ON TORSION ELEMENTS IN THE CHOW-GROUPS OF QUADRICS

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0. Introduction

In this paper we present some new example of torsion elements in the Chow-groups of projective quadric. For hyperbolic projective quadric X and any $0 \leqslant p < \dim(X)/2$, the Chow-group of cycles of codimension p (modulo rational equevalence) is a free abelian group generated by the class h^p of plane section of codimension p,. By transfer arguments it follows, that for arbitrary (smooth) projective quadric, $CH^p(X) = \mathbb{Z} \cdot h^p \oplus Tors.(CH^p(X))$.

On the other hand, for any projective quadric X of dimension more than 2, the *Picard group* $CH^1(X)$ is isomorphic to $\mathbb{Z} \cdot h$, in other words, there is no torsion. The natural question arises: can we extend this result to higher Chow-groups.

Basic here is the following conjecture due to N. Karpenko:

Let X_φ be a projective quadric, defined by quadratic form ϕ of dimension n over a field of characteristic not 2.

Conjecture 1 ([7], Conjecture 0.1).

For any p, if n is sufficiently large, then $CH^p(X_\varphi) = \mathbb{Z} \cdot h^p$.

This conjecture was supported by the following computations:

Theorem ([6], Theorem 6.1; [8], Theorem 6.1, Theorem 8.5).

Under above notations,

(a) $CH^2(X_\varphi) = \mathbb{Z}$, *for* $n > 8$;
(b) $CH^3(X_\varphi) = \mathbb{Z}$, *for* $n > 12$;
(c) $CH^4(X_\varphi) = \mathbb{Z}$, *for* $n > 24$;

Moreover, the boundaries in (a) and (b) are exact ones, and more generally:

Theorem ([7], Theorem 2.4).

For any $p > 1$ there exist a $4p$-dimensional quadratic form φ (over a suitable field F), such that $Tors.(CH^p(X_\varphi)) \neq 0$.

In [7] an attempt was made to make Conjecture 1 more precise:

Conjecture 2 ([7], Conjecture 0.2).

If $n > 4p$ for some p, then $CH^p(X_\varphi) = \mathbb{Z} \cdot h$.

The principal aim of this note is to disprove this stronger version of the Conjecture - see Corollary 2.1 .

For any $r \geqslant 2$ we will construct the quadratic form φ of dimension $6 \cdot 2^r$ (over suitable field), such that $Tors.(CH^{2^r+1}(X_{\varphi})) \neq 0$. This clearly will disprove the Conjecture 2 .

The main ingredients of the construction are: the *special pair* of O.Izhboldin (see [3], Section 9), the computations of *unramified cohomology* by B.Kahn-M.Rost-R.J.Sujatha (including the motivic interpretation of the later)(see [5], and [4], Appendix), and the *motivic cohomological operations* of V.Voevodsky (see [17]). Although, the original problem is formulated in the classical Chow-motivic language, we have to work in the bigger *triangulated category of mixed motives* of V.Voevodsky (see [16]). In this category we can use the *operations* and the decomposition of the motive of a quadric from [14], Theorem 3.1.

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1. Some preliminary computations

Under *generalized Albert form* we will understand the anisotropic part of the difference of two n-fold Pfister forms:

$$(\langle\langle a_1, \ldots, a_n\rangle\rangle - \langle\langle b_1, \ldots, b_n\rangle\rangle)_{an}.$$

Such forms were studied extensively in [1].

Proposition 1.1 .

For any r, over some field of given characteristic $\neq 2$ there exists anisotropic generalized Albert form of dimension $6 \cdot 2^r$.

Proof of Proposition 1.1

Lemma 1.2 .

Let F be a field of characteristic not 2, and q/F be an anisotropic quadratic form. Let $K := F(x_1, \ldots, x_r)$ be a purely transcendental extension, and $p := q \times \langle\langle x_1 \ldots, x_r\rangle\rangle$ be a quadratic form over K. Then p is anisotropic.

Proof

It is evidently sufficient to consider the case $r = 1$.

Suppose that $p := q \times \langle 1, -x\rangle$ is isotropic over $F(x)$. Restricting to $F((x))$, we get a contradiction with the Theorem of Springer (see [10], VI, Proposition 1.9). So, p is anisotropic.

$\square$

Lemma 1.3 .

Suppose the Pfister form $\langle\langle a, b\rangle\rangle$ over the field F is anisotropic. Then the Albert form $\langle a, b, -ab, yz, -y, -z\rangle$ is anisotropic over the field $L := F(y, z)$.

Proof

If our Albert form is isotropic over L, then it is isotropic over $F((y))((z))$, and by [10], VI, Proposition 1.9, we have that $\langle a, b, -ab, -y\rangle$ is isotropic over $F((y))$ (since $\langle y, -1\rangle$ is evidently anisotropic over the last field). Again by the Theorem of Springer we get a contradiction.

$\square$

Suppose $K = k(a, b, c, d, x_1, \ldots, x_r)$ be purely transcendental extension, generated by specified variables. Consider quadratic form $\varphi := \rho \times \langle\langle x_1, \ldots, x_r\rangle\rangle$, where $\rho := \langle a, b, -ab, cd, -c, -d\rangle$.

Lemma 1.4 .

The form φ is anisotropic.

Proof

By Lemma 1.2 , it is sufficient to prove that $\langle a, b, -ab, cd, -c, -d\rangle$ is anisotropic over $L := k(a, b, c, d)$. Take $F := k(a, b)$. Then evidently $\langle\langle a, b\rangle\rangle$ is anisotropic over F. By Lemma 1.3 we get what we need.

$\square$

Proposition 1.1 is proven.

$\square$

2. The Main Theorem

Main Theorem .

Let φ is anisotropic generalized Albert form of dimension $6 \cdot 2^r$ over the field k of characteristic 0. Then $\mathrm{Tors.\,CH}^{2^r+1}(X_\varphi) \neq 0$.

Combining the Main Theorem with Proposition 1.1 , we get:

Corollary 2.1 .

Conjecture 2 is wrong.

Proof of the Main Theorem

Since φ is anisotropic *generalized Albert form* of dimension $6 \cdot 2^r$, we have: $\varphi = \rho \times \langle\langle x_1, \ldots, x_r\rangle\rangle$, where $\rho = \langle a, b, -ab, -c, -d, cd\rangle$ is

6-dimensional *Albert form*, and $\langle\langle x_1, \dots, x_r\rangle\rangle$ is some Pfiter form (see [1]).

Denote $\psi := \langle a, b, -ab, -c, -d\rangle \times \langle\langle x_1, \dots, x_r\rangle\rangle$. Then (φ, ψ) is a *special pair* of Izhboldin (see [3], Section 9), and by condition, it is an *anisotropic special pair*.

By [3], Theorem 9.4 (3), we have that $\psi|_{k(X_\varphi)}$ is anisotropic Pfister neighbor, i.e., it is proportional to a subform (of dimension more than half) of anisotropic Pfister form π. This Pfister form corresponds to a nontrivial *pure symbol* α in $\mathrm{K}^M_{r+3}(k(X_\varphi))/2$. Under the natural map $\mathrm{K}^M_*(k(X_\varphi))/2 \to \mathrm{H}^*_{et.}(k(X_\varphi), \mathbb{Z}/2)$ (which is an isomorphism by "Milnor's conjecture" - see [17]), α goes to the nontrivial element $e^{r+3}(\mathrm{Pf}(\psi|_{k(X_\varphi)}))$ of $\mathrm{H}^{r+3}_{et.}(k(X_\varphi), \mathbb{Z}/2)$ (we are following the notations of [3], Lemma 7.4). By [3], Corollary 7.3, $e^{r+3}(\mathrm{Pf}(\psi|_{k(X_\varphi)}))$ actually belongs to the *unramied* part: $\mathrm{H}^{r+3}_{nr}(k(X_\varphi)/k, \mathbb{Z}/2)$ of $\mathrm{H}^{r+3}_{et.}(k(X_\varphi), \mathbb{Z}/2)$.

We have natural map: $\eta_2^* : \mathrm{H}^*_{et.}(k, \mathbb{Z}/2) \to \mathrm{H}^*_{nr}(k(X_\varphi)/k, \mathbb{Z}/2)$. Let us denote $\tilde{\mathrm{H}}^*_{nr}(k(X_\varphi)/k, \mathbb{Z}/2) := \mathrm{H}^*_{nr}(k(X_\varphi)/k, \mathbb{Z}/2)/image(\eta_2^*)$. For arbitrary element x from $\mathrm{H}^*_{nr}(k(X_\varphi)/k, \mathbb{Z}/2)$, we denote it's image in $\tilde{\mathrm{H}}^*_{nr}(k(X_\varphi)/k, \mathbb{Z}/2)$ as $\tilde{x}$.

Proposition 2.2 (cf. [3], Lemma 10.5, Lemma 7.4).
Under the above notations, $\tilde{e}^{r+3}(\mathrm{Pf}(\psi|_{k(X_\varphi)}))$ *is nonzero.*

Proof

Suppose, $\tilde{e}^{r+3}(\mathrm{Pf}(\psi|_{k(X_\varphi)})) = 0$. That means that $e^{r+3}(\mathrm{Pf}(\psi|_{k(X_\varphi)})) = \eta_2^{r+3}(\lambda)$ for some $\lambda \in \mathrm{H}^{r+3}_{et.}(k, \mathbb{Z}/2)$. By the results of V.Voevodsky, we have the natural identification:

$$\mathrm{K}^M_{r+3}(k)/2 = \mathrm{H}^{r+3}_{et.}(k, \mathbb{Z}/2) = I^{r+3}(W(k))/I^{r+4}(W(k)),$$

where $I \subset W(k)$ is the ideal of even-dimensional form in the Witt-ring $W(k)$ (see [17] and [11], Section 3.1). So, there exists quadratic form $q \subset I^{r+3}(W(k)) \backslash I^{r+4}(W(k))$, s.t. $\overline{q} \in I^{r+3}(W(k))/I^{r+4}(W(k))$ corresponds to λ under the identifications above. By "J-filtration Conjecture" we have $I^{r+3}(W(k)) = J^{r+3}(W(k))$ (see [11], Section 3.3, Statement 2). That means that the *degree* (see [9]) of q is $r+3$, and there exists a *Generalized Splitting Tower* of Manfred Knebusch (see [9]) $k = k_0 \subset k_1 \subset \cdots \subset k_{s-1} \subset k_s$, s.t. for all $0 \leqslant l \leqslant s-2$, $k_{l+1} = k_l(Q_l)$, where Q_l is quadric (over k_l) of dimension greater than $2^{r+3} - 2$, and $(q|_{k_{s-1}})_{anis.}$ is proportional to anisotropic $r+3$-fold Pfister form over k_{s-1}. That means that $\overline{q}|_{k_{s-1}} = \lambda|_{k_{s-1}} \in \mathrm{K}^M_{r+3}(k_{s-1})/2$ is a nonzero *pure symbol.*

Since $\dim(q_l) > 2^{r+3} > 6 \cdot 2^r = \dim(\varphi)$, by the result of Detlew Hoffmann (see [2], Theorem 1), we have that $\varphi|_{k_{s-1}}$ is anisotropic.

Let us denote $F := k_{s-1}$. We have: $\varphi|_F$ is anisotropic, and (if you want, again by [3], Theorem 9.4 (3)) $\psi|_{F(X_\varphi)}$ is a neighbor of anisotropic Pfister form $\langle\langle\lambda|_F\rangle\rangle|_{F(X_\varphi)}$ (notice that $\lambda|_F$ is a *pure symbol*). In particular, $\lambda|_{F(X_\varphi)(X_\psi)} = 0$. Since $\psi \subset \varphi$, we have $\lambda|_{F(X_\psi)} = 0$. That means that $\psi|_F$ is a Pfister neighbour of $\langle\langle\lambda|_F\rangle\rangle$ By [3], Lemma 9.4 (2), we get: $\varphi|_F$ is isotropic - a contradiction. So, $\tilde{e}^{r+3}(\mathrm{Pf}(\psi|_{k(X_\varphi)})) \neq 0$

□

Let $\gamma := \tilde{e}^{r+3}(\mathrm{Pf}(\psi|_{k(X_\varphi)})) \in \tilde{\mathrm{H}}^{r+3}_{nr}(k(X_\varphi)/k, \mathbb{Z}/2)$. By Proposition 2.2 , $\gamma \neq 0$.

Together with the map $\eta_2^m : \mathrm{H}^m_{et.}(k, \mathbb{Z}/2) \to \mathrm{H}^m_{nr.}(k(X_\varphi)/k, \mathbb{Z}/2)$ we can consider the map: $\eta^m : \mathrm{H}^m_{et.}(k, \mathbb{Q}/\mathbb{Z}(m-1)) \to \mathrm{H}^m_{nr.}(k(X_\varphi)/k, \mathbb{Q}/\mathbb{Z}(m-1))$.

From [5], Theorem 7.4 and Remark after it (and from "Milnor's conjecture", see [17]), we have an exact sequence:

$$0 \to (\mathrm{Ker}(\eta_2^m))_0 \to \mathrm{coker}(\eta_2^m) \to \mathrm{coker}(\eta^m),$$

where $(\mathrm{Ker}(\eta_2^m))_0 = \{y \in \mathrm{Ker}(\eta_2^m) \mid \{-1\} \cdot y = 0\}$.

Lemma 2.3 .

The group $\mathrm{Ker}(\eta_2^{r+3} : \mathrm{H}^{r+3}_{et.}(k, \mathbb{Z}/2) \to \mathrm{H}^{r+3}_{nr.}(k(X_\varphi)/k, \mathbb{Z}/2))$ *is zero. Consequently,* $(\mathrm{Ker}(\eta_2^{r+3}))_0 = 0$.

Proof

Suppose, our Ker is nontrivial. Since $\mathrm{H}^{r+3}_{nr.}(k(X_\varphi)/k, \mathbb{Z}/2))$ is a subgroup in $\mathrm{H}^{r+3}_{et.}(k(X_\varphi), \mathbb{Z}/2)$, in this case we get a nontrivial element $h \in \mathrm{Ker}(\mathrm{H}^{r+3}_{et.}(k, \mathbb{Z}/2) \to \mathrm{H}^{r+3}_{et.}(k(X_\varphi), \mathbb{Z}/2)$.

By [11], Lemma from Section 3, there exists field extension L/k, s.t. $h|_L$ is nonzero *pure symbol* $\{a_1, \dots, a_{r+3}\}$ for some $a_i \in L$. Then $\varphi|_L$ (up to coefficient) is a subform in the anisotropic Pfister form $\langle\langle a_1, \dots, a_{r+3}\rangle\rangle$. Suppose ρ is a complementary form, i.e.: $\varphi|_L \perp \rho = \langle\langle a_1, \dots, a_{r+3}\rangle\rangle$.

Since $\varphi|_L \in I^{r+2}(W(L))$ and $\langle\langle a_1, \dots, a_{r+3}\rangle\rangle \in I^{r+3}(W(L))$, we have that $\rho \in I^{r+2}(W(L))$. But, the $\dim(\rho) = 2^{r+3} - 6 \cdot 2^r = 2^{r+1} < 2^{r+2}$ - contradiction.

□

So, we have an exact sequence:

$$0 \to \mathrm{coker}(\eta_2^{r+3}) \to \mathrm{coker}(\eta^{r+3}).$$

Lemma 2.4 .

Let X be arbitrary quadric, then $\mathrm{coker}(\eta^m_{\mathbb{Q}_{(p)}/\mathbb{Z}_{(p)}} : \mathrm{H}^m_{et.}(k, \mathbb{Q}_{(p)}/\mathbb{Z}_{(p)}) \to \mathrm{H}^m_{nr.}(k(X), \mathbb{Q}_{(p)}/\mathbb{Z}_{(p)})) = 0$ *for any $p \neq 2$ and any m.*

Proof

Let $\mathrm{Spec}(L) \subset X$ will be arbitrary point on our quadric. Since $X_\varphi|_L$ is isotropic, it is birationally equivalent to projective space, and hence, $\mathrm{coker}(\eta^m_A : \mathrm{H}^m_{et.}(L, A) \to \mathrm{H}^m_{nr.}(L(X), A)) = 0$ for any group of coefficients A (see [12], 12.10, 7.3). Since $\mathbb{Q}_{(p)}/\mathbb{Z}_{(p)}$ is uniquely 2-divisible, by transfer arguments, we have that $\mathrm{coker}(\eta^m_{\mathbb{Q}_{(p)}/\mathbb{Z}_{(p)}} : \mathrm{H}^m_{et.}(k, \mathbb{Q}_{(p)}/\mathbb{Z}_{(p)}) \to \mathrm{H}^m_{nr.}(k(X), \mathbb{Q}_{(p)}/\mathbb{Z}_{(p)})) = 0$.

$\square$

The above Lemma shows:
$\mathrm{coker}(\eta^{r+3}) = \mathrm{coker}(\eta^{r+3}_{\mathbb{Q}_{(2)}/\mathbb{Z}_{(2)}} : \mathrm{H}^{r+3}_{et.}(k, \mathbb{Q}_{(2)}/\mathbb{Z}_{(2)}) \to \mathrm{H}^{r+3}_{nr.}(k(X_\varphi), \mathbb{Q}_{(2)}/\mathbb{Z}_{(2)}))$.

In this light, we will denote $\eta^{r+3}_{\mathbb{Q}_{(2)}/\mathbb{Z}_{(2)}}$ simply as η^{r+3}.

In [4] B.Kahn interpreted $\mathrm{coker}(\eta^*)$ in terms of *motivic cohomology* - see [4], Theorem A.1: there is an exact sequence:

$$0 \to \mathrm{coker}(\eta^{r+3}) \to \mathrm{H}^{r+5}_B(\mathcal{X}_{X_\varphi}, \mathbb{Z}_{(2)}(r+2)),$$

where $\mathcal{X}_{X_\varphi}$ is a *standard simplicial scheme*, corresponding to the morphism $X_\varphi \to \mathrm{Spec}(k)$ (see, for example, [14], Definition 2.3.1), and $\mathrm{H}^{r+5}_B(\mathcal{X}_{X_\varphi}, \mathbb{Z}_{(2)}(r+2)) = \mathrm{Hom}_{DM^-_{eff.}}(M(\mathcal{X}_{X_\varphi}), \mathbb{Z}_{(2)}(r+2)[r+5])$ - the Hom-group in the *triangulated category of mixed motives* of V.Voevodsky, where $\mathbb{Z}$ by definition is the motive of a point $(M(\mathrm{Spec}(k)))$.

Remark Notice the discrepancy in the notations for η^m in [4] in comparison to [5] (where it coinsides with our notation). η^m in [4] corresponds to η^{m+1} in [5].

This shows, that we have an exact sequence:

$$0 \to \mathrm{coker}(\eta^{r+3}_2) \to \mathrm{Hom}_{DM^-_{eff.}}(M(\mathcal{X}_{X_\varphi}), \mathbb{Z}_{(2)}(r+2)[r+5]).$$

Since we have a nontrivial element γ in $\mathrm{coker}(\eta^{r+3}_2)$, it's image $\gamma' \in \mathrm{Hom}_{DM^-_{eff.}}(M(\mathcal{X}_{X_\varphi}), \mathbb{Z}_{(2)}(r+2)[r+5])$ will also be nontrivial.

Using the fact that $\mathrm{Hom}_{DM^-_{eff.}}(M(\mathrm{Spec}(L)), \mathbb{Z}(i)[j]) = 0$ for $j > i$, where L is a field (see [17], Corollary 2.3), and the fact that $M(\mathcal{X}_{X_\varphi})|_L = M(\mathrm{Spec}(L))$ for any point $\mathrm{Spec}(L) \subset X_\varphi$ on a quadric (see [17], Lemma 3.8, or [14], Theorem 2.3.4), by usual transfer arguments we have that the groups $\mathrm{Hom}_{DM^-_{eff.}}(M(\mathcal{X}_{X_\varphi}), \mathbb{Z}_{(2)}(i)[j])$ have exponent 2 for $j > i$. That means, that we have an embedding: $\mathrm{Hom}_{DM^-_{eff.}}(M(\mathcal{X}_{X_\varphi}), \mathbb{Z}_{(2)}(r+2)[r+5]) \to \mathrm{Hom}_{DM^-_{eff.}}(M(\mathcal{X}_{X_\varphi}), \mathbb{Z}/2(r+2)[r+5])$. Let γ'' be the image of γ' under this map. We have: $\gamma'' \neq 0$.

From this point we will drop subscript $DM^-_{eff.}$ from Hom, and $M(-)$.

Together with $\mathcal{X}_{X_\varphi}$ we can consider $\tilde{\mathcal{X}}_{X_\varphi} := \mathrm{Cone}[-1](pr : \mathcal{X}_{X_\varphi} \to \mathbb{Z})$, wher pr is induced by the natural projection $\mathcal{X}_{X_\varphi} \to \mathrm{Spec}(k)$.

Now, we can use Voevodsky's *cohomological operations.*

We have *cohomological operations* Q_i of bidegree $(2^i-1)[2^{i+1}-1]$, s.t. Q_i is a differential (it's square iz zero), and for quadric X of dimension greater or equal 2^m-1, Q_m acts without cohomology on $\mathrm{Hom}(\tilde{\mathcal{X}}_X, \mathbb{Z}/2(*)[*'])$ (see [17], Theorem 3.17, Theorem 3.25 and Lemma 4.11 (notice that in the proof of the later no specific of Pfister case is used))

In particular, in our case, $Q_1, \dots, Q_{r+2}$ will act without cohomology on $\mathrm{Hom}(\tilde{\mathcal{X}}_{X_\varphi}, \mathbb{Z}/2(*)[*'])$.

Since $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2(i)[j]) = 0$ for $j > i$ (see [17], Corollary 2.3), we can identify $\mathrm{Hom}(\tilde{\mathcal{X}}_{X_\varphi}, \mathbb{Z}/2(i)[j])$ with $\mathrm{Hom}(\mathcal{X}_{X_\varphi}, \mathbb{Z}/2(i)[j])$ if $j > i$. Below we will deal only with such cohomology ($j > i$), so, we will use $\mathcal{X}_{X_\varphi}$ instead of $\tilde{\mathcal{X}}_{X_\varphi}$.

Let us consider $Q_{r-1} \circ Q_{r-2} \circ \dots \circ Q_1(\gamma'')$.

Lemma 2.5 .
$Q_{r-1} \circ Q_{r-2} \circ \dots \circ Q_1(\gamma'') \neq 0$.

Proof

Suppose that $Q_{r-1} \circ Q_{r-2} \circ \dots \circ Q_1(\gamma'') = 0$. Then there exists such $0 \leqslant i \leqslant r-2$, that $Q_i \circ \dots Q_1(\gamma'') \neq 0$, but $Q_{i+1} \circ \dots Q_1(\gamma'') = 0$.

Since Q_j has bidegree $(2^j-1)[2^{j+1}-1]$ (see [17], Theorem 3.17), we have: $y := Q_i \circ \dots Q_1(\gamma'') \in \mathrm{Hom}(\mathcal{X}_{X_\varphi}, \mathbb{Z}/2(r-i+2^{i+1})[r-i+2^{i+2}+1])$. Since $Q_{i+1}(y) = 0$ and Q_{i+1} acts without cohomology on $\mathrm{Hom}(\mathcal{X}_{X_\varphi}, \mathbb{Z}/2(*)[*'])$, we have that there exists $z \in \mathrm{Hom}(\mathcal{X}_{X_\varphi}, \mathbb{Z}/2(r-i+1)[r-i+2])$, s.t. $y = Q_{i+1}(z)$.

By [16], Lemma 6.4, [17], Theorem 4.1 (see also [4], Theorem A.1), we have an identification of $\mathrm{Hom}(\mathcal{X}_{X_\varphi}, \mathbb{Z}/2(r-i+1)[r-i+2]$ with the $\mathrm{Ker}(\mathrm{K}^M_{r-i+2}(k)/2 \to \mathrm{K}^M_{r-i+2}(k(X_\varphi))/2)$.

Since $\dim(X_\varphi) = 6{\cdot}2^r > 2^{r-i+2}$ we have by "Kahn-Rost-Sujatha Conjecture" (see [11], Statement 1 from Section 3.2) that $\mathrm{Ker}(\mathrm{K}^M_{r-i+2}(k)/2 \to \mathrm{K}^M_{r-i+2}(k(X_\varphi))/2) = 0$. We get a contradiction. So, $Q_{r-1} \circ Q_{r-2} \circ \dots \circ Q_1(\gamma'') \neq 0$.

$\square$

So, we get a nontrivial element $u := Q_{r-1} \circ Q_{r-2} \circ \dots \circ Q_1(\gamma'') \in \mathrm{Hom}(\mathcal{X}_{X_\varphi}, \mathbb{Z}/2(2^r+1)[2^{r+1}+2])$.

Since γ'' came from cohomology group with $\mathbb{Z}_{(2)}$ coefficients, and $Q_i = [\beta, q_i]$ for some operation q_i (β here is a Bockstein morphism) (see [17], Theorem 3.17(3)), we have that there exists $v \in \mathrm{Hom}(\mathcal{X}_{X_\varphi}, \mathbb{Z}(2^r+1)[2^{r+1}+2])$ s.t. $\overline{v} = u$.

The main conclusion is:

Lemma 2.6 .

The group $\mathrm{Hom}(\mathcal{X}_{X_\varphi}, \mathbb{Z}(2^r+1)[2^{r+1}+2])$ *is nontrivial.*

$\square$

Now, we need to relate the motivic cohomology of $\mathcal{X}_{X_\varphi}$ and those of X_φ. Let X^j_φ be (as usually) the variety of j-dimensional projective spaces on X_φ. By [14], Theorem 4.1, the motive $M(X_\varphi)$ of X_φ decomposes as: $M(X_\varphi) = \oplus_{i=0,\dots,2^r-1} F_\alpha(X_\rho)(i)[2i]$, where $\alpha := \{x_1, \dots, x_r\} \in \mathrm{K}^M_r(k)/2$, and $F_\alpha(X_\rho)$ is so-called *higher form* of 4-dimensional quadric X_ρ, and $F_\alpha(X_\rho)$ is an extension (in $DM^-_{eff.}$) of $M(\mathcal{X}_{X_\varphi})$, $M(\mathcal{X}_{X^{2^r}_\varphi})(2^r)[2^{r+1}]$, $M(\mathcal{X}_{X^{2^{r+1}}_\varphi})(2\cdot 2^r)[2\cdot 2^{r+1}]$, $M(\mathcal{X}_{X^{2^{r+1}}_\varphi})(3\cdot 2^r-1)[3\cdot 2^{r+1}-2]$, $M(\mathcal{X}_{X^{2^r}_\varphi})(4\cdot 2^r-1)[4\cdot 2^{r+1}-2]$ and $M(\mathcal{X}_{X_\varphi})(5\cdot 2^r-1)[5\cdot 2^{r+1}-2]$.

More precisely, there exists the following diagram in $DM^-_{eff.}(k)$ (we drop $M(-)$ from the notations):

$$\begin{array}{ccccccccc}
 & & \mathcal{X}_{X_\varphi} & & \mathcal{X}_{X^{2^r}_\varphi}(2^r)[2^{r+1}] & & & & \mathcal{X}_{X_\varphi}(5\cdot 2^r-1)[5\cdot 2^{r+1}-2] \\
 & \nearrow\star & \downarrow[1] & \nearrow\star & \downarrow[1] & \dots & & \nearrow\star & \downarrow[1] \\
F_\alpha(X_\rho) & \longleftarrow & Y_1 & \longleftarrow & Y_2 & & Y_5 & \longleftarrow & 0
\end{array},$$

(see [14], Lemma 3.23, Theorem 3.1, and use *octachedron axiom*).

In particular, we have an exact sequenses:

$$\mathrm{Hom}(F_\alpha(X_\rho), \mathbb{Z}(2^r+1)[2^{r+1}+2]) \leftarrow \mathrm{Hom}(\mathcal{X}_{X_\varphi}, \mathbb{Z}(2^r+1)[2^{r+1}+2]) \leftarrow \\ \leftarrow \mathrm{Hom}(Y_1, \mathbb{Z}(2^r+1)[2^{r+1}+1]).$$

and

$$\mathrm{Hom}(Y_2, \mathbb{Z}(2^r+1)[2^{r+1}+1]) \leftarrow \mathrm{Hom}(Y_1, \mathbb{Z}(2^r+1)[2^{r+1}+1]) \leftarrow \\ \leftarrow \mathrm{Hom}(\mathcal{X}_{X^{2^r}_\varphi}(2^r)[2^{r+1}], \mathbb{Z}(2^r+1)[2^{r+1}+1]) \leftarrow \mathrm{Hom}(Y_2, \mathbb{Z}(2^r+1)[2^{r+1}]).$$

Notice, that Y_2 is an extension of some $\mathcal{X}_P(j)[2j]$, where P are smooth projective varieties and $j \geqslant 2^{r+1} > 2^r+1$. By [17], Corollary 2.2(1), we have: $\mathrm{Hom}(Y_2, \mathbb{Z}(2^r+1)[*]) = 0$ and, consequently, $\mathrm{Hom}(Y_1, \mathbb{Z}(2^r+1)[2^{r+1}+1]) = \mathrm{Hom}(\mathcal{X}_{X^{2^r}_\varphi}(2^r)[2^{r+1}], \mathbb{Z}(2^r+1)[2^{r+1}+1])$, and the later group is equal to: $\mathrm{Hom}(\mathcal{X}_{X^{2^r}_\varphi}, \mathbb{Z}(1)[1])$.

So, we have an exact sequence:

$$\mathrm{Hom}(F_\alpha(X_\rho), \mathbb{Z}(2^r+1)[2^{r+1}+2]) \leftarrow \mathrm{Hom}(\mathcal{X}_{X_\varphi}, \mathbb{Z}(2^r+1)[2^{r+1}+2]) \leftarrow \\ \leftarrow \mathrm{Hom}(\mathcal{X}_{X^{2^r}_\varphi}, \mathbb{Z}(1)[1]),$$

where the last map is induced by the map $f : \mathcal{X}_{X_\varphi} \to \mathcal{X}_{X^{2^r}_\varphi}(2^r)[2^{r+1}+1]$, which is the composition: $\mathcal{X}_{X_\varphi} \to Y_1[1] \to \mathcal{X}_{X^{2^r}_\varphi}(2^r)[2^{r+1}+1]$ in the diagram above.

By [13], Corollary 3.2.1, $\mathrm{Hom}(\mathcal{X}_{X_\varphi^{2^r}},\mathbb{Z}(1)[1])=\mathrm{H}^0(\mathcal{X}_{X_\varphi^{2^r}},\mathcal{O}^*)=$
$=\mathrm{H}^0(\mathrm{Spec}(k),\mathcal{O}^*)=\mathrm{Hom}(\mathbb{Z},\mathbb{Z}(1)[1])$, and this isomorphism is induced by projection $pr:\mathcal{X}_{X_\varphi^{2^r}}\to\mathbb{Z}$ (as usually, I'm dropping $M(-)$).

So, we get an exact sequence:

$$\mathrm{Hom}(F_\alpha(X_\rho),\mathbb{Z}(2^r+1)[2^{r+1}+2])\leftarrow\mathrm{Hom}(\mathcal{X}_{X_\varphi},\mathbb{Z}(2^r+1)[2^{r+1}+2])\leftarrow$$
$$\leftarrow\mathrm{Hom}(\mathbb{Z},\mathbb{Z}(1)[1]),$$

where the last map is induced by the composition
$g:\mathcal{X}_{X_\varphi}\xrightarrow{f}\mathcal{X}_{X_\varphi^{2^r}}(2^r)[2^{r+1}+1]\xrightarrow{pr(2^r)[2^{r+1}+1]}\mathbb{Z}(2^r+1)[2^{r+1}+1]$.

Lemma 2.7 .

$g=0$

Proof

By the very construction of the diagram above - see [14], the proof of Theorem 3.1, the composition $Y_1[1]\to\mathcal{X}_{X_\varphi^{2^r}}(2^r)[2^{r+1}+1]\to\mathbb{Z}(2^r)[2^{r+1}+1]$ equals the composition: $Y_1[1]\to F_\alpha(X_\rho)[1]\to X_\varphi[1]\xrightarrow{\alpha_{2^r}[1]}\mathbb{Z}(2^r)[2^{r+1}+1]$, where $\alpha_{2^r}\in\mathrm{Hom}(X_\varphi,\mathbb{Z}(2^r)[2^{r+1}])=CH^{2^r}(X_\varphi)$ is given by the class of a plane section of codimension 2^r. Hence, $g=0$.

$\square$

Now, we can finish the proof of the Main Theorem .

From Lemma 2.7 it follows that the map: $\mathrm{Hom}(\mathcal{X}_{X_\varphi},\mathbb{Z}(2^r+1)[2^{r+1}+2])\leftarrow\mathrm{Hom}(\mathbb{Z},\mathbb{Z}(1)[1])$ is trivial. And we have an exact sequence:

$$\mathrm{Hom}(F_\alpha(X_\rho),\mathbb{Z}(2^r+1)[2^{r+1}+2])\leftarrow\mathrm{Hom}(\mathcal{X}_{X_\varphi},\mathbb{Z}(2^r+1)[2^{r+1}+2])\leftarrow 0$$

Since $F_\alpha(X_\rho)$ is a direct summand in the motive of X_φ, we have (by above) an exact sequence:

$$\mathrm{Hom}(X_\varphi,\mathbb{Z}(2^r+1)[2^{r+1}+2])\leftarrow\mathrm{Hom}(\mathcal{X}_{X_\varphi},\mathbb{Z}(2^r+1)[2^{r+1}+2])\leftarrow 0$$

By Lemma 2.6 , the group $\mathrm{Hom}(\mathcal{X}_{X_\varphi},\mathbb{Z}(2^r+1)[2^{r+1}+2])$ is nontrivial, on the other hand it has exponent 2 (since $2^{r+1}+2>2^r+1$). That means that the group $\mathrm{Hom}(X_\varphi,\mathbb{Z}(2^r+1)[2^{r+1}+2])$ contains a torsion element of exponent 2.

By [15], Theorem 4.2.9, we have: $\mathrm{Hom}(X_\varphi,\mathbb{Z}(2^r+1)[2^{r+1}+2])=CH^{2^r+1}(X_\varphi)$. The Main Theorem is proven.

$\square$

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