# On the kernels in Milnor's K-theory under function field extensions

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## Contents



## 0 Introduction

In this paper we will consider the following question:

Question 1.

*Let $k$ be a field of characteristic different from 2, and $Q/k$ be some quadric.*

*Is it true, that the ideal* $\mathrm{Ker}_Q := \mathrm{Ker}(\mathrm{K}_*^M(k)/2 \to \mathrm{K}_*^M(k(Q))/2)$ *in* $\mathrm{K}_*^M(k)/2$ *is generated by pure symbols ?*

The answer to our question is known to be positive in the case $Q$ - *Pfister neigbour* (i.e.: a subquadric of dimension more or equal than half in a *big Pfister quadric*) by [6], and for 2-dimensional *non-Pfister* quadric by [8].

The positive answer to our question would imply the following:
1) Conjecture of B.Kahn, M.Rost and R.J.Sujatha (see Conjecture 1 of the original version of [3], where authors asked basically the same question about the low degree part of the kernel).
2) The fact that any *conservative* quadric (i.e. such quadric, for which the

kernel above is nontrivial) is a subquadric in a *big Pfister quadric*.
3) The fact, that if some quadric $Q$ is a *normvariety* for some $h \in \mathrm{K}_m^M(k)/2$, then $h$ is a *pure symbol*. From this, on it's part, follows nice motivic description of *excellent* quadrics.

We will show, that our question is equivalent to the fact that $\mathrm{Ker}_Q$, considered as a *cyclic module* of M.Rost (see [7]) is generated by one precisely described *pure* symbol. So, in the case of positive answer to the question we would have explicit description of the $\mathrm{Ker}_Q$; and we have such a description for the *pure* part of the kernel unconditionally.

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# 1 Universal symbol

In this section we will show, that for arbitrary quadric $Q/k$ there exists some purely transcendental extension $k(A_Q)/k$, and some *pure* symbol $\alpha_Q \in \mathrm{K}_*^M(k(A_Q))/2$, s.t. for arbitrary field extension $E/k$, our quadric $Q$ has $E$-point if and only if $\alpha_Q|_{E(A_Q)} = 0$.

Statement 2.

*Suppose we have complete flag $F$ of quadrics $Q_0 \subset Q_1 \subset \cdots \subset Q_n = Q$, where $n = \dim(Q)$.*

*Then there exists purely transcendental field extension $k(A_F)/k$, and a pure symbol $\{y_0, \ldots, y_n\} \in \mathrm{K}_{n+1}^M(k(A_F))/2$, s.t. for arbitrary field extension $E/k$, $Q_i$ has $E$-point if and only if $\{y_0, \ldots, y_i\}|_{E(A_F)} = 0$.*

*Proof*

Induction on the dimension of $Q$.

1) $q = \langle 1, -a_0 \rangle$. Evidently, $\alpha_{k\sqrt{a_0}} = \{a_0\} \in \mathrm{K}_1^M(k)/2$ will have desired property.

2) Let now $q_i = \langle 1, -a_0, \ldots, -a_i \rangle$, and $F'$ is a flag: $Q_0 \subset \cdots \subset Q_{n-1}$. By inductive assumption: we have $\alpha_{F'} = \{y_0, \ldots, y_{n-1}\} \in \mathrm{K}_n^M(k(A_{F'}))/2$, which has desired property with respect to $F'$.

In particular $\alpha_{F'}|_{k(Q_{n-1})(A_{F'})} = 0$ (Since over $k(Q_{n-1})$ $Q_{n-1}$ has a point). By the "Third representation theorem" (see [5], IX, Theorem 2.8) this means, that $q_{n-1}|_{k(A_{F'})}$ is similar to a subform of $\langle\langle y_0, \ldots, y_{n-1} \rangle\rangle$, and since $q_{n-1}$ represents 1, it is a subform itself. We have: $q_{n-1}|_{k(A_{F'})} \perp q'' = \langle\langle y_0, \dots, y_{n-1}\rangle\rangle$, where $q''$ is some form, defined over $k(A_{F'})$.

From this: if for some field extension $E/k$, $\{y_0, \dots, y_{n-1}\} \neq 0$ (as an element of $\mathrm{K}_n^M(E(A_{F'}))/2$), then the following four conditions are equivalent:

a) $q|_E$ - is isotropic.

b) $q|_{E(A_{F'})}$ - is isotropic.

c) $(\langle\langle y_0, \dots, y_{n-1}\rangle\rangle \perp -(q'' \perp \langle a_n\rangle))|_{E(A_{F'})}$ is $dim(q'') + 1$-times isotropic.

d) $(q'' \perp \langle a_n\rangle)|_{E(A_{F'})}$ - is a subform in $\langle\langle y_0, \dots, y_{n-1}\rangle\rangle|_{E(A_{Q'})}$.

By the "Third representation theorem" (see [5], IX, Theorem 2.8) the last condition is equivalent to the fact: $\langle\langle y_0, \dots, y_{n-1}\rangle\rangle|_{E(A_{F'})(V_{q''\perp\langle a_n\rangle})}$ represents the "generic value" $(q'' \perp \langle a_n\rangle)(\mathfrak{v})$, where $V_{q''\perp\langle a_n\rangle}$ is the underlying $k(A_{F'})$-vector space of quadratic form $q'' \perp \langle a_n\rangle$ (defined over that field), and $\mathfrak{v}$ - coordinates in $V_{q''\perp\langle a_n\rangle}$. And this is equivalent to: $\{y_0, \dots, y_{n-1}, (q'' \perp \langle a_n\rangle)(\mathfrak{v})\} = 0 \in \mathrm{K}_n^M(E(A_{F'})(V_{q''\perp\langle a_n\rangle}))/2$.

Put: $\alpha_F = \{y_0, \dots, y_{n-1}, (q'' \perp \langle a_n\rangle)(\mathfrak{v})\}$, and $k(A_F) = k(A_{F'})(V_{q''\perp\langle a_n\rangle})$.

If $\{y_0, \dots, y_{n-1}\}|_{E(A_{F'})} \neq 0$, then we already proved that $Q_E$ has a point if and only if $\alpha_F|_{E(A_F)} = 0$.

If $\{y_0, \dots, y_{n-1}\}|_{E(A_{F'})} = 0$, then $Q_{n-1} \subset Q$ has a rational point over $E$, and, on the other hand, trivially, $\alpha_F|_{E(A_F)} = 0$.

□

Corollary 3. *All anisotropic quadrics are stably embedable, i.e., for each such quadric $Q/k$ there exists purely transcendental extension $k(A_Q)$, s.t. $Q|_{k(A_Q)} \subset Q_\alpha$, where $Q_\alpha/k(A_Q)$ is Pfister quadric.*

*Proof*

In the previous statement take: $E = k(Q)$, $k(A_Q) = k(A_F)$ and $\alpha = \alpha_F$ for some full flag $F$ of subquadrics in $Q$. Then since $Q|_{k(Q)}$ is isotropic, we have: $\alpha_Q|_{k(A_Q)(Q)} = 0$. So, $\alpha \in \mathrm{K}_{n+1}^M(k(A_Q))$ is a nontrivial representative of the kernel. And $Q|_{k(A_Q)}$ is a subquadric in $Q_{\alpha_Q}$.

□

Corollary 4.

*For anisotropic quadric $Q$, and some quadric $R$ we have, that $Q|_{k(R)}$ is isotropic if and only if $R|_{k(A_Q)}$ is a subquadric in a Pfister quadric $Q_{\alpha_Q}$.*

*Remark* By the result of D.W.Hoffmann, $Q|_{k(R)}$ is anisotropic if $\dim(R) \geq 2^k - 1 > \dim(Q)$ (for some $k$) (see [2], Theorem 1).

This shows, that $Q_{\alpha_Q}$ does not contain subquadrics of big dimension defined over $k$.

Corollary 5.

*Let $Q$, $R$ be quadrics over $k$, and $\dim(Q) = \dim(R) = n$. Let $Q^i$ (resp. $R^i$) be variety of $i$-dimensional projective planes on $Q$ (resp $R$).*

*Then the following is equivalent:*

*(1) $M(Q) = M(R)$, where $M(X)$ is a Chow motive of $X$.*

*(2) $\forall 0 \leq i \leq [n/2]$, $\forall E/k$ - field extension,* $\mathrm{Ker}(\mathrm{K}_*^M(E)/2 \to \mathrm{K}_*^M(E(Q^i))/2) = \mathrm{Ker}(\mathrm{K}_*^M(E)/2 \to \mathrm{K}_*^M(E(R^i))/2)$.

*Proof*

By [8], Proposition 5.1 we have: (1)$\Leftrightarrow$ *Universal splitting towers of M.Knebusch* for $Q$ and $R$ are equivalent (see [4], or [8], Definition 2.4.19).

That is: $M(Q) = M(R) \ \Leftrightarrow \ \forall 0 \leq i \leq [n/2]$, $Q^i$ has a rational point over $k(R^i)$, and $R^i$ has a rational point over $k(Q^i)$.

So, evidently, (1)$\Rightarrow$(2). And to prove the opposite conclusion, it is enough to show, that if $\mathrm{Ker}(\mathrm{K}_*^M(E)/2 \to \mathrm{K}_*^M(E(Q))/2) = \mathrm{Ker}(\mathrm{K}_*^M(E)/2 \to \mathrm{K}_*^M(E(R))/2)$, $\forall E/k$, then $Q$ has a rational point over $k(R)$ and $R$ has a rational point over $k(Q)$ (the case of arbitrary $0 \leq i \leq [n/2]$ can be easily obtained from this).

Take $E = k(Q)(A_R)$ (notations as above).
Then $\mathrm{Ker}(\mathrm{K}_*^M(E)/2 \to \mathrm{K}_*^M(E(Q))/2) = 0$. Hence the second kernel is zero as well, and so, $\alpha_R|_{k(Q)(A_R)} = 0 \in \mathrm{K}_*^M(E)/2$. By Statement 2, $R$ has a rational point over $k(Q)$. Similarly, $Q$ has a rational point over $k(R)$.

$\square$

*Remark* If $M(Q)$ or $M(R)$ is undecomposable, then in (2) it is enough to consider only $i = 0$ (see [8], Corollary 3.15).

# 2 Pure part of the kernel

In this section we will show that for arbitrary quadric $Q/k$ the set of *pure* symbols in $\mathrm{Ker}(\mathrm{K}_*^M(k)/2 \to \mathrm{K}_*^M(k(Q))/2)$ is generated (under the action of the semigroup of *pure* symbols from $\mathrm{K}_*^M(k)/2$) by the *specializations* of one element $\alpha_Q \in \mathrm{K}_{n+1}^M(k(A_Q))$, where $\alpha_Q$ can be taken as $\alpha_F$ for some flag $F$ (and $k(A_Q)$ as $k(A_F)$).

Let me remind (see [7]), that the *cyclic module* over $k$ is a collection of $\mathbb{Z}$-graded abelian groups $M(E)$, where $E$ runs through all finitely generated field extensions $E/k$, which are connected by the following operations:

(1) For any $\varphi : \mathrm{Spec}(E) \to \mathrm{Spec}(F)$ over $k$, the homomorphism $\varphi^* : M(F) \to M(E)$ of degree 0 ("pull-back").
(2) For any finite morphism $\varphi : \mathrm{Spec}(E) \to \mathrm{Spec}(F)$ over $k$, the homomorphism $\varphi_* : M(E) \to M(F)$ of degree 0 ("push-forward").
(3) For any $F/k$, $M(F)$ has a structure of $\mathrm{K}_*^M(F)$-module, which respects grading on both objects.
(4) For a discrete valuation $\mathfrak{v}$ on $F/k$, there is the homomorphism $\partial_{\mathfrak{v}} : M(F) \to M(k(\mathfrak{v}))$ of degree $-1$ ("derivation").

All this data should satisfy natural compatibility rules (see [7], notice, that our notations are a bit different). Typical example is $M(E) = \mathrm{K}_*^M(E)$.

It is not difficult to see, that $M(E) = \mathrm{Ker}_{Q|_E}$ will also be one (really, any submodule of *cyclic module*, closed under operations is a cyclic module; and for a cyclic module $M$, $M' := M/r$, where $r$ is some fixed integer, is also a cyclic module).

Together with a derivation we can consider "specialization" map. Namely, if $\pi$ is uniformizing parameter for $\mathfrak{v}$, then we have: $S_{\mathfrak{v}}^{\pi} : M(F) \to M(k(\mathfrak{v}))$ of degree 0, where $S_{\mathfrak{v}}^{\pi}(x) := \partial_{\mathfrak{v}}(\{-\pi\} \cdot x)$. If $\partial_{\mathfrak{v}}(x) = 0$, then $S_{\mathfrak{v}}^{\pi}(x)$ does not depend on the choice of $\pi$. If $X$ is a smooth variety, and $\mathrm{Spec}(E)$ is a point on it, given by some $\mathbb{Z}^r$ - valuation, then the composition $S_{\mathfrak{v}_r}^{\pi_r} \circ \cdots \circ S_{\mathfrak{v}_1}^{\pi_1}$ defines a specialization map from $M_{0,E}(k(X))$ to $M(E)$, where $M_{0,E}(k(X)) := \mathrm{Ker}(M(k(X)) \stackrel{\partial_D}{\to} \oplus_{\mathrm{Spec}(E) \in D-div.of X} M(k(D)))$. This map does not depend on the choice of $\mathbb{Z}^r$ - valuation, or on the uniformizers (see [7]). Notice, that the specialization map is a composition of maps of type (3) and (4).

Suppose we have a *cyclic module* $M$, some (finitely generated) field extension $E/k$, and some element $x \in M(E)$. Then it is natural to say, that *cyclic submodule generated by* $x$ is the minimal *cyclic submodule* $M_x$ of $M$, containing $x$. Since we know, that *cyclic submodules* are precisely submodules, closed under operations, we have $M_x(F)$ consists of those $y \in M(F)$, that $y$ can be obtained from $x$ using operations (1), (2), (3), (4). Similarly, we say, that *submodule weakly generated by* $x$ is the minimal submodule (not nesessarily *cyclic*) $M_x^w$ of $M$, closed under (1), (3) and (4).

Denote by $\mathrm{PKer}_Q$ the ideal in $\mathrm{K}_*^M(k)/2$, generated by *pure* symbols from $\mathrm{Ker}_Q$. Also, we will denote the same way the corresponding submodule in $\mathrm{K}_*^M/2$: $\mathrm{PKer}_Q(E) := \mathrm{PKer}_{Q|_E}$.

Proposition 6.

$\mathrm{PKer}_Q$ *coinsides with a submodule of* $\mathrm{K}_*^M/2$ *weakly generated by* $\alpha_Q$.

*Proof*

As we know from Statement 2, $\alpha_Q \in \mathrm{PKer}_Q(k(A_Q))$. Also, it is clear, that operations (1) and (3) preserve $\mathrm{PKer}_Q$.

Lemma 7.

*Operation* (4) *(acting on* $\mathrm{K}_*^M$*) maps pure symbols to pure ones.*

*Proof*

Really, for a valuation $\mathfrak{v}$ on a filed $F$, any *pure* symbol in $F$ can be written as $\beta = \{\pi^{k_1} \cdot b_1, \dots, \pi^{k_n} \cdot b_n\}$, where $\pi$ is uniformizing parameter, and $b_j$'s are units for $\mathfrak{v}$.

Applying the fact, that $\{\pi^k \cdot a, \pi^l \cdot b\} = \{\pi^k \cdot a, -\pi^{l-k} \cdot b/a\}$, we get, that $\beta = \{\pi^d \cdot c_1, c_2, \dots, c_n\}$, where $d = g.c.d.(k_j; j = 1, \dots, n)$, and $c_m$'s are units.

Then $\partial_{\mathfrak{v}}(\beta) = \{c_2^d, c_3, \dots, c_n\}$ is *pure.*

$\square$

So, we have, that $(\mathrm{K}_*^M/2)^w_{\alpha_Q} \subset \mathrm{PKer}_Q$.

Before we prove the opposite inclusion, let us generalize a bit the construction of *Universal symbol.*

Suppose we have two quadratic forms $p' \subset p''$, full flag $G$ of forms "between them": $p' = p_0 \subset p_1 \subset \cdots \subset p_s = p''$, where $\dim(p_{t+1}) - \dim(p_t) = 1$, and an inclusion of $p'$ into some Pfister form $\langle\langle\gamma\rangle\rangle$. Let $p_j = p_0 \perp \langle -c_1, \dots, -c_j\rangle$. For any $0 \le j \le t \le s$, we will denote as $G^{[j,t]}$ the sub-flag $p_j \subset \cdots \subset p_t$.

Then we can construct some purely transcendental field extension $k(A_{G,\gamma})$, and some pure symbol $\alpha_{G,\gamma}$ in the following way.

Let $\langle\langle\gamma\rangle\rangle = p_0 \perp q_{G^{[0,0]},\gamma}$, then take: $k(A_{G^{[0,1]},\gamma}) := k(V_{q_{G^{[0,0]},\gamma} \perp \langle c_1\rangle})$, and take $\alpha_{G^{[0,1]},\gamma} := \gamma \cdot \{q_{G^{[0,0]},\gamma} \perp \langle c_1\rangle(\mathfrak{v})\}$, where $q_{G^{[0,0]},\gamma} \perp \langle c_1\rangle(\mathfrak{v})$ is the *generic value* of $q_{G^{[0,0]},\gamma} \perp \langle c_1\rangle$. We have: $p_1|_{k(A_{G^{[0,1]},\gamma})} \subset \langle\langle\alpha_{G^{[0,1]},\gamma}\rangle\rangle$.

Now, by definition, $k(A_{G^{[0,i+1]},\gamma}) := k(A_{G^{[0,i]},\gamma})(A_{G^{[i,i+1]},\alpha_{G^{[0,i]},\gamma}})$, and $\alpha_{G^{[0,i+1]},\gamma} := \alpha_{G^{[i,i+1]},\alpha_{G^{[0,i]},\gamma}}$. Also, by definition: $\langle\langle\alpha_{G^{[0,l]},\gamma}\rangle\rangle = p_l|_{k(A_{G^{[0,l]},\gamma})} \perp q_{G^{[0,l]},\gamma}$.

Let us return now to the proof of Proposition 6.

Suppose we have some *pure* symbol $\beta$ in $\mathrm{Ker}(\mathrm{K}_*^M(k)/2 \to \mathrm{K}_*^M(k(Q))/2)$.

Let $F$ be full flag of quadrics: $Q_0 \subset Q_1 \subset \cdots \subset Q_n = Q$, $q_i = \langle 1, -a_0, \dots, -a_i\rangle$, which we used in the construction of $\alpha_Q = \alpha_F$. Notice, that $q_0 = \langle\langle a_0\rangle\rangle$. Then it is easy to see, that $k(A_F) = k(A_{F,\{a_0\}})$, and $\alpha_F = \alpha_{F,\{a_0\}}$.

We have $q_n = q \subset \langle\langle\beta\rangle\rangle$ for some $\beta \in \mathrm{K}_*^M(k)/2$ (since $q$ represents 1). Take $\beta_{Q_0} := \{a_0\}$. We have: $q_0 \subset \langle\langle\beta_{Q_0}\rangle\rangle \subset \langle\langle\beta\rangle\rangle$.

Suppose $q_i \subset \langle\langle\beta_{Q_i}\rangle\rangle \subset \langle\langle\beta\rangle\rangle$, $deg(\beta_{Q_i}) = j$, then $deg(\alpha_{F^{[i,n]},\beta_{Q_i}}) = j+n-i$.

We have: $\alpha_{F[i,n],\beta_{Q_i}} = \beta_{Q_i} \cdot \{y_{i+1}, \ldots, y_n\}$, where $y_l$ is a pull-back of $(q_{F[i,l-1],\beta_{Q_i}} \perp \langle a_l \rangle)(\mathfrak{v}_l)$ to $k(A_{F[i,n],\beta_{Q_i}})$.

Let $q_i \subset \langle\langle \beta_{Q_i} \rangle\rangle$, and $\langle\langle \beta_{Q_i} \rangle\rangle = q_i \perp q_{F[i,i],\beta_{Q_i}}$. We have two possibilities:
1)$q_{F[i,i],\beta_{Q_i}} \perp \langle a_{i+1} \rangle$ is isotropic, i.e. there are $t_0, z_0$, that $q_{F[i,i],\beta_{Q_i}}(t_0) + a_{i+1} z_0^2 = 0$.
2)$q_{F[i,i],\beta_{Q_i}} \perp \langle a_{i+1} \rangle$ is anisotropic.

1) In this case we have: divisor $y_{i+1} = 0$ on $A_{F[i,i+1],\beta_{Q_i}} = V_{q_{F[i,i],\beta_{Q_i}} \perp \langle a_{i+1} \rangle}$ has rational point $(t_0, z_0)$, and $q_{i+1} \subset \langle\langle \beta_{Q_i} \rangle\rangle$. So, we can construct $k(A_{F[i+1,n],\beta_{Q_i}})$, and $\alpha_{F[i+1,n],\beta_{Q_i}}$. Let $Y_{i+1} = 0$ be the preimage of the divisor $y_{i+1} = 0$ under the natural projection $A_{F[i,n],\beta_{Q_i}} \to V_{q_{F[i,i],\beta_{Q_i}} \perp \langle a_{i+1} \rangle}$.

Let $p'_l$ be the *specialization* of $q_{F[i,l],\beta_{Q_i}}$ over $k(A_{F[i,l],\beta_{Q_i}})$ at the generic point $E_l$ of the fiber of $A_{F[i,l],\beta_{Q_i}}$ over the rational point $(t_0, z_0)$ of $V_{q_{F[i,i],\beta_{Q_i}} \perp \langle a_{i+1} \rangle}$.

Then $E_l$ is naturally a purely transcendental extension of $k(A_{F[i+1,l],\beta_{Q_i}})$, and if $j : \mathrm{Spec}(E_l) \to \mathrm{Spec}(k(A_{F[i+1,l],\beta_{Q_i}}))$ is corresponding morphism, then we have: $j^*(q_{F[i+1,l],\beta_{Q_i}})$ is a nondegenerate part of $p'_l$ (it can be easily proved by induction on $l$).

Hence, $\alpha_{F[i+1,n],\beta_{Q_i}} \in \mathrm{K}^M_{j+n-i-1}(k(A_{F[i+1,n],\beta_{Q_i}}))/2$ is a specialization of the derivative $\partial_{Y_{i+1}=0}(\alpha_{F[i,n],\beta_{Q_i}})$.

We can take $\beta_{Q_{i+1}} = \beta_{Q_i}$. Then $q_{i+1} \subset \langle\langle \beta_{Q_{i+1}} \rangle\rangle \subset \langle\langle \beta \rangle\rangle$.

2) We have: $q_i \subset \langle\langle \beta_{Q_i} \rangle\rangle \subset \langle\langle \beta \rangle\rangle$, and $q_i \subset q_{i+1} \subset \langle\langle \beta \rangle\rangle$. Since $q_{F[i,i],\beta_{Q_i}} \perp \langle a_{i+1} \rangle$ is anisotropic, there exist some subform $p$ of $\langle\langle \beta \rangle\rangle$ of dimension $= \dim(\langle\langle \beta_{Q_i} \rangle\rangle) + 1$, which contains both $\langle\langle \beta_{Q_i} \rangle\rangle$ and $q_{i+1}$. That means, that there exist $(t_0, z_0)$, s.t. $p = \langle\langle \beta_{Q_i} \rangle\rangle \perp \langle q_{F[i,i],\beta_{Q_i}}(t_0) + a_{i+1} z_0^2 \rangle$.

Take: $\beta_{Q_{i+1}} := \beta_{Q_i} \cdot \{q_{F[i,i],\beta_{Q_i}}(t_0) + a_{i+1} z_0^2\} \in \mathrm{K}^M_{j+1}(k)/2$. Then $q_{i+1} \subset \langle\langle \beta_{Q_{i+1}} \rangle\rangle \subset \langle\langle \beta \rangle\rangle$.

Evidently, $\beta_{Q_{i+1}}$ is a specialization of $\alpha_{F[i,i+1],\beta_{Q_i}} \in \mathrm{K}^M_{j+1}(k(V_{q_{F[i,i],\beta_{Q_i}} \perp \langle a_{i+1} \rangle}))/2$ at the rational point $(t_0, z_0)$.

Also, we have natural identification: $k(A_{F[i+1,l],\beta_{Q_{i+1}}})$ with $E_l$, where $E_l$ is defined precisely as in the case (1). Under this identification, $q_{F[i+1,l],\beta_{Q_{i+1}}}$ is isomorphic to the specialization of $q_{F[i,l],\beta_{Q_i}}$ at the generic point $(E_l)$ of the fiber of the natural projection $A_{F[i,l],\beta_{Q_i}} \to V_{q_{F[i,i],\beta_{Q_i}} \perp \langle a_{i+1} \rangle}$ over the point $(t_0, z_0)$.

Hence, $\alpha_{F[i+1,n],\beta_{Q_{i+1}}} \in \mathrm{K}^M_{j+n-i}(k(A_{F[i+1,n],\beta_{Q_{i+1}}}))/2$ is a *specialization* of $\alpha_{F[i,n],\beta_{Q_i}} \in \mathrm{K}^M_{j+n-i}(k(A_{F[i,n],\beta_{Q_i}}))/2$.

Continuing this way, we get: $q = q_n \subset \langle\langle \beta_{Q_n} \rangle\rangle \subset \langle\langle \beta \rangle\rangle$, and for each $0 \leq i < n$, $\alpha_{F[i+1,n],\beta_{Q_{i+1}}} \in \mathrm{K}^M_{j+n-i}(k(A_{F[i+1,n],\beta_{Q_{i+1}}}))/2$ is obtained from $\alpha_{F[i,n],\beta_{Q_i}} \in \mathrm{K}^M_{j+n-i}(k(A_{F[i,n],\beta_{Q_i}}))/2$ using derivations and specializations.

But, evidently, $k(A_{F[n,n],\beta_{Q_n}}) = k$, and $\alpha_{F[n,n],\beta_{Q_n}} = \beta_{Q_n}$.

On the other hand, as we saw, $\alpha_{F[0,n],\beta_{Q_0}} = \alpha_F$.

Hence, $\beta$ is divisible by some pure symbol, obtained from $\alpha_F$ using derivations and specializations.

So, $\mathrm{PKer}_Q \subset (\mathrm{K}^M_*/2)^w_{\alpha_Q}$.

Proposition is proven.

$\square$

*Remark 1* In particular we see, that for any *pure symbol* $\beta \in \mathrm{Ker}(\mathrm{K}^M_*(k)/2 \to \mathrm{K}^M_*(k(Q))/2)$, $\beta$ is divisible by a *pure* symbol from the kernel of degree $\leq \dim(Q) + 1$. This result was known to M.Knebusch, as E.R.Gentile and D.B.Shapiro point out (see [1], Remark after Corollary 8).

*Remark 2* Actually, we could make our *universal symbol* $\alpha_Q$ a bit more "universal". Namely, we can consider $F(\mathbb{P}^{n+1})$ - variety of full flags in $\mathbb{P}^{n+1} \supset Q$.

Then taking the generic point $k(F(\mathbb{P}^{n+1}))$ of later variety with generic flag $F_{gen}$, we would get corresponding *universal symbol* $\alpha_{gen,Q} := \alpha_{F_{gen}} \in \mathrm{K}^M_{n+1}(k(F(\mathbb{P}^{n+1}))(A_{F_{gen}}))/2$.

This way, $\alpha_{gen,Q}$ does not depend on anything but $Q$ itself, and our usual $\alpha_{F_0}$ is just a specialization of $\alpha_{gen,Q}$ at the generic point of the fiber of $A_F$ over $k$-rational point $F_0$ of $F(\mathbb{P}^{n+1})$ (notice, that we can consider $A_F$ as a fibration over $F(\mathbb{P}^{n+1})$ with rational fibers).

From Proposition 6 it follows that $(\mathrm{K}^M_*/2)^w_{\alpha_{F_0}} = (\mathrm{K}^M_*/2)^w_{\alpha_{gen,Q}}$. Really, the first module is a submodule of the later (since $\alpha_{F_0}$ is a specialization of $\alpha_{gen,Q}$), and the later module is, evidently, a submodule in $\mathrm{PKer}_Q$.

Since the same is true for any extension $E/k$, we get: $(\mathrm{K}^M_*/2)_{\alpha_{F_0}} = (\mathrm{K}^M_*/2)_{\alpha_{gen,Q}}$ as well (see [7]).

In the light of Proposition 6, our Question 1 is equivalent to the following:

Question 8.

*Is the following true?*

*a)* $\mathrm{Ker}_Q = (\mathrm{K}^M_*/2)_{\alpha_Q}$.

*b)* $\mathrm{PKer}_Q$ *is stable under transfers.*

Notice, that Question 8)b) is trivial for $\dim(Q) \leq 2$; to the contrary - Question 8)a) for $\dim(Q) = 0$ is more or less equivalent to Milnor's conjecture (on e'tale cohomology). Hopefully, due to V.Voevodsky's methods we know that the answer is positive for $Q$ - *Pfister quadric* (see [6]). Natural strategy for Question 8)a) would be to reduce the case of arbitrary quadric $Q/k$ to that of *Pfister* one $Q_{\alpha_Q}/k(A_Q)$, using the universality of the symbol $\alpha_Q$.

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