HIGHER ORDER LAPLACIANS II.

LAPLACIAN COMMUTING WITH

THE HIGHER ORDERS

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## Introduction

We continue the study of higher order Laplacians introduced in the previous paper [9]. We slightly modify this notion because now we use unnormed integrals w.r.t. a kernel function $H(p,q)$ . The infinitesimal generators (the higher order Laplacians) are denoted by $\square_H^{(k)}$ .

In this paper the normal analycity plays an important role. A Riemannian manifold is defined to be normal analytic if it is real analytic in the normal coordinates. A kernel function $H(p,q)$ is normal analytic if, for any $p$ , the kernel function $H_p(\cdot) = H(p,\cdot)$ is analytic in the normal coordinates defined around $p$ .

In the first chapter we prove the <u>Basic Theorem</u> of the operators $\square_H^{(k)}$ asserting that for a symmetric $(H(p,q) = H(p,q))$ normal analytic function $H$ the Laplacian commutes with the operators $\square_H^{(k)}$ if and only if:

1) on any geodesics $\gamma$ the kernel function $H^2/\omega$ (where $\omega$ is the Riemann–density in normal coordinates) is depending on the geodesics distance $r(p,q)$ ; i.e. a function $\phi_\gamma$ exists such that $H^2/\omega$ is of the form $\phi_\gamma(r(p,q))$ on $\gamma$ ,

2) the kernel function $\mathscr{K}(p,q) = H(p,q)/\omega(p,q)$ satisfies the ultrahyperbolic equation

$$(\Delta_p \mathscr{K})(p,q) = (\Delta_q \mathscr{K})(p,q) .$$

In Chapter 2 we prove that all the spaces satisfying the curvature condition

(*) $$\nabla_i\rho_{jk} + \nabla_i\rho_{ki} + \nabla_k\rho_{ij} = 0$$

for the Ricci tensor $\rho_{ij}$ are normal analytic. As all the Einstein metrics satisfy the condition (*) so this is a generalization of the Kazdan–De Turck Theorem [1].

We show too, that the condition (*) holds if and only if the Laplacian $\Delta$ commutes with the second Willmore's operator $\Delta^{(2)} = \square_1^{(2)}$.

From these theorems, for example, the normal analycity of the D'Atri spaces (where the geodesics involutions are volume preserving) follows. Furthermore a space is D'Atri space if and only if the Laplacian commutes with the operators $\square_\omega^{(k)}$.

Involving also the other invariants $\sigma_p^{(i)}(q)$ of the Jacobian field in the considerations, similar theorems are proved also for the so called (i)–D'Atri spaces.

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## § 1. The Basic Theorem of the higher order Laplacians

Let $H(p,q) : M^n \times M^n \longrightarrow \mathbb{R}$ be a $C^\infty$–kernel function on a $C^\infty$–Riemannian manifold $M^n$. For a unit vector $\dot{e}_p \in T_p(M^n)$ the $e_p(r)$ denotes the arc–wise parametrized geodesics with the tangent vector $\dot{e}_p$ at $p = e_p(0)$. In the first part of this paper–series we introduced the averaging operator $E_{H;p;r}$ along the geodesics spheres $S_{p;r}$ using the normalized function $H(p,e_p(r))/\int H(p,e_p(r))d\dot{e}_p$ as weight function, where $d\dot{e}_p$ means the normalized euclidean measure of the unit tangent vectors in the tangent space $T_p$. The higher order Laplacians $\Delta_H^{(k)}$ w.r.t. $H$ were defined by the even derivatives w.r.t. $r$ as follows:

$$(1.1) \qquad \Delta_{H;p}^{(k)}\varphi := \frac{\partial^{2k}E_{H;p;r}(\varphi)}{\partial r^{2k}} /r = 0 ,$$

where $\varphi$ is a $C^{\infty}$–function on $M^n$.

In this paper we deal mostly with the unnormed averaging $UE_{H;p;r}$ defined by

$$(1.2) \qquad UE_{H;p;r}(\varphi) = \int \varphi(e_p(r))H(p,e_p(r))d\dot{e}_p ,$$

and the operators $\square_H^{(k)}$ are defined by the 2k–th derivative:

$$(1.3) \qquad \square_{H;p}^{(k)}\varphi := \frac{\partial^{2k}UE_{H;p;r}}{\partial r^{2k}} /r = 0 .$$

This operator can be written also in the following form:

$$(1.4) \qquad \square_{H;p}^{(k)} = \sum_{a=0}^{2k} \begin{bmatrix} 2k \\ a \end{bmatrix} \int H_{\dot{e}_p}^{(a)} \nabla_{\dot{e}_p}^{(2k-a)} d\dot{e}_p ,$$

where $\nabla^{(i)}_{\dot e}$ is the covariant derivative of i–th order furthermore $H^{(a)}_{\dot e_p} := \nabla^{(a)}_{\dot e_p} H = \nabla\nabla \dots \nabla H(\dot e_p, \dots, \dot e_p)$ . Notice, that these operators are of the form

(1.5) $$\square^{(k)}_{H;p} = D^{(k)}_{H;p} + \lambda^{(k)}_{H;p} ,$$

where $D^{(k)}_{H;p}$ is a differential operator furthermore

(1.6) $$\lambda^{(k)}_{H;p} = \int H^{(2k)}_{\dot e_p} d\dot e_p$$

is a scalar operator (multiplication with the function $\lambda^{(k)}_H$) .

It is plain that the operators $\square^{(k)}_H$ are self adjoint in the $L^2(M^n)$–Hilbert space if and only if the kernel function H is symmetric : $H(p,q) = H(q,p)$ .

Let $f(r) : \mathbb{R} \longrightarrow \mathbb{R}$ be a $C^\infty$–even function (i.e. $f(r) = f(-r)$) . It defines the radial kernel function $F(p,q) = f(r(p,q))$ on $M^n$ , where $r(p,q) = r_p(q)$ means the geodesics distance between p and q . For the product–kernel–function

(1.7) $$(fH)(p,q) := F(p,q)H(p,q)$$

we have

(1.8) $$\square^{(k)}_{(fH)} = \sum_{a=0}^{k} \begin{bmatrix} 2k \\ 2a \end{bmatrix} f^{(2a)}(0)\square^{(k-a)}_H ,$$

where $f^{(2a)}(0)$ is the (2a)–th derivative of f at 0 .

In this chapter we investigate mainly the commutativity of the Laplacian $\Delta = \nabla^i\nabla_i$ with the operators $\square_H^{(k)}$ . We draw into these considerations also the convolution $*$ defined on functions as usual by.

$$(1.9) \qquad H * \varphi(p) = \int H(p,q)\varphi(q)dq \,.$$

First notice that the Taylor formula gives the expression

$$(1.10) \qquad UE_{H;p;r}(\varphi) = \sum_k \frac{r^k}{k!} \square_{H;p}^{(k)}(\varphi)$$

for the unnormed averaging $UE_{H;p;r}(\varphi) = \int \varphi(e_p(r))H(e_p(r))d\dot{e}_p$ . This formula is the so called <u>pre–Pizzetti–formula</u>. If the functions $\varphi$ and $H$ are normal analytic, then the right side converges for small values of $r$ .

Let $\omega_p(q) = \sqrt{\det|g^{ij}|}$ be the Riemannian density in a normal coordinate neighbourhood around $p$ . If $f$ in (1.7) is a function of compact support whose supporting radius (the infimum of the positive 0–places) is less than the injectivity radius of the manifold at a point $p$ , then the function

$$(1.11) \qquad f\,\mathscr{H}_p := \frac{F_pH_p}{\omega_p}$$

is well defined around the point $p$ . In a normal coordinate neighbourhood around $p$ we get

$$(1.12) \qquad (f\,\mathscr{H}) * \varphi(p) = \Omega_{n-1}\int f(r)r^{n-1}\Big(\int \varphi(e_p(r))H(e_p(r))d\dot{e}_p\Big)dr$$

therefore by the Taylor series (1.10) we have

(1.13) $$(f\,\mathscr{H}) * \varphi = \Omega_{n-1} \sum_k \frac{1}{(2k)!} \left(\int f(r) r^{n+2k+1} dr\right) \square_H^{(k)} \varphi ,$$

where $\Omega_{n-1}$ is the volume of the $(n-1)$–dimensional euclidean unit sphere. If the functions $\varphi$ and $H$ are normal anylytic, then the right side of (1.13) is convergent for the functions $f$ with small supporting radius.

For a fixed radius $r = R$, the averaging operator $UE_{H;p;R}$ can be generated from the convolutions operators of the form $(f\,\mathscr{H}) *$ as follows.

Let $f_n$ be a function–series which tends to the Dirac function

(1.14) $$\frac{1}{\Omega_{n-1} R^{n-1}} \delta_R .$$

Then the operator series $(f_n \mathscr{H}) *$ tends (on the continuous functions) to the operator $UE_{H;p:R}$.

Lemma 1.1 Let $H(p,q)$ be a normal analytic kernel function on a normal analytic space. Then the Lapalcian $\Delta$ commutes with the operators $\square_H^{(k)}$ ; $k = 0,1,2,3 \ldots$ if and only if it commutes with the convolution operators $(f\,\mathscr{H}) *$ for any $f$.

Proof It is enough to testify the commutativity on normal anylytic functions $\varphi$. In this case the function

$$\phi_{p;R} := (\Delta_p UE_{H;p;R} - UE_{H;p;R} \Delta_p) \varphi$$

is analytic w.r.t. the variable $R$ at any point $p$.

If $\Delta$ commutes with the operators $\Box_H^{(k)}$, then $\phi_{p;R} = 0$ holds for any R by the Pizzetti–formula (1.10). Therefore the $\Delta$ commuts with the operators $UE_{H;p;R}$ for any fixes R. Using Riemannian summs for the integral $(f\,\mathscr{H})* = \int f(r)UE_{H;p;r}dr$ we get the commutativity with the operators $(f\,\mathscr{H})*$ as well.

Conversely, if $\Delta$ commuts with the operators $(f\,\mathscr{H})*$ then it commutes also with the operators $UE_{H;p;R}$ by the approximation procedure described at (1.14).

Therefore the $\Delta$ commutes with the operators:

$$\Box_{H;p}^{(k)} = \frac{\partial^{2k} UE_{H;p;R}}{\partial R^{2k}} / R = 0 \tag{1.15}$$

as well.

Q.e.d.

Lemma 1.2 The Laplacian $\Delta$ commutes with a convolution operator $G*$ if and only if the kernel function $G(x,y)$ satisfies the ultrahyperbolic equation

$$(\Delta_x G)(x,y) = (\Delta_y G)(x,y)\,, \tag{1.16}$$

where $\Delta_x$ (resp. $\Delta_y$) means the Laplacian's action w.r.t. x (resp. w.r.t. y).

Proof The commutativity

$$\int \Delta_x G(x,y)\varphi(y)dy = \int G(x,y)\Delta_y\varphi(y)dy$$

$$\underset{\text{Stokes}}{=} \int (\Delta_y G(x,y))\varphi(y)dy \tag{1.17}$$

satisfies if and only if (1.16) holds.

Q.e.d.

Now we are in the position to formulate <u>the Basic Theorem</u> of the operators $\square_H^{(k)}$.

<u>Theorem 1.1</u> Let $H$ be a symmetric $(H(x,y) = H(y,x))$ normal analytic kernel function on a normal analytic space such that $H(p,p) \neq 0$ for any $p$. Then the Laplacian commutes with the operators $\square_H^{(k)}$ if and only if

1) for any geodesics $\gamma$ a function $\phi_\gamma : \mathbb{R}_+ \longrightarrow R$ exists such that the function $H^2(x,y)/\omega(x,y)$ is of the form $\phi_\gamma(r(x,y))$ on $\gamma$, i.e. it depends only on the geodesics distance $r(x,y)$ on any geodesics,

2) the kernel function $H(x,y)/\omega(x,y)$ satisfies the ultrahyperbolic equation

$$\left[\Delta_x \frac{H}{\omega}\right](x,y) = \left[\Delta_y \frac{H}{\omega}\right](x,y) . \tag{1.18}$$

<u>Proof</u> By the Lemmas 1.1 and 1.2 the $\Delta$ commutes with the operators $\square_H^{(k)}$ if and only if all the functions $f\ \mathcal{H}(x,y)$ satisfy the ultrahyperbolic equation

$$(\Delta_x f\ \mathcal{H})(x,y) = (\Delta_y f\ \mathcal{H})(x,y) . \tag{1.19}$$

On the other hand we have

$$\Delta_y(F_x \mathscr{H}_x)(y) = (\Delta_y F_x)(y)\mathscr{H}_x(y) + 2F_x'(y)\mathscr{H}_x'(y) + F_x(y)(\Delta_y \mathscr{H}_x)(y) =$$

$$\left[f''(r_x(y)) + \frac{n-1}{r_x(y)} f'(r_x(y))\right]\mathscr{H}_x(y) + \tag{1.20}$$

$$+ f'(r_x(y)) \left[\left[\frac{\omega_x'(y)}{\omega_x(y)}\right] \mathscr{H}_x(y) + 2\mathscr{H}_x'(y)\right] + f(r_x(y))(\Delta_y \mathscr{H}_x)(y)$$

where the coma means derivation from the radial direction furthermore we used also the classical formula

$$(\Delta_y F_x)(y) = f''(r_x(y)) + \left[\frac{n-1}{r_x(y)} + \frac{\omega_x'(y)}{\omega_x(y)}\right] f'(r_x(y)) \,. \tag{1.21}$$

Similarly we get

$$(\Delta_x F^y \mathscr{H}^y)(x) = \left[f''(r^y(x)) + \frac{n-1}{r^y(x)} f'(r^y(x))\right]\mathscr{H}^y(x) +$$

$$+ f'(r^y(x) \left[\frac{(\omega^y)'(x)}{\omega^y(x)} \mathscr{H}^y(x) + 2(\mathscr{H}^y)'(x)\right] + f(r^y(x))(\Delta_y \mathscr{H}^y)(x) \,, \tag{1.22}$$

where $\mathscr{H}_x(y) = \mathscr{H}^y(x) = \mathscr{H}(x,y)$ . Notice that the second expression in (1.20) resp. (1.22) can be written in the following form (using $\mathscr{H}(x,y) := H(x,y)/\omega(x,y)$)

$$\frac{\omega_x'(y)}{\omega_x(y)} \mathscr{H}_x(y) + 2\mathscr{H}_x'(y) = \frac{H_x}{\omega_x}(y)\left[\ell n \frac{H_x^2}{\omega_x}\right]'(y) \tag{1.24}$$

$$\frac{(\omega^y)'(x)}{\omega^y(x)} \mathscr{H}^y(x) + 2(\mathscr{H}^y)'(x) = \frac{H^y(x)}{\omega^y(x)}\left[\ell n \frac{(H^y)^2}{\omega^y}\right]'(x) \,. \tag{1.25}$$

The $\omega(x,y)$ is an analytic symmetric kernel function (see the remarks at (1.34)) therefore also the kernel–functions $\mathscr{H}(x,y)$ and

$$Z(x,y) := \ell n \, \frac{H^2(x,y)}{\omega(x,y)} \tag{1.26}$$

are smooth a symmetric kernel functions.

The equation (1.19) satisfies for any f if and only if

1) On any geodesics $\gamma$ the symmetric kernel function $Z_\gamma(x,y)$ satisfies the equation

$$Z'_{\gamma x}(y) = (Z_\gamma^y)'(x) \,, \tag{1.27}$$

2) the kernel function $\mathscr{H}$ satisfies the ultrahyperbolic equation

$$(\Delta_x \mathscr{H})(x,y) = (\Delta_y \mathscr{H})(x,y) \,.$$

If we write the kernel function $Z_\gamma$ in an arc–wise parametrization of $\gamma$ in the form $Z_\gamma(t,s) : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, then (1.27) is equivalent with the equation

$$\frac{\partial Z_\gamma}{\partial s}(t,s) = -\frac{\partial Z_\gamma}{\partial t}(t,s) \,. \tag{1.28}$$

The proof of the Theorem can be finished by the following Lemma

Lemma 1.3 The general symmetric solution of (1.28) are just the functions of the form:

$$Z_\gamma(t,s) = \phi_\gamma(|t-s|)\,, \tag{1.29}$$

where $\phi_\gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a function of one variable.

Proof For the functions of the form (1.29) the equation (1.28) obviously holds.

Conversely, if $Z_\gamma$ is a symmetric solution of (1.28) then it satisfies also the hyperbolic equation

$$\frac{\partial^2 Z_\gamma}{\partial s^2} = -\frac{\partial^2 Z_\gamma}{\partial s\,\partial t} = -\frac{\partial^2 Z_\gamma}{\partial t\,\partial s} = \frac{\partial^2 Z_\gamma}{\partial t^2}\ . \tag{1.30}$$

Therefore the $Z_\gamma$ is of the form

$$Z_\gamma = \phi(s-t) + \psi(s+t)\ . \tag{1.31}$$

For such function the (1.28) gives

$$\phi'(s-t) + \psi'(s+t) = \phi'(s-t) - \psi'(s+t) \tag{1.32}$$

therefore $\psi' = 0$ ; $\psi = \text{constant}$ and so $Z_\gamma(s,t) = \phi_\gamma(s-t)$ follows. From the symmetry $Z_\gamma(s,t) = Z_\gamma(t,s)$ we get $Z_\gamma(s,t) = \phi_\gamma(|s-t|)$ , which proves the Lemma completely.

Q.e.d.

The above theorem will be used mainly to the kernel functions defined by the several invariants of the Jacobian field. More precisely let $A_{p;r}$ be the Jacobian endomorphism field along a geodesics $e_p(r)$ defined by

$$A''_{p;r} + R_{\dot{e}_p(r)} \circ A_{p;r} = 0\ ;\ A_{p;0} = 0\ ;\ A'_{p;0} = \mathrm{Id}\ ,$$

where $R_{\dot{e}_p(r)}(\,.\,) = R(\,.\,,\dot{e}_p(r))\dot{e}_p(r)$ is the Jacobian curvature operator field along $e_p(r)$ acting in the $(n-1)$–dimensional subspace standing orthogonal to $\dot{e}_p(r)$. The invariants $\sigma_p^{(i)}(e_p(r)) = \sigma_p^{(i)}(q)$ ; $q = e_p(r)$ ; of $A_{p;q}$, defined by

$$\det(A_{p;q} + \lambda \mathrm{Id}) = \lambda^{n-1} + \sigma_p^{(1)}(q)\lambda^{n-2} + \dots + \sigma_p^{(n-1)}(q)\ ,$$

determine local kernel functions which are symmetric by the well known property

$$A_{p,q} = A^*_{q;p}$$

of the Jacobian. The $\sigma_p^{(n-1)} = \det A_p = \theta_p$ is the polar–density function, i.e.

$$\omega_p(q) = r_p^{n-1}(q)\theta_p(q)\ ,$$

where $\omega_p$ is the Riemannian density introduced earlier. The assymptotic behaviour of $\sigma_p^{(i)}(e_p(r))$ is:

$$\sigma_p^{(i)}(e_p(r)) = \begin{bmatrix} n-1 \\ i \end{bmatrix} r^i + \text{higher order terms}\ . \tag{1.33}$$

These functions are not smooth at the diagonal points (p;p) in general. But if we normalize these functions in the following way:

$$\overset{*}{\sigma}{}^{(i)}_{p}(q) := \frac{1}{\begin{bmatrix} n-1 \\ i \end{bmatrix} r^{i}_{p}(q)} \; \sigma^{(i)}_{p}(q) , \tag{1.34}$$

then the functions $\overset{*}{\sigma}{}^{i}_{p}(q)$ are smooth functions in a neighbourhood of the diagonal $\{(p,p)\}$ . Also the properties $\overset{*}{\sigma}{}^{(i)}_{p}(q) = \overset{*}{\sigma}{}^{(i)}_{q}(p)$ ; $\overset{*}{\sigma}{}^{(i)}_{p}(p) = 1$ ; $\overset{*}{\sigma}{}^{(n-1)} = \omega$ satisfy obviously.

The explicite expression of the operators $\Box\left\{\begin{smallmatrix} k \\ \overset{*}{\sigma}(1) \end{smallmatrix}\right\}$ can be computed by the power series of the Jacobian field. Also notice, that for the constant kernel functions $H(p;q) = 1$ , the operators $\Box\left\{\begin{smallmatrix} k \\ H=1 \end{smallmatrix}\right\}$ are just the Willmore operators $\Delta^{(k)}$ introduced in the first part of this paper–series. Using the recursion formula (2.10) of this previous paper as well as the power series method, we get the following formulas by an easy computation

$$\Delta^{(1)} = \frac{1}{n} \Delta , \tag{1.35}$$

$$\Delta^{(2)} = \frac{1}{n(n+2)} (3\Delta^2 + 2\rho^{ij}\nabla_i\nabla_j + 2(\nabla_j \rho^{ij})\nabla_i) , \tag{1.36}$$

$$\Box\left\{\begin{smallmatrix} 1 \\ \overset{*}{\sigma}(1) \end{smallmatrix}\right\} = \frac{1}{n} \left[ \Delta - \frac{1}{3(n-1)} R \right] , \tag{1.37}$$

$$\Box\left\{\begin{smallmatrix} 2 \\ \overset{*}{\sigma}(1) \end{smallmatrix}\right\} = \Delta^{(2)} + \frac{1}{n(n+2)} \left[ -\frac{2}{n-1} R\Delta - \frac{4}{n-1} \rho^{ij}\nabla_i\nabla_j - \frac{1}{n-1} (\nabla^i R)\nabla_i + \right. \tag{1.38}$$

$$\left. + \frac{1}{5(n-1)} \left[ - 6\Delta R + \rho_{ab}\rho^{ab} + \frac{3}{2} R_{abcd}R^{abcd} \right] \right] ,$$

(1.39) $$\Box^{(1)}_{\overset{*}{\sigma}(n-1)=\omega} = \frac{1}{n}(\Delta - \frac{1}{3}R)\,,$$

(1.40) $$\Box^{(2)}_{\omega} = \Delta^{(2)} + \frac{1}{n(n+2)}\Big[-2R\Delta - 4\rho^{ij}\nabla_i\nabla_j - (\nabla^i R)\nabla_i - $$
$$- \frac{6}{5}\Delta R + \frac{1}{3}R^2 + \frac{4}{5}\rho_{ab}\rho^{ab} + \frac{1}{5}R_{abcd}R^{abcd}\Big]\,.$$

In these formulas $R_{abcd} = \langle \nabla_b\nabla_a - \nabla_a\nabla_b\partial_c, \partial_d\rangle$ is the curvature tensor furthermore $\rho_{ab} = R_a{}^p{}_{bp}$ resp. $R = \rho^a_a$ is the Ricci curvature resp. the curvature scalar.

§ 2. Analycity at commuting higher order Laplacians

In this chapter we consider the Riemannian spaces satisfying the curvature condition:

(2.1) $$\nabla_i\rho_{jk} + \nabla_j\rho_{ki} + \nabla_k\rho_{ij} = 0$$

for the Ricci curvature $\rho_{ij}$. All the Einstein manifolds satisfy this condition, so the following theorem is a generalization of the Kazdan–De Turck theorem.

Theorem 2.1 All the Riemannian metrics satisfying the curvature condition (2.1) are real analytic in the harmonic resp. normal coordinate neighbourhoods.

Proof Using the method of harmonic coordinates (developed in the proof of the Kazdan–De Turck theorem [1]) we have to prove only that the symbol

$$h_{ij} \longrightarrow \xi_i|\xi|^2 h_{jk} + \xi_j|\xi|^2 h_{ki} + \xi_k|\xi|^2 h_{ij} \tag{2.2}$$

of the equation (2.1) is injective. This property obiously satisfies, as if

$$\xi_i h_{jk} + \xi_j h_{ki} + \xi_k h_{ij} = 0$$

for any $\xi$ , then

$$\begin{aligned} |\xi|^2 \xi^j \xi^k h_{jk} = 0; 2|\xi|^2 \xi^j h_{jk} + \xi^i \xi^j h_{ij} \xi_k = 0 \,; \\ |\xi|^2 h_{jk} + \xi^i \xi_j h_{ik} + \xi^i \xi_k h_{ij} = 0 \,, \end{aligned} \tag{2.3}$$

from which $h_{jk} = 0$ follows. This proves the injectivity of (2.2) and the theorem completely.

Q.e.d.

Theorem 2.2 A Riemannian space satisfies the condition (2.1) if and only if the Laplacian commutes with the second Willmore's operator $\Delta^{(2)}$ .

The equation (2.1) holds also in the case, if the Laplacian commutes with the operators $\underset{\overset{*}{\sigma}(i)}{\Box^{(1)}}$ , $\underset{\overset{*}{\sigma}(i)}{\Box^{(2)}}$ . More precisely this commutativity is equivalent with the conditions: (2.1) and $\underset{\overset{*}{\sigma}(i)}{\Lambda^{(2)}}$ = constant .

Proof By (1.36) the operators $\Delta$ , $\Delta^{(2)}$ commutes if and only if the Laplacian commutes with the operator

$$\nabla_i \rho^{\ell i} \nabla_\ell = \rho^{\ell i} \nabla_i \nabla_\ell + (\nabla_i \rho^{\ell i}) \nabla_\ell \,. \tag{2.4}$$

Using the Ricci identities, this last commutativity is equivalent with the equation

$$(2.5)\qquad \nabla_s(\nabla^s\rho^{\ell i} + \nabla^i\rho^{\ell s})\nabla_i\nabla_\ell + \nabla_s(\nabla_i\nabla^s\rho^{\ell i} - R_p{}^{\ell s}{}_i\rho^{pi} + \rho^{ps}\rho_p^\ell)\nabla_\ell = 0\,.$$

The highest (third) order therm of the differential operator on the left side is

$$(2.6)\qquad (\nabla^s\rho^{\ell i} + \nabla^\ell\rho^{is} + \nabla^i\rho^{s\ell})\nabla_\ell\nabla_i\nabla_s\,,$$

so the equation (2.1) follows for the spaces satisfying the commutativity $\Delta\Delta^{(2)} = \Delta^{(2)}\Delta$.

Conversely, if (2.1) holds, then

$$(2.7)\qquad 0 = \nabla_s R + 2\nabla^\ell\rho_{s\ell} = 2\nabla_s R = \nabla_\ell\rho_s^\ell\,,$$

so we have to prove, that $\Delta$ commutes with $\rho^{\ell i}\nabla_i\nabla_\ell$. This commutativity is equivalent with (2.1) and with the following equations

$$(2.8)\qquad \nabla_i\nabla^i\rho^{\ell s} + 2\rho^{ij}R_i{}^{\ell s}{}_j + 2\rho_i^\ell\rho^{is} = 0$$

$$(2.9)\qquad \tfrac{4}{3}(\nabla^s\rho^{ij})R_{\ell ijs} - 2\rho^{ij}\nabla_\ell\rho_{ij} = 0\,.$$

We show (using the Ricci identities) that the equations (2.8) and (2.9) follow from (2.1). In fact, the equation (2.8) follows from (2.1) by the following computation.

$$0 = \nabla_i \nabla^i \rho_{\ell s} + \nabla^i \nabla_\ell \rho_{si} + \nabla^i \nabla_s \rho_{i\ell} =$$

(2.10)
$$= \nabla_i \nabla^i \rho_{\ell s} + 2\rho^{ij} R_i{}^{\ell s}{}_j + 2\rho^\ell_i \rho^{is} + \nabla_s \nabla^i \rho_{\ell i} + \nabla_\ell \nabla^i \rho_{si} =$$

$$= \nabla_i \nabla^i \rho_{\ell s} + 2\rho^{ij} R_i{}^{\ell s}{}_j + 2\rho^\ell_i \rho^{is} ,$$

because $\nabla_s \nabla^i \rho_{\ell i} = \nabla_\ell \nabla^i \rho_{si} = 0$ by (2.7). The equation (2.9) follows from (2.1) by

(2.11)
$$0 = \nabla^\ell \nabla^i \nabla_i \rho_{\ell s} + \nabla^\ell \nabla^i \nabla_\ell \rho_{si} + \nabla^\ell \nabla^i \nabla_s \rho_{i\ell} =$$

$$= \nabla^i \nabla^\ell \nabla_i \rho_{\ell s} - R_{ps\ell p} \nabla^i \rho^{\ell p} + 2\nabla^\ell (-\rho^{ij} R_{i\ell sj} + \rho_{i\ell} \rho^i_s) =$$

$$= \nabla^i \nabla_i \nabla^\ell \rho_{\ell s} + 4R_{sp\ell i} \nabla^i \rho^{p\ell} - 6\rho^{p\ell} \nabla \rho_{p}{}^{\ell} ,$$

which is just (2.9) by (2.7). This proves the first statement completely.

For the second statement we have to notice, that the operator $\square^{(1)}_{\overset{*}{\sigma}(i)}$ is of the form $P_i \Delta + Q_i R$ , where $P_i$ and $Q_i$ are constant; therefore from $\Delta \square^{(1)}_{\overset{*}{\sigma}(i)} = \square^{(1)}_{\overset{*}{\sigma}(i)} \Delta$ it $R =$ constant follows. So the $\Delta$ commutes also with $\square^{(2)}_{\overset{*}{\sigma}(i)}$ iff it commutes with an operator of the form

$$\rho^{ab} \nabla_a \nabla_b + \phi_{(i)} ,$$

where the function $\phi_{(i)}$ is the constant time of the function $\Lambda^{(2)}_{\overset{*}{\sigma}(i)}$ defined in (1.6). From this commutativity we get an equation similar to (2.5), for which the highest order term on the left side is (2.6) again. So also the spaces with the properties

(2.12) $$\Delta \underset{\overset{*}{\sigma}(i)}{\square^{(1)}} = \underset{\overset{*}{\sigma}(i)}{\square^{(1)}} \Delta \; ; \; \Delta \underset{\overset{*}{\sigma}(i)}{\square^{(2)}} = \underset{\overset{*}{\sigma}(i)}{\square^{(2)}} \Delta$$

satisfy the curvature condition (2.1). More precisely (2.12) satisfies iff beside (2.1) also $\underset{\overset{*}{\sigma}(i)}{\Lambda^{(2)}}$ = constant holds.

Q.e.d.

We have to mention that the half part of the theorem was proved also by O. Kawalski in [7]. In fact, he proved that the commutativity $\Delta\Delta^{(2)} = \Delta^{(2)}\Delta$ implies the condition (2.1), but the equivalentness of these conditions (i.e. the conversed statement) is not proved there.

## § 3. D'Atri spaces

A Riemannian space is called to be a D'Atri space if the geodesics involutions are volume preserving. In such spaces the odd order derivatives $\omega^{(2k+1)}_{\dot{e}_p}$ of the density function $\omega_p$ vanish. Specially

(3.1) $$\omega^{(3)}_{\dot{e}_p} = -\frac{1}{2}\left[\nabla_{\dot{e}_p}\rho\right]\left[\dot{e}_p, \dot{e}_p\right] = 0$$

follows, which is equivalent with the condition

(3.2) $$\nabla_i\rho_{jk} + \nabla_j\rho_{ki} + \nabla_k\rho_{ij} = 0 \; .$$

Therefore the D'Atri spaces are normal analytic manifolds by Theorem 2.1.

Let $\omega_x^g(y)$ be the restriction of the density function $\omega_x(y)$ onto the geodesics $g$. If these functions are of the form $\omega_x^g(y) = \phi_g(r(x,y))$ then the space is obviously a D'Atri space. The conversed statement is also true, i.e. this property characterizes the D'Atri spaces.

In fact, a D'Atri space is a normal analytic therefore the function $\omega_x^g(y)$ is an analytic, symmetric and central symmetric kernel function (double function) on the geodesics $g$. Such functions are always of the form $\omega_x^g(y) = \phi_g(r(x,y))$ proved by O. Kowalski and L. Vanhecke [8] (Theorem 2.5). So we have

<u>Theorem 3.1</u> The D'Atri spaces are normal analytic manifolds and these are characterized by the property, where the density function $\omega_x^g(y)$ depends only on $r(x,y)$ (i.e. it is of the form $\omega_x^g(y) = \phi_g(r(x,y))$ on any geodesics $g$.

Combining this theorem with the Basic Theorem 1.1 we have

<u>Theorem 3..2</u> A space is a D'Atri space if and only if the Laplacian commutes with the operators $\square^{(k)}_{\omega=\overset{*}{\sigma}(n-1)}$.

<u>Proof</u> If the space is a D'Atri space then it is normal analytic and also the density $\omega$ is normal analytic function (by the previous theorem). Furthermore the functions $\omega^2/\omega = \omega$ resp. $\omega/\omega = 1$ satisfy the conditions 1 resp. 2 of Theorem 1.1, therefore the Laplacian commutes with the operators $\square^{(k)}_{\omega}$.

Conversely, if the $\Delta$ commutes with the operators $\square_{\omega}^{(k)}$ , then the metric is normal analytic by the Theorem 2.1 and 2.2 furthermore the $\omega^{g}$ is of the form $\omega_{p}^{g}(q) = \phi_{g}(r(p,q))$ by Theorem 1.1. I.e. the space is a D'Atri space.

Q.e.d.

A Riemannian manifold is defined to be an <u>(i)–D'Atri space</u> if the kernel functions $(\overset{*}{\sigma}{}_{p}^{(i)})^{2}/\omega_{p}$ are central symmetric at any point $p$ .

Using the same argument as before we get

<u>Theorem 3.3</u> The (i)–D'Atri spaces are normal analytic spaces. A space is an (i)–D'Atri space if and only if the kernel function $(\overset{*}{\sigma}{}_{x}^{(i)}(y))^{2}/\omega$ is of the form $\phi_{\gamma}(r(x,y))$ on any geodesics $\gamma$ .

<u>Theorem 3.4</u> The Laplacian commutes with the operators $\square_{\overset{*}{\sigma}{}^{(i)}}^{(k)}$ if and only if the space is an (i)–D'Atri space satisfying the ultrahyperbolic equation

$$\Delta_{x}\frac{\overset{*}{\sigma}{}^{(i)}}{\omega}(x,y) = \Delta_{y}\frac{\overset{*}{\sigma}{}^{(i)}}{\omega}(x,y)\,. \tag{3.3}$$

The Willmore's commutative spaces (or probabilistic commutative spaces) are defined by commuting Willmore's operators. All these spaces are D'Atri spaces by the following theorem

Theorem 3.5 The Laplacian commutes with the Willmore's operator if and only if the space is a D'Atri space satisfying the ultrahyperbolic equation

$$\Delta_x \frac{1}{\omega}(x,y) = \Delta_y \frac{1}{\omega}(x,y) . \tag{3.4}$$

F. Tricerri and L. Vanhecke investigated [10] homogeneous Riemannian manifolds G/H with commuting invariant differential operators. They proved that all these spaces are D'Atri spaces. This result is a special case of the above theorems, raather more a stronger theorem can be stated: All these spaces are (i)–D'Atri spaces (for any index i) satisfying also the ultrahyperbolic equations:

$$\Delta_x \frac{\overset{*}{\sigma}{}^{(i)}}{\omega}(x,y) = \Delta_y \frac{\overset{*}{\sigma}{}^{(i)}}{\omega}(x,y) . \tag{3.5}$$

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