SYMPLECTIC AND POISSON STRUCTURES OF CERTAIN MODULI SPACES. II. PROJECTIVE REPRESENTATIONS OF COCOMPACT PLANAR DISCRETE GROUPS

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SYMPLECTIC AND POISSON STRUCTURES OF CERTAIN MODULI SPACES. II. PROJECTIVE REPRESENTATIONS OF COCOMPACT PLANAR DISCRETE GROUPS

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ABSTRACT. Let G be a Lie group with a biinvariant metric, not necessarily positive definite. It is shown that a certain construction carried out in an earlier paper for the fundamental group of a closed surface may be extended to an arbitrary infinite orientation preserving cocompact planar discrete group of euclidean or non-euclidean motions π and yields (i) a symplectic structure on a certain smooth manifold $\mathcal{M}$ containing the space $\mathrm{Hom}(\pi, G)$ of homomorphisms and, furthermore, (ii) a hamiltonian G-action on $\mathcal{M}$ preserving the symplectic structure together with a momentum mapping in such a way that the reduced space equals the space $\mathrm{Rep}(\pi, G)$ of representations. More generally, the construction also applies to certain spaces of projective representations. For G compact, the resulting spaces of representations inherit structures of *stratified symplectic space* in such a way that the strata have finite symplectic volume. For example, MEHTA-SESHADRI moduli spaces of semistable holomorphic parabolic bundles with rational weights or spaces closely related to them arise in this way by *symplectic reduction in finite dimensions*.

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Introduction

Let π be a finitely generated orientation preserving infinite cocompact planar discrete group of euclidean or hyperbolic motions. So π acts by isometries on the euclidean or upper half plane (as appropriate) and the orbit space Σ is a compact orientable Riemann surface. In the hyperbolic case, π is also called a (cocompact) *Fuchsian group*. Let G be a Lie group with a biinvariant metric, not necessarily positive definite. The set of representations or more generally projective representations of π in G is known to inherit additional structure, under suitable circumstances, cf. e. g. [6], [20], [21], [24], [25], [27], [30], [32], [35], [36], [39], [42], [43], [44]. In this paper we shall study the symplectic or more generally *Poisson* geometry of such representations. More specifically, extending a certain construction carried out in an earlier paper [32] for the fundamental group of a closed surface, we shall obtain certain *extended representation spaces* (see below for a precise definition). In [32] the theory has been made for arbitrary finite presentations but it has been applied only to the standard presentation of the fundamental group of a closed surface. However, the general approach in [32] applies to an arbitrary group of the kind π and yields the following, see (2.9) below for a more precise statement.

Theorem. *There is a smooth symplectic manifold $\mathcal M$ containing $\mathrm{Hom}(\pi,G)$ (as a deformation retract) together with a hamiltonian G-action on $\mathcal M$ and momentum mapping in such a way that the reduced space equals the space $\mathrm{Rep}(\pi,G)$ of representations of π in G. More generally, spaces of projective representations of π are obtained by symplectic reduction at appropriate non-zero values of the momentum mapping.*

The (smooth) symplectic manifold $\mathcal M$ together with the G-action and momentum mapping is what we mean by an *extended representation space*. The second clause of the Theorem will be made precise in Section 3 below. Our result, apart from being interesting in its own right, reveals some interesting and attractive geometric properties of these twisted representation spaces, which have been spelled out in [32] for the special case considered there. For example, it implies that, for G compact, the resulting twisted representation spaces inherit a structure of stratified symplectic space. In particular it entails that *symplectic reduction, applied to a certain smooth finite dimensional symplectic manifold with a hamiltonian action of the unitary group* yields certain stratified symplectic spaces containing as top stratum the stable part of the MEHTA-SESHADRI [39] moduli spaces of semistable parabolic vector bundles with rational weights. In fact the two spaces are presumably globally homeomorphic but details have not been worked out yet. For parabolic degree zero, they *are* known to be homeomorphic, by a result of MEHTA-SESHADRI, cf. (4.1) and (4.3) in [39].

The projective representations mentioned above, also referred to as *twisted represention̲s* below, are representations of certain central extensions of π. These extensions include in particular all fundamental groups of Seifert fibered spaces with empty boundary (as 3-manifolds) which are Eilenberg-Mac Lane spaces and have orientable decomposition surface, cf. [40]; see Section 3 below for details. Because of their relevance for Floer homology, $\mathrm{SU}(N)$-representation spaces of Seifert fibered homology 3-spheres have been studied in [24], [25], [27], and [36]. Among others, our approach compactifies spaces of irreducible representations of fundamental groups of the Seifert fibered spaces belonging to the above class, including the homology 3-spheres just mentioned, as symplectic manifolds to stratified symplectic spaces.

The group π is determined by its genus ℓ and its torsion numbers $m_1, \ldots, m_n$. The basic idea is that the construction carried out in our paper [32] for a surface group still works for such a group π when the free group on the generators is replaced by a free product of a free group of rank 2ℓ together with cyclic groups $Q_1, \ldots, Q_n$ of finite orders respectively $m_1, \ldots, m_n$. The key observation is that π, being assumed infinite but cocompact, is a two-dimensional Poincaré duality group over the reals [26]. In particular, the second homology group $\mathrm{H}_2(\pi, \mathbf{R})$ is a one-dimensional real vector space and, starting from a generator, we can carry out a construction of closed 2-form on an extended moduli space similar to that in [32] for the case of an ordinary surface group. The infinitesimal structure is then handled by means of a corresponding length two *projective* resolution of the reals $\mathbf{R}$ in the category of left $\mathbf{R}\pi$-modules, and the fact that π is a Poincaré duality group over the reals entails the nondegeneracy of the resulting 2-form, much as in [32].

I am indebted to A. Weinstein for discussions. Any unexplained notation is the same as that in our paper [32].

1. Poincaré duality for cocompact planar groups

The group π is given by a presentation

$$\mathcal{P} = \langle x_1, y_1, \ldots, x_\ell, y_\ell, z_1, \ldots, z_n; r, r_1, \ldots, r_n \rangle,$$

where

$$r = \Pi[x_j, y_j] z_1 \ldots z_n, \quad r_j = z_j^{m_j}, \quad m_j \geq 2.$$

Since π is assumed cocompact there are no parabolic elements in the Fuchsian case. The hypothesis that π is infinite is equivalent to the requirement that the *measure*

$$\mu(\pi) = 2\ell - 2 + \sum \left(1 - \frac{1}{m_j}\right)$$

be non-negative. For example the group with the smallest positive measure is given by the presentation

$$\langle z_1, z_2, z_3; z_1 z_2 z_3, z_1^2, z_2^3, z_3^7 \rangle.$$

All this is classical and may be found in standard textbooks, see e. g. [37].

The cohomology of π is well known, cf. [33], [41], [44]. In fact, for an arbitrary ground ring R, the Fox calculus, applied to the presentation $\mathcal{P}$, yields a free resolution

$$\mathbf{R}(\mathcal{P}): \ldots \to R_j(\mathcal{P}) \xrightarrow{\partial_j} \ldots \to R_2(\mathcal{P}) \xrightarrow{\partial_2} R_1(\mathcal{P}) \xrightarrow{\partial_1} R\pi$$

of R in the category of left $R\pi$-modules, cf. our paper [33]. We recall that, for $j \geq 3$, $R_j(\mathcal{P}) = R\pi[r_1, \ldots, r_n]$, the free left $R\pi$-module on the relators $r_1, \ldots, r_n$, while $R_2(\mathcal{P}) = R\pi[r, r_1, \ldots, r_n]$ and $R_1(\mathcal{P}) = R\pi[x_1, y_1, \ldots, x_\ell, y_\ell, z_1, \ldots, z_n]$. Moreover the boundary operators ∂_j are given by the Fox calculus; in particular ∂_2 is given by the matrix of Fox derivatives, which roughly amounts to reexpressing the elements

$r-1, r_1-1, \dots, r_n-1$ of the group ring $R\pi$ in terms of x_1-1, y_1-1, z_1-1 etc., and ∂_1 maps the generators x_j etc. to the elements x_j-1 etc. of $R\pi$. In particular, the chain complex calculating the homology of π with values in R looks like

$$\dots \to R[r, r_1, \dots, r_n] \xrightarrow{\overline{\partial}_2} R[x_1, y_1, \dots, x_\ell, y_\ell, z_1, \dots, z_n] \xrightarrow{0} R,$$

where the operator $\overline{\partial}_2$ satisfies the formulas

$$\overline{\partial}_2(r) = z_1 + \dots + z_n$$
$$\overline{\partial}_2(r_j) = m_j z_j, \quad 1 \le j \le n.$$

Henceforth we denote by m the least common multiple of $m_1, \dots, m_n$. The 2-chain

$$b = mr - \frac{m}{m_1} r_1 - \dots - \frac{m}{m_n} r_n$$

is then a 2-cycle, and a closer look at the resolution reveals that $\mathrm{H}_2(\pi, \mathbf{Z})$ is then infinite cyclic, generated by the class of b. Likewise, $\mathrm{H}_2(\pi, \mathbf{R})$ is a one-dimensional real vector space; we shall take $\kappa = \frac{1}{m}[b] \in \mathrm{H}_2(\pi, \mathbf{R})$ as its generator. Moreover, the group π is in fact a *two-dimensional Poincaré duality group* over the reals having fundamental class $\kappa \in \mathrm{H}_2(\pi, \mathbf{R})$, that is, for every $\mathbf{R}\pi$-module A, cap product with κ yields a natural isomorphism

$$\cap\kappa \colon \mathrm{H}^*(\pi, A) \to \mathrm{H}_{2-*}(\pi, A),$$

cf. [26]. In particular, $\mathrm{H}^2(\pi, \mathbf{R})$ is also a one-dimensional real vector space. The group π is thus of type FP_2 over the reals, cf. e.g. [28]. Moreover it has $\mathrm{H}^1(\pi, \mathbf{R}\pi)$ zero. In fact, π has a torsion free subgroup of finite index and hence a torsion free normal subgroup of finite index, necessarily a surface group, say τ. The cohomology $\mathrm{H}^1(\pi, \mathbf{R}\pi)$ then amounts to the invariants $\mathrm{H}^0(Q, \mathrm{H}^1(\tau, \mathbf{R}\pi))$, with reference to the quotient group $Q = \pi/\tau$ which is finite. However, as a τ-module, $\mathbf{R}\pi$ decomposes as a finite direct sum of copies of $\mathbf{R}\tau$, and $\mathrm{H}^1(\tau, \mathbf{R}\tau)$ is zero since τ is a surface group. Hence $\mathrm{H}^1(\pi, \mathbf{R}\pi)$ is zero. It is well known that a finitely presented group of type FP_2 having $\mathrm{H}^1(\pi, \mathbf{R}\pi)$ zero is a two-dimensional duality group. The above considerations show that the dualizing module is in fact that of the reals, with trivial π-module structure. Hence π is a two-dimensional Poincaré duality group over the reals.

For a ring R containing the rationals, the free resolution $\mathbf{R}(\mathcal{P})$ projects onto a projective resolution of length 2. For later reference, we spell it out for $R = \mathbf{R}$, the reals. Let $j = 1, \dots, n$, and write Q_j for the finite cyclic subgroup of π generated by z_j; it has exact order m_j, see e. g. [33]. Since Q_j is a finite group, its augmentation ideal $IQ_j \subseteq \mathbf{R}Q_j$ is a projective $\mathbf{R}Q_j$-module whence the induced $\mathbf{R}\pi$-module $\mathbf{R}\pi \otimes_{\mathbf{R}Q_j} IQ_j$ is projective. We now consider the beginning

$$\mathbf{R}\pi[r, r_1, \dots, r_n] \xrightarrow{\partial_2} \mathbf{R}\pi[x_1, y_1, \dots, x_\ell, y_\ell, z_1, \dots, z_n] \xrightarrow{\partial_1} \mathbf{R}\pi$$

of our free resolution $\mathbf{R}(\mathcal{P})$ of the reals and divide out the generators $r_1, \dots, r_n$ of $\mathbf{R}\pi[r, r_1, \dots, r_n]$ and their ∂_2-images in $\mathbf{R}\pi[x_1, y_1, \dots, x_\ell, y_\ell, z_1, \dots, z_n]$. This yields a projective resolution

$$\mathbf{P}(\mathcal{P}) \colon P_2(\mathcal{P}) \xrightarrow{\partial_2} P_1(\mathcal{P}) \xrightarrow{\partial_1} \mathbf{R}\pi$$

of $\mathbf{R}$ in the category of left $\mathbf{R}\pi$-modules with $P_2(\mathcal{P}) = \mathbf{R}\pi[r]$ and

$$P_1(\mathcal{P}) = \mathbf{R}\pi[x_1, y_1, \ldots, x_\ell, y_\ell] \oplus \mathbf{R}\pi \otimes_{\mathbf{R}Q_1} IQ_1 \oplus \cdots \oplus \mathbf{R}\pi \otimes_{\mathbf{R}Q_n} IQ_n.$$

Notice, for $n = 0$, the group π is just a surface group in the usual sense and $\mathbf{P}(\mathcal{P})$ boils down to the usual free resolution for a surface group.

For illustration, when $\ell = 1$ and $n = 1$, we have $r = [x, y]z$ and

$$\partial_2[r] = (1 - xyx^{-1})[x] + (x - [x, y])[y] + [x, y][z].$$

2. Representation spaces of planar discontinuous groups

Write F for the free group on the generators in $\mathcal{P}$. As in [32], for a group Π we denote by $(C_*(\Pi, R), \partial)$ the chain complex of its inhomogeneous reduced normalized bar resolution over R. Pick $c \in C_2(F, \mathbf{R})$ so that $\partial c = [r] - \frac{1}{m_1}[r_1] - \cdots - \frac{1}{m_n}[r_n] \in C_1(F, \mathbf{R})$; this is certainly possible. Modulo the normal closure N of the relators, c will then represent the class κ. As in [32], let O be the subspace of the Lie algebra g where the exponential mapping is regular. Denote by ρ the word map from $G^{2\ell+n} = \operatorname{Hom}(F, G)$ to G^{1+n} for the presentation $\mathcal{P}$, and write $\mathcal{H}(\mathcal{P}, G)$ for the smooth manifold determined by the requirement that a pull back diagram

$$\begin{array}{ccc}
\mathcal{H}(\mathcal{P}, G) & \xrightarrow{\widehat{\rho}} & O \times O^n \\
{\scriptstyle\eta}\big\downarrow & & \big\downarrow{\scriptstyle\exp} \\
\operatorname{Hom}(F, G) & \xrightarrow[\rho]{} & G \times G^n
\end{array}$$

of spaces results, where the induced map from $\mathcal{H}(\mathcal{P}, G)$ to O^{1+n} is denoted by $\widehat{\rho}$. The construction in our paper [32] yields (i) a 2-chain $c \in C_2(F, \mathbf{R})$ ([32] Lemma 2), (ii) the equivariant closed 2-form $\omega_{c,\mathcal{P}} = \eta^*(\omega_c) - \widehat{\rho}^* B$ on $\mathcal{H}(\mathcal{P}, G)$ ([32] Theorem 1), and (iii) a smooth equivariant map $\mu\colon \mathcal{H}(\mathcal{P}, G) \to g^*$ whose adjoint $\mu^\sharp$ from g to $C^\infty(\mathcal{H}(\mathcal{P}, G))$ satisfies the identity

$$\delta_G(\omega_{c,\mathcal{P}}) = d\mu^\sharp\colon g \to \Omega^1(\mathcal{H}(\mathcal{P}, G))$$

on $\mathcal{H}(\mathcal{P}, G)$ ([32] Theorem 2). In fact, we can at first carry out these constructions with reference to b and thereafter divide by m. In particular, $\omega_{c,\mathcal{P}} - \mu^\sharp$ is an equivariantly closed form in $(\Omega_G^{*,*}(\mathcal{H}(\mathcal{P}, G)); d, \delta_G)$ of total degree 2. Thus, cf. [1] and what is said in Section 5 of [32], μ formally satisfies the property of being a momentum mapping for the 2-form $\omega_{c,\mathcal{P}}$ on $\mathcal{H}(\mathcal{P}, G)$, with reference to the obvious G-action, except that $\omega_{c,\mathcal{P}}$ is not necessarily non-degenerate. When the standard homotopy operator on forms on g is taken, the relevant map ψ from g to g^* is in fact the adjoint of the chosen 2-form $\cdot$ on g, cf. the remark in Section 1 of [32]. The map μ then amounts to the composite of $\widehat{\rho}$ with the sum map from O^{n+1} to g, combined with the adjoint of the given 2-form $\cdot$ on g.

Let $\phi \in \operatorname{Hom}(F, G)$, and suppose that $\phi(r)$ and each $\phi(r_j)$ lie in the centre of G. Then the composite of ϕ with the adjoint representation of G induces a structure of a π-module on g, and we write g_ϕ for g, viewed as as π-module in this way. Recall that the 2-form $\cdot$ on g and the homology class κ determine the alternating 2-form

$$\omega_{\kappa,\cdot,\phi}\colon \mathrm{H}^1(\pi, g_\phi) \otimes \mathrm{H}^1(\pi, g_\phi) \xrightarrow{\cup} \mathrm{H}^2(\pi, \mathbf{R}) \xrightarrow{\cap\kappa} \mathbf{R} \tag{2.1}$$

on $\mathrm{H}^1(\pi, g_\phi)$. Application of the functor $\mathrm{Hom}_{\mathbf{R}\pi}(\cdot, g_\phi)$ to the free resolution $\mathbf{R}(\mathcal{P})$ yields the chain complex

$$\mathbf{C}(\mathcal{P}, g_\phi)\colon \mathrm{C}^0(\mathcal{P}, g_\phi) \xrightarrow{\delta^0_\phi} \mathrm{C}^1(\mathcal{P}, g_\phi) \xrightarrow{\delta^1_\phi} \mathrm{C}^2(\mathcal{P}, g_\phi), \tag{2.2}$$

cf. [32] (4.1), computing the group cohomology $\mathrm{H}^*(\pi, g_\phi)$ in degrees 0 and 1; we recall that there are canonical isomorphisms

$$\mathrm{C}^0(\mathcal{P}, g_\phi) \cong g, \quad \mathrm{C}^1(\mathcal{P}, g_\phi) \cong g^{2\ell+n}, \quad \mathrm{C}^2(\mathcal{P}, g_\phi) \cong g^{1+n}.$$

To explain the geometric significance of this chain complex, denote by α_ϕ the smooth map from G to $\mathrm{Hom}(F, G)$ which assigns $x\phi x^{-1}$ to $x \in G$, and write $R_\phi\colon g^{2\ell+n} \to \mathrm{T}_\phi\mathrm{Hom}(F, G)$ and $R_{\rho\phi}\colon g^{1+n} \to \mathrm{T}_{\rho\phi}G^{1+n}$ for the corresponding operations of right translation. The tangent maps $\mathrm{T}_e\alpha_\phi$ and $\mathrm{T}_\phi\rho$ make commutative the diagram

$$\begin{array}{ccccc} \mathrm{T}_eG & \xrightarrow{\mathrm{T}_e\alpha_\phi} & \mathrm{T}_\phi\mathrm{Hom}(F, G) & \xrightarrow{\mathrm{T}_\phi\rho} & \mathrm{T}_{\rho(\phi)}G^{1+n} \\ {\scriptstyle \mathrm{Id}}\uparrow & & {\scriptstyle \mathrm{R}_\phi}\uparrow & & {\scriptstyle \mathrm{R}_{\rho(\phi)}}\uparrow \\ g & \xrightarrow[\delta^0_\phi]{} & g^{2\ell+n} & \xrightarrow[\delta^1_\phi]{} & g^{1+n}, \end{array} \tag{2.3}$$

cf. [32] (4.2). The commutativity of the diagram (2.3) shows at once that right translation identifies the kernel of the derivative $\mathrm{T}_\phi\rho$ with the kernel of the coboundary operator δ^1_ϕ from $\mathrm{C}^1(\mathcal{P}, g_\phi)$ to $\mathrm{C}^2(\mathcal{P}, g_\phi)$, that is, with the vector space $\mathrm{Z}^1(\pi, g_\phi)$ of g_ϕ-valued 1-cocycles of π; this space does *not* depend on a specific presentation $\mathcal{P}$, whence the notation. We note that $\mathrm{C}^1(\mathcal{P}, g_\phi) = Z^1(F, g_\phi)$, the space of g_ϕ-valued 1-cocycles for F. Pick $\widehat{\phi} \in \mathcal{H}(\mathcal{P}, G)$ so that the canonical map η from $\mathcal{H}(\mathcal{P}, G)$ to $G^{2\ell+n}$ sends $\widehat{\phi}$ to ϕ. Then, likewise, right translation identifies the restriction of the 2-form $\omega_{c,\mathcal{P}}$ to the kernel of the derivative $\mathrm{T}_{\widehat{\phi}}\widehat{\rho}$ with the 2-form on $\mathrm{Z}^1(\pi, g_\phi)$ obtained as the composite of $\omega_{\kappa,\cdot,\phi}$ with the projection from $\mathrm{Z}^1(\pi, g_\phi)$ to $\mathrm{H}^1(\pi, g_\phi)$, cf. Corollary 4.8 of [32].

To obtain the space $\mathrm{Hom}(\pi, G)$ as the zero locus of a momentum mapping in the *usual* sense, we must cut the space $\mathrm{Hom}(F, G)$ to size, in the following way: Denote by $F^\natural$ the group given by the presentation

$$\langle x_1, y_1, \ldots, x_\ell, y_\ell, z_1, \ldots, z_n; z_1^{m_1}, \ldots, z_n^{m_n}\rangle.$$

Its space $\mathrm{Hom}(F^\natural, G)$ of homomorphisms decomposes as

$$\mathrm{Hom}(F^\natural, G) \cong G^{2\ell} \times \mathrm{Hom}(Q_1, G) \times \cdots \times \mathrm{Hom}(Q_n, G),$$

and the generators induce an embedding of $\mathrm{Hom}(F^\natural, G)$ into $\mathrm{Hom}(F, G)$ as a smooth submanifold with a finite number of connected components. We shall say more about these connected components later. By construction, the word map ρ induces a smooth map r from $\mathrm{Hom}(F^\natural, G)$ to G so that $\mathrm{Hom}(\pi, G) = r^{-1}(e)$, the pre-image of the neutral element e of G. This map merely depends on the relator r whence we denote it by that same symbol. Application of the functor $\mathrm{Hom}_{\mathbf{R}\pi}(\cdot, g_\phi)$ to the projective resolution $\mathbf{P}(\mathcal{P})$ then yields the chain complex

$$\mathbf{C}(\mathbf{P}(\mathcal{P}), g_\phi)\colon \mathrm{C}^0(\mathbf{P}(\mathcal{P}), g_\phi) \xrightarrow{\delta^0_\phi} \mathrm{C}^1(\mathbf{P}(\mathcal{P}), g_\phi) \xrightarrow{\delta^1_\phi} \mathrm{C}^2(\mathbf{P}(\mathcal{P}), g_\phi), \tag{2.4}$$

computing the group cohomology $\mathrm{H}^*(\pi, g_\phi)$ in all degrees; by construction, there are canonical isomorphisms

$$\mathrm{C}^0(\mathbf{P}(\mathcal{P}), g_\phi) \cong g, \quad \mathrm{C}^2(\mathbf{P}(\mathcal{P}), g_\phi) \cong g.$$

Moreover, the canonical projection from $\mathbf{R}(\mathcal{P})$ to $\mathbf{P}(\mathcal{P})$ induces a canonical injection of $\mathbf{C}(\mathbf{P}(\mathcal{P}), g_\phi)$ into $\mathbf{C}(\mathcal{P}, g_\phi)$ identifying the former with a subcomplex of the latter.

We now suppose that ϕ lies in $\mathrm{Hom}(F^\natural, G)$ viewed as a subspace of $\mathrm{Hom}(F, G)$, that is, $\phi(r_j)$ is trivial for $j = 1, \dots, n$ but, beware, $\phi(r)$ is still admitted to be an arbitrary element of the centre of G. The following observation will be crucial.

Proposition 2.5. *The tangent maps $\mathrm{T}_e\alpha_\phi$ and $\mathrm{T}_\phi r$ make commutative the diagram*

$$\begin{tikzcd} \mathrm{T}_e G \arrow[r, "\mathrm{T}_e\alpha_\phi"] & \mathrm{T}_\phi \mathrm{Hom}(F^\natural, G) \arrow[r, "\mathrm{T}_\phi r"] & \mathrm{T}_{r(\phi)} G \\ g \arrow[u, "\mathrm{Id}"] \arrow[r, "\delta^0_\phi"'] & \mathrm{C}^1(\mathbf{P}(\mathcal{P}), g_\phi) \arrow[u, "\mathrm{R}_\phi"] \arrow[r, "\delta^1_\phi"'] & g, \arrow[u, "\mathrm{R}_{r(\phi)}"'] \end{tikzcd} \tag{2.6}$$

having its vertical arrows isomorphisms of vector spaces.

In fact, the diagram (2.3) restricts to the diagram (2.6).

The next step is to carry out a construction of a space similar to that of the space denoted by $\mathcal{H}(\mathcal{P}, G)$ in [32]. To this end, write $\mathcal{H}^\natural(\mathcal{P}, G)$ for the space determined by the requirement that a pull back diagram

$$\begin{tikzcd} \mathcal{H}^\natural(\mathcal{P}, G) \arrow[r, "\widehat{r}"] \arrow[d, "\eta"'] & O \arrow[d, "\exp"] \\ \mathrm{Hom}(F^\natural, G) \arrow[r, "r"'] & G \end{tikzcd} \tag{2.7}$$

results, where the induced map from $\mathcal{H}^\natural(\mathcal{P}, G)$ to O is denoted by $\widehat{r}$. The space $\mathcal{H}^\natural(\mathcal{P}, G)$ is a smooth manifold and the induced map η from $\mathcal{H}^\natural(\mathcal{P}, G)$ to $\mathrm{Hom}(F^\natural, G)$ is a smooth codimension zero immersion whence $\mathcal{H}^\natural(\mathcal{P}, G)$ has the same dimension as $\mathrm{Hom}(F^\natural, G)$; moreover the above injection of $\mathrm{Hom}(\pi, G)$ into $\mathrm{Hom}(F, G)$ induces a canonical injection of $\mathrm{Hom}(\pi, G)$ into $\mathcal{H}^\natural(\mathcal{P}, G)$. Further, $\mathcal{H}^\natural(\mathcal{P}, G)$ may be viewed as a subspace of $\mathcal{H}(\mathcal{P}, G)$ in a canonical way, and, for the present ϕ, the above chosen $\widehat\phi \in \mathcal{H}(\mathcal{P},G)$ actually lies in $\mathcal{H}^\natural(\mathcal{P},G)$. The commutativity of the diagram (2.6) then shows that right translation identifies the kernel of the derivative $\mathrm{T}_\phi r$ with the vector space $\mathrm{Z}^1(\pi,g_\phi)$ of g_ϕ-valued 1-cocycles of π, and the same is true of the kernel of the derivative

$$\mathrm{T}_{\widehat\phi}\widehat r\colon \mathrm{T}_{\widehat\phi}\mathcal{H}^\natural(\mathcal{P},G) \to \mathrm{T}_{\widehat r\widehat\phi}O.$$

Still by Corollary 4.8 of [32], right translation identifies the restriction of the 2-form $\omega_{c,\mathcal{P}}$ to the kernel of the derivative $\mathrm{T}_{\widehat\phi} r$ with the 2-form on $\mathrm{Z}^1(\pi,g_\phi)$ obtained as the composite of $\omega_{\kappa,\cdot,\phi}$ with the projection from $\mathrm{Z}^1(\pi,g_\phi)$ to $\mathrm{H}^1(\pi,g_\phi)$.

Suppose that the given 2-form $\cdot$ on g is non-degenerate. Write Z for the centre of G and z for its Lie algebra. By Poincaré duality in the cohomology of π, for every ϕ in the pre-image $r^{-1}(z) \subseteq \mathcal{H}(\mathcal{P},G)$ of $z \subseteq O$, in particular, for every $\phi \in \mathrm{Hom}(\pi,G)$, the 2-form $\omega_{\kappa,\cdot,\phi}$ is then symplectic. Notice that when π is torsion free, that is, a surface group, this form boils down to that considered by GOLDMAN [6] and reexamined in [32].

Next we consider the restriction map from $(\Omega_G^{*;*}(\mathcal{H}(\mathcal{P},G));d,\delta_G)$ to $(\Omega_G^{*;*}(\mathcal{H}^\natural(\mathcal{P},G));d,\delta_G)$ induced by the injection of $\mathcal{H}^\natural(\mathcal{P},G)$ into $\mathcal{H}(\mathcal{P},G)$. Abusing notation, we write $\omega_{c,\mathcal{P}}$ and $\mu^\sharp$ for the classes in $(\Omega_G^{*;*}(\mathcal{H}^\natural(\mathcal{P},G));d,\delta_G)$ obtained by restriction to $(\Omega_G^{*;*}(\mathcal{H}(\mathcal{P},G));d,\delta_G)$ of the corresponding classes denoted by the same symbol. It is clear that the identity $\delta_G(\omega_{c,\mathcal{P}}) = d\mu^\sharp$ passes to $(\Omega_G^{*;*}(\mathcal{H}^\natural(\mathcal{P},G));d,\delta_G)$. In other words, for every $X \in g$, we have

$$\omega_{c,\mathcal{P}}(X_{\mathcal{H}^\natural(\mathcal{P},G)},\cdot\,) = d(X\circ\mu), \tag{2.8}$$

that is, formally the momentum mapping property is satisfied, perhaps up to a sign depending on the choice of conventions which is unimportant for the geometry of the situation. The rest of the construction is now exactly the same as that in our paper [32], except that we work with the space $\mathcal{H}^\natural(\mathcal{P},G)$ instead of $\mathcal{H}(\mathcal{P},G)$ and with $\mathbf{P}(\mathcal{P})$ instead of $\mathbf{R}(\mathcal{P})$. In fact, the formal momentum mapping property (2.8), together with the symplecticity of the 2-form $\omega_{\kappa,\cdot,\phi}$ at every $\phi \in r^{-1}(z)$, implies that $\omega_{c,\mathcal{P}}$ has maximal rank equal to $\dim\mathcal{H}^\natural(\mathcal{P},G)$ at every point $\widehat\phi$ of $\mathcal{H}^\natural(\mathcal{P},G)$ in the pre-image $\widehat r^{-1}(z)$ of z, in particular, at every point of $\mathrm{Hom}(\pi,G)$. In fact, given a point $\widehat\phi$ of $\widehat r^{-1}(z)$, with the notation $\phi=\eta(\widehat\phi)$, the symplecticity of the 2-form $\omega_{\kappa,\cdot,\phi}$ implies that the 2-form $\omega_{c,\mathcal{P}}$ on the tangent space $\mathrm{T}_{\widehat\phi}\mathcal{H}^\natural(\mathcal{P},G)\cong \mathrm{C}^1(\mathbf{P}(\mathcal{P}),g_\phi)$, restricted to the subspace $\mathrm{Z}^1(\pi,g_\phi)$ of 1-cocycles, has degeneracy space equal to the subspace $\mathrm{B}^1(\pi,g_\phi)$ of 1-coboundaries, and the momentum mapping property then implies that the 2-form $\omega_{c,\mathcal{P}}$ on the whole space $\mathrm{C}^1(\mathbf{P}(\mathcal{P}),g_\phi)$ is non-degenerate. Let $\mathcal{M}^\natural(\mathcal{P},G)$ be the subspace of $\mathcal{H}^\natural(\mathcal{P},G)$ where the 2-form $\omega_{c,\mathcal{P}}$ is non-degenerate; this is an open G-invariant subset containing the pre-image $\widehat r^{-1}(z)$. Summarizing, we obtain the following.

Theorem 2.9. *The space $\mathcal{M}^\natural(\mathcal{P},G)$ is a smooth G-manifold (having in general more than one connected component), the 2-form $\omega_{c,\mathcal{P}}$ is a G-invariant symplectic structure on it, and the restriction*

$$\mu = -\psi\circ r\colon \mathcal{M}^\natural(\mathcal{P},G) \to g^*$$

is a momentum mapping in the usual sense.

Notice for a surface group the smooth manifold $\mathcal{M}^\natural(\mathcal{P}, G)$ boils down to the one denoted by $\mathcal{M}(\mathcal{P}, G)$ in [32]. We already remarked that when the constructions in [32] are carried out by means of the standard homotopy operator on forms on g, the map ψ from g to g^* is the adjoint of the chosen 2-form $\cdot$ on g. Whatever homotopy operator has been taken, symplectic reduction then yields the space $\mathrm{Rep}(\pi, G)$ of representations of π in G.

3. Twisted representation spaces

The construction of the universal central extension of the fundamental group of a closed surface generalizes in the following way to our group π: Let F be the free group on the generators of the presentation $\mathcal{P}$, N the normal closure of $r, r_1, \ldots, r_n$ in F, and $\Gamma = F/[F, N]$; then the kernel $N/[F, N]$ of the canonical projection from Γ to π decomposes into a direct sum of $n+1$ infinite cyclic groups, generated by $[r] = r[F, N] \in \Gamma$ and $[r_1] = r_1[F, N], \ldots, [r_n] = r_n[F, N]$; this is an immediate consequence of the statement of the *Identity Theorem* for the presentation $\mathcal{P}$, see e. g. our paper [33]. Notice the decomposition of $N/[F, N]$ depends on the presentation $\mathcal{P}$, and *not* merely on π. A closer look, cf. the formulas in our paper [33], shows that the second cohomology group $\mathrm{H}^2(\pi, \mathbf{Z})$ admits the following description: Write ζ for the $\mathbf{Z}$-valued "cocycle" which assigns 1 to $[r]$ and 0 to the other generators of $N/[F, N]$ and, for $1 \leq j \leq n$, write ζ_j for the $\mathbf{Z}$-valued "cocycle" which assigns 1 to $[r_j]$ and 0 to the other generators of $N/[F, N]$. The group $\mathrm{H}^2(\pi, \mathbf{Z})$ decomposes into a direct sum of an infinite cyclic group, generated by the class of ζ, and n finite cyclic groups of orders respectively $m_1, \ldots, m_n$, generated by the classes of $\zeta_1, \ldots, \zeta_n$, respectively. In particular, with the integers as coefficients, unlike for the case of the fundamental group of a surface, there is no canonical choice of universal central extension. On the other hand, the second cohomology group $\mathrm{H}^2(\pi, \mathbf{R})$ is a one-dimensional real vector space and hence there is a *universal central extension*

$$0 \to \mathbf{R} \to \Gamma_{\mathbf{R}} \to \pi \to 1 \tag{3.1}$$

of π by the reals which does *not* depend on the choice of presentation. In fact, the latter is determined by the requirement that a diagram of the kind

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N/[F, N] & \longrightarrow & \Gamma & \longrightarrow & \pi & \longrightarrow & 1 \\ & & {\scriptstyle\lambda}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\mathrm{Id}} & & \\ 0 & \longrightarrow & \mathbf{R} & \longrightarrow & \Gamma_{\mathbf{R}} & \longrightarrow & \pi & \longrightarrow & 1 \end{array} \tag{3.2}$$

be commutative, where $\lambda[r] = 1$ while, for $j = 1, \ldots, n$, $\lambda[r_j]$ may be chosen arbitrarily; whatever choice of the values $\lambda[r_j]$, the resulting extensions of π by $\mathbf{R}$ will be *congruent*. In other words, if the two extensions

$$0 \to \mathbf{R} \to \Gamma_1 \to \pi \to 1, \quad 0 \to \mathbf{R} \to \Gamma_2 \to \pi \to 1$$

arise from homomorphisms λ_1 and λ_2, respectively, as in (3.2) above, λ_1 and λ_2 determine a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{R} & \longrightarrow & \Gamma_1 & \longrightarrow & \pi & \longrightarrow & 1 \\ & & {\scriptstyle\mathrm{Id}}\big\downarrow & & \big\downarrow & & {\scriptstyle\mathrm{Id}}\big\downarrow & & \\ 0 & \longrightarrow & \mathbf{R} & \longrightarrow & \Gamma_2 & \longrightarrow & \pi & \longrightarrow & 1 \end{array}$$

of group extensions. Write Z for the centre of G, let z be the Lie algebra of Z, and let $X \in z$. When G is connected, z coincides with the zentre of g but in general z equals the invariants for the induced action of the group π_0 of components of G on the centre of g. Let $\mathrm{Hom}_X(\Gamma_{\mathbf{R}}, G)$ denote the space of homomorphisms ϕ from $\Gamma_{\mathbf{R}}$ to G having the property that $\phi(t[r]) = \exp(tX)$; we assume X chosen so that $\mathrm{Hom}_X(\Gamma_{\mathbf{R}}, G)$ is non-empty. This assumption is topological in nature. We briefly explain this below. Let $\mathrm{Rep}_X(\Gamma_{\mathbf{R}}, G) = \mathrm{Hom}_X(\Gamma_{\mathbf{R}}, G)/G$, the resulting *twisted representation space*. The space $\mathrm{Hom}_X(\Gamma_{\mathbf{R}}, G)$ and hence $\mathrm{Rep}_X(\Gamma_{\mathbf{R}}, G)$ is unambigously defined, independently of the choice of presentation etc., since so is the central extension (3.1). The space $\mathrm{Rep}_X(\Gamma_{\mathbf{R}}, G)$ is one of *projective* representations of our group π.

The choice of generators in $\mathcal{P}$ identifies the space $\mathrm{Hom}_X(\Gamma_{\mathbf{R}}, G)$ with the pre-image of $\exp(X) \in G$, for the word map r from $\mathrm{Hom}(F^{\natural}, G)$ to G induced by the relator r, and we can play a similar game as before, with the same choice of $c \in C_2(F)$ so that $\partial c = r$ represents $\kappa \in \mathrm{H}_2(\pi, \mathbf{R})$. More precisely, since the centre of G is contained in O, the space $\mathrm{Hom}_X(\Gamma_{\mathbf{R}}, G)$ arises as the pre-image of $X \in z \subseteq O$ under the map $\hat{r}$ from $\mathcal{H}^{\natural}(F, G)$ to O. Furthermore, in view of the Corollary to Lemma 1 in [32], the map ψ from g to g^* is regular at every point of the centre of g, in fact, the restriction of ψ to the centre equals the adjoint of the given 2-form whence the space $\mathrm{Hom}_X(\Gamma_{\mathbf{R}}, G)$ equals the pre-image of the adjoint $X^{\sharp} \in g^*$ of X under the momentum mapping μ from $\mathcal{M}(\mathcal{P}, G)$ to g^*. Consequently the space $\mathrm{Rep}_X(\Gamma_{\mathbf{R}}, G)$ is the corresponding reduced space, for the coadjoint orbit in g^* consisting of the single point $X^{\sharp}$. We already pointed out that, when the standard homotopy operator on forms on g is taken, the map ψ from g to g^* is in fact the adjoint of the chosen 2-form $\cdot$ on g.

The ambiguity with the choice of universal central extension over the integers is of course resolved by such a choice. Extensions of this kind have recently become of interest in the literature. We therefore explain briefly the resulting representation theory from our point of view: An arbitrary central extension $\Gamma_{(b,\beta_1,\ldots,\beta_n)}$ of π by the integers is given by a presentation

$$\langle x_1, y_1, \ldots, x_\ell, y_\ell, z_1, \ldots, z_n, h; [h, x_j], [h, y_j], [h, z_j], rh^{-b}, r_j h^{-\beta_j} \rangle$$

where

$$r = \Pi[x_j, y_j] z_1 \ldots z_n, \quad r_j = z_j^{m_j}$$

as before and where the parameters $b, \beta_1, \ldots, \beta_n$ are arbitrary integers; they correspond of course to the decomposition of $\mathrm{H}^2(\pi, \mathbf{Z})$ mentioned earlier, and different choices of these parameters may lead to the same group. In particular, a group of the kind $\Gamma_{(1,\beta_1,\ldots,\beta_n)}$ fits into a central extension

$$0 \longrightarrow \mathbf{Z} \longrightarrow \Gamma_{(1,\beta_1,\ldots,\beta_n)} \longrightarrow \pi \longrightarrow 1$$

in such a way that the class $[r]$ of r in $\Gamma_{(1,\beta_1,\ldots,\beta_n)}$ is identified with a generator of $\mathbf{Z}$. Let $\mathrm{Hom}_X(\Gamma_{(1,\beta_1,\ldots,\beta_n)}, G)$ denote the space of homomorphisms ϕ from $\Gamma_{(1,\beta_1,\ldots,\beta_n)}$ to G having the property that $\phi([r]) = \exp(X)$. Let $\mathrm{Rep}_X(\Gamma_{(1,\beta_1,\ldots,\beta_n)}, G) = \mathrm{Hom}_X(\Gamma_{(1,\beta_1,\ldots,\beta_n)}, G)/G$, the resulting *twisted representation space*. The choice of generators identifies the space $\mathrm{Rep}_X(\Gamma_{\mathbf{R}}, G)$ with the space $\mathrm{Rep}_X(\Gamma_{(1,\beta_1,\ldots,\beta_n)}, G)$, whatever $(\beta_1, \ldots, \beta_n)$.

When the parameters $b, \beta_1, \ldots, \beta_n, m_1, \ldots, m_n$ satisfy certain numerical conditions, $\Gamma_{(b,\beta_1,\ldots,\beta_n)}$ is the fundamental group of a *Seifert fibre space* which is (i) closed (as a 3-manifold), (ii) is an Eilenberg-Mac Lane space, and has (iii) orientable decomposition surface, cf. [40]. By *symplectic reduction in finite dimensions*, we thus obtain in particular spaces of representations of fundamental groups of *all* Seifert fibre spaces belonging to the class described above. The significance of this remark has already been spelled out in the Introduction.

In our situation, the parameter X is a certain topological characteristic class of a principal V-bundle over the orbit space Σ with structure group G associated with the representations in $\mathrm{Rep}_X(\Gamma_{\mathbf{R}}, G)$, cf. [30].

4. The connected components

The reduced spaces arising from the above construction will in general have more than one connected component. We now explain briefly how these components arise and how they can be labelled.

Let C denote a finite cyclic group. Then the space $\mathrm{Hom}(C, G)$ has finitely many connected components, whence the above space $\mathrm{Hom}(F^\natural, G)$ and therefore the various representation spaces will have finitely many connected components, certainly more than one if π has elliptic elements.

We describe some of the details in a special case: Suppose G compact and connected, and let T be a maximal torus in G, of rank say r. Then the space $\mathrm{Hom}(C, T)$ is a finite set, consisting of $r|C|$ points. Consequently the space $\mathrm{Hom}(C, G)$ will have $r|C|$ connected components, each one of the form G/K for a closed subgroup K of G. The connected components, in turn, correspond to conjugacy classes of elements in G. Thus a connected component of $\mathrm{Hom}(F^\natural, G)$ and therefore of the various representation spaces will be determined by a choice of conjugacy class, for each one of the generators $z_1, \ldots, z_n$ of finite order.

We now consider the special case of $G = U(n)$, the unitary group. A choice of X so that $\mathrm{Hom}_X(\Gamma_{\mathbf{R}}, G)$ is non-empty will correspond to a certain holomorphic rank n vector bundle ζ on Σ. Let $p_1, \ldots, p_n$ be the distinct points on Σ arising as the images of the fixed points of the elliptic transformations $z_1, \ldots, z_n$ in π. A choice of conjugacy class for each z_j corresponds to picking a flag and a rational weight, for each p_j; in other words a connected component corresponds to a parabolic structure with rational weights on ζ. Our twisted moduli space $\mathrm{Rep}_X(\Gamma_{\mathbf{R}}, G)$ will then contain as top stratum a homeomorphic image of the stable part of the corresponding MEHTA-SESHADRI [39] moduli space of semistable parabolic rank n holomorphic vector bundles with rational weights of degree determined by X. It is likely that in fact our construction yields all of these moduli spaces, not just the stable part. We already pointed out that, for parabolic degree zero, the spaces *are* known to be homeomorphic, by a result of MEHTA-SESHADRI, cf. (4.1) and (4.3) in [39]. What is *new* here is that important geometric information about the MEHTA-SESHADRI moduli spaces is obtained by *symplectic reduction, applied to a smooth finite dimensional symplectic manifold with a hamiltonian action of the finite dimensional Lie group* $\mathrm{U}(n)$.

5. Applications

Suppose G compact. Recall that the notion of stratified symplectic space has been introduced in [22].

Theorem 5.1. *With respect to the decomposition according to G-orbit types, each connected component of the space* $\mathrm{Rep}(\pi, G)$ *and, more generally, each connected component of a twisted representation space* $\mathrm{Rep}_X(\Gamma_{\mathbf{R}}, G)$ *inherits a structure of stratified symplectic space.*

In fact, the argument for the main result of [22] shows that each connected component of a reduced space of the kind considered inherits a structure of stratified symplectic space. In the setting of [22] the hypothesis of properness is used *only* to guarantee that the reduced space is in fact connected. In our situation, we know a priori that the reduced space is connected.

Theorem 5.2. *Each stratum of (a connected component of) the space* $\mathrm{Rep}(\pi, G)$ *and, more generally, each stratum of (a connected component of) a twisted representation space* $\mathrm{Rep}_X(\Gamma_{\mathbf{R}}, G)$ *has finite symplectic volume.*

The proof follows the same pattern as that for the argument for (3.9) in [22]. There the unreduced symplectic manifold is assumed compact. However the compactness of the zero level set suffices; in our situation the zero level set *is* compact. In fact, it suffices to prove the statement for the local model in [22] which looks like the reduced space of a unitary representation of a compact Lie group, for the corresponding unique momentum mapping having the value zero at the origin. For the local model there is no difference between (3.9) in [22] and our situation. Once the statement is established for the local model, that of Theorem 5.2 follows since the reduced space may be covered by finitely many open sets having a local model of the kind described.

We mention two other consequences:

Corollary 5.3. *Each connected component has a unique open, connected, and dense stratum.*

In fact, this follows at once from [22] (5.9). Likewise [22] (5.11) entails the following.

Corollary 5.4. *For each connected component, the reduced Poisson algebra is symplectic, that is, its only Casimir elements are the constants.*

6. Elliptic surfaces

Recall an elliptic surface is a smooth compact complex surface M with a proper surjective holomorphic map $f: M \to \Sigma$ onto a complex curve Σ such that the generic fibre is an elliptic curve. Let $\{p_1, \dots, p_n\} \subseteq \Sigma$ be the finite set of non-regular values. Then Σ may be viewed as an orbifold, to each point p_j the multiplicity of its fibre being attached. The map f induces an isomorphism from the fundamental group $\pi = \pi_1(M)$ onto the orbifold fundamental group of (Σ, S). Results of BAUER [24] relate moduli spaces of parabolic bundles on Σ of parabolic degree zero with moduli spaces of degree zero vector bundles on M. Bauer's construction avoids DONALDSON'S [29] solution of the KOBAYASHI-HITCHIN conjecture. In a follow up paper [34] we shall study certain moduli spaces over elliptic surfaces from the symplectic point of view and in particular offer a somewhat more intrinsic construction of the moduli space $\mathrm{Rep}_X(\Gamma_{\mathbf{R}}, G)$ with its stratified symplectic structure, viewed as a moduli space of Einstein-Hermitian connections on a certain projectively flat bundle over M. In particular, this will give the symplectic counterpart of Bauer's relationship together with (i) a compactification thereof and (ii) an extension to arbitrary parabolic degree. A choice of projective embedding of M will then again correspond to a choice of universal central extension of π over the integers. Essentially different choices of λ in (3.2) will then correspond to topologically inequivalent line bundles.

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