$

where Q *is a quotient of the subgroup of elements of order* 2 *in the group* $K_3(R)$.

The second objective of the paper is to compute explicitely products in the algebraic K-theory of the ring of integers $\mathbb{Z}$. First of all, we determine in low dimensions the products $K_i(\mathbb{Z}) \star K_k(\mathbb{Z})$, the homology groups of $SL(\mathbb{Z})$ and $St(\mathbb{Z})$, and the Hurewicz homomorphism (see Proposition 5.1). Secondly, we consider maps

$$K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) \xrightarrow{\star} K_{i+k}(\mathbb{Z}) \longrightarrow K_{i+k}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}$$

for all positive integers i and k , where the second arrow is the tensor product of $K_{i+k}(\mathbb{Z})$ with the inclusion of $\mathbb{Z}$ into the ring of 2-adic integers $\hat{\mathbb{Z}_2}$. We call these maps 2-adic products for $K_*(\mathbb{Z})$ and continue to denote them by the symbol $\star$. We are able to calculate all such 2-adic products (see Theorems 5.6, 5.7, Corollary 5.8, Remark 5.9 and Theorem 8.2):

The 2-adic product $\star : K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) \longrightarrow K_{i+k}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}$ *is trivial except if* $i \equiv k \equiv 1$ *mod 8 or* $i \equiv 1$ *mod 8 and* $k \equiv 2$ *mod 8 (or* $i \equiv 2$ *mod 8 and* $k \equiv 1$ *mod 8). If* $i \equiv k \equiv 1$ *mod 8 or* $i \equiv 1$ *mod 8 and* $k \equiv 2$ *mod 8, then its image is cyclic of order 2.*

Almost all these products are deduced from a topological argument based on results by M. Bökstedt [11], V. Voevodsky [37] and C. Weibel [41].

Observe that the 2-adic products in the K-theory of $\mathbb{Z}$ have a very small image. On the other hand, we observe in Theorem 6.4 that the image of the product

$$\star : K_1(E) \otimes K_{2n-1}(E) \longrightarrow K_{2n}(E)$$

is huge when E is a cyclotomic field and n and odd integer.

The last part of the paper is devoted to the construction of a 2-adic version of the cyclotomic elements of C. Soulé (see [34], Sections 4.3 and 4.4, p. 239–240). For any odd integer $n \geq 3$, we define the group of 2-adic cyclotomic elements in the 2-adic K-theory of $\mathbb{Z}$, $\mathcal{C}(n-1) \in K_{2n-1}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}$, and the group of 2-adic cyclotomic elements in étale cohomology, $\mathcal{C}^{\text{ét}}(n-1) \in H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \hat{\mathbb{Z}_2}(n))$, and establish the isomorphism (see Corollary 7.11)

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \hat{\mathbb{Z}_2}(n)) \cong \mathcal{C}^{\text{ét}}(n-1) \oplus \mathbb{Z}/2\,.$$

Since the case of the prime 2 is not often covered in the literature, we include explicit computations with the 2-adic cyclotomic elements. The use of the 2-adic cyclotomic elements enables us to complete our investigation of the 2-adic products in $K_*(\mathbb{Z})$ (see Theorem 8.2). The crucial point of our argument is the proof of the 2-adic Quillen-Lichtenbaum conjecture by V. Voevodsky [37]:

The Dwyer-Friedlander map at the prime 2

$$K_i(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2} \longrightarrow K_i^{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}])$$

is an isomorphism for all integers $i \geq 2$.

We actually derive this result (see Theorem 8.1) from Weibel's computation [41] of the 2-torsion of $K_*(\mathbb{Z})$ and from a result of W. Dwyer and E. Friedlander [18]. We also give another proof of some of the above vanishing results on the 2-adic products for $K_*(\mathbb{Z})$ (see Proposition 8.5).

We finally mention in Proposition 8.7 an interesting relationship between products in algebraic K-theory and the kernel of the Dwyer-Friedlander map.

For any odd prime l and any integer $n \geq 2$, the image of the product map

$$\star : K_{2n-1}(\mathbb{Z}) \otimes K_{2n-1}(\mathbb{Z}) \longrightarrow K_{4n-2}(\mathbb{Z})$$

is contained in the kernel of the Dwyer-Friedlander map $K_{4n-2}(\mathbb{Z}) \to K^{\text{ét}}_{4n-2}(\mathbb{Z}[\frac{1}{l}])$.

The paper is organized as follows. In Section 1, we give a new construction of the Whitehead exact sequence for spectra. Section 2 presents another approach of the study of the Hurewicz homomorphism for spectra using the so-called Postnikov cofibrations. Section 3 is devoted to general results on the relations between products in algebraic K-theory and the kernel of the stable and of the non-stable Hurewicz homomorphism. Section 4 provides the above exact sequences involving the K-groups and the homology groups of the linear groups in dimensions ≤ 5. Section 5 contains the results on the calculation of products in the algebraic K-theory of the ring of integers $\mathbb{Z}$ which follow from the topological argument. Section 6 concerns the product in the K-theory of cyclotomic fields. In Section 7, we construct the various groups of 2-adic cyclotomic elements. We finally discuss in Section 8 the Dwyer-Friedlander map (at the prime 2) relating the algebraic K-theory of $\mathbb{Z}$ to its étale K-theory and compute the last product in $K_*(\mathbb{Z})$.

Throughout the paper, all rings are supposed to have an identity. We write $R^\times$ for the group of units in a ring R. For a prime number l, $\hat{\mathbb{Z}_l}$ denotes the ring of l-adic integers, and G_l the l-torsion subgroup of an abelian group G. We consider all ordinary homology groups with (trivial) coefficients in $\mathbb{Z}$ except if explicitly mentioned. The étale cohomology groups will be denoted by $H^i_{\text{ét}}$ with étale sheaves as coefficients. For an abelian group G, $K(G,s)$ is the Eilenberg-MacLane space having all homotopy groups trivial except for G in dimension s and $H(G)$ the Eilenberg-MacLane spectrum having all homotopy groups trivial except for G in dimension 0. If X is any CW-complex or any CW-spectrum and i any integer, we write $\alpha_i : X \to X[i]$ for its i-th Postnikov section (i.e., $\pi_k X[i] = 0$ for $k > i$ and $(\alpha_i)_* : \pi_k X \overset{\cong}{\to} \pi_k X[i]$ for $k \leq i$) and $\gamma_i : X(i) \to X$ for the fiber of α_i; in other words, $X(i)$ is the i-connected cover of X. For $j \geq i+1$, $X(i,j]$ denotes $X(i)[j]$, whose homotopy groups are $\pi_k X(i,j] = 0$ if $k \leq i$ or $k > j$ and $\pi_k X(i,j] \cong \pi_k X$ if $i+1 \leq k \leq j$.

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1. The Whitehead exact sequence for spectra

Let S be the sphere spectrum and $S \xrightarrow{\alpha_0} S[0] = H(\mathbb{Z})$ its 0-th Postnikov section. By taking the smash product of any spectrum X with the cofibration $S(0) \xrightarrow{\gamma_0} S \xrightarrow{\alpha_0} S[0]$, where $S(0)$ is 0-connected, one obtains the cofibration of spectra

$$X \wedge S(0) \xrightarrow{\mathrm{id}\wedge\gamma_0} X \wedge S \simeq X \xrightarrow{\mathrm{id}\wedge\alpha_0} X \wedge H(\mathbb{Z}),$$

whose homotopy exact sequence is the long Whitehead exact sequence

$$\cdots \longrightarrow \pi_i(X \wedge S(0)) \xrightarrow{\bar{\chi}_i} \pi_i X \xrightarrow{\bar{h}_i} H_i X \xrightarrow{\bar{\nu}_i} \pi_{i-1}(X \wedge S(0)) \longrightarrow \cdots$$

of X; here i is any integer, $\bar{\nu}_i$ is the connecting homomorphism, $\bar{\chi}_i$ is induced by $(\mathrm{id} \wedge \gamma_0)$ and $\bar{h}_i$ by $(\mathrm{id} \wedge \alpha_0)$, i.e., $\bar{h}_i$ is the stable Hurewicz homomorphism. The groups $\pi_i(X \wedge S(0))$ are usually denoted by $\Gamma_i(X)$: that definition coincides actually with the homotopy groups of the fiber of the Dold-Thom map (see [16]) and it was recently proved in [31] that they are isomorphic to the groups introduced in the original paper [43] by J.H.C. Whitehead.

Now, let us assume that the spectrum X is $(r-1)$-connected for some integer r. The advantage of the above approach is that one can compute the groups $\Gamma_i(X)$ with the Atiyah-Hirzebruch spectral sequence for the $S(0)$-homology of X:

$$E^2_{s,t} \cong H_s(X; \pi_t S(0)) \Longrightarrow \Gamma_{s+t}(X).$$

Notice that $E^2_{s,t} = 0$ if $s \leq r-1$ or $t \leq 0$. This implies in particular that $\Gamma_i(X) = 0$ for $i \leq r$ (Hurewicz theorem) and that $(\rho_1\rho_2 \cdots \rho_{i-r})\Gamma_i(X) = 0$ for $i \geq r+1$, where ρ_k denotes the exponent of the homotopy group $\pi_k S$ for $k \geq 1$ (see also [31] for another proof and [4] for corresponding results for the generalized Hurewicz homomorphisms). The first interesting Gamma group of an $(r-1)$-connected spectrum X is

$$\Gamma_{r+1}(X) \cong E^2_{r,1} \cong \pi_r X \otimes \pi_1 S$$

(this was in fact established a long time ago by J.H.C. Whitehead, see for instance Section 14 of [43]). Our first goal is to understand the homomorphism $\bar{\chi}_{r+1} : \Gamma_{r+1}(X) \cong \pi_r X \otimes \pi_1 S \to \pi_{r+1} X$. Let us start with the following general result on the external product $\pi_i X \otimes \pi_k S \xrightarrow{\wedge} \pi_{i+k}(X \wedge S) \cong \pi_{i+k} X$ (see [36], p. 270 for the definition of the external product).

Lemma 1.1. *Let X be any spectrum, i and k two integers with $k \geq 1$. Then the image of the external product $\wedge : \pi_i X \otimes \pi_k S \to \pi_{i+k} X$ is contained in the kernel of the stable Hurewicz homomorphism $\bar{h}_{i+k}$: $\pi_{i+k} X \to H_{i+k} X$ for all integers i and all positive integers k.*

Proof. The commutative diagram

$$\begin{array}{ccc}
\pi_i X \otimes \pi_k S(0) & \underset{\cong}{\xrightarrow{(\mathrm{id})_* \otimes (\gamma_0)_*}} & \pi_i X \otimes \pi_k S \\
\Big\downarrow \wedge & & \Big\downarrow \wedge \\
\Gamma_{i+k}(X) & \xrightarrow{\bar{\chi}_{i+k}} & \pi_{i+k} X
\end{array}$$

shows that the image of $\wedge : \pi_i X \otimes \pi_k S \to \pi_{i+k} X$ is contained in $\mathrm{image}\, \bar{\chi}_{i+k} \cong \ker \bar{h}_{i+k}$. Another proof of this fact is given by Lemma 1 of [5]. □

In the case where X is $(r-1)$-connected and $i = r$, $k = 1$, we have the following exactness result:

Proposition 1.2. *For an $(r-1)$-connected spectrum X, the homomorphism $\bar{\chi}_{r+1} : \Gamma_{r+1}(X) \to \pi_{r+1}X$ in the Whitehead exact sequence is exactly the external product $\wedge : \pi_r X \otimes \pi_1 S \to \pi_{r+1}X$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
H_rX \otimes H_1S(0) & \xrightarrow[\cong]{\wedge} & H_{r+1}(X \wedge S(0)) & & \\
\uparrow \cong & & \uparrow \cong & & \\
\pi_rX \otimes \pi_1S(0) & \xrightarrow{\wedge} & \pi_{r+1}(X \wedge S(0)) & = & \Gamma_{r+1}(X) \\
\cong \downarrow {\scriptstyle (\mathrm{id})_* \otimes (\gamma_0)_*} & & \downarrow {\scriptstyle \bar{\chi}_{r+1}} & & \\
\pi_rX \otimes \pi_1S & \xrightarrow{\wedge} & \pi_{r+1}(X \wedge S) & \cong & \pi_{r+1}X\,.
\end{array}$$

The top horizontal homomorphism is an isomorphism by Künneth formula and the two top vertical arrows, which are Hurewicz homomorphisms are isomorphisms since X is $(r-1)$-connected, $S(0)$ is 0-connected and $X \wedge S(0)$ is r-connected. Consequently, the external product in the middle of the diagram is an isomorphism. The homomorphism $(\mathrm{id})_* \otimes (\gamma_0)_*$ is an isomorphism because $(\gamma_0)_* : \pi_1S(0) \xrightarrow{\cong} \pi_1S$. Therefore, $\bar{\chi}_{r+1}$ is exactly the external product $\pi_rX \otimes \pi_1S \xrightarrow{\wedge} \pi_{r+1}X$. □

Corollary 1.3. *For any $(r-1)$-connected spectrum X, the following sequence is exact:*

$$\cdots \longrightarrow \Gamma_{r+2}(X) \xrightarrow{\bar{\chi}_{r+2}} \pi_{r+2}X \xrightarrow{\bar{h}_{r+2}} H_{r+2}X \xrightarrow{\bar{\nu}_{r+2}} \pi_rX \otimes \pi_1S \xrightarrow{\wedge} \pi_{r+1}X \xrightarrow{\bar{h}_{r+1}} H_{r+1}X \longrightarrow 0\,.$$

2. Postnikov cofibrations

The purpose of this section is to present another approach of the study of the Hurewicz homomorphism. For an $(r-1)$-connected spectrum X, consider for all integers $i \geq r+1$ the cofibrations of spectra

$$\Sigma^i H(\pi_iX) \xrightarrow{\gamma_{i-1}} X[i] \xrightarrow{\alpha_{i-1}} X[i-1]\,,$$

where α_{i-1} is the $(i-1)$-st Postnikov section of $X[i]$: let us call them the Postnikov cofibrations of X. The associated homology exact sequences are

$$\cdots \longrightarrow H_{i+1}X[i] \xrightarrow{(\alpha_{i-1})_*} H_{i+1}X[i-1] \xrightarrow{\bar{\partial}} \underbrace{H_i(\Sigma^iH(\pi_iX))}_{\cong \pi_iX} \xrightarrow{(\gamma_{i-1})_*} \underbrace{H_iX[i]}_{\cong H_iX} \xrightarrow{(\alpha_{i-1})_*} H_iX[i-1] \longrightarrow 0\,,$$

and it is easy to check that $(\gamma_{i-1})_*$ is the stable Hurewicz homomorphism $\bar{h}_i$. Thus, we obtain the following

Proposition 2.1. *Let X be an $(r-1)$-connected spectrum and i an integer $\geq r+1$. There is an exact sequence*

$$\cdots \longrightarrow H_{i+1}X[i] \xrightarrow{(\alpha_{i-1})_*} H_{i+1}X[i-1] \xrightarrow{\bar{\partial}} \pi_iX \xrightarrow{\bar{h}_i} H_iX \xrightarrow{(\alpha_{i-1})_*} H_iX[i-1] \longrightarrow 0\,.$$

Now let us try to understand the homomorphism $\bar{\partial}$ for the cases $i = r+1$ and $i = r+2$.

Proposition 2.2. *(a) For any $(r-1)$-connected spectrum X, there is an exact sequence*

$$0 \longrightarrow H_{r+2}X[r+1] \xrightarrow{\bar{\varphi}} \pi_rX \otimes \pi_1S \xrightarrow{\bar{\partial}} \pi_{r+1}X \xrightarrow{\bar{h}_{r+1}} H_{r+1}X \longrightarrow 0\,.$$

(b) The homomorphism $\bar{\partial}$ is again exactly the external product $\wedge : \pi_rX \otimes \pi_1S \to \pi_{r+1}X$.

Proof. Let us look at the Postnikov cofibration of X for $i = r+1$,

$$\Sigma^{r+1} H(\pi_{r+1}X) \xrightarrow{\gamma_r} X[r+1] \xrightarrow{\alpha_r} X[r] \simeq \Sigma^r H(\pi_r X)\,,$$

and take its homology exact sequence

$$\underbrace{H_{r+2}(\Sigma^{r+1}H(\pi_{r+1}X))}_{=0} \longrightarrow H_{r+2}X[r+1] \xrightarrow{\bar{\varphi}} H_{r+2}(\Sigma^r H(\pi_r X)) \xrightarrow{\bar{\partial}}$$

$$\pi_{r+1}X \xrightarrow{\bar{h}_{r+1}} H_{r+1}X \longrightarrow \underbrace{H_{r+1}(\Sigma^r H(\pi_r X))}_{=0}\,,$$

where $\bar{\varphi}$ is written for $(\alpha_r)_*$. Since $\Sigma^r H(\pi_r X)$ is an Eilenberg-MacLane spectrum, it is clear that

$$H_{r+2}(\Sigma^r H(\pi_r X)) \cong \Gamma_{r+1}(\Sigma^r H(\pi_r X)) \cong \pi_r X \otimes \pi_1 S\,,$$

and we get assertion (a). Then, consider the map $\alpha_0 : S \to H(\mathbb{Z})$ and denote by ζ the composition

$$(\mathrm{id} \wedge \alpha_0)\,\gamma_r : \Sigma^{r+1}H(\pi_{r+1}X) \longrightarrow X[r+1] \simeq X[r+1] \wedge S \longrightarrow X[r+1] \wedge H(\mathbb{Z})\,,$$

and by F its fiber. By smashing with $H(\mathbb{Z})$ the cofibration obtained by looping the base spectrum of the cofibration $\Sigma^{r+1}H(\pi_{r+1}X) \xrightarrow{\gamma_r} X[r+1] \xrightarrow{\alpha_r} \Sigma^r H(\pi_r X)$, we get the commutative diagram

$$\begin{tikzcd}
\Sigma^{r-1}H(\pi_r X) \wedge H(\mathbb{Z}) \arrow[r] & \Sigma^{r+1}H(\pi_{r+1}X) \wedge H(\mathbb{Z}) \arrow[r, "\gamma_r \wedge \mathrm{id}"] & X[r+1] \wedge H(\mathbb{Z}) \\
F \arrow[u] \arrow[r] \arrow[d] & \Sigma^{r+1}H(\pi_{r+1}X) \arrow[u, "\mathrm{id}\wedge\alpha_0"'] \arrow[r, "\zeta"] \arrow[d, "\gamma_r"'] & X[r+1] \wedge H(\mathbb{Z}) \arrow[u, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"'] \\
X[r+1] \wedge S(0) \arrow[r, "\mathrm{id}\wedge\gamma_0"] & X[r+1] \arrow[r, "\mathrm{id}\wedge\alpha_0"] & X[r+1] \wedge H(\mathbb{Z})
\end{tikzcd}$$

in which all rows are cofibrations. Then look at their homotopy exact sequences

$$\begin{tikzcd}
H_{r+2}X[r+1] \arrow[r] & \overbrace{H_{r+1}(\Sigma^{r-1}H(\pi_r X))}^{\cong H_{r+2}(\Sigma^r H(\pi_r X))} \arrow[r, "\bar{\partial}"] & \overbrace{H_{r+1}(\Sigma^{r+1}H(\pi_{r+1}X))}^{\cong \pi_{r+1}X} \arrow[r, "(\gamma_r\wedge\mathrm{id})_*", "=\bar{h}_{r+1}"'] & \overbrace{H_{r+1}X[r+1]}^{\cong H_{r+1}X} \\
H_{r+2}X[r+1] \arrow[u, "="'] \arrow[r] \arrow[d, "="'] & \pi_{r+1}F \arrow[u] \arrow[r] \arrow[d] & \pi_{r+1}X \arrow[u, "\mathrm{Hurewicz}", "\cong"'] \arrow[r, "\zeta_*"] \arrow[d, "(\gamma_r)_*", "\cong"'] & H_{r+1}X[r+1] \arrow[u, "="'] \arrow[d, "="'] \\
H_{r+2}X[r+1] \arrow[r] & \underbrace{\Gamma_{r+1}(X[r+1])}_{\cong \Gamma_{r+1}(X)} \arrow[r, "\bar{\chi}_{r+1}"] & \underbrace{\pi_{r+1}X[r+1]}_{\cong \pi_{r+1}X} \arrow[r, "\bar{h}_{r+1}"] & \underbrace{H_{r+1}X[r+1]}_{\cong H_{r+1}X}\,.
\end{tikzcd}$$

Observe that the three horizontal arrows on the left of the diagram are injective and conclude by the five lemma that the two vertical arrows starting from $\pi_{r+1}F$ are isomorphisms: assertion (b) can then be deduced from Proposition 1.2. $\square$

Remark 2.3. It follows from Corollary 1.3 and Proposition 2.2 that the cokernel of $\bar{h}_{r+2} : \pi_{r+2}X \to H_{r+2}X$ is isomorphic to $\mathrm{image}\,\bar{\nu}_{r+2} = \ker(\wedge : \pi_r X \otimes \pi_1 S \to \pi_{r+1}X) \cong H_{r+2}X[r+1]$ for any $(r-1)$-connected spectrum X.

Similarly, we can investigate the stable Hurewicz homomorphism in dimension $r+2$. Consider the Postnikov cofibration of an $(r-1)$-connected spectrum X for $i = r+2$ and its homology exact sequence

$$\cdots \longrightarrow H_{r+3}X[r+1] \xrightarrow{\bar{\psi}} \underbrace{H_{r+2}(\Sigma^{r+2}H(\ pi_{r+2}X))}_{\cong \pi_{r+2}X} \xrightarrow{\bar{h}_{r+2}} \underbrace{H_{r+2}X[r+2]}_{\cong H_{r+2}X} \longrightarrow H_{r+2}X[r+1] \longrightarrow 0\,,$$

where $\bar{\psi}$ is written for $\bar{\partial}$. The next two lemmas describe the group $H_{r+3}X[r+1]$ and the homomorphism $\bar{\psi}$.

Lemma 2.4. *There is an exact sequence*

$$\cdots \longrightarrow \pi_{r+1}X \otimes \pi_1 S \xrightarrow{\bar{\theta}} H_{r+3}X[r+1] \xrightarrow{\bar{\eta}} {}_2(\pi_r X) \longrightarrow 0\,,$$

where ${}_2(\pi_r X)$ *denotes the subgroup of elements of order dividing* 2 *in the group* $\pi_r X$.

Proof. Let us look again at the cofibration

$$\Sigma^{r+1}H(\pi_{r+1}X) \xrightarrow{\gamma_r} X[r+1] \xrightarrow{\alpha_r} \Sigma^r H(\pi_r X)$$

and at its homology exact sequence

$$\cdots \longrightarrow H_{r+3}(\Sigma^{r+1}H(\pi_{r+1}X)) \xrightarrow{\bar{\theta}} H_{r+3}X[r+1] \xrightarrow{\bar{\eta}} H_{r+3}(\Sigma^r H(\pi_r X)) \longrightarrow 0\,,$$

where $\bar{\theta}$ and $\bar{\eta}$ are the homomorphisms induced by γ_r and α_r respectively. It turns out that $H_{r+3}(\Sigma^{r+1}H(\pi_{r+1}X)) \cong \Gamma_{r+2}(\Sigma^{r+1}H(\pi_{r+1}X)) \cong \pi_{r+1}X \otimes \pi_1 S$ because of the results of Section 1 and that $H_{r+3}(\Sigma^r H(\pi_r X)) \cong {}_2(\pi_r X)$, according to Théorème 2 of [14]. $\square$

Lemma 2.5. *The composition* $\bar{\psi}\,\bar{\theta} : \pi_{r+1}X \otimes \pi_1 S \to \pi_{r+2}X$ *is the external product* $\wedge$.

Proof. The obvious map $X(r) \to X$ provides the commutative diagram of cofibrations

$$\begin{array}{ccccc}
\Sigma^{r+2}H(\pi_{r+2}X) & \longrightarrow & X(r, r+2] & \longrightarrow & X(r, r+1] \;\simeq\; \Sigma^{r+1}H(\pi_{r+1}X) \\
\big\downarrow \text{id} & & \big\downarrow & & \big\downarrow \gamma_r \\
\Sigma^{r+2}H(\pi_{r+2}X) & \longrightarrow & X[r+2] & \longrightarrow & X[r+1]
\end{array}$$

which induces the commutative square

$$\begin{array}{ccc}
\underbrace{H_{r+3}(\Sigma^{r+1}H(\pi_{r+1}X))}_{\cong \pi_{r+1}X \otimes \pi_1 S} & \longrightarrow & \underbrace{H_{r+2}(\Sigma^{r+2}H(\pi_{r+2}X))}_{\cong \pi_{r+2}X} \\
\big\downarrow \bar{\theta} & & \big\downarrow = \\
H_{r+3}X[r+1] & \xrightarrow{\bar{\psi}} & \underbrace{H_{r+2}(\Sigma^{r+2}H(\pi_{r+2}X))}_{\cong \pi_{r+2}X}\,.
\end{array}$$

Then, the statement of Proposition 2.2 for the r-connected spectrum $X(r)$ shows that the top horizontal arrow is the external product. $\square$

We may summarize our results on the stable Hurewicz homomorphism $\bar{h}_{r+2}$ as follows.

Proposition 2.6. *Let X be an $(r-1)$-connected spectrum.*
(a) There is an exact sequence

$$\cdots \longrightarrow H_{r+3}X[r+1] \xrightarrow{\bar{\psi}} \pi_{r+2}X \xrightarrow{\bar{h}_{r+2}} H_{r+2}X \longrightarrow H_{r+2}X[r+1] \longrightarrow 0\,.$$

(b) The kernel of $\bar{h}_{r+2}$ fits into the short exact sequence

$$0 \longrightarrow \wedge(\pi_{r+1}X \otimes \pi_1 S) \longrightarrow \ker \bar{h}_{r+2} \longrightarrow Q \longrightarrow 0\,,$$

where Q is a quotient of ${}_2(\pi_r X)$.

Remark 2.7. Since $\pi_i X[r+1] = 0$ for $i \geq r+2$, $H_{r+3}X[r+1] \cong \Gamma_{r+2}(X[r+1])$. The Postnikov section $X \to X[r+1]$ induces a map f between the Atiyah-Hirzebruch spectral sequences

$$H_s(X; \pi_t S(0)) \Longrightarrow \Gamma_{s+t}(X) \quad \text{and} \quad H_s(X[r+1]; \pi_t S(0)) \Longrightarrow \Gamma_{s+t}(X[r+1])\,.$$

The lines $s+t = r+2$ in these spectral sequences give the following picture:

$$\begin{tikzcd}
H_{r+2}(X;\pi_1 S) \arrow[r, "d^2"] \arrow[d, "f_1"] & \pi_r X \otimes \pi_2 S \arrow[r] \arrow[d, "f_2", "\cong"'] & \Gamma_{r+2}(X) \arrow[r] \arrow[d, "f_3"] & H_{r+1}(X;\pi_1 S) \arrow[r] \arrow[d, "f_4", "\cong"'] & 0 \\
H_{r+2}(X[r+1];\pi_1 S) \arrow[r, "d^2"] & \pi_r X[r+1] \otimes \pi_2 S \arrow[r] & \Gamma_{r+2}(X[r+1]) \arrow[r] & H_{r+1}(X[r+1];\pi_1 S) \arrow[r] & 0\,.
\end{tikzcd}$$

By the universal coefficient theorem, one has $H_{r+2}(X;\pi_1 S) \cong (H_{r+2}X \otimes \pi_1 S) \oplus \mathrm{Tor}(H_{r+1}X, \pi_1 S)$ and $H_{r+2}(X[r+1];\pi_1 S) \cong (H_{r+2}(X[r+1]) \otimes \pi_1 S) \oplus \mathrm{Tor}(H_{r+1}X, \pi_1 S)$; thus, one can check that f_1 is surjective because of Whitehead's theorem and deduce from the five lemma that

$$\Gamma_{r+2}(X) \cong \Gamma_{r+2}(X[r+1]) \cong H_{r+3}X[r+1]\,.$$

Moreover, one can show with the argument of the proof of Proposition 2.2 (b) that the homomorphism $\bar{\psi}$ of Proposition 2.6 (a) is actually $\bar{\chi}_{r+2} : \Gamma_{r+2}(X) \to \pi_{r+2}X$ of Corollary 1.3. Consequently, the part

$$H_{r+3}X[r+1] \xrightarrow{\bar{\psi}} \pi_{r+2}X \xrightarrow{\bar{h}_{r+2}} H_{r+2}X$$

of the sequence given by Proposition 2.6 (a) is a piece of the Whitehead exact sequence.

Remark 2.8. It follows from Lemma 2.4 and the previous remark that the group $\Gamma_{r+2}(X)$ is described by the exact sequence

$$\cdots \longrightarrow \pi_{r+1}X \otimes \pi_1 S \xrightarrow{\bar{\theta}} \Gamma_{r+2}(X) \xrightarrow{\bar{\eta}} {}_2(\pi_r X) \longrightarrow 0\,,$$

and in particular that its exponent divides 4 (this was already known by [10], Section 4).

Remark 2.9. All exact sequences introduced in Sections 1 and 2 are obviously natural in X.

3. Products and Hurewicz homomorphisms in algebraic K-theory

If R is any ring, let us denote by X_R the connective K-theory spectrum of R, i.e., a (-1)-connected Ω-spectrum whose 0-th space is the infinite loop space $BGL(R)^+ \times K_0(R)$. We shall also consider the $(r-1)$-connected spectra $X_R(r-1)$ for $r \geq 0$, i.e., the fiber of the Postnikov section $X_R \to X_R[r-1]$, and call γ_{r-1} the obvious map $X_R(r-1) \to X_R$. Observe that $K_i(R) \cong \pi_i X_R(r-1)$ for $i \geq r$. Remember that the infinite loop spaces corresponding to $X_R(0)$, $X_R(1)$ and $X_R(2)$ are $BGL(R)^+$, $BE(R)^+$ and $BSt(R)^+$ respectively. If R and R' are two rings, there is a pairing $\mu : X_R \wedge X_{R'} \to X_{R\otimes R'}$ and the product in algebraic K-theory is defined as follows

$$\star : K_i(R) \otimes K_k(R') \cong \pi_i X_R \otimes \pi_k X'_R \xrightarrow{\wedge} \pi_{i+k}(X_R \wedge X_{R'}) \xrightarrow{\mu_*} \pi_{i+k} X_{R\otimes R'} \cong K_{i+k}(R \otimes R')$$

for any two integers $i \geq 0$ and $k \geq 0$ (see for instance [22], Proposition 2.4.2). We shall actually concentrate our attention to the special case where R' is the ring of integers $\mathbb{Z}$: the goal of Sections 3 and 4 is to investigate the relationships between the image of the product

$$\star : K_i(R) \otimes K_k(\mathbb{Z}) \longrightarrow K_{i+k}(R \otimes \mathbb{Z}) \cong K_{i+k}(R)$$

and the kernel of the stable and the non-stable Hurewicz homomorphism.

Remember that $X_{\mathbb{Z}}$ is a ring spectrum and let us call $j : S \to X_{\mathbb{Z}}$ its identity. Notice that j corresponds to the map $B\Sigma_\infty^+ \to BGL(\mathbb{Z})^+$ given by the inclusion of the infinite symmetric group Σ_∞ into $GL(\mathbb{Z})$. This map j induces an isomorphism $j_* : \pi_1 S \xrightarrow{\cong} \pi_1 X_{\mathbb{Z}} \cong K_1(\mathbb{Z})$ and the image of $j_* : \pi_k S \to K_k(\mathbb{Z})$ for $k \geq 2$ is described in [25] and [29]. For any ring R, the above pairing μ provides then X_R with an $X_{\mathbb{Z}}$-module structure. Let us first translate the results of Sections 1 and 2 in terms of algebraic K-theory.

Proposition 3.1. *Let R be a ring, i and k two integers with $i \geq 0$, $k \geq 1$, and consider an element $x \in K_i(R)$ and an element $y \in K_k(\mathbb{Z})$ belonging to the image of $j_* : \pi_k S \to K_k(\mathbb{Z})$.*

(a) For all $r \leq i$, $x \star y$ is an element of the kernel of the stable Hurewicz homomorphism $\bar{h}_{i+k} : K_{i+k}(R) \to H_{i+k} X_R(r-1)$.

(b) If $k \leq i-1$, then $x \star y$ is an element of the kernel of the non-stable Hurewicz homomorphisms $h_{i+k} : K_{i+k}(R) \to H_{i+k} E(R)$ and $h_{i+k} : K_{i+k}(R) \to H_{i+k} GL(R)$.

(c) If $i \geq 3$ and $k \leq i-1$, then $x \star y$ is an element of the kernel of $h_{i+k} : K_{i+k}(R) \to H_{i+k} St(R)$.

Proof. The first assertion is a consequence of Lemma 1.1 and the commutativity of the diagram

$$\begin{tikzcd}
\pi_i X_R(r-1) \otimes \pi_k S \arrow[r, "\wedge"] \arrow[d, "\cong"', "(\gamma_{r-1})_* \otimes \mathrm{id}"] & \pi_{i+k} X_R(r-1) \arrow[r, "\bar{h}_{i+k}"] \arrow[d, "\cong"', "(\gamma_{r-1})_*"] & H_{i+k} X_R(r-1) \\
\pi_i X_R \otimes \pi_k S \arrow[r, "\wedge"] \arrow[d, "\cong \otimes j_*"] & \pi_{i+k} X_R \arrow[d, "\cong"] & \\
K_i(R) \otimes K_k(\mathbb{Z}) \arrow[r, "\star"] & K_{i+k}(R), &
\end{tikzcd}$$

where the bottom square commutes because X_R is an $X_{\mathbb{Z}}$-module. In order to prove the last two assertions, consider the $(i-1)$-connected cover $BGL(R)^+(i-1)$ of the CW-complex $BGL(R)^+$, for $i \geq k+1 \geq 2$. The iterated homology suspension $\sigma : H_{i+k} BGL(R)^+(i-1) \to H_{i+k} X_R(i-1)$, which is an isomorphism since $k \leq i-1$, and the commutative diagram

$$\begin{tikzcd}
K_{i+k}(R) \arrow[r, "h_{i+k}"] \arrow[d, "="] & H_{i+k} BGL(R)^+(i-1) \arrow[d, "\sigma"', "\cong"] \\
K_{i+k}(R) \arrow[r, "\bar{h}_{i+k}"] & H_{i+k} X_R(i-1)
\end{tikzcd}$$

show that $h_{i+k} : K_{i+k}(R) \to H_{i+k}BGL(R)^+(i-1)$ fulfills $h_{i+k}(x \star y) = 0$ according to (a) for $r = i$. Since $i \geq 2$, assertion (b) then follows from the composition with the obvious homomorphism

$$H_{i+k}BGL(R)^+(i-1) \longrightarrow H_{i+k}BE(R)^+ \longrightarrow H_{i+k}BGL(R)^+ .$$

If $i \geq 3$, this homomorphism factors even through $H_{i+k}BSt(R)^+$ and we get (c). □

Now, let us consider the case $k = 1$ and $i = r$: the fact that $j_* : \pi_1 S \to K_1(\mathbb{Z})$ is an isomorphism implies the following result, where $X_R(i-1, i+1]$ is written for $X_R(i-1)[i+1]$.

Theorem 3.2. *Let R be any ring.*
(a) For any integer $i \geq 0$, there is a natural exact sequence

$$K_{i+2}(R) \xrightarrow{\bar{h}_{i+2}} H_{i+2}X_R(i-1) \xrightarrow{\bar{\nu}_{i+2}} K_i(R) \otimes K_1(\mathbb{Z}) \xrightarrow{\star} K_{i+1}(R) \xrightarrow{\bar{h}_{i+1}} H_{i+1}X_R(i-1) \longrightarrow 0 .$$

Moreover, $\ker(\star) = \operatorname{image} \bar{\nu}_{i+2} \cong H_{i+2}X_R(i-1, i+1]$.
(b) For any integer $i \geq 2$, the image of $\star : K_i(R) \otimes K_1(\mathbb{Z}) \to K_{i+1}(R)$ is contained in the kernel of the non-stable Hurewicz homomorphisms $h_{i+1} : K_{i+1}(R) \to H_{i+1}E(R)$ and $h_{i+1} : K_{i+1}(R) \to H_{i+1}GL(R)$.
(c) For any integer $i \geq 3$, the image of $\star : K_i(R) \otimes K_1(\mathbb{Z}) \to K_{i+1}(R)$ is contained in the kernel of $h_{i+1} : K_{i+1}(R) \to H_{i+1}St(R)$.

Proof. Assertion (a) follows from Corollary 1.3 and Remark 2.3 for the spectrum $X_R(i-1)$ since the diagram

$$\begin{array}{ccc}
K_i(R) \otimes \pi_1 S & \xrightarrow{\wedge} & \pi_{i+1} X_R \\
\cong \big\downarrow \mathrm{id} \otimes j_* & & \big\downarrow \cong \\
K_i(R) \otimes K_1(\mathbb{Z}) & \xrightarrow{\star} & K_{i+1}(R)
\end{array}$$

commutes again because of the $X_{\mathbb{Z}}$-module structure of X_R. Assertions (b) and (c) are direct consequences of Proposition 3.1. (b) and (c). □

It is possible to obtain a similar information on the stable and the non-stable Hurewicz homomorphism in any dimension $i \geq r+1$. Proposition 2.1 provides the exact sequence

$$\cdots \longrightarrow H_{i+1}X_R(r-1, i] \xrightarrow{(\alpha_{i-1})_*} H_{i+1}X_R(r-1, i-1] \xrightarrow{\bar{\partial}}$$
$$K_i(R) \xrightarrow{\bar{h}_i} H_iX_R(r-1) \xrightarrow{(\alpha_{i-1})_*} H_iX_R(r-1, i-1] \longrightarrow 0 .$$

Proposition 3.3. *Let R be any ring, i and r positive integers such that $r \leq i \leq 2r-1$, then the kernel of the non-stable Hurewicz homomorphism*

$$h_i : K_i(R) \to H_iBGL(R)^+(r-1)$$

is exactly the image of $\bar{\partial}$.

Proof. Let us consider the homology exact sequence of the Postnikov cofibration

$$\Sigma^i(H(K_i(R))) \xrightarrow{\gamma_{i-1}} X_R(r-1, i] \xrightarrow{\alpha_{i-1}} X_R(r-1, i-1] ,$$

and the corresponding homology exact sequence obtained from the Serre spectral sequence of the fibration of CW-complexes

$$K(K_i(R), i) \longrightarrow BGL(R)^+(r-1, i] \longrightarrow BGL(R)^+(r-1, i-1] .$$

We obtain the commutative diagram

$$\begin{array}{ccccc}
H_{i+1}X_R(r-1,i-1] & \xrightarrow{\bar\partial} & K_i(R) & \xrightarrow{\bar h_i} & H_iX_R(r-1) \\
\uparrow \sigma & & \uparrow = & & \uparrow \\
H_{i+1}BGL(R)^+(r-1,i-1] & \xrightarrow{\partial} & K_i(R) & \xrightarrow{h_i} & H_iBGL(R)^+(r-1)\,,
\end{array}$$

where the horizontal sequences are exact and the three vertical arrows are iterated suspensions. The left iterated homology suspension σ is surjective if $i+1\le 2r$ and even an isomorphism if $i+1\le 2r-1$ (see [42], p. 382); consequently we may conclude that $\operatorname{image}\partial = \operatorname{image}\bar\partial$. $\square$

4. Products and the non-stable Hurewicz homomorphism in low dimensions

The purpose of this section is to study the relationships between the algebraic K-theory of a ring R and the integral homology of its linear groups in low dimensions. In dimension 2, the following isomorphisms are known (see [3]):

$$K_2(R)\cong H_2E(R) \quad \text{and} \quad H_2GL(R)\cong K_2(R)\oplus \Lambda^2(K_1(R))\,.$$

Let us start by looking at dimensions 3 and 4. Let $\Gamma(-)$ be the quadratic functor defined on abelian groups by J.H.C. Whitehead in Section 5 of [43]: if Y is a simply connected CW-complex, then $\Gamma_3(Y)$ turns out to be isomorphic to $\Gamma(\pi_2Y)$.

Theorem 4.1. *For any ring R, there is a natural exact sequence*

$$K_4(R)\xrightarrow{h_4} H_4E(R)\xrightarrow{\nu_4}\Gamma(K_2(R))\xrightarrow{\chi_3} K_3(R)\xrightarrow{h_3} H_3E(R)\longrightarrow 0$$

and $\ker h_3$ is isomorphic to the image of the product homomorphism $\star : K_2(R)\otimes K_1(\mathbb{Z})\to K_3(R)$. In particular, $H_3E(R)\cong K_3(R)/(K_2(R)\star K_1(\mathbb{Z}))$.

Proof. Consider the exact sequence given by Proposition 2.2 for $X = X_R(1)$ and $r=2$, and also the corresponding exact sequence obtained from the Serre spectral sequence of the fibration of CW-complexes

$$K(K_3(R),3)\longrightarrow BE(R)^+[3]\longrightarrow K(K_2(R),2)\,.$$

We get the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H_4X_R(1,3] & \xrightarrow{\bar\varphi} & \overbrace{H_4(\Sigma^2H(K_2(R)))}^{K_2(R)\otimes K_1(\mathbb{Z})} & \xrightarrow{\star} & K_3(R) & \xrightarrow{\bar h_3} & H_3X_R(1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow \sigma & & \uparrow = & & \uparrow & & \\
0 & \longrightarrow & H_4BE(R)^+[3] & \xrightarrow{\varphi} & H_4(K(K_2(R),2)) & \xrightarrow{\partial} & K_3(R) & \xrightarrow{h_3} & H_3BE(R)^+ & \longrightarrow & 0\,,
\end{array}$$

where the vertical arrows are iterated suspensions. According to the proof of Proposition 3.3, the iterated homology suspension σ is surjective and thus $\operatorname{image}\partial = K_2(R)\star K_1(\mathbb{Z})$. We may also look at the Whitehead exact sequence (see [43]) of the space $BE(R)^+$:

$$\cdots \longrightarrow \Gamma_4(BE(R)^+) \xrightarrow{\chi_4} K_4(R) \xrightarrow{h_4} H_4E(R) \xrightarrow{\nu_4} \Gamma_3(BE(R)^+) \xrightarrow{\chi_3} K_3(R) \xrightarrow{h_3} H_3E(R) \longrightarrow 0\,.$$

It turns out that $H_4(K(K_2(R),2)) \cong \Gamma_3(K(K_2(R),2)) \cong \Gamma(K_2(R)) \cong \Gamma_3(BE(R)^+)$ and the argument of the proof of Proposition 2.2 shows again that one can replace a piece of that Whitehead exact sequence by the bottom sequence of the above diagram. This proves the assertion. $\square$

Remark 4.2. This extends the result for fields given in [35], Corollary 5.2, to the case of any ring R.

In order to understand the 4-dimensional and 5-dimensional Hurewicz homomorphisms, let us use exactly the same idea (but now for $r=3$) for the 2-connected CW-complex $BSt(R)^+$, respectively the 2-connected K-theory spectrum $X_R(2)$.

Theorem 4.3. *Let R be any ring.*

(a) There is a natural exact sequence

$$H_6X_R[2,4] \xrightarrow{\bar\psi} K_5(R) \xrightarrow{h_5} H_5St(R) \xrightarrow{\nu_5} K_3(R)\otimes K_1(\mathbb{Z}) \xrightarrow{\star} K_4(R) \xrightarrow{h_4} H_4St(R) \longrightarrow 0\,.$$

In particular, $H_4St(R) \cong K_4(R)/(K_3(R)\star K_1(\mathbb{Z}))$.

(b) There is a natural exact sequence

$$K_4(R)\otimes K_1(\mathbb{Z}) \xrightarrow{\bar\theta} H_6X_R[2,4] \xrightarrow{\bar\eta} {}_2(K_3(R)) \longrightarrow 0\,.$$

(c) The composition $\bar\psi\,\bar\theta$ is the product map $\star : K_4(R)\otimes K_1(\mathbb{Z}) \to K_5(R)$. Consequently, there is a natural short exact sequence

$$0 \longrightarrow K_4(R)\star K_1(\mathbb{Z}) \longrightarrow \ker h_5 \longrightarrow Q \longrightarrow 0\,,$$

where Q is a quotient of ${}_2(K_3(R))$.

Proof. As in the previous proof, we use Proposition 2.2, but consider in this case the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H_5X_R[2,4] & \xrightarrow{\bar\varphi} & \overbrace{H_5(\Sigma^3 H(K_3(R)))}^{K_3(R)\otimes K_1(\mathbb{Z})} & \xrightarrow{\star} & K_4(R) & \xrightarrow{\bar h_4} & H_4X_R(2) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle\sigma} & & \uparrow{\scriptstyle =} & & \uparrow & & \\
0 & \longrightarrow & H_5BSt(R)^+[4] & \xrightarrow{\varphi} & H_5(K(K_3(R),3)) & \xrightarrow{\partial} & K_4(R) & \xrightarrow{h_4} & H_4BSt(R)^+ & \longrightarrow & 0\,.
\end{array}$$

However, this time, σ is even an isomorphism. Observe that $H_5BSt(R)^+[4]$ is isomorphic to the kernel of $\star : K_3(R)\otimes K_1(\mathbb{Z}) \to K_4(R)$. The Whitehead exact sequence of $BSt(R)^+$ is

$$\cdots \longrightarrow \Gamma_5(BE(R)^+) \xrightarrow{\chi_5} K_5(R) \xrightarrow{h_5} H_5St(R) \xrightarrow{\nu_5} \Gamma_4(BSt(R)^+) \xrightarrow{\chi_4} K_4(R) \xrightarrow{h_4} H_4St(R) \longrightarrow 0$$

and it is easy to check that $\Gamma_4(BSt(R)^+) \cong K_3(R)\otimes K_1(\mathbb{Z}) \cong H_5(K(K_3(R),3))$. In order to understand the kernel of h_5, let us use the exact sequence given by Proposition 2.6 for $r=3$, $i=5$, and the exact sequence coming from the homology Serre spectral sequence of the fibration of CW-complexes

$$K(K_5(R),5) \longrightarrow BSt(R)^+[5] \longrightarrow BSt(R)^+[4]\,.$$

We obtain the commutative diagram

$$\begin{array}{ccccccccc}
H_6X_R(2,4] & \xrightarrow{\bar\psi} & K_5(R) & \xrightarrow{\bar h_5} & H_5X_R(2) & \longrightarrow & H_5X_R(2,4] & \longrightarrow & 0 \\
\uparrow{\scriptstyle\sigma} & & \uparrow{\scriptstyle =} & & \uparrow & & \uparrow & & \\
H_6BSt(R)^+[4] & \xrightarrow{\psi} & K_5(R) & \xrightarrow{h_5} & H_5BSt(R)^+ & \longrightarrow & H_5BSt(R)^+[4] & \longrightarrow & 0\,,
\end{array}$$

where the vertical arrows are iterated suspensions. According to the proof of Proposition 3.3, the iterated homology suspension σ is surjective and therefore $\mathrm{image}\,\psi = \mathrm{image}\,\bar\psi$. On the other hand, the group $H_6X_R(2,4]$ and the image of $\bar\psi$ may be described by Proposition 2.6 and Lemmas 2.4 and 2.5. □

The following corollary follows from the five lemma and the argument of the proofs of Theorems 4.1 and 4.3.

Corollary 4.4. *For any ring R, the iterated homology suspensions $H_3E(R) \cong H_3BE(R)^+ \to H_3X_R(1)$ and $H_4St(R) \cong H_4BSt(R)^+ \to H_4X_R(2)$ are isomorphisms.*

Remark 4.5. Observe that $h_4 : K_4(R) \to H_4St(R)$ is an isomorphism up to 2-torsion. This produces the following consequence of Proposition 9 of [8]. Let l be an odd prime, ξ_l a primitive l-root of unity of order l. Let $R = \mathbb{Z}[\xi_l + \xi_l^{-1}]$ be the ring of integers of the maximal real subfield of the cyclotomic field $\mathbb{Q}(\xi_l)$. The vanishing of the group $H_4St(R)$ in this case would imply the Kummer-Vandiver conjecture for the prime l .

5. Products in the algebraic K-theory of the ring of integers $\mathbb{Z}$

This section is devoted to the study of products in the algebraic K-theory of the ring of integers $\mathbb{Z}$:

$$\star : K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) \longrightarrow K_{i+k}(\mathbb{Z})\,.$$

Let us start by describing the results on low-dimensional products given by Section 4 in the case where $R = \mathbb{Z}$.

Proposition 5.1.
(a) The product homomorphism $\star : K_i(\mathbb{Z}) \otimes K_1(\mathbb{Z}) \to K_{i+1}(\mathbb{Z})$ is an isomorphism if $i = 1$, injective if $i = 2$, and trivial if $i \not\equiv 1$ or 2 mod 8.
(b) The product homomorphism $\star : K_i(\mathbb{Z}) \otimes K_2(\mathbb{Z}) \to K_{i+2}(\mathbb{Z})$ is trivial if $i \not\equiv 1$ mod 8.
(c) $H_4SL(\mathbb{Z}) \cong \mathbb{Z}/2$ and $H_4St(\mathbb{Z}) = 0$.
(d) There is a short exact sequence

$$0 \longrightarrow K_5(\mathbb{Z}) \xrightarrow{h_5} H_5St(\mathbb{Z}) \xrightarrow{\nu_5} \mathbb{Z}/2 \longrightarrow 0\,.$$

Proof. The assertion (a) is well known for $i = 1$. Theorem 4.1 produces the exact sequence

$$K_4(\mathbb{Z}) \xrightarrow{h_4} H_4SL(\mathbb{Z}) \xrightarrow{\nu_4} \underbrace{\Gamma(\mathbb{Z}/2)}_{\mathbb{Z}/4} \xrightarrow{\chi_3} \underbrace{K_3(\mathbb{Z})}_{\mathbb{Z}/48} \xrightarrow{h_3} \underbrace{H_3SL(\mathbb{Z})}_{\mathbb{Z}/24} \longrightarrow 0$$

(see [1], [20] and [43], Sections 5 and 13) and asserts that the product $\star : K_2(\mathbb{Z}) \otimes K_1(\mathbb{Z}) \to K_3(\mathbb{Z})$ is injective. Recently, C. Weibel deduced from the work of V. Voevodsky [37] the complete calculation of the

2-torsion of the algebraic K-theory of $\mathbb{Z}$ (see Table 1 of [41]). This, together with an argument of J. Rognes, shows that $K_4(\mathbb{Z})=0$ and implies that $H_4SL(\mathbb{Z})$ is cyclic of order 2. Moreover, $K_i(\mathbb{Z})$ is a finite odd torsion group if i is a positive integer $\equiv 0$, 4, or 6 mod 8. Therefore, $K_i(\mathbb{Z})\star K_1(\mathbb{Z})=0$ if $i\equiv 0$, 4, or 6 mod 8 or if $i+1\equiv 0$, 4, or 6 mod 8. This gives (a), and (b) follows from (a) since $K_2(\mathbb{Z})=K_1(\mathbb{Z})\star K_1(\mathbb{Z})$. Note that the first author proved the triviality of $\star : K_3(\mathbb{Z})\otimes K_1(\mathbb{Z})\to K_4(\mathbb{Z})$ in [5] before Weibel's proof of the vanishing of $K_4(\mathbb{Z})$.

Now, let us apply Theorem 4.3. The map $\bar{\psi}$ is actually the connecting homomorphism of the homology exact sequence of the cofibration

$$\Sigma^5(H(K_5(\mathbb{Z})) \longrightarrow X_{\mathbb{Z}}(2,5] \longrightarrow X_{\mathbb{Z}}(2,4]\,.$$

It is of course possible to consider the analogous cofibration for the sphere spectrum S

$$\Sigma^5(H(\pi_5(S)) \longrightarrow S(2,5] \longrightarrow S(2,4]\,.$$

The identity $j : S\to X_{\mathbb{Z}}$ of the ring spectrum $X_{\mathbb{Z}}$ induces the commutative diagram

$$\begin{tikzcd}
H_6S(2,4] \arrow[r] \arrow[d, "j_*"] & \pi_5(S) \,=\, 0 \arrow[d] \\
H_6X_{\mathbb{Z}}(2,4] \arrow[r, "\bar{\psi}"] & K_5(\mathbb{Z})
\end{tikzcd}$$

which shows that $\bar{\psi}\, j_*=0$. Now, look at the following commutative diagram where the bottom homomorphism is given by the second assertion of Theorem 4.3:

$$\begin{tikzcd}
H_6S(2,4] \arrow[r, "\cong"] \arrow[d, "j_*"] & H_6(\Sigma^3H(\pi_3(S))) \,\cong\, {}_2(\pi_3(S)) \arrow[d, "\cong"] \\
H_6X_{\mathbb{Z}}(2,4] \arrow[r, "\bar{\eta}"] & H_6(\Sigma^3H(K_3(\mathbb{Z}))) \,\cong\, {}_2(K_3(\mathbb{Z}))\,.
\end{tikzcd}$$

The top horizontal arrow is an isomorphism because the vanishing of $\pi_4(S)$ exhibits an equivalence $S(2,4]\simeq \Sigma^3H(\pi_3(S))$. The right vertical arrow is an isomorphism since the homomorphism $\pi_3(S)\to K_3(\mathbb{Z})$, induced by j, is injective (remember that $\pi_3(S)\cong \mathbb{Z}/24$ and $K_3(\mathbb{Z})\cong\mathbb{Z}/48$). Therefore, there exists a splitting $\tau : {}_2(K_3(\mathbb{Z}))\to H_6X_{\mathbb{Z}}(2,4]$ of $\bar{\eta}$ such that τ is the composition of an isomorphism ${}_2(K_3(\mathbb{Z}))\xrightarrow{\cong} H_6S(2,4]$ with j_*. It then follows from the vanishing of the composition $\bar{\psi}\,j_*$ that $\bar{\psi}\,\tau=0$. Consequently, the group Q of Theorem 4.3 (c) is trivial if $R=\mathbb{Z}$ since $Q=\mathrm{image}\,(\bar{\eta}\,\tau)$, and $\ker h_5\cong \mathrm{image}\,(\bar{\psi}\,\bar{\theta})$ is the image of the product map $\star : K_4(\mathbb{Z})\otimes K_1(\mathbb{Z})\to K_5(\mathbb{Z})$. Consequently, there is an exact sequence

$$0\longrightarrow K_4(\mathbb{Z})\star K_1(\mathbb{Z}) \xrightarrow{\bar{\psi}\,\bar{\theta}} K_5(\mathbb{Z}) \xrightarrow{h_5} H_5St(\mathbb{Z}) \xrightarrow{\nu_5} K_3(\mathbb{Z})\otimes K_1(\mathbb{Z}) \xrightarrow{\star} K_4(\mathbb{Z}) \xrightarrow[\cong]{h_4} H_4St(\mathbb{Z}) \longrightarrow 0\,.$$

The fact that $K_4(\mathbb{Z})=0$ provides the vanishing of $H_4St(\mathbb{Z})$ and the short exact sequence (d) (which was already given in Section 6 of [6])

$$0\longrightarrow K_5(\mathbb{Z}) \xrightarrow{h_5} H_5St(\mathbb{Z}) \xrightarrow{\nu_5} \mathbb{Z}/2\longrightarrow 0\,.$$

According to Theorem 1 of [21], Theorem 1 of [32] and Table 1 of [41], $K_5(\mathbb{Z})\cong\mathbb{Z}\oplus T$, where T is a finite abelian 3-group: thus, $H_5St(\mathbb{Z})\cong\mathbb{Z}\oplus T$ and h_5 is multiplication by 2 on the infinite cyclic factor of $K_5(\mathbb{Z})$. $\square$

Corollary 5.2. *Let R be a ring such that the homomorphism $\ell_* : K_3(\mathbb{Z}) \to K_3(R)$ (induced by the obvious map $\ell : \mathbb{Z} \to R$) induces an isomorphism $\ell_* \otimes \mathrm{id} : K_3(\mathbb{Z}) \otimes \mathbb{Z}/2 \xrightarrow{\cong} K_3(R) \otimes \mathbb{Z}/2$ (for instance, if $R = \mathbb{Q}$ or any localization of $\mathbb{Z}$). Then the non-stable Hurewicz homomorphism $h_4 : K_4(R) \to H_4St(R)$ is an isomorphism.*

Proof. This follows from the commutativity of the diagram

$$\begin{tikzcd}
K_3(\mathbb{Z}) \otimes K_1(\mathbb{Z}) \arrow[r, "\star"] \arrow[d, "\ell_* \otimes \mathrm{id}", "\cong"'] & K_4(\mathbb{Z}) = 0 \arrow[d, "\ell_*"] & & \\
K_3(R) \otimes K_1(\mathbb{Z}) \arrow[r, "\star"] & K_4(R) \arrow[r, "h_4"] & H_4St(R) \arrow[r] & 0 ,
\end{tikzcd}$$

where the rows are the exact sequences given by Theorem 4.3 (a). □

The argument of the proof of Proposition 5.1 (d) produces also the next two corollaries

Corollary 5.3. *Let R be a ring such that the composition of $j : S \to X_{\mathbb{Z}}$ with the obvious map $X_{\mathbb{Z}} \to X_R$ induces an isomorphism ${}_2(\pi_3(S)) \xrightarrow{\cong} {}_2(K_3(R))$. Then $K_4(R) \star K_1(\mathbb{Z}) \cong \ker(h_5 : K_5(R) \to H_5St(R))$.*

Corollary 5.4. *If R is a ring as in Corollary 5.3 (for instance, if $R = \mathbb{Z}$ or any localization of $\mathbb{Z}$), then the iterated homology suspension $H_5St(R) \cong H_5BSt(R)^+ \to H_5X_R(2)$ is an isomorphism.*

Proof. By hypothesis, one has actually the exact sequences

$$\begin{tikzcd}
H_6(\Sigma^4(H(K_4(R)))) \arrow[r, "\bar\psi\,\bar\theta"] & K_5(R) \arrow[r, "\bar h_5"] & H_5X_R(2) \arrow[r] & H_5X_R(2,4] \arrow[r] & 0 \\
H_6K(K_4(R),4) \arrow[r, "\psi\theta"] \arrow[u, "\sigma"] & K_5(R) \arrow[r, "h_5"] \arrow[u, "="] & H_5BSt(R)^+ \arrow[r] \arrow[u] & H_5BSt(R)^+[4] \arrow[r] \arrow[u, "\sigma'"] & 0 ,
\end{tikzcd}$$

and the iterated homology suspension σ is clearly an isomorphism; the same is true for σ' because of the commutativity of

$$\begin{tikzcd}
H_5X_R(2,4] \arrow[r, "\cong"] & H_5(\Sigma^3 H(K_3(R))) \\
H_5BSt(R)^+[4] \arrow[r, "\cong"] \arrow[u, "\sigma'"] & H_5(K(K_3(R),3)) . \arrow[u, "\cong"]
\end{tikzcd}$$

The assertion then follows from the five lemma. □

In the remainder of this section, we consider maps

$$K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) \xrightarrow{\star} K_{i+k}(\mathbb{Z}) \longrightarrow K_{i+k}(\mathbb{Z}) \otimes \hat{\mathbb{Z}}_2$$

for all positive integers i and k, where the second arrow is the tensor product of $K_{i+k}(\mathbb{Z})$ with the inclusion $\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}}_2$. We call these maps 2-adic products for $K_*(\mathbb{Z})$ and continue to denote them by the symbol $\star$. Again because of Table 1 of [41], $K_i(\mathbb{Z})$ is a finite odd torsion group if i is a positive integer $\equiv 0$, 4, or

6 mod 8 and $\cong \mathbb{Z} \oplus$ (finite odd torsion group) if $i \equiv 5$ mod 8; thus, the only 2-adic products which can be non trivial are the following:

$$\begin{aligned}
K_{8s+1}(\mathbb{Z}) \otimes K_{8t+1}(\mathbb{Z}) &\xrightarrow{\star} K_{8(s+t)+2}(\mathbb{Z}) \otimes \mathbb{Z}\hat{_2} \\
K_{8s+1}(\mathbb{Z}) \otimes K_{8t+2}(\mathbb{Z}) &\xrightarrow{\star} K_{8(s+t)+3}(\mathbb{Z}) \otimes \mathbb{Z}\hat{_2} \\
K_{8s+2}(\mathbb{Z}) \otimes K_{8t+5}(\mathbb{Z}) &\xrightarrow{\star} K_{8(s+t)+7}(\mathbb{Z}) \otimes \mathbb{Z}\hat{_2} \\
K_{8s+2}(\mathbb{Z}) \otimes K_{8t+7}(\mathbb{Z}) &\xrightarrow{\star} K_{8(s+t+1)+1}(\mathbb{Z}) \otimes \mathbb{Z}\hat{_2} \\
K_{8s+3}(\mathbb{Z}) \otimes K_{8t+7}(\mathbb{Z}) &\xrightarrow{\star} K_{8(s+t+1)+2}(\mathbb{Z}) \otimes \mathbb{Z}\hat{_2} \\
K_{8s+5}(\mathbb{Z}) \otimes K_{8t+5}(\mathbb{Z}) &\xrightarrow{\star} K_{8(s+t+1)+2}(\mathbb{Z}) \otimes \mathbb{Z}\hat{_2}
\end{aligned}$$

for s and $t \geq 0$. We shall determine five of these products in the present section and the remaining one in Section 8.

The inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces a map $\lambda : BGL(\mathbb{Z})^+ \to BO$ and the induced homomorphism

$$\lambda_* : K_*(\mathbb{Z}) \longrightarrow \pi_* BO$$

is a ring homomorphism since it can be written as the composition $\lambda_* : K_*(\mathbb{Z}) \to K_*(\mathbb{C}) \to \pi_* BO$, where both arrows are ring homomorphisms (see [13], p. 50 and p. 58). One can understand the kernel of λ_* at the prime 2 by the following argument. If p is a prime $\equiv 3$ or 5 mod 8, M. Bökstedt introduced in [11] (see also [26] and Section 4 of [18]) a space $J(p)$ which is defined by the pull-back diagram

$$\begin{tikzcd}
J(p) \arrow[r, "\lambda'"] \arrow[d, "f_p"] & BO \arrow[d, "c"] \\
F\Psi^p \arrow[r, "b"] & BU,
\end{tikzcd}$$

where $F\Psi^p$ is the fiber of $(\Psi^p - 1) : BU \to BU$ (recall that $F\Psi^p \simeq BGL(\mathbb{F}_p)^+$ by Theorem 7 of [27]), b the Brauer lifting and c the complexification. The fibers of the horizontal maps are homotopy equivalent to the unitary group $U \simeq SU \times S^1$. More precisely, Bökstedt was interested in the covering space $JK(\mathbb{Z}, p)$ of $J(p)$ corresponding to the cyclic subgroup of order 2 of $\pi_1 J(p) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. After completion at the prime 2, he constructed a map

$$\tilde{\varphi} : (BGL(\mathbb{Z})^+)\hat{_2} \longrightarrow JK(\mathbb{Z}, p)\hat{_2}$$

which induces a split surjection on all homotopy groups. Let us write $\widehat{\lambda}$ and $\widehat{\lambda'}$ for the 2-completion of the maps λ and λ' respectively: it turns out that the composition

$$(BGL(\mathbb{Z})^+)\hat{_2} \xrightarrow{\tilde{\varphi}} JK(\mathbb{Z}, p)\hat{_2} \longrightarrow J(p)\hat{_2} \xrightarrow{\widehat{\lambda'}} BO\hat{_2}$$

is exactly $\widehat{\lambda}$. Recall that the localization exact sequence in K-theory implies that

$$(BGL(\mathbb{Z}[\tfrac{1}{2}])^+)\hat{_2} \simeq (BGL(\mathbb{Z})^+)\hat{_2} \times (S^1)\hat{_2}\,.$$

Therefore, $\tilde{\varphi}$ provides a map

$$\varphi : (BGL(\mathbb{Z}[\tfrac{1}{2}])^+)\hat{_2} \longrightarrow J(p)\hat{_2}$$

which also induces a split surjection on all homotopy groups. Since the 2-torsion of $K_*(\mathbb{Z})$ is known by Table 1 of [41], it is easy to check that $\tilde{\varphi}$ and φ are actually homotopy equivalences. Consequently, we obtain (see also Corollary 8 of [41]):

Proposition 5.5. *For all primes $p \equiv 3$ or $5 \mod 8$, there is a pull-back diagram*

$$\begin{tikzcd}
(BGL(\mathbb{Z})^+)\hat{}_2 \times (S^1)\hat{}_2 \arrow[r, "\widehat{\lambda'}"] \arrow[d, "\widehat{f_p}"] & BO\hat{}_2 \arrow[d, "\widehat{c}"] \\
(F\Psi^p)\hat{}_2 \arrow[r, "\widehat{b}"] & BU\hat{}_2 \,.
\end{tikzcd}$$

Consequently, there is a fibration

$$SU\hat{}_2 \xrightarrow{\eta} (BGL(\mathbb{Z})^+)\hat{}_2 \xrightarrow{\widehat{\lambda}} BO\hat{}_2 \,.$$

This fibration induces the long exact sequence

$$\cdots \longrightarrow \pi_i SU \otimes \mathbb{Z}\hat{}_2 \xrightarrow{\eta_*} K_i(\mathbb{Z}) \otimes \mathbb{Z}\hat{}_2 \xrightarrow{\widehat{\lambda}_*} \pi_i BO \otimes \mathbb{Z}\hat{}_2 \longrightarrow \pi_{i-1} SU \otimes \mathbb{Z}\hat{}_2 \longrightarrow \cdots \,.$$

Remember that $\pi_i SU = 0$ if i is even and $\pi_i SU \cong \mathbb{Z}$ if i is odd ≥ 3.

Theorem 5.6. *The 2-adic products*

$$K_{8s+3}(\mathbb{Z}) \otimes K_{8t+7}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t+1)+2}(\mathbb{Z}) \otimes \mathbb{Z}\hat{}_2$$
$$K_{8s+5}(\mathbb{Z}) \otimes K_{8t+5}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t+1)+2}(\mathbb{Z}) \otimes \mathbb{Z}\hat{}_2$$

are trivial for all integers s and $t \geq 0$.

Proof. For any product mentioned in the statement of the theorem, let us consider the commutative diagram

$$\begin{tikzcd}
K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) \arrow[r, "\star"] \arrow[d, "\lambda_* \otimes \lambda_*"] & K_{i+k}(\mathbb{Z}) \arrow[r] \arrow[d, "\lambda_*"] & K_{i+k}(\mathbb{Z}) \otimes \mathbb{Z}\hat{}_2 \arrow[d, "\widehat{\lambda}_*"] \\
\pi_i BO \otimes \pi_k BO \arrow[r] & \pi_{i+k} BO \arrow[r] & \pi_{i+k} BO \otimes \mathbb{Z}\hat{}_2 \,,
\end{tikzcd}$$

where the bottom left horizontal arrow is the product map in $\pi_* BO$ and where the right horizontal arrows denote the tensor product with $\mathbb{Z}\hat{}_2$. Let $x \in K_i(\mathbb{Z})$ and $y \in K_k(\mathbb{Z})$. One has clearly $\lambda_*(y) = 0$ since $\pi_k BO = 0$ for $k = 8t+5$ or $k = 8t+7$. Thus, $\lambda_*(x \star y) = \lambda_*(x)\lambda_*(y) = 0$. This shows that the 2-adic product $x \star y \in K_{i+k}(\mathbb{Z}) \otimes \mathbb{Z}\hat{}_2$ belongs to the kernel of $\widehat{\lambda}_*$, and consequently to the image of $\eta_* : \pi_{i+k} SU \otimes \mathbb{Z}\hat{}_2 \to K_{i+k}(\mathbb{Z}) \otimes \mathbb{Z}\hat{}_2$. Since $i+k$ is even, the group $\pi_{i+k} SU$ is trivial and $x \star y$ vanishes. $\square$

In the next theorem, we look at the groups $K_{8s+1}(\mathbb{Z})$ for $s \geq 0$. Remember that $K_1(\mathbb{Z}) \cong \mathbb{Z}/2$ and that $K_{8s+1}(\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus$ (finite odd torsion group) for $s \geq 1$ by Table 1 of [41]. The fibration given by Proposition 5.5 provides the exact sequence

$$0 \longrightarrow \pi_{8s+1} SU \otimes \mathbb{Z}\hat{}_2 \cong \mathbb{Z}\hat{}_2 \xrightarrow{\eta_*} K_{8s+1}(\mathbb{Z}) \otimes \mathbb{Z}\hat{}_2 \xrightarrow{\widehat{\lambda}_*} \pi_{8s+1} BO \otimes \mathbb{Z}\hat{}_2 \cong \mathbb{Z}/2 \longrightarrow 0$$

if $s \geq 1$ (if $s = 0$, $\pi_1 SU = 0$). Let us write x_s for the element of order 2 in $K_{8s+1}(\mathbb{Z})$. We denote by y_s, for $s \geq 1$, the generator of the infinite cyclic summand of $K_{8s+1}(\mathbb{Z})$ whose image under the 2-adic completion $K_{8s+1}(\mathbb{Z}) \to K_{8s+1}(\mathbb{Z}) \otimes \mathbb{Z}\hat{}_2$ is exactly the image of a generator of $\pi_{8s+1} SU \otimes \mathbb{Z}\hat{}_2$ under the homomorphism $\eta_* : \pi_{8s+1} SU \otimes \mathbb{Z}\hat{}_2 \cong \mathbb{Z}\hat{}_2 \to K_{8s+1}(\mathbb{Z}) \otimes \mathbb{Z}\hat{}_2$.

Theorem 5.7. *Consider the 2-adic product*

$$K_{8s+1}(\mathbb{Z})\otimes K_{8t+1}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t)+2}(\mathbb{Z})\otimes \hat{\mathbb{Z_2}}$$

for any integers s, $t\geq 0$.
(a) If s *and* t *are* ≥ 1, *then* $y_s\star y_t=0$.
(b) If $s\geq 0$ *and* $t\geq 1$, *then* $x_s\star y_t=0$.
(c) For s *and* $t\geq 0$, $x_s\star x_t$ *is the generator of* $K_{8(s+t)+2}(\mathbb{Z})\otimes \hat{\mathbb{Z_2}}\cong \mathbb{Z}/2$.

Proof. The commutativity of the square

$$\begin{array}{ccc} K_{8s+1}(\mathbb{Z}) & \longrightarrow & K_{8s+1}(\mathbb{Z})\otimes \hat{\mathbb{Z_2}} \\ \Big\downarrow \lambda_* & & \Big\downarrow \widehat{\lambda}_* \\ \pi_{8s+1}BO & \xrightarrow{\cong} & \pi_{8s+1}BO\otimes \hat{\mathbb{Z_2}}, \end{array}$$

where the horizontal arrows denote the tensor product with $\hat{\mathbb{Z_2}}$, and the definition of y_s show that $\lambda_*(y_s)=0$. We deduce similarily that $\lambda_*(y_t)=0$. This implies the vanishing of $\widehat{\lambda}_*(y_s\star y_t)$ and $\widehat{\lambda}_*(x_s\star y_t)$. The fact that $\pi_{8(s+t)+2}SU=0$ then enables us to deduce the triviality of the products $y_s\star y_t$ and $x_s\star y_t$ in $K_{8(s+t)+2}(\mathbb{Z})\otimes\hat{\mathbb{Z_2}}$. The image of x_s under λ_* is the generator c_s of $\pi_{8s+1}BO\cong\mathbb{Z}/2$ and it is known that c_sc_t is non trivial in $\pi_{8(s+t)+2}BO$ (see [36], p. 304). Therefore, it follows from the equality $\lambda_*(x_s\star x_t)=c_sc_t$ that the product $x_s\star x_t$ does not vanish. □

Corollary 5.8. *The products*

$$K_{8s+2}(\mathbb{Z})\otimes K_{8t+5}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t)+7}(\mathbb{Z})\otimes \hat{\mathbb{Z_2}}$$
$$K_{8s+2}(\mathbb{Z})\otimes K_{8t+7}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t+1)+1}(\mathbb{Z})\otimes \hat{\mathbb{Z_2}}$$

are trivial for all s *and* $t\geq 0$.

Proof. According to Theorem 5.7,

$$K_{8s+2}(\mathbb{Z})\otimes\hat{\mathbb{Z_2}}\cong (K_1(\mathbb{Z})\star K_{8s+1}(\mathbb{Z}))\otimes\hat{\mathbb{Z_2}}.$$

This implies the assertion because the products $K_{8s+1}(\mathbb{Z})\otimes K_{8t+5}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t)+6}(\mathbb{Z})\otimes\hat{\mathbb{Z_2}}=0$ and $K_{8s+1}(\mathbb{Z})\otimes K_{8t+7}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t+1)}(\mathbb{Z})\otimes\hat{\mathbb{Z_2}}=0$ are obviously trivial (see Table 1 of [41]). □

Remark 5.9. The last interesting product is

$$K_{8s+1}(\mathbb{Z})\otimes K_{8t+2}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t)+3}(\mathbb{Z})\otimes\hat{\mathbb{Z_2}}.$$

As in the previous corollary, we have

$$(K_{8s+1}(\mathbb{Z})\star K_{8t+2}(\mathbb{Z}))\otimes\hat{\mathbb{Z_2}}\cong (K_1(\mathbb{Z})\star K_{8s+1}(\mathbb{Z})\star K_{8t+1}(\mathbb{Z}))\otimes\hat{\mathbb{Z_2}}$$

and Theorem 5.7 asserts that the image of this product is at most a cyclic group of order 2. We shall determine it exactly in Theorem 8.2.

6. Products in the algebraic K-theory of cyclotomic fields

The results of Section 5 indicate that the 2-adic products in $K_*(\mathbb{Z})$ are trivial or have a very small image. In this section, we show that in the case of products in the K-theory of number fields, the image of product maps can be quite big. In the proof, we use the methods of [8] and [9]. Let us consider an odd prime number l, a positive integer m, and the cyclotomic field $E = \mathbb{Q}(\xi_{l^m})$ obtained from $\mathbb{Q}$ by adding a primitive root of unity ξ_{l^m} of order l^m. Our goal is to show that for n odd, the product homomorphism

$$\star : K_1(E) \otimes K_{2n-1}(E)_l \longrightarrow K_{2n}(E)_l$$

has a big image.

If R is a commutative ring, X_R is a ring spectrum with respect to $\mu : X_R \wedge X_R \to X_{R\otimes R} \to X_R$, where the first map is the pairing which was also called μ at the beginning of Section 3 (see [22], Proposition 2.4.2) and the second is induced by the multiplication $R \otimes R \to R$. The product structure of $K_*(R)$, also denoted by $\star$, is given by the composition

$$\star : K_i(R) \otimes K_k(R) \cong \pi_i X_R \otimes \pi_k X_R \xrightarrow{\wedge} \pi_{i+k}(X_R \wedge X_R) \xrightarrow{\mu_*} \pi_{i+k} X_R \cong K_{i+k}(R)\,.$$

Recall that the K-theory with $\mathbb{Z}/l^m$-coefficients (for a prime l and a positive integer m, with $m \geq 2$ if $l = 2$) may be defined by $K_k(R;\mathbb{Z}/l^m) = \pi_k(M \wedge X_R)$, where M is the mod l^m Moore spectrum (i.e., such that $H_0 M \cong \mathbb{Z}/l^m$, $H_k M = 0$ for $k \neq 0$). Notice that M is a ring spectrum with identity i_M and product μ_M (see [30], p. 22). We also consider the following products (see also [13]):

$$\star : K_i(R)\otimes K_k(R;\mathbb{Z}/l^m) \cong \pi_i X_R \otimes \pi_k(M\wedge X_R) \xrightarrow{\wedge} \pi_{i+k}(M\wedge X_R\wedge X_R) \xrightarrow{\mu_*} \pi_{i+k}(M\wedge X_R) \cong K_{i+k}(R;\mathbb{Z}/l^m)$$

and

$$\star : K_i(R;\mathbb{Z}/l^m) \otimes K_k(R;\mathbb{Z}/l^m) \cong \pi_i(M \wedge X_R) \otimes \pi_k(M \wedge X_R) \xrightarrow{\wedge}$$
$$\pi_{i+k}(M \wedge M \wedge X_R \wedge X_R) \xrightarrow{(\mu_M\wedge\mu)_*} \pi_{i+k}(M \wedge X_R) \cong K_{i+k}(R;\mathbb{Z}/l^m)\,.$$

Remark 6.1. Since M is a ring spectrum, the diagram

$$\begin{array}{ccccc}
S \wedge S & \xrightarrow{\mathrm{id}\wedge i_M} & S \wedge M & \xrightarrow{i_M\wedge \mathrm{id}} & M \wedge M \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \mu_M \\
S & \xrightarrow{i_M} & M & \xrightarrow{\mathrm{id}} & M
\end{array}$$

commutes and implies the compatibility of the three products, i.e., the commutativity of the diagram

$$\begin{array}{ccccc}
K_i(R) \otimes K_k(R) & \xrightarrow{\mathrm{id}\otimes\mathrm{red}} & K_i(R) \otimes K_k(R;\mathbb{Z}/l^m) & \xrightarrow{\mathrm{red}\otimes\mathrm{id}} & K_i(R;\mathbb{Z}/l^m) \otimes K_k(R;\mathbb{Z}/l^m) \\
\downarrow \star & & \downarrow \star & & \downarrow \star \\
K_{i+k}(R) & \xrightarrow{\mathrm{red}} & K_{i+k}(R;\mathbb{Z}/l^m) & \xrightarrow{\mathrm{id}} & K_{i+k}(R;\mathbb{Z}/l^m)\,,
\end{array}$$

where red is the map which is induced on K-theory by the reduction of coefficients mod l^m.

For any ring R, there is the following Bockstein long exact sequence

$$\cdots \longrightarrow K_k(R) \xrightarrow{l^m} K_k(R) \longrightarrow K_k(R;\mathbb{Z}/l^m) \xrightarrow{\mathfrak{b}} K_{k-1}(R) \longrightarrow \cdots\,,$$

where $\mathfrak{b}$ is the Bockstein homomorphism.

Lemma 6.2. *For any $x \in K_i(R)$ and any $y \in K_k(R;\mathbb{Z}/l^m)$, one has $\mathfrak{b}(x \star y) = x \star \mathfrak{b}(y) \in K_{i+k-1}(R)$.*

Proof. The Bockstein homomorphism $\mathfrak{b}$ is induced by the obvious map $\varepsilon : M \to \Sigma^{-1}S$ which fits into the commutative diagram

$$\begin{array}{ccc} S \wedge M & \xrightarrow{\simeq} & M \\ \big\downarrow{\scriptstyle \mathrm{id}\wedge\varepsilon} & & \big\downarrow{\scriptstyle \varepsilon} \\ S \wedge \Sigma^{-1}S & \xrightarrow{\simeq} & \Sigma^{-1}S \end{array}$$

and provides the commutativity of

$$\begin{array}{ccc} K_i(R) \otimes K_k(R;\mathbb{Z}/l^m) & \xrightarrow{\star} & K_{i+k}(R;\mathbb{Z}/l^m) \\ \big\downarrow{\scriptstyle \mathrm{id}\otimes\mathfrak{b}} & & \big\downarrow{\scriptstyle \mathfrak{b}} \\ K_i(R) \otimes K_{k-1}(R) & \xrightarrow{\star} & K_{i+k-1}(R) \end{array}$$

and the statement of the lemma. □

This lemma implies the formula

$$\mathfrak{b}(Tr_{E/\mathbb{Q}}(u \star \beta_m^{\star n})) = Tr_{E/\mathbb{Q}}(u \star \mathfrak{b}(\beta_m^{\star n})),$$

where u is any element $\in K_1(E) = E^\times$, $\beta_m = \beta(\xi_{l^m}) \in K_2(E;\mathbb{Z}/l^m)$ is the Bott element (see Definition 2.7.2 of [40]) and $Tr_{E/\mathbb{Q}}$ is the transfer map (see [28], Section 4). Using this equality, we can rewrite the definition of the Stickelberg pseudosplitting homomorphism Λ from [7], Section IV.1, or [9], Definition 3.2, as follows.

Definition 6.3. There is a homomorphism

$$\Lambda : \bigoplus_p K_{2n-1}(\mathbb{F}_p)_l \longrightarrow K_{2n}(\mathbb{Q})_l\ ,$$

where $\Lambda = \prod_p \Lambda_p$ and $\Lambda_p : K_{2n-1}(\mathbb{F}_p)_l \to K_{2n}(\mathbb{Q})_l$ is given by the formula

$$\Lambda_p(\kappa_p) = \begin{cases} Tr_{E/\mathbb{Q}}(\lambda_b(\mathfrak{p}) \star \mathfrak{b}({\beta_k}^{\star n})^{b^n\gamma_l}) & \text{if } l \text{ does not divide } n, \\ Tr_{E/\mathbb{Q}}(\lambda_b(\mathfrak{p}) \star \mathfrak{b}({\beta_k}^{\star n})^{nb^n\gamma_l}) & \text{if } l \text{ divides } n, \end{cases}$$

where b is a natural number such that $(b, w_{n+1}(\mathbb{Q})) = 1$ and κ_p is a generator of the group $K_{2n-1}(\mathbb{F}_p)_l$. In addition, $\lambda_b(\mathfrak{p}) \in E^\times$ are the twisted Gauss sums (see [8], Definition 3) and $\gamma_l = 1 + l^n + l^{2n} + l^{3n} \ldots = \frac{1}{1-l^n} \in \hat{\mathbb{Z}_l}$.

Theorem 6.4. *Let I be the image of the map*

$$K_1(E) \otimes K_{2n-1}(E)_l \xrightarrow{\star} K_{2n}(E)_l \xrightarrow{Tr_{E/\mathbb{Q}}} K_{2n}(\mathbb{Q})_l\ ,$$

where n is an odd integer. Then the exponent of the group $K_{2n}(\mathbb{Q})_l/I$ divides the number $(\#K_{2n}(\mathbb{Z})_l)^2$.

Proof. Consider the localization sequence

$$0 \longrightarrow K_{2n}(\mathbb{Z})_l \longrightarrow K_{2n}(\mathbb{Q})_l \xrightarrow{\partial} \bigoplus_p K_{2n-1}(\mathbb{F}_p)_l \longrightarrow 0\,.$$

By Definition 6.3, we see that $\operatorname{image}\Lambda \subseteq I$. On the other hand, by Proposition 2 of [8], the composition $\partial\,\Lambda$ acts by raising into the power with exponent an integer $|(b^{n+1}-1)\zeta_{\mathbb{Q}}(-n)|_l^{-1}$ (recall that the Gauss sums used in the construction of Λ depend on b), where $\zeta_{\mathbb{Q}}(s)$ is the Riemann zeta function. Consider now $x \in K_{2n}(\mathbb{Q})_l$ and $z = \partial(x) \in \bigoplus_p K_{2n-1}(\mathbb{F}_p)_l$. The computations above show that for every b,

$$\frac{\Lambda(z)}{x^{|(b^{n+1}-1)\zeta_{\mathbb{Q}}(-n)|_l^{-1}}} \in K_{2n}(\mathbb{Z})_l\,.$$

The greatest common divisor of all $|(b^{n+1}-1)|_l^{-1}$ over the integers b which are relatively prime to $w_{n+1}(\mathbb{Q})$ equals the number $|w_{n+1}(\mathbb{Q})|_l^{-1}$, by Lemma 2.3 of [15]. Since the integer $|w_{n+1}(\mathbb{Q})\zeta_{\mathbb{Q}}(-n)|_l^{-1}$ divides the number t of elements in the group $K_{2n}(\mathbb{Z})_l$, it follows that $x^{t^2} \in \operatorname{image}\Lambda \subseteq I$. □

Remark 6.5. Observe that in an abelian group the product of a torsion element and a nontorsion element is again nontorsion. Hence, every torsion element in $K_{2n-1}(E)$ can be written as a quotient of two nontorsion elements. Consequently, the subgroup of $K_1(E)\star K_{2n-1}(E)$ generated by the subset $\{x\star y \mid x \in K_1(E)\,,\, y \in K_{2n-1}(E)\,,\, y \text{ nontorsion }\}$ contains the subgroup $K_1(E)\star K_{2n-1}(E)_l$ of $K_{2n}(E)$.

Remark 6.6. It follows from Theorem 6.4 and Theorem 3.4 of [9] that the image of $K_1(E)\star K_{2n-1}(E)_l$ under the transfer $Tr_{E/\mathbb{Q}} : K_{2n}(E) \to K_{2n}(\mathbb{Q})$ contains the group of divisible elements $D(n)_l \cong K_{2n}^{\text{ét}}(\mathbb{Z}[\frac{1}{l}])$ from Section 5.2 of [8] (see also [7], Section IV.3).

7. 2-adic cyclotomic elements in $K_{2n-1}(\mathbb{Z})\otimes\hat{\mathbb{Z}_2}$

In this section, we discuss a 2-adic version of cyclotomic elements (see [34], Sections 4.3 and 4.4, p. 239–240). We construct the subgroup of cyclotomic elements in the 2-adic K-group of $\mathbb{Z}$

$$K_{2n-1}(\mathbb{Z})\otimes\hat{\mathbb{Z}_2} \cong K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}])\otimes\hat{\mathbb{Z}_2} \cong \varprojlim K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2^m)\,,$$

where the inverse limit is taken over reduction maps

$$r_m : K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2^{m+1}) \longrightarrow K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2^m)\,,$$

and the subgroup of étale 2-adic cyclotomic elements in

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\hat{\mathbb{Z}_2}(n)) \cong \varprojlim H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2^m(n))\,.$$

Since the case of the prime 2 is not often discussed in the literature, we shall make explicit computations. We assume that n is an odd integer and use the following notation:

- ξ_{2^m} $= \exp(\frac{2\pi i}{2^m})$ is a primitive root of unity of order 2^m, where m is a positive integer,
- E_m $= \mathbb{Q}(\xi_{2^m})$ is the corresponding cyclotomic field,
- R_m $= \mathbb{Z}[\xi_{2^m}]$ is the ring of integers of E_m,
- R'_m $= \mathbb{Z}[\frac{1}{2}, \xi_{2^m}]$,
- $\mathbb{Z}/2^m(n)$ is the n-th Tate twist of the sheaf of roots of unity on $\operatorname{Spec} R'_k$ for $k \geq 1$,
- $\langle g_1, g_2, ...g_r\rangle$ denotes the subgroup generated by elements g_1, $g_2, \ldots, g_r$ of a given abelian group G,
- C_m^+ $= \langle -1, , \xi_{2^m}^{\frac{1-a}{2}}\dfrac{1-\xi_{2^m}^a}{1-\xi_{2^m}}\,,\, 1 < a < 2^{m-1}\,,\, a \text{ odd}\rangle$ is the group of real cyclotomic units in $E_m^\times$ (see [38], p. 145),
- C_m denotes the subgroup of $R_m^\times$ generated by C_m^+ and ξ_{2^m},
- C'_m is the subgroup of $R_m^{'\times}$ generated by C_m and $1-\xi_{2^m}^a$ for $1 \leq a < 2^m$ and a odd.

Remark 7.1. It is easy to show that C'_m is generated by ξ_{2^m} and $1-\xi_{2^m}^a$ for $1 \leq a < 2^m$ and a odd. Observe that the group C'_m is also generated by $1-\xi_{2^m}^a$ and $\xi_{2^m}(1-\xi_{2^m}^a)$ for $1 \leq a < 2^m$ and a odd.

Example 7.2. The group C'_2 is generated by $1-i$ and $1+i$ because $i(1-i) = 1+i$ and $i(1+i) = -(1-i)$.

Remark 7.3. We leave to the reader to check that

$$N_{E_m/\mathbb{Q}}(\xi_{2^m}^b(1-\xi_{2^m}^a)) = 2$$

for all a, $b \in \mathbb{Z}$ with $1 \leq a < 2^m$ and a odd, where $N_{E_m/\mathbb{Q}}$ denotes the norm map down to $\mathbb{Q}$.

Let us assume for a moment that $m \geq 2$ and consider the map

$$\alpha_m : C'_m \otimes \mathbb{Z}/2^m(n-1) \longrightarrow K_{2n-1}(R'_m;\mathbb{Z}/2^m)$$

given by the formula

$$\alpha_m(c \otimes \xi_{2^m}^{\otimes(n-1)}) = c \star \beta_m^{\star(n-1)},$$

where $c \in C'_m \subseteq K_1(R'_m)$ and $\beta_m = \beta(\xi_{2^m}) \in K_2(R'_m;\mathbb{Z}/2^m)$ is the Bott element (see Definition 2.7.2 of [40]). The product $\star$ used in that formula is explained at the beginning of Section 6. We shall also use the transfer map in K-theory (see [28], Section 4)

$$Tr_{E_m/\mathbb{Q}} : K_{2n-1}(R'_m;\mathbb{Z}/2^m) \longrightarrow K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2^m).$$

Definition 7.4. Let n be any odd integer ≥ 3.
(a) Let $\mathcal{C}_m(n-1) \subseteq K_{2n-1}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2^m)$ denote the image of the composition $Tr_{E_m/\mathbb{Q}}\,\alpha_m$ for $m \geq 2$.
(b) We define $\mathcal{C}_1(n-1) \subseteq K_{2n-1}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2)$ to be the group $r_1\mathcal{C}_2(n-1)$, where

$$r_1 : K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/4) \longrightarrow K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2)$$

is the reduction of coefficients on K-groups.
(c) Let $\mathcal{C}(n-1) = \varprojlim \mathcal{C}_m(n-1)$ which is a subgroup of

$$\varprojlim K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2^m) \cong K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}]) \otimes \hat{\mathbb{Z}_2} \cong K_{2n-1}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}.$$

We call $\mathcal{C}(n-1)$ the group of 2-adic cyclotomic elements in $K_{2n-1}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}$.

Similarly we can define 2-adic cyclotomic elements inside the étale cohomology group $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(n))$. Since $\text{Pic}(R'_m)_2$ is a trivial group by [39], Satz C, p. 244, it follows from the Kummer exact sequence that we have the isomorphism

$$H^1_{\text{ét}}(R'_m;\mathbb{Z}/2^m(n)) \cong H^1_{\text{ét}}(R'_m;\mathbb{Z}/2^m(1)) \otimes \mathbb{Z}/2^m(n-1) \cong R'^{\times}_m \otimes \mathbb{Z}/2^m(n-1).$$

Hence we get the composition of maps

$$\alpha_m^{\text{ét}} : C'_m \otimes \mathbb{Z}/2^m(n-1) \longrightarrow R'^{\times}_m \otimes \mathbb{Z}/2^m(n-1) \xrightarrow{\cong} H^1_{\text{ét}}(R'_m;\mathbb{Z}/2^m(n)),$$

where the first map is induced by the natural injection. Note that the maps α_m and $\alpha_m^{\text{ét}}$ are only defined for $m \geq 2$.

Lemma 7.5. *The map $\alpha_m^{\text{ét}}$ is an isomorphism for all $m \geq 2$.*

Proof. It is sufficient to prove that $R'^{\times}_m/2^m \cong C'_m/2^m$. It follows from the commutative diagram of embeddings of groups

$$\begin{array}{ccc} WU_m & \longrightarrow & R_m^{\times} \\ \uparrow & & \uparrow \\ WC_m^{+} & \longrightarrow & C_m\,, \end{array}$$

where $W = \langle \xi_{2^m} \rangle$ and U_m is the group of units of the maximal real subfield of E_m, that the index $[R_m^{\times} : C_m]$ divides the product of indices $[R_m^{\times} : WU_m][WU_m : WC_m^{+}]$. By [38], Corollary 4.13, p. 39 and Theorem 8.2, p. 145, we know that $[R_m^{\times} : WU_m] = 1$ and that $[WU_m : WC_m^{+}] = [U_m : C_m^{+}]$ is the class number which is odd (see [39], Satz C, p. 244). Thus, we obtain $R_m^{\times}/2^m \cong C_m/2^m$ and, by the definition given at the beginning of this section, $R'^{\times}_m/2^m \cong C'_m/2^m$. □

Definition 7.6. Let n be any odd integer ≥ 3.
(a) If $m \geq 2$, we define $\mathcal{C}_m^{\text{ét}}(n-1)$ to be the image of the composition

$$C'_m \otimes \mathbb{Z}/2^m(n-1) \xrightarrow[\cong]{\alpha_m^{\text{ét}}} H^1_{\text{ét}}(R'_m; \mathbb{Z}/2^m(n)) \xrightarrow{Tr_{E_m/\mathbb{Q}}} H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}/2^m(n))\,,$$

where $Tr_{E_m/\mathbb{Q}}$ denotes the transfer map in étale cohomology.
(b) For $m = 1$ we put $\mathcal{C}_1^{\text{ét}}(n-1)$ to be the image of $\mathcal{C}_2^{\text{ét}}(n-1)$ via the reduction of coefficients

$$r_1 : H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}/4(n)) \longrightarrow H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}/2(n))\,.$$

(c) Taking inverse limits with respect to m above we obtain a subgroup $\mathcal{C}^{\text{ét}}(n-1)$ of

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \hat{\mathbb{Z}_2}(n)) \cong \varprojlim H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}/2^m(n))$$

which we call the subgroup of étale 2-adic cyclotomic elements.

Observe that for $m \geq k \geq 2$ the norm map $N_{E_m/E_k} : C'_m \to C'_k$ is surjective by Remark 7.1. For $m \geq k > 2$, we have the following commutative diagram with exact rows which comes from the naturality of the Bockstein long exact sequence in étale cohomology:

$$\begin{array}{ccccc} H^1_{\text{ét}}(R'_m; \mathbb{Z}/2^{k-1}(n)) & \longrightarrow & H^1_{\text{ét}}(R'_m; \mathbb{Z}/2^{k}(n)) & \xrightarrow{r'} & H^1_{\text{ét}}(R'_m; \mathbb{Z}/2(n)) \\ \downarrow{\scriptstyle Tr_{E_m/E_2}} & & \downarrow{\scriptstyle Tr_{E_m/E_2}} & & \downarrow{\scriptstyle Tr_{E_m/E_2}} \\ H^1_{\text{ét}}(R'_2; \mathbb{Z}/2^{k-1}(n)) & \longrightarrow & H^1_{\text{ét}}(R'_2; \mathbb{Z}/2^{k}(n)) & \longrightarrow & H^1_{\text{ét}}(R'_2; \mathbb{Z}/2(n))\,. \end{array} \tag{7.1}$$

For $m \geq k \geq 2$, in a similar way to the proof of Lemma 7.5 we obtain the isomorphism

$$H^1_{\text{ét}}(R'_m; \mathbb{Z}/2^k(n)) \cong C'_m \otimes \mathbb{Z}/2^k(n-1)\,. \tag{7.2}$$

By the projection formula in étale cohomology (see [33], Lemma 6) the right vertical map in Diagram (7.1) is surjective. In addition, the reduction of coefficients map r' in the top exact sequence is surjective. By induction over k we can assume that the left vertical map is surjective. Hence, it follows that the middle vertical map in (7.1) is surjective (see Diagram 2.6 of [9]). Let us put

$$B_m = \begin{cases} r_1(\alpha_2^{\text{ét}}(C'_2 \otimes \mathbb{Z}/4(n-1))) & \text{if } m = 1, \\ Tr_{E_m/E_2}(\alpha_m^{\text{ét}}(C'_m \otimes \mathbb{Z}/2^m(n-1))) & \text{if } m \geq 2, \end{cases} \tag{7.3}$$

where $r_1 : H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2},i];\mathbb{Z}/4(n)) \to H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2},i];\mathbb{Z}/2(n))$ is the reduction of coefficients and $i = \xi_4$. From the surjectivity of the transfer maps in Diagram (7.1) we obtain the isomorphism

$$(7.4) \qquad H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2},i];\hat{\mathbb{Z}_2}(n)) \cong \varprojlim H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2},i];\mathbb{Z}/2^m(n)) \cong \varprojlim B_m\,,$$

where the inverse limit is taken with respect to the reduction of coefficients in étale cohomology.

Lemma 7.7. *For any $k \geq 1$, $m \geq 1$ and n odd, $H^1_{\text{ét}}(R'_k;\mathbb{Z}/2^m(n)) \cong H^1_{\text{ét}}(R'_k;\hat{\mathbb{Z}_2}(n))/2^m$.*

Proof. This is an exercise on long exact sequences in cohomology which are associated with the sequence of Galois modules

$$0 \longrightarrow \hat{\mathbb{Z}_2}(n) \xrightarrow{2^m} \hat{\mathbb{Z}_2}(n) \longrightarrow \mathbb{Z}/2^m(n) \longrightarrow 0$$

and the sequence of sheaves

$$0 \longrightarrow \mathbb{Z}/2(1) \longrightarrow \mathbb{G}_{\text{m}} \xrightarrow{2} \mathbb{G}_{\text{m}} \longrightarrow 0\,,$$

where $\mathbb{G}_{\text{m}}$ is the mulplicative group scheme on $\operatorname{Spec} R'_k$. The sequence of Galois modules induces a long exact sequence in étale cohomology, which gives the short exact sequence

$$0 \longrightarrow H^1_{\text{ét}}(R'_k;\hat{\mathbb{Z}_2}(n))/2^m \longrightarrow H^1_{\text{ét}}(R'_k;\mathbb{Z}/2^m(n)) \longrightarrow H^2_{\text{ét}}(R'_k;\hat{\mathbb{Z}_2}(n))[2^m] \longrightarrow 0\,.$$

Here, $H^2_{\text{ét}}(R'_k;\hat{\mathbb{Z}_2}(n))[2^m]$ denotes the subgroup $\{x \in H^2_{\text{ét}}(R'_k;\hat{\mathbb{Z}_2}(n)) \,|\, 2^m x = 0\}$. We want to prove that the group $H^2_{\text{ét}}(R'_k;\hat{\mathbb{Z}_2}(n))[2^m]$ vanishes. Since the $\hat{\mathbb{Z}_2}$-module $H^2_{\text{ét}}(R'_k;\hat{\mathbb{Z}_2}(n))$ is finitely generated, we see that $H^2_{\text{ét}}(R'_k;\hat{\mathbb{Z}_2}(n)) = 0$ if and only if $H^2_{\text{ét}}(R'_k;\hat{\mathbb{Z}_2}(n))/2 = 0$. The same long exact sequence in étale cohomology with $m = 1$ gives the short exact sequence

$$0 \longrightarrow H^2_{\text{ét}}(R'_k;\hat{\mathbb{Z}_2}(n))/2 \longrightarrow H^2_{\text{ét}}(R'_k;\mathbb{Z}/2(n)) \longrightarrow H^3_{\text{ét}}(R'_k;\hat{\mathbb{Z}_2}(n))[2] \longrightarrow 0\,.$$

In addition, by triviality of the Galois action on $\mathbb{Z}/2(1)$, we get the isomorphism

$$H^2_{\text{ét}}(R'_k;\mathbb{Z}/2(n)) \cong H^2_{\text{ét}}(R'_k;\mathbb{Z}/2(1)) \otimes \mathbb{Z}(n-1)\,.$$

Now consider the long exact sequence in cohomology which is induced by the above exact sequence of sheaves: it gives the short exact sequence

$$0 \longrightarrow \operatorname{Pic}(R'_k)/2 \longrightarrow H^2_{\text{ét}}(R'_k;\mathbb{Z}/2(1)) \longrightarrow \operatorname{Br}(R'_k)[2] \longrightarrow 0\,,$$

since $H^1_{\text{ét}}(R'_k;\mathbb{G}_{\text{m}}) = \operatorname{Pic}(R'_k)$ and $H^2_{\text{ét}}(R'_k;\mathbb{G}_{\text{m}}) = \operatorname{Br}(R'_k)$ by [23], p. 107 and p. 109. We know from [39], Satz C, p. 244, that $\operatorname{Pic}(R'_k)/2 = 0$. On the other hand, it follows from class field theory (see Example 2.22(f), p. 109 of [23]) that:

$$\operatorname{Br}(R'_k) \ \text{ is } \ \mathbb{Z}/2 \ \text{ if } \ k = 1 \ \text{ and vanishes for } k \geq 2\,,$$
$$H^3_{\text{ét}}(R'_k;\mathbb{Z}/2(n)) \ \text{ is } \ \mathbb{Z}/2 \ \text{ if } \ k = 1 \ \text{ and vanishes for } k \geq 2\,.$$

This implies that $H^2_{\text{ét}}(R'_k;\mathbb{Z}/2(n)) \cong H^3_{\text{ét}}(R'_k;\hat{\mathbb{Z}_2}(n))[2]$. Consequently, we get $H^2_{\text{ét}}(R'_k;\hat{\mathbb{Z}_2}(n))/2 = 0$ and may deduce the lemma. □

By Theorem 10 of [41] and Theorem 1 (i) of [19], we know that

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2},i];\hat{\mathbb{Z}_2}(n)) \cong \hat{\mathbb{Z}_2} \oplus \mathbb{Z}/4\,.$$

Hence, by Lemma 7.7 applied for $k = m = 2$ we obtain:

$$(7.5) \qquad H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2},i];\mathbb{Z}/4(n)) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4\,.$$

On the other hand by (7.2) the map $\alpha_2^{\text{ét}}$ gives an isomorphism

$$(7.6) \qquad H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2},i];\mathbb{Z}/4(n)) \cong C'_2 \otimes \mathbb{Z}/4(n-1)\,.$$

According to Example 7.2, the group C_2' has two generators: $1-i$ and $1+i$. The module $\mathbb{Z}/4(n-1)$ has a canonical generator $\xi_4^{\otimes(n-1)}$. Hence, using the isomorphism (7.6) we see that the elements

$$(1-i)\otimes\xi_4^{\otimes(n-1)} \quad \text{and} \quad (1+i)\otimes\xi_4^{\otimes(n-1)}$$

generate $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2},i];\mathbb{Z}/4(n))$. We will use these elements below to contruct explicitly a generator of the $\hat{\mathbb{Z}_2}$-module $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(n))$. For $m\geq 2$ we check directly that

$$\begin{aligned} N_{E_m/E_2}(1-\xi_{2^m}) &= 1-i \\ N_{E_m/E_2}(1-\xi_{2^m}^{2^m-1}) &= 1+i\,. \end{aligned} \tag{7.7}$$

Hence, by the definition of the inverse system (7.3) we obtain the two elements

$$e_1 = (b_m^{(1)}),\quad e_2 = (b_m^{(2)}) \in \varprojlim B_m \cong H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2},i];\hat{\mathbb{Z}_2}(n))\,,$$

where $b_m^{(1)}$ and $b_m^{(2)}$ are the elements of B_m defined as follows:

$$\begin{aligned} b_m^{(1)} &= \begin{cases} r_1(\alpha_2^{\text{ét}}((1-i)\otimes\xi_4^{\otimes(n-1)})) & \text{if } m=1,\\ Tr_{E_m/E_2}(\alpha_m^{\text{ét}}((1-\xi_{2^m})\otimes\xi_{2^m}^{\otimes(n-1)})) & \text{if } m\geq 2,\end{cases} \\ b_m^{(2)} &= \begin{cases} r_1(\alpha_2^{\text{ét}}((1+i)\otimes\xi_4^{\otimes(n-1)})) & \text{if } m=1,\\ Tr_{E_m/E_2}(\alpha_m^{\text{ét}}((1-\xi_{2^m}^{2^m-1})\otimes\xi_{2^m}^{\otimes(n-1)})) & \text{if } m\geq 2.\end{cases} \end{aligned} \tag{7.8}$$

Lemma 7.8. *The elements e_1 and e_2 are nontorsion elements in $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2},i];\hat{\mathbb{Z}_2}(n))$.*

Proof. If both e_1 and e_2 were torsion elements, then their images in the group (see Lemma 7.7)

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2},i];\mathbb{Z}/4(n)) \cong \mathbb{Z}/4\oplus\mathbb{Z}/4$$

under the reduction of coefficients mod 4 would be contained in the same $\mathbb{Z}/4$ summand of that group. This is impossible because the images of e_1 and e_2 generate the group $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2},i],\mathbb{Z}/4(n))$ by construction. It follows that at least one of them is nontorsion. On the other hand the complex conjugation τ acts on the group $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2},i];\hat{\mathbb{Z}_2}(n))$. The automorphism τ sends e_1 to e_2 and e_2 to e_1. Indeed,

$$b_1^{(1)\tau} = r_1(\alpha_2^{\text{ét}}((1-i)^{\tau(-1)^{n-1}}\otimes\xi_4^{\otimes(n-1)})) = r_1(\alpha_2^{\text{ét}}((1+i)\otimes\xi_4^{\otimes(n-1)})) = b_1^{(2)}$$

because $(1-i)^\tau = 1+i$ and n is odd by assumption. For $m\geq 2$ we have

$$b_m^{(1)\tau} = Tr_{E_m/E_2}(\alpha_m^{\text{ét}}((1-\xi_{2^m})^{\tau(-1)^{n-1}}\otimes\xi_{2^m}^{\otimes(n-1)})) = Tr_{E_m/E_2}(\alpha_m^{\text{ét}}((1-\xi_{2^m}^{2^m-1})\otimes\xi_{2^m}^{\otimes(n-1)})) = b_m^{(2)}$$

because $(1-\xi_{2^m})^\tau = 1-\xi_{2^m}^{2^m-1}$ and n is odd by assumption. Hence, we obtain that $e_1^\tau = e_2$. In a similar way we prove that $e_2^\tau = e_1$. It follows that both elements must be nontorsion. $\square$

Observe that by the isomorphism (7.2) and Definition 7.6 we have:

$$\mathcal{C}^{\text{ét}}(n-1) \cong \text{image}\left(H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2},i];\hat{\mathbb{Z}_2}(n)) \xrightarrow{Tr_{E_2/E_1}} H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\hat{\mathbb{Z}_2}(n))\right).$$

Recall that for n odd,

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\hat{\mathbb{Z}_2}(n)) \cong \hat{\mathbb{Z}_2}\oplus\mathbb{Z}/2$$

by Theorem 1 (i) of [19] and Table 1 of [41], and consider the surjective map

$$\phi : H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\hat{\mathbb{Z}_2}(n)) \longrightarrow H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2(n)) \cong \langle -1\,,\,2\rangle\otimes\mathbb{Z}/2(n-1)\,,$$

where $\langle -1\,,\,2\rangle = \mathbb{Z}[\frac{1}{2}]^\times/2$.

Lemma 7.9. *The generator of the 2-torsion subgroup of* $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(n))$ *is sent by the map* ϕ *to* $(-1)\otimes\xi_2^{\otimes(n-1)}$.

Proof. Let $S_0=\{2\}$ be the set containing only the prime number 2, $\mathbb{Q}_{S_0}$ the maximal extension of $\mathbb{Q}$ unramified outside 2 and $G_{S_0}=G(\mathbb{Q}_{S_0}/\mathbb{Q})$. We know that

$$H^*_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathcal{F})\cong H^*(G_{S_0};\mathcal{F})$$

for $\mathcal{F}=\mathbb{Z}/2^m(n)$ or $\mathcal{F}=\hat{\mathbb{Z}_2}(n)$ (see [24], Proposition 2.9, p. 209). We have the commutative diagram

$$\begin{array}{ccc} H^0(G_{S_0};\mathbb{Z}/2(n)) & \xrightarrow{\partial} & H^1(G_{S_0};\mathbb{Z}/2(n)) \\ \big\downarrow = & & \big\downarrow = \\ \mathbb{Z}/2(n) & \longrightarrow & \mathrm{Hom}(G(\mathbb{Q}(i,\sqrt{2})/\mathbb{Q}),\mathbb{Z}/2(n)) , \end{array} \tag{7.9}$$

where the upper horizontal arrow is the connecting homomorphism in the long cohomology exact sequence associated with the exact sequence of G_{S_0}-modules

$$0\longrightarrow\mathbb{Z}/2(n)\longrightarrow\mathbb{Z}/4(n)\longrightarrow\mathbb{Z}/2(n)\longrightarrow 0 .$$

It is easy to observe that ∂ is an embedding. The Galois group $G(\mathbb{Q}(i,\sqrt{2})/\mathbb{Q})\cong\mathbb{Z}/2\oplus\mathbb{Z}/2$ is generated by the complex conjugation τ and the automorphism σ such that $\sigma(i)=i$ and $\sigma(\sqrt{2})=-\sqrt{2}$. Observe that the lower horizontal arrow in Diagram (7.9) sends the generator $\xi_2^{\otimes n}$ of $\mathbb{Z}/2(n)$ to a homomorphism $f:G(\mathbb{Q}(i,\sqrt{2})/\mathbb{Q})\to\mathbb{Z}/2(n)$ such that $f(\tau)=\xi_2^{\otimes n}$ and $f(\sigma)=1^{\otimes n}$. Let $\mathcal{O}_{S_0}$ denote the ring of integers in $\mathbb{Q}_{S_0}$ and consider the following commutative diagram:

$$\begin{array}{ccc} H^0(G_{S_0};\mathcal{O}_{S_0}^\times)/2 & \underset{\cong}{\xrightarrow{\partial'}} & H^1(G_{S_0};\mathbb{Z}/2(1)) \\ \big\downarrow = & & \big\downarrow = \\ \langle -1\,,\,2\rangle & \xrightarrow{\cong} & \mathrm{Hom}(G(\mathbb{Q}(i,\sqrt{2})/\mathbb{Q}),\mathbb{Z}/2(1)) . \end{array} \tag{7.10}$$

The upper horizontal arrow ∂' in Diagram (7.10) is the connecting homomorphism in the long cohomology exact sequence associated with the exact sequence

$$0\longrightarrow\mathbb{Z}/2(1)\longrightarrow\mathcal{O}_{S_0}^\times\xrightarrow{2}\mathcal{O}_{S_0}^\times\longrightarrow 0 .$$

Tensoring the inverse map ${\partial'}^{-1}$ with $\mathbb{Z}/2(n-1)$ we get the isomorphism

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2(n))\cong\langle -1\,,\,2\rangle\otimes\mathbb{Z}/2(n-1) \tag{7.11}$$

which we constantly use. Hence, the isomorphism (7.11) is just the inverse of the lower horizontal arrow in Diagram (7.10) tensored with $\mathbb{Z}/2(n-1)$ which we can compute explicitly. We are actually only interested in the image of (-1) tensored with $\xi_2^{\otimes(n-1)}$. We observe that $(-1)\otimes\xi_2^{\otimes(n-1)}$ is sent to a homomorphism f such that $f(\tau)=\xi_2^{\otimes n}$ and $f(\sigma)=1\otimes\xi_2^{\otimes(n-1)}=1^{\otimes n}$. Consider the commutative diagram

$$\begin{array}{ccc} H^0_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2(n)) & \xrightarrow{\partial} & H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2^m(n)) \\ \big\downarrow = & & \big\downarrow \\ H^0_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2(n)) & \xrightarrow{\partial} & H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2^{m-1}(n)) , \end{array} \tag{7.12}$$

where the horizontal arrows are the connecting homomorphisms in the long exact sequences associated with the exact sequences

$$0 \longrightarrow \mathbb{Z}/2^{j-1}(n) \longrightarrow \mathbb{Z}/2^j(n) \longrightarrow \mathbb{Z}/2(n) \longrightarrow 0$$

for $j = m$, $m+1$. The horizontal arrows in Diagram (7.12) are injective. In this way we get a commutative diagram

$$\begin{array}{ccc} H^0_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2(n)) & \longrightarrow & H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(n)) \\ \downarrow = & & \downarrow \\ H^0_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2(n)) & \xrightarrow{\partial} & H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2(n)) \cong \langle -1\,,\,2\rangle \otimes \mathbb{Z}/2(n-1) \end{array} \tag{7.13}$$

which implies that $H^0_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2(n))$ maps isomorphically to the torsion subgroup of $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(n))$. On the other hand, the diagrams (7.9), (7.10), (7.12), (7.13) show that the generator of $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(n))_2$ maps to the element $(-1)\otimes \xi_2^{\otimes(n-1)}$ in $\langle -1\,,\,2\rangle \otimes \mathbb{Z}/2(n-1)$. □

Lemma 7.10. *The elements e_1 and e_2 are sent by the transfer map Tr_{E_2/E_1} to the same nontorsion element e whose image $\widetilde{e}$ in the $\hat{\mathbb{Z}_2}$-module $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(n))/\text{torsion} \cong \hat{\mathbb{Z}_2}$ is a generator. In addition, the element e is sent to $2\otimes\xi_2^{\otimes(n-1)}$ via the map ϕ of Lemma 7.9.*

Proof. Since the transfer map Tr_{E_2/E_1} commutes with the complex conjugation, we have:

$$Tr_{E_2/E_1}(e_2) = Tr_{E_2/E_1}(e_1^\tau) = (Tr_{E_2/E_1}(e_1))^\tau = Tr_{E_2/E_1}(e_1)\,.$$

Let us put $e = Tr_{E_2/E_1}(e_1) \in \mathcal{C}^{\text{ét}}(n-1)$. First we prove that e is nontorsion in the group $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(n))$. The composition

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\hat{\mathbb{Z}_2}(n))/\text{torsion} \xrightarrow{\chi} H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2},i];\hat{\mathbb{Z}_2}(n))/\text{torsion} \xrightarrow{Tr_{E_2/E_1}} H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\hat{\mathbb{Z}_2}(n))/\text{torsion}\,,$$

where χ is induced by the inclusion $\mathbb{Z}[\frac{1}{2}] \hookrightarrow \mathbb{Z}[\frac{1}{2},i]$, is multiplication by 2 because of the basic property of the transfer. Let g be a generator of the group $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(n))/\text{torsion}$. Then $\chi(g) = a\,\widetilde{e_1}$ for some $a \in \hat{\mathbb{Z}_2}$, where $\widetilde{e_1}$ denotes the image of e_1 in the group $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2},i];\hat{\mathbb{Z}_2}(n))/\text{torsion}$. Hence, $2g = a\,Tr_{E_2/E_1}(\widetilde{e_1})$ and we see that $a = 2^i u$, where $i=0$ or 1 and $u \in (\hat{\mathbb{Z}_2})^\times$. It follows that the element $Tr_{E_2/E_1}(e_1) = e$ is nontorsion. In order to check that $\widetilde{e}$ generates $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(n))/\text{torsion}$ we consider the image of e in the group

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\hat{\mathbb{Z}_2}(n))/2 \cong H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2(n)) \cong \langle -1\,,\,2\rangle \otimes \mathbb{Z}/2(n-1)$$

(see Lemma 7.7). It then follows by the projection formula in étale cohomology that the image of e equals

$$Tr_{E_2/E_1}(\alpha_2^{\text{ét}}((1-i)\otimes\xi_2^{\otimes(n-1)})) = 2\otimes\xi_2^{\otimes(n-1)} \in \langle -1\,,\,2\rangle \otimes \mathbb{Z}/2(n-1),$$

where $\xi_2 = -1$. By Lemma 7.9, the image of the generator of the module $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(n))_2$ is equal to $(-1)\otimes\xi_2^{\otimes(n-1)}$. Hence, the element $\widetilde{e}$ must be a generator of the $\hat{\mathbb{Z}_2}$-module $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(n))/\text{torsion}$. □

Corollary 7.11. *For any odd integer n , there is a natural isomorphism of $\hat{\mathbb{Z}_2}$-modules*

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\hat{\mathbb{Z}_2}(n)) \cong \mathcal{C}^{\text{ét}}(n-1) \oplus \mathbb{Z}/2\,.$$

We keep writing $\widetilde{e}$ for the generator of $\mathcal{C}^{\text{ét}}(n-1)$ which is the image via the last isomorphism of the generator of $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(n))/\text{torsion}$ introduced in Lemma 7.10.

Proof. This follows from Lemmas 7.9 and 7.10. □

8. 2-adic cyclotomic elements and products in $K_*(\mathbb{Z})$

We start this section with a proof of the Quillen-Lichtenbaum conjecture for $\mathbb{Z}$ at the prime 2 wich uses Weibel's computation [41] and the topological approach of Bökstedt [11] and Dwyer-Friedlander [18]. Since the decisive step in the proof of the conjecture was made by V. Voevodsky in [37], we attribute the theorem to him.

Theorem 8.1 (Voevodsky). *The Dwyer-Friedlander map at the prime* 2
$$K_i(\mathbb{Z})\otimes\hat{\mathbb{Z}_2}\longrightarrow K_i^{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}])$$
is an isomorphism for all integers $i\geq 2$.

Proof. Proposition 5.5 above and Proposition 4.2 of [18] show immediately that the map
$$(BGL(\mathbb{Z}[\tfrac{1}{2}])^+)\hat{_2}\simeq(BGL(\mathbb{Z})^+)\hat{_2}\times(S^1)\hat{_2}\longrightarrow\widehat{K}^{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}])$$
which induces the Dwyer-Friedlander map is a homotopy equivalence (here $\widehat{K}^{\text{ét}}(\mathbb{Z}[\frac{1}{2}])$ denotes the étale K-theory space of the ring $\mathbb{Z}[\frac{1}{2}]$, see [17], Definition 4.3, p. 256). This provides the assertion. □

With the next result we complete the investigation of the 2-adic products in $K_*(\mathbb{Z})$ (see Section 5, in particular Remark 5.9).

Theorem 8.2. *The image of the 2-adic product map*
$$\star: K_{8s+1}(\mathbb{Z})\otimes K_{8t+2}(\mathbb{Z})\longrightarrow K_{8(s+t)+3}(\mathbb{Z})\otimes\hat{\mathbb{Z}_2}$$
is cyclic of order 2 *for every* s *and* $t\geq 0$.

Proof. If $s=t=0$, observe that it is known, and also given by Proposition 5.1 (a), that the product map $\star: K_1(\mathbb{Z})\otimes K_2(\mathbb{Z})\to K_3(\mathbb{Z})$ is injective. In the general case, according to Remark 5.9, the image of the product map is at most a cyclic group of order 2. To finish the proof we must show that the product map
$$\star: K_{8s+1}(\mathbb{Z})\otimes K_{8t+2}(\mathbb{Z})\longrightarrow K_{8(s+t)+3}(\mathbb{Z})\otimes\hat{\mathbb{Z}_2}$$
is nontrivial. The étale K-theory has a well-defined product which is compatible with the Dwyer-Friedlander map. We denote the product in algebraic and in étale K-theory by the same symbol $\star$. Hence, we have the following commutative diagram (see [17], Proposition 4.4, p. 256)

$$\begin{array}{ccc} K_{8s+1}(\mathbb{Z})\otimes K_{8t+2}(\mathbb{Z}) & \xrightarrow{\ \star\ } & K_{8(s+t)+3}(\mathbb{Z}) \\ \downarrow & & \downarrow \\ K^{\text{ét}}_{8s+1}(\mathbb{Z}[\frac{1}{2}])\otimes K^{\text{ét}}_{8t+2}(\mathbb{Z}[\frac{1}{2}]) & \xrightarrow{\ \star\ } & K^{\text{ét}}_{8(s+t)+3}(\mathbb{Z}[\frac{1}{2}])\,. \end{array}\tag{8.1}$$

The Dwyer-Friedlander spectral sequence ([17], Proposition 5.1, p. 260) provides the isomorphisms
$$K^{\text{ét}}_{8s+1}(\mathbb{Z}[\tfrac{1}{2}])\cong H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\hat{\mathbb{Z}_2}(4s+1))\tag{8.2}$$
$$K^{\text{ét}}_{8t+2}(\mathbb{Z}[\tfrac{1}{2}])\cong H^2_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\hat{\mathbb{Z}_2}(4t+2))\tag{8.3}$$
and the short exact sequence
$$0\longrightarrow H^3_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\hat{\mathbb{Z}_2}(4(s+t)+3))\longrightarrow K^{\text{ét}}_{8(s+t)+3}(\mathbb{Z}[\tfrac{1}{2}])\longrightarrow H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\hat{\mathbb{Z}_2}(4(s+t)+2))\longrightarrow 0\,.\tag{8.4}$$

The spectral sequence is compatible with the products in étale K-theory and the cup product in étale cohomology by [17], Proposition 5.4, p.262. Hence, we get the commutative diagram

$$(8.5)\qquad \begin{array}{ccc} H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(4s+1))\otimes H^2_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(4t+2)) & \xrightarrow{\ \cup\ } & H^3_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(4(s+t)+3)) \\ \downarrow & & \downarrow \\ K^{\text{ét}}_{8s+1}(\mathbb{Z}[\frac{1}{2}])\otimes K^{\text{ét}}_{8t+2}(\mathbb{Z}[\frac{1}{2}]) & \xrightarrow{\ \star\ } & K^{\text{ét}}_{8(s+t)+3}(\mathbb{Z}[\frac{1}{2}])\,. \end{array}$$

The isomorphisms (8.2) and (8.3), the exact sequence (8.4) and Diagram (8.5) show that the composition of the upper horizontal arrow with the right vertical arrow in Diagram (8.1) has image contained in $H^3_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(4(s+t)+3))$ considered as subgroup of $K^{\text{ét}}_{8(s+t)+3}(\mathbb{Z}[\frac{1}{2}])$ via the exact sequence (8.4). We know by Table 1 of [41] that the 2-torsion subgroups of $K_{8s+1}(\mathbb{Z})$ and of $K_{8t+2}(\mathbb{Z})$ are cyclic of order 2 for all s, $t\geq 0$. Let x_s denote the generator of the summand $\mathbb{Z}/2$ in the group $K_{8s+1}(\mathbb{Z})$ and z_t the generator of $\mathbb{Z}/2$ in $K_{8t+2}(\mathbb{Z})$. We want to show that the element $x_s\star z_t$ does not vanish. In order to simplify the notation, if M is a Galois module and k an integer, let us write $M(k)$ for $M\otimes\mathbb{Z}/16(k)$. Then consider the commutative diagram

$$(8.6)\qquad \begin{array}{ccc} H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(4s+1))\otimes H^2_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(4t+2)) & \xrightarrow{\ \cup\ } & H^3_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(4(s+t)+3)) \\ \downarrow & & \downarrow \\ H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/16(1))(4s)\otimes H^2_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/16(2))(4t) & \xrightarrow{\ \cup\ } & H^3_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/16(3))(4(s+t))\,, \end{array}$$

where the vertical arrows are the reduction of the coefficients mod 16. We shall show that the image of $x_s\star z_t$ in the group $H^3_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/16(4(s+t)+3))$ via Diagrams (8.1), (8.5) and (8.6) is nontrivial. Because the Dwyer-Friedlander map is an isomorphism according to Theorem 8.1, we can check that the maps coming from the Dwyer-Friedlander spectral sequence

$$(8.7)\qquad \begin{array}{ccc} K_{8s+1}(\mathbb{Z})\otimes\hat{\mathbb{Z}_2} & \longrightarrow & H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(4s+1)) \\ K_{8t+2}(\mathbb{Z})\otimes\hat{\mathbb{Z}_2} & \longrightarrow & H^2_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(4t+2)) \end{array}$$

are isomorphisms. It follows that the elements x_s and z_t map to the 2-torsion elements in the cohomology groups via the above maps. Let us call their images x'_s and z'_t respectively. We also have an isomorphism $K_2(\mathbb{Z})\cong H^2_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(2))$. Since $H^3_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\hat{\mathbb{Z}_2}(k))=0$ for k even (see [19], proof of Corollary 1), this implies that the map

$$(8.8)\qquad K_2(\mathbb{Z})/16\longrightarrow H^2_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/16(2))$$

is an isomorphism. On the other hand, $K_2(\mathbb{Z})$ is a cyclic group of order 2 generated by the Steinberg symbol $\{-1,-1\}$. Obviously the map (8.8) commutes with products, so $\{-1,-1\}$ maps by (8.8) to $(-1)\cup(-1)$, which is nontrivial. Both target and source of the map (8.8) are isomorphic to $\mathbb{Z}/2$. So we observe that z'_t goes to an element of the form $[(-1)\cup(-1)]\otimes w_t$ in $H^2_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/16(2))(4t)$, where w_t stands for a generator of the module $\mathbb{Z}/16(4t)$. Now let us have a look at the image of x'_s in $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/16(1))(4s)$. We know by (7.11) that

$$(8.9)\qquad H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/16(1))=\langle -1,2\rangle/16\cong\mathbb{Z}/2\oplus\mathbb{Z}/16\,.$$

The $\mathbb{Z}/2$ summand above is generated by -1 and the $\mathbb{Z}/16$ summand by 2. The group $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/16(1))$ has only three nontrivial elements of order 2, namely -1, 8 and -8. Via the natural reduction map $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/16(1)) \longrightarrow H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2(1))$ the element -1 goes to -1, 8 goes to 2 and -8 goes to -2. Hence we deduce from Lemma 7.9 that x'_s maps to $(-1) \otimes w_s \in H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/16(1))(4s)$, where w_s is a generator of $\mathbb{Z}/16(4s)$. By Diagrams (8.1), (8.5) and (8.6), the product $x_s \star z_t$ maps to the element $[(-1) \cup (-1) \cup (-1)] \otimes w_{s+t}$ in $H^3_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/16(3))(4(s+t))$, where $w_{s+t} = w_s \otimes w_t$ is a generator of $\mathbb{Z}/16(4(s+t))$. But the element $(-1)\cup(-1)\cup(-1)$ is the image of the symbol $\{-1,-1,-1\}$ which comes from $(-1) \otimes \{-1,-1\}$ via the product map $K_1(\mathbb{Z}) \otimes K_2(\mathbb{Z}) \longrightarrow K_3(\mathbb{Z})$. The injectivity of that map (see Proposition 5.1 (a)) implies that $\{-1,-1,-1\} \neq 0$. Consider finally the commutative diagram

$$
\begin{array}{ccc}
K_3^M(\mathbb{Z}[\frac{1}{2}])/16 & \longrightarrow & H^3_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/16(3)) \\
\downarrow & & \downarrow \\
K_3^M(\mathbb{Q})/16 & \xrightarrow{\cong} & H^3_{\text{ét}}(\mathbb{Q};\mathbb{Z}/16(3))\,,
\end{array}
\tag{8.10}
$$

where the horizontal arrows are the Galois symbol maps. The lower horizontal map is an isomorphism by Theorem 5.1 of [37]. The element $\{-1,-1,-1\}$ considered in $K_3^M(\mathbb{Z}[\frac{1}{2}])/16$ maps nontrivially to $K_3^M(\mathbb{Q})/16$. Its image in $H^3_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/16(3))$ is equal to $(-1)\cup(-1)\cup(-1)$ and must be nontrivial by Diagram (8.10). This proves that $x_s \star z_t$ does not vanish. $\square$

Our next goal in this section is to use the 2-adic cyclotomic elements constructed in Section 7 to reprove parts of Theorem 8.1 and of our results on the other 2-adic products in $K_*(\mathbb{Z})$ given in Section 5.

Proposition 8.3. *The natural homomorphism*

$$K_{2n-1}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2} \longrightarrow H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\hat{\mathbb{Z}_2}(n))$$

of Dwyer and Friedlander provides a surjective homomorphism (respectively an isomorphism)

$$\mathcal{C}(n-1) \longrightarrow \mathcal{C}^{\text{ét}}(n-1)\,.$$

when $n \geq 3$ *and* $n \equiv 1 \mod 4$ *(respectively* $n \equiv 3 \mod 4$*).*

Proof. Let F be a number field containing $\mathbb{Q}(i)$ and S a finite set of prime ideals containing prime ideals over 2 in the ring of integers $\mathcal{O}_F$ of F. The ring $\mathcal{O}_S$ denotes as usual the ring of S-integers in F. By [17], Proposition 4.4, p. 256, there is a surjective homomorphism

$$K_{2n-1}(\mathcal{O}_S) \otimes \hat{\mathbb{Z}_2} \longrightarrow K^{\text{ét}}_{2n-1}(\mathcal{O}_S) \xrightarrow{\cong} H^1_{\text{ét}}(\mathcal{O}_S;\hat{\mathbb{Z}_2}(n)) \tag{8.11}$$

obtained by taking the inverse limit of the homomorphisms

$$K_{2n-1}(\mathcal{O}_S;\mathbb{Z}/2^m) \longrightarrow K^{\text{ét}}_{2n-1}(\mathcal{O}_S;\mathbb{Z}/2^m) \xrightarrow{\cong} H^1_{\text{ét}}(\mathcal{O}_S;\mathbb{Z}/2^m(n))\,. \tag{8.12}$$

The maps (8.12) commute with products for $m \geq 2$, and with transfers and reductions of coefficients for $m \geq 1$. For $m \geq 2$ we have the following commutative diagram:

$$
\begin{array}{ccc}
C'_m \otimes \mathbb{Z}/2^m(n-1) & \xrightarrow{\alpha_m} & K_{2n-1}(R'_m;\mathbb{Z}/2^m) \\
\downarrow{=} & & \downarrow \\
C'_m \otimes \mathbb{Z}/2^m(n-1) & \xrightarrow[\cong]{\alpha_m^{\text{ét}}} & K^{\text{ét}}_{2n-1}(R'_m;\mathbb{Z}/2^m)\,.
\end{array}
\tag{8.13}
$$

Consequently, by Definitions 7.4 and 7.6 we obtain a surjective map

$$\mathcal{C}(n-1) \longrightarrow \mathcal{C}^{\text{ét}}(n-1)\,. \tag{8.14}$$

If $n \equiv 3 \mod 4$, then $K_{2n-1}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2} \cong \hat{\mathbb{Z}_2}$ by Table 1 of [41], and the map (8.14) is also injective. $\square$

Now, let us formulate the following corollary the proof of which does not use Theorem 8.1.

Corollary 8.4. *For any integer $i \geq 2$, the homomorphism*

$$K_i(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}/\text{torsion} \longrightarrow K_i^{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}])/\text{torsion}$$

induced by the Dwyer-Friedlander map is an isomorphism.

Proof. For $i = 2n-1$ with n odd ≥ 3, Proposition 8.3. implies that the map is surjective since

$$K_{2n-1}^{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}])/\text{torsion} \cong \mathcal{C}^{\text{ét}}(n-1) \cong \hat{\mathbb{Z}_2}$$

by Corollary 7.11. The map is injective because

$$K_{2n-1}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}/\text{torsion} \cong \hat{\mathbb{Z}_2}$$

by [12], and a surjective homomorphism of $\hat{\mathbb{Z}_2}$-modules $\hat{\mathbb{Z}_2} \to \hat{\mathbb{Z}_2}$ must be injective. This holds also in dimensions $i = 2n-1$, with n even, and i even because the corresponding algebraic and étale K-groups are finite according to [12] and [19], Theorem 1. $\square$

Now let us deduce parts of Theorems 5.6 and 5.7 from a simple calculation of Steinberg symbols and Weibel's computation of the 2-torsion of $K_*(\mathbb{Z})$ (Table 1 of [41]). First of all observe that by Corollary 8.4, Corollary 7.11 and Proposition 8.3, we have the following sequence of isomorphisms for any odd $n \geq 3$:

$$K_{2n-1}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}/\text{torsion} \cong K_{2n-1}^{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}])/\text{torsion} \cong H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \hat{\mathbb{Z}_2}(n))/\text{torsion} \cong \mathcal{C}^{\text{ét}}(n-1)\,. \tag{8.15}$$

Consider the generator $\widetilde{e}$ of $\mathcal{C}^{\text{ét}}(n-1)$ given by Corollary 7.11. It is clear that there is a nontorsion element $b_n \in K_{2n-1}(\mathbb{Z})$ which is sent by (8.15) onto $\widetilde{e}$. We observe that for n odd ≥ 3, the element b_n can be chosen in such a way that the image of b_n in $K_{2n-1}(\mathbb{Z})/\text{torsion} \cong \mathbb{Z}$ is an odd multiple of a generator. Hence, we can assume for our computations that the image of b_n in $K_{2n-1}(\mathbb{Z})/\text{torsion}$ is a generator.

Proposition 8.5. *Assume that n and m are odd integers ≥ 3.*
(a) The product map

$$\star : K_1(\mathbb{Z}) \otimes K_{2m-1}(\mathbb{Z}) \longrightarrow K_{2m}(\mathbb{Z})$$

sends the subgroup $\langle -1 \rangle \otimes \langle b_m \rangle$ to zero.
(b) The 2-adic product map

$$\star : K_{2n-1}(\mathbb{Z}) \otimes K_{2m-1}(\mathbb{Z}) \longrightarrow K_{2(n+m-1)}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2}$$

sends the subgroup $\langle b_n \rangle \otimes \langle b_m \rangle$ to zero.

Proof. Since we are only interested in the image of the product (a) in the 2-torsion part of $K_{2m}(\mathbb{Z})$, it is sufficient to consider the following commutative diagram

$$\begin{array}{ccc} K_1(\mathbb{Z})/4 \otimes K_{2m-1}(\mathbb{Z})/4 & \xrightarrow{\ \star\ } & K_{2m}(\mathbb{Z})_2 \\ \downarrow & & \downarrow \\ K_1(\mathbb{Z})/4 \otimes K_{2m-1}(\mathbb{Z};\mathbb{Z}/4) & \xrightarrow{\ \star\ } & K_{2m}(\mathbb{Z};\mathbb{Z}/4)\,. \end{array} \tag{8.16}$$

According to Table 1 of [41], $K_{2m}(\mathbb{Z}) \otimes \hat{\mathbb{Z}_2} \cong \mathbb{Z}/2$ if $m \equiv 1 \mod 4$ and the right vertical arrow is a well defined embedding. Hence, to prove claim (a) we need to show that $\langle -1\rangle \otimes \langle b_m\rangle$ maps to zero in $K_{2m}(\mathbb{Z};\mathbb{Z}/4)$ via the homomorphisms in Diagram (8.16). The left vertical arrow in Diagram (8.16) maps the element $(-1)\otimes b_m$ to

$$(-1)\otimes Tr_{E_2/E_1}(c\star\beta_2^{\star(m-1)})$$

for some $c \in C_2'$. Sending it further to $K_{2m}(\mathbb{Z};\mathbb{Z}/4)$ along the lower horizontal arrow and using the projection formula and associativity of the product for K-groups with $\mathbb{Z}/4$ coefficients we find out that its image equals

$$Tr_{E_2/E_1}(((-1)\star c)\star\beta_2^{\star(m-1)}).$$

The product $\star : K_1(R)\otimes K_1(R) \to K_2(R)$ with integral coefficients is just the Steinberg symbol $\{\,,\}$ for any commutative ring with identity R (see [22], Proposition 2.2.3). Hence, we get:

$$\begin{aligned}(-1)\star(1-i) &= \{-1,1-i\} = \{i,1-i\}^2 = 1\\ (-1)\star(1+i) &= \{-1,1+i\} = \{-i,1+i\}^2 = 1.\end{aligned}$$

But the group C_2' is generated by $1-i$ and $1+i$, hence $(-1)\star c = 1$. This proves (a) for $m \equiv 1 \mod 4$ and there is nothing to prove if $m \equiv 3 \mod 4$ since $K_{2m}(\mathbb{Z})_2 = 0$ by [41].

In order to prove (b) consider the commutative diagram

(8.17)
$$\begin{array}{ccc} K_{2n-1}(\mathbb{Z})/4\otimes K_{2m-1}(\mathbb{Z})/4 & \xrightarrow{\star} & K_{2(n+m-1)}(\mathbb{Z})_2 \\ \downarrow & & \downarrow \\ K_{2n-1}(\mathbb{Z};\mathbb{Z}/4)\otimes K_{2m-1}(\mathbb{Z};\mathbb{Z}/4) & \xrightarrow{\star} & K_{2(n+m-1)}(\mathbb{Z};\mathbb{Z}/4)\,. \end{array}$$

It is enough to show that the image of $b_n\otimes b_m$ in the upper left corner of Diagram (8.17) maps to the trivial element in the lower right corner on its journey through the diagram. Indeed, $b_n\otimes b_m$ is sent to

$$Tr_{E_2/E_1}(c_1\star\beta_2^{\star(n-1)})\otimes Tr_{E_2/E_1}(c_2\star\beta_2^{\star(m-1)})$$

via the left vertical arrow in Diagram (8.17) for some $c_1, c_2 \in C_2'$. Sending the last element along the bottom horizontal arrow and using projection formula we get the element

$$Tr_{E_2/E_1}(c_1\star\beta_2^{\star(n-1)}\star c_2c_2^{\tau}\star\beta_2^{\star(m-1)}),$$

where τ denotes the complex conjugation. Since β_2 commutes with elements $c \in K_1(R)$, we obtain the following equality:

$$c_1\star\beta_2^{\star(n-1)}\star c_2c_2^{\tau}\star\beta_2^{\star(m-1)} = \pm c_1\star c_2c_2^{\tau}\star\beta_2^{\star(n+m-2)}.$$

On the other hand, we have the following equalities involving Steinberg symbols:

(8.18)
$$\begin{aligned}&(1-i)\star(1-i) = \{1-i,1-i\} = \{1-i,-1\} = \{1-i,i\}^2 = 1\\ &(1-i)\star(1+i) = \{1-i,1+i\} = \{1-i,i(1-i)\} = \{1-i,i\}\{1-i,1-i\} = 1\\ &(1+i)\star(1-i) = \{1-i,1+i\}^{-1} = 1\\ &(1+i)\star(1+i) = \{1+i,1+i\} = \{1+i,i(1-i)\}\\ &\qquad = \{1+i,i\}\{1+i,1-i\} = \{1+i,i\} = \{1+i,-1\} = 1\end{aligned}$$

The group C_2' is generated by $1-i$ and $1+i$ (see Example 7.2). Hence, by bilinearity of the Steinberg symbol and by (8.18) we get $c_1\star c_2c_2^{\tau} = \{c_1, c_2c_2^{\tau}\} = 1$ and it follows that

$$Tr_{E_2/E_1}(c_1\star\beta_2^{\star(n-1)}\star c_2c_2^{\tau}\star\beta_2^{\star(m-1)}) = 1\,.$$

So $b_n \otimes b_m$ maps to zero in $K_{2(n+m-1)}(\mathbb{Z})_2$ via the horizontal arrow of Diagram (8.17) because the right vertical arrow is injective. $\square$

Remark 8.6. We can prove claim (a) in Proposition 8.5 without even knowing the 2-torsion of the group $K_{2m}(\mathbb{Z})$. In order to see this, fix $s > 1$ such that 2^s annihilates the group $K_{2m}(\mathbb{Z})_2$. Consider the commutative diagram

$$\begin{array}{ccc} K_1(\mathbb{Z}) \otimes K_{2m-1}(\mathbb{Z}) & \xrightarrow{\star} & K_{2m}(\mathbb{Z})_2 \\ \downarrow & & \downarrow \\ K_1(\mathbb{Z})/2^s \otimes K_{2m-1}(\mathbb{Z};\mathbb{Z}/2^s) & \xrightarrow{\star} & K_{2m}(\mathbb{Z};\mathbb{Z}/2^s)\,, \end{array}$$

where the right vertical arrow is injective by assumptions. Now we argue in the same way as in the proof of (a) in Proposition 8.5. The crucial point is to prove that the element $Tr_{E/\mathbb{Q}}((-1)\star u_s \star \beta_s^{\star(m-1)}) \in K_{2m}(\mathbb{Z};\mathbb{Z}/2^s)$ is trivial for any $u_s \in C'_s$. But we know the generators of the group C'_s by Remark 7.1, so it is enough to observe the following equalities on Steinberg symbols:

$$\{-1,\, \xi_{2^s}\} = \{-\xi_{2^s},\, \xi_{2^s}\}^{2^{s-1}} = 1 \quad \text{and} \quad \{-1,\, 1-\xi_{2^s}^a\} = \{\xi_{2^s}^a,\, 1-\xi_{2^s}^a\}^{2^{s-1}} = 1\,.$$

Let us conclude this section by the following observation about the relationships between products in algebraic K-theory and the Dwyer-Friedlander map.

Proposition 8.7. *For any odd prime l and any integer $n \geq 2$, the image of the product map*

$$\star : K_{2n-1}(\mathbb{Z}) \otimes K_{2n-1}(\mathbb{Z}) \longrightarrow K_{4n-2}(\mathbb{Z})$$

is contained in the kernel of the Dwyer-Friedlander map $K_{4n-2}(\mathbb{Z}) \to K_{4n-2}^{\text{ét}}(\mathbb{Z}[\frac{1}{l}])$.

Proof. The products in algebraic K-theory and étale K-theory commute with the Dwyer-Friedlander map. Observe that

$$K_{2n-1}^{\text{ét}}(\mathbb{Z}[\tfrac{1}{l}]) = \begin{cases} \hat{\mathbb{Z}_l} & \text{if } n \text{ is odd,} \\ \mathbb{Z}/|w_n(\mathbb{Q})|_l^{-1} & \text{if } n \text{ is even.} \end{cases}$$

Hence $K_{2n-1}^{\text{ét}}(\mathbb{Z}[\frac{1}{l}]) \cong H_{\text{ét}}^1(\mathbb{Z}[\frac{1}{l}];\mathbb{Z}_l(n))$ is cyclic. But the product in étale K-theory is just the cup product in étale cohomology. This shows that the product

$$K_{2n-1}^{\text{ét}}(\mathbb{Z}[\tfrac{1}{l}]) \otimes K_{2n-1}^{\text{ét}}(\mathbb{Z}[\tfrac{1}{l}]) \longrightarrow K_{4n-2}^{\text{ét}}(\mathbb{Z}[\tfrac{1}{l}])$$

is zero because the cup product $H_{\text{ét}}^1 \otimes H_{\text{ét}}^1 \to H_{\text{ét}}^2$ is anticommutative. $\square$

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Dominique Arlettaz
Institut de mathématiques, Université de Lausanne, CH-1015 Lausanne, Switzerland
e-mail: dominique.arlettaz@ima.unil.ch

Grzegorz Banaszak
Department of Mathematics, Adam Mickiewicz University, 60769 Poznań, Poland
e-mail: BANASZAK@math.amu.edu.pl

Wojciech Gajda
current address: Max-Planck-Institut für Mathematik, 53235 Bonn, Germany
permanent address: Department of Mathematics, Adam Mickiewicz University, 60769 Poznań, Poland
e-mail: GAJDA@math.amu.edu.pl