REPRESENTATION THEORY OF HOMOTOPY TYPES WITH AT MOST TWO NON TRIVIAL HOMOTOPY GROUPS

Hans-Joachim Baues and Yuri Drozd

It is a classical result of Postnikov [15] that homotopy types X with at most two non trivial homotopy groups $\pi_m X = A$ and $\pi_n X = B, 2 \le m < n$, are classified by the k-invariant

$$k_X \in H^{n+1}(K(A,m), B) \tag{1}$$

Here the cohomology group of the Eilenberg-Mac Lane space $K(A,m)$ was computed by Eilenberg-Mac Lane [11] and Cartan [5]. Let p be a prime and let $\mathbb{Z}_p \subset \mathbb{Q}$ be the smallest subring of $\mathbb{Q}$ containing $1/q$ for all primes q with $q \neq p$. We consider finitely generated $\mathbb{Z}_p$-modules A and B and the stable range $n < 2m-1$. Hence X is a p-local space with at most two non trivial homotopy groups in a stable range. Then the homotopy type of X admits a product decomposition

$$X \simeq X_1 \times \ldots \times X_j \tag{2}$$

where all X_i with $1 \le i \le j$ are indecomposable and this decomposition is unique up to permutation. We classify in this paper the indecomposable factors in (2) by the following result.

Theorem (A). *Let $n - m \le (p+1)(2p-2)$. Then the classification of indecomposable factors in* (2) *is*

$$\begin{cases} \textit{wild} & \textit{for } n-m = (p+1)(2p-2), \\ \textit{tame} & \textit{for } n-m = t(2p-2),\ 1 \le t \le p, \\ \textit{tame} & \textit{for } p = 2 \textit{ and } n-m = 3, \textit{ and} \\ \textit{essentially finite} & \textit{otherwise.} \end{cases}$$

For the tame and essentially finite cases we compute below a complete list of all possible indecomposable factors. Moreover we obtain a corresponding result as in theorem (A) also for all cases $q = n - m > (p+1)(2p-2)$. But for this one has to use the explicit computation of the homology $H_q K(A,m)$ by Cartan [5].

Let $\mathbf{A}^k$ be the *stable homotopy category* of $(m-1)$-connected $(m+k)$-dimensional finite CW-complexs, $m \ge k+1$. Then $\mathbf{A}^k$ is an additive category and there is the old problem to classify the indecomposable objects in $\mathbf{A}^k$ or equivalently to determine the "representation type" of $\mathbf{A}^k$. J.H.C. Whitehead [16] and Chang [6] showed that $\mathbf{A}^2$ has essentially finite representation type; compare the books of Hilton [13], [14]. The representation type of $\mathbf{A}^2$ can actually be deduced from theorem (A). Moreover Baues-Hennes [2] showed that $\mathbf{A}^3$ has tame representation type; a complete list of indecomposable objects in $\mathbf{A}^3$ is given in [2]; compare also the book [1]. Now theorem (A) implies

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Corollary (B). $\mathbf{A}^k$ *has wild representation type for* $k \geq 6$.

Hence only the representation type of $\mathbf{A}^4$ and $\mathbf{A}^5$ remain unknown. Theorem (A) and the classification in [1] indicate the complexity of the categories $\mathbf{A}^4$ and $\mathbf{A}^5$. It is, however, amazing that the spaces in $\mathbf{A}^4$ with torsion free homology still have finite representation type, see Baues-Drozd [3]. Moreover Henn [12] showed that the p-local version of $\mathbf{A}^k$ has a tame representation type for $k \leq 4p-5, p$ odd. Theorem (A) overlaps with the result of Baues-Hennes [2] for $p = 2$ and $n - m = 2$ and with the result of Henn [12] for $p = 3$ and $n - m = 4$.

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§ 1 The decomposition problem

Let $\mathbf{C}$ be an additive category with zero object $*$ and biproducts $A \oplus B$. An object X in $\mathbf{C}$ is *decomposable* if there exists an isomorphism $X \cong A \oplus B$ where A and B are not isomorphic to $*$. A *decomposition* of X is an isomorphism

(1.1)
$$X = A_1 \oplus \ldots \oplus A_n \quad \text{with} \quad n < \infty,$$

where A_i is indecomposable for all $i \in \{1, \ldots, n\}$. The decomposition of X is *unique* if $B_1 \oplus \ldots \oplus B_m \cong X \cong A_1 \oplus \ldots \oplus A_n$ implies $m = n$ and that there is a permutation σ with $B_{\sigma(i)} \cong A_i$. A morphism f in $\mathbf{C}$ is indecomposable if F is indecomposable in the category of pairs in $\mathbf{C}$; objects in this category of pairs are the morphisms f in $\mathbf{C}$ and morphisms $(\alpha, \beta) : f \to g$ are pairs of morphisms in $\mathbf{C}$ satisfying $g\alpha = \beta f$. The *decomposition problem* in $\mathbf{C}$ can be described by the following task: find a complete list of indecomposable isomorphism types in $\mathbf{C}$ and describe the possible decomposition of objects in $\mathbf{C}$. This problem is considered by representation theory. We say that the decomposition problem in $\mathbf{C}$ is wild or equivalently that $\mathbf{C}$ has *wild representation type* if the solution of the decomposition problem would imply a solution of the following kind of problem.

(1.2) Problem. Let k be a field and consider the following additive category $\mathbf{V}^{\alpha,\beta}$. Objects are finite dimensional k-vector spaces V together with two endomorphism $\alpha_V, \beta_V : V \to V$. Morphisms are k-linear maps $f : V \to W$ satisfying $f\alpha_V = \alpha_W f$ and $f\beta_V = \beta_W f$. The decomposition problem in $\mathbf{V}^{\alpha,\beta}$ for any field k is termed a "wild problem of representation theory".

On the other hand we say that $\mathbf{C}$ has *tame representation type* if a complete list of indecomposable objects of $\mathbf{C}$ is computed. If the number of objects in this list which satisfy a given finiteness restraint is finite then we say that $\mathbf{C}$ has *essentially finite* representation type. If the list is finite then $\mathbf{C}$ has *finite* representation type.

For example consider the category of finitely generated (*f.g.*) abelian groups; this category has essentially finite representation type since the list of indecomposable objects is given by the indecomposable cyclic groups $\mathbb{Z}$ and $\mathbb{Z}/p^i$ where p is a prime and $i \geq 1$. The *finiteness restraint* is given by a number N which bounds the order of the torsion subgroup. In fact, it is well known that a *f.g.* abelian group A admits a decomposition

(1.3)
$$A \cong C_1 \oplus \ldots \oplus C_t$$

where C_i for $i = 1, \dots, t$ is an indecomposable cyclic group and this decomposition is unique. We say that t is the *rank* of A and that the *f.g.* abelian group A is

$$\begin{cases} \textit{torsion}: & \text{if } C_i \in \{\mathbb{Z}/p^j,\ j \geq 1,\ p \text{ prime}\} \\ p\textit{-torsion}: & \text{if } C_i \in \{\mathbb{Z}/p^j,\ j \geq 1\} \\ \textit{elementary}: & \text{if } C_i \in \{\mathbb{Z}/p,\ p \text{ prime}\} \\ p\textit{-elementary}: & \text{if } C_i \in \{\mathbb{Z}/p\} \\ \textit{free}: & \text{if } C_i \in \{\mathbb{Z}\} \\ p\textit{-primary}: & \text{if } C_i \in \{\mathbb{Z},\ \mathbb{Z}/p^j,\ j \geq 1\} \end{cases}$$

§ 2 The classification theorem

Let p be a prime. We say that a simply connected CW-complex X is <u>p-local finite type</u> if all homotopy groups or equivalently all homology groups of X are finitely generated $\mathbb{Z}_p$-modules. Moreover the p-<u>local dimension</u> of X satisfies $\dim_p(X) \leq n$ if $H_i X = 0$ for $i > n$ and if $H_n X$ is a free $\mathbb{Z}_p$-module. Let $2 \leq m$. We define homotopy categories together with a Postnikov functor

$$\phi : \mathbf{K}(m,n)_p \to \mathbf{k}(m,n)_p \tag{2.1}$$

Here the objects of $\mathbf{k}(m,n)_p$ are CW-complexes X with at most two non trivial homotopy groups $\pi_m X = A$ and $\pi_n X = B$ such that A and B are finitely generated $\mathbb{Z}_p$-modules. Moreover the objects of $\mathbf{K}(m,n)_p$ are p-local finite type CW-complexes Y with $\dim_p(Y) \leq n+1$ and $\pi_i Y = 0$ for $m < i < n$. The functor ϕ in (2.1) carries Y to the $n-th$ section of the Postnikov tower of Y. The categories in (2.1) are both additive categories and ϕ is an additive functor. The biproduct of $X_1, X_2 \in \mathbf{k}(m,n)_p$ is the product $X_1 \times X_2$ of spaces and the biproduct of $Y_1, Y_2 \in \mathbf{K}(m,n)_p$ is the one point union $Y_1 \vee Y_2$ of spaces.

(2.2) Proposition. *Objects $X \in \mathbf{k}(m,n)_p$ and $Y \in \mathbf{K}(m,n)_p$ with $2 \leq m < n < 2m-1$ have unique decompositions*

$$X = X_1 \times \ldots \times X_t$$
$$Y = Y_1 \vee \ldots \vee Y_r$$

where the X_i and Y_j are indecomposable. Moreover the additive functor ϕ yields a bijection of the set of indecomposable homotopy types in $\mathbf{k}(m,n)_p$ and the set of indecomposable homotopy types Y in $\mathbf{K}(m,n)_p$ with $Y \neq S^{n+1}_p$ where S^{n+1}_p is the p-local sphere of dimension $n+1$.

Proof. The unique decomposition is well known, see for example [17]. Hence the bijection is obtained by a theorem of J.H.C. Whitehead on trees of homotopy types see for examples 10.4.4 in [1]. Given an indecomposable $X \in \mathbf{k}(m,n)_p$ we obtain the inverse $\phi^{-1}X = Y$ by the unique indecomposable $Y \in \mathbf{K}(m,n)_p$ for which there exists a set E such that $Y \vee \bigvee_E S^{n+1}_p$ is homotopy equivalent to p-local $(n+1)$-skeleton of X. q.e.d.

The proposition shows that the representation types of the categories $\mathbf{k}(m,n)_p$ and $\mathbf{K}(m,n)_p$ coincide. For the classification of objects in these topological categories we use the following algebraic category; see 3.6.1 [1].

(2.3) Definition. Let $\mathbf{Ab}$ be the category of abelian groups and let $E, F : \mathbf{Ab} \to \mathbf{Ab}$ be functors. Let p be a prime. Then the category $\mathbf{S}(E,F)_p$ is defined as follows. Objects are triple of *f.g.* $\mathbb{Z}_p$-modules (A, R, H) where H is free together with a chain complex

$$S = (H \xrightarrow{b} F(A) \xrightarrow{\partial} R \xrightarrow{\delta} E(A) \to 0) \tag{1}$$

which is exact in $F(A)$ and $E(A)$. Moreover $\delta\partial = 0$. Morphisms $S \to S'$ are triple (a, r, h) of homomorphisms in $\mathbf{Ab}$ for which the following diagram commutes.

$$\begin{array}{ccccccccc} H & \longrightarrow & F(A) & \longrightarrow & R & \xrightarrow{\delta} & E(A) & \longrightarrow & 0 \\ \downarrow h & & \downarrow a_* & & \downarrow r & & \downarrow a_* & & \\ H' & \longrightarrow & F(A') & \longrightarrow & R' & \xrightarrow[\delta']{} & E(A') & \longrightarrow & 0 \end{array} \tag{2}$$

We introduce a natural equivalence relation $\sim$ on $\mathbf{S}(E,F)_p$ by setting $(a,r,h) \sim (a',r',h')$ if $a = a', h = h'$ and

$$r_* = r'_* : \ker(\partial) \to \ker(\partial') \tag{3}$$

We associate with an object S the exact sequence

$$H \xrightarrow{b} F(A) \xrightarrow{\partial} \ker(\delta) \xrightarrow{j} \operatorname{cok}(\partial) \xrightarrow{\delta} E(A) \longrightarrow 0 \tag{4}$$

where j is given by $\ker(\delta) \to R \to \operatorname{cok}(\partial)$. Here $\ker(\delta)$ denotes the kernel of δ and $\operatorname{cok}(\partial)$ denotes the cokernel of ∂. If E and F are additive functors then $\mathbf{S}(E,F)_p$ and also the quotient category $\mathbf{S}(E,F)_p/\sim$ are additive categories. We call S an *Eilenberg-Mac Lane object* if $A = 0$ or $B = \ker(\delta) = 0$ or even $A = B = 0$.

We say that a functor $\varphi : \mathbf{C} \to \mathbf{K}$ between categories is a *detecting functor* if φ reflects isomorphisms and is full and representative. This implies that φ induces a bijection between isomorphism types in $\mathbf{C}$ and $\mathbf{K}$ respectively. For example the quotient functor

$$q : \mathbf{S}(E,F)_p \to \mathbf{S}(E,F)_p/\sim \tag{2.4}$$

is a detecting functor since q is a linear extension of categories, see 3.6.1 (6) [1]. Moreover we prove in 6.5.2 [1] the following result.

(2.5) Classification theorem. *Let p be a prime and let $2 \le m < n < 2m-1$. Then there is an additive detecting functor*

$$\Lambda : \mathbf{K}(m,n)_p \to \mathbf{S}(E,F)_p/\sim$$

where E and F are additive functors given by the homology

$$\begin{cases} E(A) = H_n K(A,m) \\ F(A) = H_{n+1} K(A,m) \end{cases}$$

of an Eilenberg-Mac Lane space $K(A,m)$.

Given $Y \in \mathbf{K}(m,n)_p$ we have the k-invariant

$$
(2.6)\qquad
\begin{aligned}
k_Y &\in H^{n+1}(K(A,m),B)\\
&\overset{\psi}{\cong} \mathrm{Hom}(H_{n+1}K(A,m),B)\oplus \mathrm{Ext}(H_nK(A,m),B)\\
&= \mathrm{Hom}(F(A),B)\oplus \mathrm{Ext}(E(A),B)
\end{aligned}
$$

with $A=\pi_m Y$ and $B=\pi_n Y$. Here the isomorphism ψ in (2.6) can be chosen to be natural in A and B if the assumptions in (2.5) hold, see Decker [7]. Hence the k-invariant k_Y yields the coordinates

$$
(2.7)\qquad
\begin{cases}
k_Y^1 \in \mathrm{Hom}(F(A),B)\\
k_Y^2 \in \mathrm{Ext}(E(A),B)
\end{cases}
$$

Now the functor Λ in (2.5) has the property that for $S=\Lambda(Y)$ there is a natural commutative diagram

(2.8)

$$
\begin{tikzcd}
 & H_{n+1}K(A,m) \arrow[d, equal] & B \arrow[d, equal] & & H_nK(A,m) \arrow[d, equal] & \\
H \arrow[r] \arrow[d, equal] & F(A) \arrow[r, "\partial"] \arrow[d, equal] & \ker\delta \arrow[r, "j"] \arrow[d, equal] & \mathrm{cok}(\partial) \arrow[r, "\delta"] \arrow[d, equal] & E(A) \arrow[r] \arrow[d, equal] & 0 \\
H_{n+1}Y \arrow[r] & \Gamma_n Y \arrow[r] & \pi_n Y \arrow[r] & H_n Y \arrow[r] & \Gamma_{n-1}Y \arrow[r] & 0
\end{tikzcd}
$$

Here the bottom row is part of the exact sequence of J.H.C. Whitehead, see 6.5.3 [1]. Moreover k_Y^1 coincides with ∂ in (2.8) and k_Y^2 coincides with the extension element $\{R\}\in \mathrm{Ext}(E(A),B)$ given by the chain complex $S=\Lambda(Y)$. This shows that the k-invariant k_Y can be readily deduced from the chain complex $S=\Lambda(Y)$.

We say that an additive functor $E:\mathbf{Ab}\to\mathbf{Ab}$ is *elementary* if there exist elementary *f.g.* abelian groups E_1 and E_2 (see (1.3)) such that

$$
(2.9)\qquad E(A) = A\otimes E_1 \oplus A * E_2
$$

where we use the tensor product $A\otimes B$ and the torsion product $A*B$ of abelian groups A,B. Moreover E is *p-elementary* if E_1 and E_2 are p-elementary.

(2.10) Proposition. *Consider the functor* $\mathbf{Ab}\to\mathbf{Ab}$ *which carries A to the homology $H_nK(A,m)$ of the Eilenberg-Mac Lane space $K(A,m)$. For $m<n<2m$ this functor is elementary.*

In fact, Cartan showed in theorem 2 of section 11 [5] that for $q<m$

$$
H_{m+q}K(A,m) = A\otimes E_1 \oplus A*E_2
$$

with $E_1=\bigoplus\{E_1^p,\ p \text{ prime}\}$ and $E_2=\bigoplus\{E_2^p,\ p\text{ prime}\}$. Here E_1^p (resp. E_2^p) is the $\mathbb{Z}/p$-vector space with the basis consisting of "admissible p-words $(a_1,a_2,\dots)$ of the first (resp. second) kind of stable degree q such that $a_1\equiv 0 \pmod{2p-2}$". In particular we can deduce from Cartan's result the following addendum which is of importance for the proof of theorem (A).

(2.11) Addendum. *Let p be a prime and let $m < n < 2m$ and $m - n \leq (p+1)(2p-2)$. Then*

$$\mathbb{Z}/p * H_n K(A,m) = \begin{cases} A \otimes (\mathbb{Z}/p \oplus \mathbb{Z}/p) & \text{if} \quad m-n = (p+1)(2p-2) \\ A \otimes \mathbb{Z}/p & \text{if} \quad m-n = t(2p-2),\ 1 \leq t \leq p \\ A * \mathbb{Z}/p & \text{if} \quad m-n = t(2p-2)+1,\ 1 \leq t \leq p \\ 0 & \text{otherwise} \end{cases}$$

The addendum is relevant for the application of theorem (2.5) since one readily obtains the following fact.

(2.12) Lemma. *Let E and F be elementary functors $\mathbf{Ab} \to \mathbf{Ab}$ and let p be a prime. Then there is a canonical isomorphism of additive categories*

$$\mathbf{S}(E,F)_p = \mathbf{S}(\mathbb{Z}/p * E, \mathbb{Z}/p * F)_p$$

where the functors $\mathbb{Z}/p * E$ and $\mathbb{Z}/p * F$ are p-elementary.

Using (2.4), (2.9) and (2.12) we see that the classification of indecomposable objects in $\mathbf{K}(m,n)_p$ is obtained by the classification of indecomposable objects in $\mathbf{S}(E,F)_p$ where E and F are p-elementary functors. We prove below the following result.

(2.13) Theorem. *Let E and F be p-elementary functors given by (E_1, E_2) and (F_1, F_2) respectively. Then the objects in $\mathbf{S}(E,F)_p$ have a unique decomposition and the representation type of $\mathbf{S}(E,F)_p$ is*

- *essentially finite if at most one of the groups E_1, E_3, F_1, F_2 is $\mathbb{Z}/p$ and all the others are trivial,*
- *tame if $E(A) = A \otimes \mathbb{Z}/p$ and $F(A) = A * \mathbb{Z}/p$,*
- *tame if $E(A) = A * \mathbb{Z}/p$ and $F(A) = A \otimes \mathbb{Z}/p$,*
- *wild otherwise.*

For the tame cases the classification of indecomposable objects is given in § 3 below. Using (2.11) the theorem implies theorem (A) in the introduction. Theorem (2.13) is proved in section § 4.

§ 3 The indecomposable objects

We first consider the tame cases of theorem (2.13) and theorem (A) in the introduction.

For the description of indecomposable objects we use certain words. Let L be a set, the elements of which are called "letters". A word with letters in L is an element in the free monoid generated by L. Such a word a is written $a = a_1 a_2 \dots a_n$ with $a_i \in L, n \geq 0$; for $n = 0$ this is the empty word ϕ. Let $b = b_1 \dots b_k$ be a word. We write $w = \dots b$ if there is a word a with $w = ab$, similarly we write $w = b \dots$ if there is a word c with $w = bc$. A *subword* of an infinite sequence $\dots a_{-2} a_{-1} a_0 a_1 a_2 \dots$ with $a_i \in L, i \in \mathbb{Z}$, is a finite connected subsequence $a_n a_{n+1} \dots a_{n+k}, n \in \mathbb{Z}$.

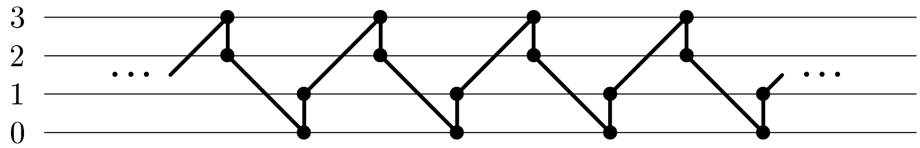
(3.1) Definition. Let p be a prime. We define a collection of finite words $w = w_1 w_2 \dots w_k$. The letters w_i of w are the symbols ξ, η or natural numbers $s_i, r_i, i \in \mathbb{Z}$,

which are powers of p. We write the letters s_i as upper indices, the letters r_i as lower indices since we have to distinguish between these numbers. For example $\xi^2\eta_8\xi^4\eta$ is such a word for $p = 2$. The *basic sequence* is defined by

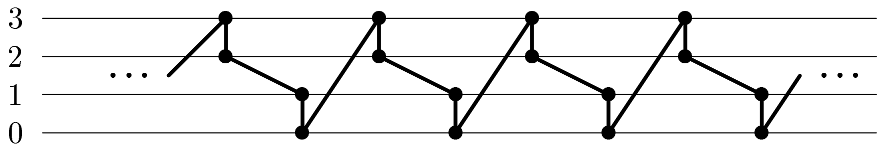
$$\dots^{s_0}\eta_{r_0}\xi^{s_1}\eta_{r_1}\xi^{s_2}\eta_{r_2}\dots \tag{1}$$

This is the infinite product $\dots a(-1)a(0)a(1)a(2)\dots$ of words $a(i) = \xi^{s_i}\eta_{r_i}$ with $i \in \mathbb{Z}$.

The basic sequence (1) can be visualized by infinite graphs in two ways, either by the tensor graph or by the torsion graph:



Tensor graph



Torsion graph

The letters s_i, resp. r_i in the basic sequence correspond to vertical edges conecting levels 2 and 3 resp. the levels 0 and 1. The letters η and ξ correspond in both graphs to diagonal edges: In the tensor graph η connects level 0 and 2 and ξ connects level 1 and 3; in the torsion graph η connects level 1 and 2 and ξ connects level 0 and 3. We consider connected finite subgraphs of the tensor graph and the torsion graph. These subgraphs have an initial vertex at the left hand side and a final vertex at the right hand side.

A *p-tensor word* is any finite subword w of the basic sequence of the form

$$w = a\dots \quad \text{where} \quad a =^{i_1} \eta \quad \text{or} \quad a = \eta \tag{2}$$

This is the same as any connected finite subgraph of the tensor graph containing η with initial vertex at level 2 or 3. Moreover $w \neq^p \eta$ and $w \neq^p \eta_r\xi$ since these words correspond to indecomposable Eilenberg-Mac Lane objects.

A *p-torsion word* is any finite subword w of the basic sequence (1) satisfying

$$\begin{cases} w = a\dots \quad \text{and} \quad w = \dots c \quad \text{where} \\ a = \xi \quad \text{or} \quad a =^{s_i} \eta \quad \text{or} \quad a = \eta \quad \text{and} \\ c = \eta_{r_j} \quad \text{or} \quad c = \xi \quad \text{or} \quad c = \xi^{s_j} \end{cases} \tag{3}$$

This is the same as any connected finite subgraph of the torsion graph containing ξ or η with the property that both the initial vertex and the final vertex are not at level 1. Moreover $w \neq \xi$ and $w \neq^p \eta_t \xi$ since these words correspond to indecomposable Eilenberg-Mac Lane objects.

The *vertices of level* i of a tensor word (resp. torsion word) are defined by the vertices of level i of the corresponding graph; $i \in \{0, 1, 2, 3\}$. We also write $\mid x \mid = i$ if x is a vertex of level i. We call x an ∞ *-vertex* if x is not a vertex of a vertical edge. Such an ∞-vertex of a tensor word w or a torsion word w (if it exists) automatically is the initial or the final vertex of the corresponding subgraph. Let $\beta_i(w)$ be the number of ∞-vertices of level i in the word w; this is the *Betti number* of w satisfying $0 \leq \beta_i(w) \leq 1$ and $\beta_1(w) = 0$. Let $r_\alpha \dots r_\beta$ and $s_\mu \dots s_\nu$ be the words of lower indices and of upper indices respectively given by w. We define the *homology* of w by

$$
\begin{cases}
H_0(w) = \mathbb{Z}_p^{\beta_0(w)} \oplus \mathbb{Z}/r_\alpha \oplus \dots \oplus \mathbb{Z}/r_\beta \\
H_1(w) = 0 \\
H_2(w) = \mathbb{Z}_p^{\beta_2(w)} \oplus \mathbb{Z}/s_\mu \oplus \dots \oplus \mathbb{Z}/s_\nu \\
H_3(w) = \mathbb{Z}_p^{\beta_3(w)}
\end{cases}
\tag{3.2}
$$

It is in the definition of the Betti numbers of w where the difference between the tensor graph and the torsion graph plays an essential role.

(3.3) Definition. We associate with a p-tensor word w a chain complex

$$S(w) \in \mathbf{S}(\otimes\mathbb{Z}/p, *\mathbb{Z}/p)_p$$

Here $S(w)$ is given by $A = H_0(w)$ and the chain complex

$$H_3(w) \xrightarrow{b} A * \mathbb{Z}/p \xrightarrow{\partial} R \xrightarrow{\delta} A \otimes \mathbb{Z}/p \longrightarrow 0$$

with $\operatorname{cok}(\partial) = H_2(w)$. The abelian group R is given as follows. Let $s_\mu \dots s_\nu$ be the word of those upper indices of w which are exponents of ξ; that is for $\mu \leq i \leq \nu$ the word ξ^{s_i} is a subword of w. Then

$$R^\xi(w) = \mathbb{Z}/s_\mu \cdot p \oplus \dots \oplus \mathbb{Z}/s_\nu \cdot p$$

is defined where the order of the cyclic summands is the product $s_i \cdot p$ of s_i and the prime p. Moreover we define

$$R_1^\eta(w) = \begin{cases} \mathbb{Z}_p & \text{if } w = \eta \dots \\ \mathbb{Z}/s & \text{if } w = {}^s\eta \dots \\ 0 & \text{otherwise} \end{cases}$$

$$R_2^\eta(w) = \begin{cases} \mathbb{Z}/p & \text{if } w = \dots {}^s\eta_u \\ 0 & \text{otherwise} \end{cases}$$

Using these pieces we obtain the direct sum

$$R = R^\xi(w) \oplus R_1^\eta(w) \oplus R_2^\eta(w)$$

which is part of the chain complex above. The maps b, ∂, δ in (1) are all given by the canonical maps

$$\mathbb{Z}_p \to \mathbb{Z}/p \quad \text{and} \quad \mathbb{Z}/p \to \mathbb{Z}/p^t \to \mathbb{Z}/p$$

where the cyclic summands correspond to indices in w which are neighbours via ξ or η respectively. For example we have the following list. We write $E(r) = E(\mathbb{Z}/r)$ and $E(\infty) = E(\mathbb{Z}_p)$ and similarly $F(r) = F(\mathbb{Z}/r)$.

w	$H_3(w)$	$\to$	$A * \mathbb{Z}/p$ $\Vert$ $F(A)$	$\to$	R	$\to$	$A \otimes \mathbb{Z}/p$ $\Vert$ $E(A)$
η	0		0		$\mathbb{Z}_p$	$\to$	$E(\infty)$
$s \neq p, {}^s\eta$	0		0		$\mathbb{Z}/s$	$\to$	$E(\infty)$
${}^s\eta_r$	0		0		$\mathbb{Z}/s$	$\to$	$E(r)$
	0		$F(r)$	$=$	$\mathbb{Z}/p$		0
$s \neq p, {}^s\eta_r\xi$	0		0		$\mathbb{Z}/s$	$\to$	$E(r)$
	$\mathbb{Z}_p$	$\to$	$F(r)$		0		0
${}^s\eta_r\xi^t$	0		0		$\mathbb{Z}/s$	$\to$	$E(r)$
	0		$F(r)$	$\to$	$\mathbb{Z}/t \cdot p$		0
${}^s\eta_r\xi^t\eta$	0		0		$\mathbb{Z}/s$	$\to$	$E(r)$
	0		$F(r)$	$\to$	$\mathbb{Z}/t \cdot p$	$\to$	$E(\infty)$
${}^s\eta_r\xi^t\eta_u$	0		0		$\mathbb{Z}/s$	$\to$	$E(r)$
	0		$F(r)$	$\to$	$\mathbb{Z}/t \cdot p$	$\to$	$E(u)$
	0		$F(u)$	$=$	$\mathbb{Z}/p$		0
${}^s\eta_r\xi^t\eta_u\xi$	0		0		$\mathbb{Z}/s$	$\to$	$E(r)$
	0		$F(r)$	$\to$	$\mathbb{Z}/t \cdot p$	$\to$	$E(u)$
	$\mathbb{Z}_p$	$\to$	$F(u)$		0		0
${}^s\eta_r\xi^t\eta_u\xi^v$	0		0		$\mathbb{Z}/s$	$\to$	$E(r)$
	0		$F(r)$	$\to$	$\mathbb{Z}/t \cdot p$	$\to$	$E(u)$
	0		$F(u)$	$\to$	$\mathbb{Z}/t \cdot v$		0

If we omit the initial upper index s in w on the left hand side of the list then we have to replace $\mathbb{Z}/s$ in the list by $\mathbb{Z}_p$.

(3.4) Definition. We associate with a p-torsion word w a chain complex

$$S(w) \in \mathbf{S}(*\mathbb{Z}/p, \otimes\mathbb{Z}/p)_p$$

Here $S(w)$ is again given by $A = H_0(w)$ and the chain complex

$$H_3(w) \xrightarrow{b} A \otimes \mathbb{Z}/p \xrightarrow{\partial} R \xrightarrow{\delta} A * \mathbb{Z}/p \to 0$$

with $\mathrm{cok}\,(\partial) = H_2(w)$. The abelian group R is defined by

$$R = R^{\xi}(w) \oplus R_1^{\eta}(w) \oplus \bar{R}_2^{\eta}(w)$$

where $R^{\xi}(w)$ and $R_1^{\eta}(w)$ are given as in (3.3) and where $\bar{R}_2^{\eta}(w)$ is the following modification of $R_2^{\eta}(w)$ in (3.3):

$$\bar{R}_2^{\eta}(w) = \begin{cases} \mathbb{Z}/p & \text{if } w = \dots \eta_u \\ 0 & \text{otherwise} \end{cases}$$

The maps b, ∂ and δ are again defined by canonical maps as in (3.3). For example we have the following list.

w	$H_3(w)$	$\to$	$A \otimes \mathbb{Z}/p$ $\|$ $F(A)$	$\to$	R	$\to$	$A * \mathbb{Z}/p$ $\|$ $E(A)$
ξ^u	0		$F(\infty)$	$\to$	$\mathbb{Z}/u \cdot p$		0
$\xi^u \eta_t$	0		$F(\infty)$	$\to$	$\mathbb{Z}/u \cdot p$	$\to$	$E(t)$
	0		$F(t)$	$=$	$\mathbb{Z}/p$		0
$\xi^u \eta_t \xi$	0		$F(\infty)$	$\to$	$\mathbb{Z}/u \cdot p$	$\to$	$E(t)$
	$\mathbb{Z}_p$	$\to$	$F(t)$		0		0
$\xi^u \eta_t \xi^v$	0		$F(\infty)$	$\to$	$\mathbb{Z}/u \cdot p$	$\to$	$E(t)$
	0		$E(t)$	$\to$	$\mathbb{Z}/v \cdot p$		0
${}^s\eta_t$	0		0		$\mathbb{Z}/s$	$\to$	$E(t)$
	0		$F(t)$	$=$	$\mathbb{Z}/p$		0
$s \neq p, {}^s\, \eta_t \xi$	0		0		$\mathbb{Z}/s$	$\to$	$E(t)$
	$\mathbb{Z}_p$	$\to$	$F(t)$		0		0
${}^s\eta_t \xi^u$	0		0		$\mathbb{Z}/s$	$\to$	$E(t)$
	0		$F(t)$	$\to$	$\mathbb{Z}/u \cdot p$		0

$$\begin{array}{lccccccc}
 & 0 & & 0 & & \mathbb{Z}/s & \to & E(t) \\
{}^s\eta_t\xi^u\eta_v & 0 & & F(t) & \to & \mathbb{Z}/u\cdot p & \to & E(v) \\
 & 0 & & F(v) & = & \mathbb{Z}/p & & 0
\end{array}$$

$$\begin{array}{lccccccc}
 & 0 & & 0 & & \mathbb{Z}/s & \to & E(t) \\
{}^s\eta_t\xi^u\eta_v\xi & 0 & & F(t) & \to & \mathbb{Z}/u\cdot p & \to & E(v) \\
 & \mathbb{Z}_p & \to & F(v) & & 0 & & 0
\end{array}$$

$$\begin{array}{lccccccc}
 & 0 & & 0 & & \mathbb{Z}/s & \to & E(t) \\
{}^s\eta_t\xi^u\eta_v\xi^y & 0 & & F(t) & \to & \mathbb{Z}/u\cdot p & \to & E(v) \\
 & 0 & & F(v) & \to & \mathbb{Z}/y\cdot p & & 0
\end{array}$$

Again if we omit the initial upper index s in the word w on the left hand side of the list then we have to replace $\mathbb{Z}/s$ in the list by $\mathbb{Z}_p$.

(3.5) Definition. A *p-cyclic word* is defined by a pair (w, φ) where w is a subword of the basic sequence of the form $(q \geq 1)$

$$w = \xi^{s_1}\eta_{r_1}\xi^{s_2}\eta_{r_2}\dots\xi^{s_q}\eta_{r_q} = a(1)\dots a(q)$$

and where φ is an automorphism of a finite dimensional $\mathbb{Z}/p$-vector space $V = V(\varphi)$. Two cyclic words (w, φ) and (w', φ') are *equivalent* if w' is a cyclic permutation of w, that is, $w' = a(i)\dots a(q)a(1)\dots a(i-1)$, and if there is an isomorphism $\psi : V(\varphi) \cong V(\varphi')$ with $\varphi = \psi^{-1}\varphi'\psi$. A cyclic word (w, φ) is a *special p-cyclic word* if φ is an indecomposable automorphism and if w is not of the form $w = w'w'\dots w'$ where the right-hand side is a j-fold power of a word w' with $j > 1$: We define the *homology* of a p-cyclic word by

$$H_i(w, \varphi) = \bigoplus_v H_i(w)$$

Here $v = \dim V(\varphi)$ and the right hand side is the v-fold direct sum of the homology $H_i(w)$ defined in (3.2).

(3.6) Definition. We associate with a p-cyclic word (w, φ) a chain complex

$$S(w, \varphi) \in \mathbf{S}(\otimes\mathbb{Z}/p, *\mathbb{Z}/p)_p$$
$$S(w, \varphi) \in \mathbf{S}(*\mathbb{Z}/p, \otimes\mathbb{Z}/p)_p$$

which is given as follows. Let v be the dimension of $V(\varphi)$ and let $U^v = U \oplus \dots \oplus U$ be the v-fold dircet sum of the abelian group U. Then $S(w, \varphi)$ is given by

$$A = H_0(w)^v, R = R^\xi(w)^v, H = 0$$

and by the chain complex

$$0 \to F(H_0(w))^v \xrightarrow{\partial} R^\xi(w)^v \xrightarrow{\delta^v} E(H_0(w))^v \to 0$$

where $H_0(w)$ is a p-torsion group. The chain complex is given by the direct sum of the following lines:

$$\begin{array}{ccccc}
F(r_q)^v & \xrightarrow{\partial^\varphi} & (\mathbb{Z}/s_1 \cdot p)^v & \xrightarrow{\delta^v} & E(r_1)^v \\
F(r_1)^v & \xrightarrow{\partial^v} & (\mathbb{Z}/s_2 \cdot p)^v & \xrightarrow{\delta^v} & E(r_2)^v \\
\vdots & & & & \\
F(r_{q-1})^v & \xrightarrow{\partial^v} & (\mathbb{Z}/s_q \cdot p)^v & \xrightarrow{\delta^v} & E(r_q)^v
\end{array}$$

Here ∂^v and δ^v denote the v-fold direct sum of the canonical maps. The map ∂^φ, however, is given by the composite

$$F(r_q)^v \overset{\varphi}{\cong} F(r_q)^v \xrightarrow{\partial^v} (\mathbb{Z}/s_1 \cdot p)^v$$

where φ is the automorphism of $V(\varphi) = F(r_q)^v$ given by the cyclic word (w, φ).

It is easy to compute the abelian group $B = \ker(\delta)$ from the description of the chain complexes $S(w)$ and $S(w, \varphi)$ above. Recall that a chain complex S with $A = 0$ or $B = 0$ is termed on Eilenberg-Mac Lane object; see (2.3). The indecomposable Eilenberg-Mac Lane objects are completely described by elementary cyclic groups A or B and by $(H = \mathbb{Z}, A = B = 0)$ respectively.

(3.7) Theorem. *Let p be a prime. A complete list of indecomposable objects in $\mathbf{S}(\otimes\mathbb{Z}/p, *\mathbb{Z}/p)_p$ is given by the following objects:*

- *indecomposable Eilenberg-Mac Lane objects,*
- *chain complexes $S(w)$ where w is a p-tensor word,*
- *chain complexes $S(w, \varphi)$ where (w, φ) is a special p-cyclic word.*

A complete list of indecomposable objects in $\mathbf{S}(*\mathbb{Z}/p, \otimes\mathbb{Z}/p)_p$ is given by the following objects:

- *indecomposable Eilenberg-Mac Lane objects,*
- *chain complexes $S(w)$ where w is a p-torsion word,*
- *chain complexes $S(w, \varphi)$ where (w, φ) is a special p-cyclic word.*

For two objects S, S' in these lists there is an isomorphism $S \cong S'$ if and only if there are equivalent special p-cyclic words $(w, \varphi) \sim (w', \varphi')$ with $S = S(w, \varphi)$ and $S' = S(w', \varphi')$.

In theorem (A) in the introduction we need the category $\mathbf{S}(\otimes\mathbb{Z}/p, *\mathbb{Z}/p)_p$ for all tame cases $n - m = t(2p - 2)$ with $1 \le t \le p$. Also the category $\mathbf{S}(*\mathbb{Z}/p, \otimes\mathbb{Z}/p)_p$ is needed in theorem (A) for the tame case $p = 2$ and $n - m = 3$. More precisely we get the following two corollaries of (3.7). Let $M(C, n)$ be the Moore space of the abelian group C and let $K(C, m)_p^{n+1}$ be the indecomposable part of the p-local $(n+1)$-skeleton of the Eilenberg-Mac Lane space $K(C, m)$; compare (2.2). Moreover let $X(w)$, resp. $X(w, \varphi)$, be the space corresponding to $S(w)$, resp. $S(w, \varphi)$, via the detecting functor Λ in (2.5). As abelian group C is an <u>elementary</u> $\mathbb{Z}_p$-<u>module</u> if $C = \mathbb{Z}_p$ or if $C = \mathbb{Z}/p^i$ with $i \ge 1$.

(3.8) Corollary. *Let p be a prime and $2 \le m < n$ with $n < 2m - 1$ and $n - m = t(2p-2)$ with $1 \le t \le p$. Then a complete list of indecomposable objects in $\mathbf{K}(m, n)_p$ is given by the following objects.*

- $S_p^{n+1}, M(C, n), K(C, m)_p^{n+1}$ *where C is an elementary $\mathbb{Z}_p$-module,*

- *spaces $X(w)$ where w is a p-tensor word,*
- *spaces $X(w,\varphi)$ is a special p-cyclic word.*

(3.9) Corollary. *Let $m \geq 5$. Then a complete list of indecomposable objects in $\mathbf{K}(m, m+3)_2$ is given by the following objects with $p = 2$.*

- *S_p^{m+4}, $M(C, m+3)$, $K(C,m)_p^{m+4}$ where C is an elementary $\mathbb{Z}_p$-module,*
- *spaces $X(w)$ where w is a p-torsion word,*
- *spaces $X(w,\varphi)$ where (w,φ) is a special p-cyclic word.*

Here $X(w)$ and $X(w,\varphi)$ are again defined by the detecting functor Λ in (2.5) with $p = 2$ and $5 \leq m < n = m+3$.

The homology of the spaces $X(w)$ is related to the homology of the word w by the following formula.

$$H_i(X(w)) = \begin{cases} H_0(w) & \text{for } i = m \\ H_2(w) & \text{for } i = n \\ H_3(w) & \text{for } i = n+1 \end{cases} \tag{3.10}$$

This also holds if we replace w by (w,φ). The classification of indecomposable objects by words or graphs as above thus shows by (3.10) immediately part of the homology of the corresponding spaces.

§ 4 Proof of Theorem 2.13

We are going to interpret the category $\mathbf{S} = \mathbf{S}(E,F)_p$ in (2.3) in terms of "matrix problems", namely, matrices over a bimodule (cf. [8, 9]). Consider the category $\mathbf{M}_p^2 = \mathbf{M}_p \times \mathbf{M}_p$, where $\mathbf{M}_p$ is the category of finitely generated $\mathbb{Z}_p$-modules and the bimodule $\mathbf{U}$ over it, i.e. the bifunctor $\mathbf{U} : (\mathbf{M}_p^2)^{\mathrm{op}} \times \mathbf{M}_p^2 \to \mathbf{Ab}$, defined as follows:

$$\mathbf{U}((A,R),(A',R')) = \mathrm{Hom}(R, E(A')) \oplus \mathrm{Hom}(F(A), R').$$

We write $\mathbf{U}[A,R]$ for $\mathbf{U}((A,R),(A,R))$. Recall that the category $\mathrm{EL}(\mathbf{U})$ of matrices over $\mathbf{U}$ or elements of $\mathbf{U}$ has the set of objects $\bigcup_{(A,R)} \mathbf{U}[A,R]$. A morphism from an element $(\delta,\partial) \in \mathbf{U}[A,R]$ to $(\delta',\partial') \in \mathbf{U}[A',R']$ is by definition a pair (α,β), where $\alpha : A \to A', \beta : R \to R'$ such that $\alpha_*\delta = \delta'\beta$ and $\beta\partial = \partial'\alpha_*$.

A complex from $\mathbf{S}$ is just a triple (δ,∂,h), where (δ,∂) is an object from $\mathrm{EL}(\mathbf{U})$ with the property that δ is surjective and $\delta\partial = 0$, while h is an epimorphism $H \to \ker\partial$ for some free abelian group H. Hence, essentially the classification of objects from $\mathbf{S}$ and from $\mathrm{EL}(\mathbf{U})$ are the same.

Denote by C_m the cyclic group of order p^m if $m \in \mathbb{N}$ and $C_m = \mathbb{Z}_p$ if $m = \infty$. The following propositions are evident.

(4.1) Proposition.

1. *$C_m \otimes \mathbb{Z}/p = \mathbb{Z}/p$.*
2. *$C_m * \mathbb{Z}/p = \mathbb{Z}/p$ if $m \in \mathbb{N}$ and $C_\infty * \mathbb{Z}/p = 0$.*
3. *If $m \geq n$, an epimorphism $C_m \to C_n$ induces an isomorphism $C_m \otimes \mathbb{Z}/p \to C_n \otimes \mathbb{Z}/p$ and zero map on $C_m * \mathbb{Z}/p$.*
4. *If $m \leq n < \infty$, a monomorphism $C_m \to C_n$ induces an isomorphism $C_m * \mathbb{Z}/p \to C_n * \mathbb{Z}/p$ and zero map on $C_m \otimes \mathbb{Z}/p$.*

(4.2) Proposition.

1. $\mathrm{Hom}(C_m, \mathbb{Z}/p) = \mathbb{Z}/p$.
2. $\mathrm{Hom}(\mathbb{Z}/p, C_m) = \mathbb{Z}/p$ *if* $m \in \mathbb{N}$ *and* $\mathrm{Hom}(\mathbb{Z}/p, C_\infty) = 0$.
3. *If* $m > 1$ *then* $\phi\psi = 0$ *for each* $\psi : \mathbb{Z}/p \to C_m$ *and* $\phi : C_m \to \mathbb{Z}/p$.
4. *If* $m \geq n$, *an epimorphism* $C_m \to C_n$ *induces an isomorphism* $\mathrm{Hom}(C_n, \mathbb{Z}/p) \to \mathrm{Hom}(C_m, \mathbb{Z}/p)$ *and zero map on* $\mathrm{Hom}(\mathbb{Z}/p, C_m)$.
5. *If* $m \leq n < \infty$, *a monomorphism* $C_m \to C_n$ *induces an isomorphism* $\mathrm{Hom}(\mathbb{Z}/p, C_m) \to \mathrm{Hom}(\mathbb{Z}/p, C_n)$ *and zero map on* $\mathrm{Hom}(C_m, \mathbb{Z}/p)$.

Consider the first tame cases of Theorem 2.13. Let $E = \_ \otimes \mathbb{Z}/p$ and $F = \_ * \mathbb{Z}/p$. If $A = \bigotimes_m C_m^{a_m}$ and $R = \bigotimes_m C_m^{r_m}$, an element $\delta \in \mathrm{Hom}(R, E(A))$ can be considered as a block matrix of the form:

$$\tag{1} \begin{pmatrix} \delta_{11} & \delta_{12} & \dots & \delta_{1\infty} \\ \delta_{21} & \delta_{22} & \dots & \delta_{2\infty} \\ \dots & \dots & \dots & \dots \\ \delta_{\infty 1} & \delta_{\infty 2} & \dots & \delta_{\infty\infty} \end{pmatrix}$$

where δ_{ij} is a matrix with entries in $\mathbb{Z}/p$ of size $a_i \times r_j$. Just in the same way, an element $\partial \in \mathrm{Hom}(F(A), R)$ can be considered as a block matrix of the form:

$$\tag{2} \partial = \begin{pmatrix} \partial_{11} & \partial_{12} & \dots \\ \partial_{21} & \partial_{22} & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

where ∂_{ij} is a matrix with entries in $\mathbb{Z}/p$ of size $r_i \times a_j$. Remark that both $F(C_\infty)$ and $\mathrm{Hom}(F(A), C_\infty)$ are zeros.

A homomorphism $\alpha : A \to A'$ is also given by a matrix

$$\tag{3} \alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1\infty} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2\infty} \\ \dots & \dots & \dots & \dots \\ \alpha_{\infty 1} & \alpha_{\infty 2} & \dots & \alpha_{\infty\infty} \end{pmatrix}$$

where α_{ij} is a matrix with entries in $\mathrm{Hom}(C_j, C_i)$ of size $a'_i \times a_j$. It is an isomorphism if and only if all diagonal blocks α_{ii} are invertible. Just in the same way a homomorphism $\beta : R \to R'$ is given by an analogous block matrix (β_{ij}). For a matrix α of the form (3) put

$$\alpha^+ = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1\infty} \\ 0 & \alpha_{22} & \dots & \alpha_{2\infty} \\ 0 & 0 & \dots & \alpha_{3\infty} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha_{\infty\infty} \end{pmatrix}$$

and

$$\alpha^- = \begin{pmatrix} \alpha_{11} & 0 & 0 & \dots & 0 \\ \alpha_{21} & \alpha_{22} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{\infty 1} & \alpha_{\infty 2} & \dots & \dots & \alpha_{\infty\infty} \end{pmatrix}$$

Now the objects from $\mathrm{EL}(\mathbf{U})$ are just pairs (δ, ∂), where δ and ∂ are matrices of the forms (1) and (2) respectively, while a homomorphism from this object to another one (δ', ∂') is a pair of matrices (α, β) of the form (3) such that $\alpha^+\delta = \delta'\beta^-$ and $\beta^+\partial = \partial'\alpha^-$. Therefore elements from $\mathrm{EL}(\mathbf{U})$ coincide with representations of a bunch of chains in the sense of [4] (cf. also [10] for the special case of chains, just what we need). Namely, this bunch consists of the chains $E_1 \simeq F_1\mathbb{N} \cup \{\infty\}, E_2 \simeq F_2 \simeq \mathbb{N}$ with the equivalence relation $\sim$ such that $m \in E_2$ is equivalent to the same $n \in E_1$. Thus we can use the list of indecomposable representation given in [4] which implies that this classification is tame. Taking into consideration for a pair (δ, ∂) to represent an object from $\mathbf{S}$ and adding, when necessary, a homomorphism $H \to F(A)$, we obtain the following list of indecomposable objects from $\mathbf{S}$.

Consider finite sequences $w = (m_1, m_2, \dots, m_n)$, where n is even and m_i are non-negative integers or ∞ such that

$$\tag{4} \begin{aligned} &m_i \neq 0, \quad \text{except possibly } m_n, \\ &m_i \neq \infty, \quad \text{except possibly } m_1 \text{ and } m_n, \\ &m_i \neq 1, \quad \text{for odd } i, \text{ except possibly } m_1 \text{ and } m_{n-1} \text{ if } m_n = 0. \end{aligned}$$

For each sequence w we define the object $S(w)$ as follows:

$$\tag{5} \begin{aligned} &A = \bigoplus_{i\,\mathrm{odd}} C_{m_i}, \\ &R = \bigoplus_{i\,\mathrm{even}} C_{m_i}, \\ &H = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even and } m_n \neq \infty \\ 0 & \text{otherwise} \end{cases} \\ &h \quad \text{is the surjection } \mathbb{Z}_p \to C_{m_n} \ (\text{if } H \neq 0) \end{aligned}$$

while ∂ and δ are given by the following diagram:

$$\begin{array}{ccccc} & & C_{m_1} & \to & E(C_{m_2}) \\ F(C_{m_2}) & \to & C_{m_3} & \to & E(C_{m_4}) \\ F(C_{m_4}) & \to & C_{m_5} & \to & E(C_{m_6}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F(C_{m_{n-2}}) & \to & C_{m_{n-1}} & \to & E(C_{m_n}) \end{array}$$

where all "left" arrows are monomorphisms and all "right" ones are epimorphisms. In this case H and h are uniquely defined. Namely if $m_n = 0$ or $m_n = \infty$ then $H = 0$; otherwise $H = \mathbb{Z}_p$ and h is the natural epimorphism $\mathbb{Z}_p \to F(C_m) = \mathbb{Z}/p$.

Suppose now that $m_i \notin \{0, 1, \infty\}$ for all i and the sequence w is non-periodic, i.e. there exists no shorter sequence v such that w is just a repetition of several copies of v. Then for each polynomial $f(T) \in \mathbb{Z}/p[T], f(T) \neq T^d$, which is a power of an irreducible one, define the object $S(w, f)$ as follows:

$$\tag{6} \begin{aligned} &H = 0, \\ &A = \bigoplus_{i\,\mathrm{odd}} C^d_{m_i}, \\ &R = \bigoplus_{i\,\mathrm{even}} C^d_{m_i}, \end{aligned}$$

where $d = \deg f$, while ∂ and δ are given by the following diagram:

$$\begin{array}{ccccc}
F(C^d_{m_n}) & \stackrel{\phi}{\to} & C^d_{m_1} & \to & E(C^d_{m_2}) \\
F(C^d_{m_2}) & \to & C^d_{m_3} & \to & E(C^d_{m_4}) \\
F(C^d_{m_4}) & \to & C^d_{m_5} & \to & E(C^d_{m_6}) \\
\multicolumn{5}{c}{\dotfill} \\
F(C^d_{m_{n-2}}) & \to & C^d_{m_{n-1}} & \to & E(C^d_{m_n})
\end{array}$$

In this diagram all mappings, except ϕ, are given by matrices εI, where I is the identity matrix, while ε is injection for "right" arrows and surjection for "left" ones. The mapping ϕ is given by the matrix $\varepsilon\Phi$, where Φ is the Frobenius matrix with characteristic polynomial $f(T)$. In this case $H = 0$.

(4.3) Proposition. *All objects $S(w)$ and $S(w, f)$ defined above are indecomposable and every indecomposable object in $\mathbf{S}$ is isomorphic either to $S(w)$ or to $S(w, f)$ for some choice of w and f. Moreover, the only isomorphisms between these objects are $S(w, f) \cong S(w', f)$, where w' is obtained from w by a cyclic permutation mapping m_1 to m_k with odd k.*

The case $E = \_ * \mathbb{Z}/p, F = \_ \otimes \mathbb{Z}/p$ is quite analogous, except that the following values of m_i (forbidden above) are possible:

$$m_1 = 0,\ m_2 = \infty$$
$$M_{n-1} = 1,\ m_n = \infty.$$

Comparing the chain complexes $S(w)$ in (4.3) and (3.3) we see that (3.7) is a consequence of (4.3).

If $E = 0$ or $F = 0$, while the other functor is $\_ \otimes \mathbb{Z}/p$ or $\_ * \mathbb{Z}/p$, the situation is quite trivial: the indecomposable objects can only be one of the following forms:

$$C_m \to E(C_k);$$
$$F(C_m) \to C_k;$$
$$\mathbb{Z} \to F(C_m)$$

(the first one if $E \neq 0$, the second and the third if $F \neq 0$). In particular, these cases are essentially finite.

We now prove that all other cases are wild. For instance, let $E = F = \_ \otimes \mathbb{Z}/p$ (other cases are similar or even easier to handle). Then an object from $\mathrm{EL}(\mathbf{U})$ is given by two matrices: δ as in (1) and ∂ as in (2). Two such objects are isomorphic if and only if there are matrices α, β of the form (3) such that all α_{ii} are invertible, $\alpha^+\delta = \delta'\beta^-$ and $\beta^+\partial = \partial'\alpha^+$. If there are no direct summands C_1 in R and the rows of δ are linear independent as well as the columns of ∂, the pair (∂, δ) defines an object from $\mathbf{S}$ (with $H = 0$).

Put now $A = \bigoplus_{i=1}^3 C_i^n, R = \bigoplus_{i=2}^4 C_i^n$,

$$\delta = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \qquad \partial = \begin{pmatrix} 0 & 0 & I \\ 0 & I & I \\ I & X & Y \end{pmatrix},$$

where I is $n \times n$ identity matrix, X, Y are any $n \times n$ matrices. This gives an object $S(X,Y) \in \mathbf{S}$. If $S(X,Y) \cong S(X',Y')$ then $\alpha^+ = \beta^-$, whence $\alpha^+ = \beta^- = \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3)$. Then $\beta^+\partial = \partial'\alpha^+$ implies that $\alpha_1 = \alpha_2 = \alpha_3, \beta_{ij} = 0$ for $i \neq j$ and $\alpha_3 X = X'\alpha_3, \alpha_3 Y = Y'\alpha_3$, i.e. the pairs (X,Y) and (X',Y') are conjugate. Therefore, $\mathbf{S}$ is wild.

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