Counter-examples to Sister Beiter's cyclotomic coefficient conjecture

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Abstract

Let $\Phi_n(x)$ denote the nth cyclotomic polynomial. In 1968 Sister Marion Beiter conjectured that $a_n(k)$, the coefficient of x^k in $\Phi_n(x)$, satisfies $|a_n(k)| \leq (p+1)/2$ in case $n = pqr$ with $p < q < r$ primes. Since then several results towards establishing her conjecture have been proved (for example $|a_n(k)| \leq 3p/4$). Here we show that, nevertheless, Beiter's conjecture is false for every $p \geq 11$. We also prove that given any $\epsilon > 0$ there exist infinitely many triples (p_j, q_j, r_j) with $p_1 < p_2 < \ldots$ consecutive primes such that $|a_{p_j q_j r_j}(n_j)| > (2/3 - \epsilon)p_j$ for $j \geq 1$.

1 Introduction

The nth cyclotomic polynomial $\Phi_n(x)$ is defined by

$$\Phi_n(x) = \prod_{\substack{1 \leq j \leq n \\ (j,n)=1}} (x - \zeta_n^j) = \sum_{k=0}^{\varphi(n)} a_n(k) x^k,$$

where φ is Euler's totient function and ζ_n a primitive nth root of unity. For the k not in the range $[0, \varphi(n)]$ we put $a_n(k) = 0$. The coefficients $a_n(k)$ are known to be integral. The study of the $a_n(k)$ began with the startling observation that for small n we have $|a_n(k)| \leq 1$ (it thus seems, as D. Lehmer [10] worded it, that the primitive roots of unity conspire to achieve this smallness). The first counter-example to $|a_n(k)| \leq 1$ occurs for $n = 105$: $a_{105}(7) = -2$. Note that 105 is the smallest odd integer having three prime factors. If $\omega_1(n)$ denotes the number of odd prime factors of n, then it is well-known that if $\omega_1(n) \leq 2$, then $\Phi_n(x)$ is *flat*, that is all its coefficients satisfy $|a_n(k)| \leq 1$. Thus $n = 105$ is the first candidate integer for $\Phi_n(x)$ to be non-flat. We see that with respect to the smallness of the coefficients the first non-trivial case arises when $\omega_1(n) = 3$. In this case some authors say that $\Phi_n(x)$ is *ternary*. Then we write $n = pqr$ with $2 < p < q < r$.

We define the *height* of $\Phi_n(x)$ to be $\max\{|a_n(k)| : 0 \leq k \leq \varphi(n)\}$ and denote it by $A(n)$. In 1968 Sister Marion Beiter [3] put forward the following conjecture (which she repeated in 1971 [4]).

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Conjecture 1 (Sister Marion Beiter, 1968). *If $2 < p < q < r$ are primes, then $A(pqr) \leq \frac{p+1}{2}$.*

Note that $A(2qr) = 1$. In case either q or $r \equiv \pm 1 (\mathrm{mod}\ p)$ Beiter proved her conjecture. This result was extended by Bachman [1].

Theorem 1 (Gennady Bachman, 2003). *If either q or r is congruent to ± 1 or ± 2 modulo p, then $A(pqr) \leq (p+1)/2$. If q or r is congruent to $(p \pm 1)/2$ modulo p, then $A(pqr) \leq (p+3)/2$.*

In a further paper Beiter [4] points out that her conjecture is true for $p \leq 5$ and shows that $A(pqr) \leq p - \lfloor p/4 \rfloor$, thus improving on a result from Bang [2] proved in 1895, to the effect that $A(pqr) \leq p - 1$. The best known general upper bound to date is due to Bachman [1], who proved that $A(pqr) \leq p - \lceil p/4 \rceil$. In the same paper Bachman showed:

Theorem 2 (Gennady Bachman, 2003). *Let q^* and r^*, $0 < q^*, r^* < p$ be the inverses of q and r modulo p respectively. Set $a = \min(q^*, r^*, p - q^*, p - r^*)$. Then $A(pqr) \leq \min(\frac{p-1}{2} + a, p - a)$.*

H. Möller [12] indicated for every prime $p > 3$ a cyclotomic polynomial $\Phi_{pqr}(x)$ having a coefficient equal to $(p+1)/2$. This shows that Beiter's conjecture is best possible, if true. In particular Möller proved:

Theorem 3 (Herbert Möller, 1971). *Let $3 < p < q < r$ be prime numbers satisfying $q \equiv 2 (\mathrm{mod}\ p)$ and $r = (mpq - 1)/2$ for some integer m. Then*

$$a_{pqr}(\frac{1}{2}(p-1)(qr+1)) = \frac{p+1}{2}.$$

Earlier Emma Lehmer [11] had shown that for q and r as in the latter Theorem we have $a_{pqr}(\frac{1}{2}(p-3)(qr+1)) = (p-1)/2$. On combining Möller's result with Theorem 1 we infer that for his choice of p, q and r we have $A(pqr) = (p+1)/2$.

Let $M(p)$ be the maximum of the heights of the ternary cyclotomic polynomials, where p is the smallest prime factor of n. The case $p = 3$ was investigated in detail by Beiter [5], who found that $M(3) = 2$. Beiter's conjecture in combination with $M(3) = 2$ and Möller's result leads to the following conjecture.

Conjecture 2 *For $p > 2$ we have $M(p) = \frac{p+1}{2}$.*

We will show that our main result, presented below, can be used to infer that Beiter's conjecture is 'very false'.

Theorem 4 *Let p be a prime. Given an $1 \leq \beta \leq p$ we let β^* be the unique integer $1 \leq \beta^* \leq p - 1$ with $\beta\beta^* \equiv 1 (\mathrm{mod}\ p)$.*

Let $\mathcal{B}_-(p)$ be the set of integers β satisfying

$$1 \leq \beta \leq \frac{p-3}{2},\ p \leq \beta + 2\beta^* + 1,\ \beta > \beta^*. \tag{1}$$

For every prime $q \equiv \beta (\mathrm{mod}\ p)$ with $q > q_-(p)$ and $\beta \in \mathcal{B}_-(p)$, there exists a prime $r_- > q$ and an integer n_- such that $a_{pqr_-}(n_-) = \beta_- - p$, where $q_-(p), r_-$

and n_- can be explicitly given.

Let $\mathcal{B}_+(p)$ be the set of integers β satisfying

$$1 \le \beta \le \frac{p-3}{2},\ \beta+\beta^* \ge p,\ \beta^* \le 2\beta, \tag{2}$$

For every prime $q \equiv \beta \pmod p$ with $q > q_+(p)$ and $\beta \in \mathcal{B}_+(p)$ there exists a prime $r_+ > q$ and an integer n_+ such that $a_{pqr_+}(n_+) = p-\beta$, where $q_+(p), r_+$ and n_+ can be explicitly given. In case $\beta \in \mathcal{B}_+(p)$ and $\beta+\beta^* = p$, then $A(pqr_+) = p-\beta$.

Put $\mathcal{B}(p) = \mathcal{B}_-(p) \cup \mathcal{B}_+(p)$. If $\mathcal{B}(p)$ is non-empty, then

$$M(p) \ge p - \min\{\mathcal{B}(p)\} > \frac{p+1}{2},$$

and so Beiter's conjecture is false for the prime p.

Explicit choices of $q_-(p), r_-$ and n_- are given in Theorem 10 and explicit choices of $q_+(p), r_+$ and n_+ in Theorem 11, see also Tables 7 and 8.

Note that the sets $\mathcal{B}_-(p)$ and $\mathcal{B}_+(p)$ are disjoint. For $p \le 11$ the set $\mathcal{B}(p)$ turns out to be empty. For $11 \le p \le 73$ it is given in Table 1. The underlined element is $(p-3)/2$ and for this range always turns out to be in $\mathcal{B}(p)$. The final column gives a lower bound for $M(p)$. The table shows that Beiter's conjecture is false for $11 \le p \le 73$.

Proposition 1 *For $p \ge 11$, $\mathcal{B}(p)$ is non-empty and $\max\{\mathcal{B}(p)\} = (p-3)/2$.*

Proof. Consider $\beta = (p-3)/2$. If $p \equiv 1 \pmod 3$, then $\beta^* = 2(p-1)/3$ and one checks that $\beta \in \mathcal{B}_+(p)$. If $p \equiv 2 \pmod 3$, then $\beta^* = (p-2)/3$ and one checks that $\beta \in \mathcal{B}_-(p)$. □

Table 1: The sets $\mathcal{B}_-(p)$, $\mathcal{B}_+(p)$ and $\mathcal{B}(p)$

p	$\mathcal{B}_-(p)$	$\mathcal{B}_+(p)$	$\mathcal{B}_-(p) \cup \mathcal{B}_+(p) = \mathcal{B}(p)$	$p - \min \mathcal{B}(p)$
11	$\{\underline{4}\}$	$\emptyset$	$\{4\}$	7
13	$\emptyset$	$\{\underline{5}\}$	$\{5\}$	8
17	$\{\underline{7}\}$	$\emptyset$	$\{7\}$	10
19	$\emptyset$	$\{\underline{8}\}$	$\{8\}$	11
23	$\{\underline{10}\}$	$\{9\}$	$\{9, 10\}$	14
29	$\{\underline{13}\}$	$\{12\}$	$\{12, 13\}$	17
31	$\{13\}$	$\{\underline{14}\}$	$\{13, 14\}$	18
37	$\emptyset$	$\{\underline{17}\}$	$\{17\}$	20
41	$\{18, \underline{19}\}$	$\{17\}$	$\{17, 18, 19\}$	24
43	$\{18\}$	$\{19, \underline{20}\}$	$\{18, 19, 20\}$	25
47	$\{\underline{22}\}$	$\{18, 20\}$	$\{18, 20, 22\}$	29
53	$\{\underline{25}\}$	$\{22, 23, 24\}$	$\{22, 23, 24, 25\}$	31
59	$\{23, 26, \underline{28}\}$	$\{27\}$	$\{23, 26, 27, 28\}$	36
61	$\{25, 28\}$	$\{27, \underline{29}\}$	$\{25, 27, 28, 29\}$	36
67	$\emptyset$	$\{26, 30, \underline{32}\}$	$\{26, 30, 32\}$	40
71	$\{32, 33, \underline{34}\}$	$\{27, 29, 30, 31\}$	$\{27, 29, 30, 31, 32, 33, 34\}$	44
73	$\{33\}$	$\{27, 30, 34, \underline{35}\}$	$\{27, 30, 33, 34, 35\}$	46

From Theorem 4, Proposition 1 and Dirichlet's theorem on arithmetic progressions the following result is inferred.

Theorem 5 *Suppose $p \geq 11$ is a prime. Then the set of q for which $A(pqr) > (p+1)/2$ for some prime r has a positive lower density $\underline{\delta}$ satisfying*

$$\underline{\delta} \geq \frac{|\mathcal{B}_+(p)| + |\mathcal{B}_-(p)|}{p-1} \geq \frac{1}{p-1} > 0,$$

and hence Beiter's conjecture is false for this prime.

The elementary method of proof of Proposition 1 allows one to prove, e.g., that if $p \equiv 23 (\mathrm{mod}\ 24)$ and $p > 23$, then $M(p) \geq (5p-3)/8$ and more generally it allows one to indicate for every $\epsilon > 0$ an arithmetic progression such that for all primes larger than some explicit number in this progression we have $M(p) \geq (\frac{2}{3} - \epsilon)p$ (Proposition 4). On invoking a result from the theory of inverses modulo p it can be even shown that the latter lower bound holds for *all* primes p sufficiently large.

Theorem 6 *Let $\epsilon > 0$. Then $\frac{2}{3}(1-\epsilon)p \leq M(p) \leq \frac{3}{4}p$ for every sufficiently large prime p.*

In Table 2 we give for some small primes p intervals $[a, b]$ such that $a \leq M(p) \leq b$. The number $b = p - \lceil p/4 \rceil$ and $a = |a_{pqr}(n)|$, showing that $M(p) \geq a$. The values of a are the largest known to us and were found by extensive computer calculation.

Table 2: `Interval for` $M(p)$

p	$(p+1)/2$	q	r	n	$M(p)$ interval	$[2p/3]$	$\delta(p)$
3	2	5	7	7	[2, 2]	2	0
5	3	7	11	119	[3, 3]	3	0
7	4	11	37	963	[4, 5]	4	≥ 0
11	6	19	601	34884	[7, 8]	7	≥ 1
13	7	31	1097	137160	[8, 9]	8	≥ 1
17	9	29	41	4801	[10, 12]	11	≥ 1
19	10	53	859	318742	[12, 14]	12	≥ 2
23	12	41	4903	1583731	[14, 17]	15	≥ 2
29	15	127	7793	8915220	[18, 21]	19	≥ 3
31	16	89	4519	4424131	[19, 23]	20	≥ 3
37	19	47	1217	743670	[22, 27]	24	≥ 3
41	21	71	97	96529	[26, 30]	27	≥ 5
43	22	53	2963	2358548	[26, 32]	28	≥ 4
47	24	347	12113	64756445	[29, 35]	31	≥ 5
53	27	61	17377	18037438	[33, 39]	35	≥ 6
59	30	67	21247	27047555	[37, 44]	39	≥ 7
61	31	191	30203	126913006	[38, 45]	40	≥ 7
67	34	191	91127	417817361	[42, 50]	44	≥ 8
71	36	311	13327	91183645	[44, 53]	47	≥ 8
73	37	83	4241	9156474	[46, 54]	48	≥ 9

The last column gives information about the difference $\delta(p) := M(p) - (p+1)/2$. In case $\delta(p) > 0$ the associated p, q and r give rise to a counter-example to Beiter's conjecture.

If $\beta \in \mathcal{B}_-(p)$, then $p \leq 3\beta$ and $p - \beta \leq 2p/3$. If $\beta \in \mathcal{B}_+(p)$, then $p \leq \beta + \beta^* \leq 3\beta$ and so $\beta \geq p/3$ and hence $p - \beta \leq 2p/3$. Thus Theorem 4 only allows one to find counter-examples $\leq 2p/3$ to Beiter's conjecture. Extensive numerical computations gave many counter-examples not covered by Theorem 4, but all of them are $\leq 2p/3$. Thus the strongest corrected version of Beiter's conjecture which we can presently neither disprove nor prove is as follows.

Conjecture 3 (Corrected Beiter conjecture). *We have $M(p) \leq 2p/3$.*

Note that it implies that Beiter's original conjecture is correct for $p = 7$. This is at present still an open problem. If $A(7qr) > 4$, then we must have $q \equiv \pm 3 (\mathrm{mod}\ 7)$ by Theorem 1.

Our final result deals with some apparent variations of $M(p)$. Let $M_+(p)$ and $M_-(p)$ be the maximum, respectively minimum of the coefficients of the ternary cyclotomic polynomials with p the smallest prime factor of n.

Theorem 7 *We have $M_-(p) = M_+(p) = M(p)$.*

For a nice survey of properties of coefficients of cyclotomic polynomials see Thangadurai [15].

1.1 Some results of Nathan Kaplan

Using the identity

$$\Phi_{pqr}(x) = (1 + x^{pq} + x^{2pq} + \cdots)(1 + x + \cdots + x^{p-1} - x^q - \cdots - x^{q+p-1})\Phi_{pq}(x^r),$$

Kaplan [8] proved the following lemma.

Lemma 1 (Nathan Kaplan, 2007). *Let $n \geq 0$ be an integer. Put*

$$b_i = \begin{cases} a_{pq}(i) & \text{if } ri \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$a_{pqr}(n) = \sum_{m=0}^{p-1} b_{f(m)} - \sum_{m=0}^{p-1} b_{f(m+q)}, \tag{3}$$

where $f(m)$ is the unique value $0 \leq f(m) < pq$ such that $f(m) \equiv r^{-1}(n - m) (\mathrm{mod}\ pq)$.

Since the $a_{pq}(i)$ are easily computed, Kaplan's lemma is actually useful. Indeed, his lemma plays a crucial role in our counter-example constructions. A nice feature of the lemma is that it works for every $n \geq 0$. Thus if it shows that $|a_{pqr}(n)| \neq 0$, then we know that $n \leq \varphi(pqr)$. In our counter-example constructions this saves us from checking that for the chosen n we have $n \leq \varphi(pqr)$.

The next lemma gives the values of $a_{pq}(i)$. For a proof see e.g. Lam and Leung [9] or Thangadurai [15],

Lemma 2 *Let $p < q$ be odd primes. Let ρ and σ be the (unique) non-negative integers for which $(p-1)(q-1) = \rho p + \sigma q$ Let $0 \le m < pq$. Then either $m = \alpha_1 p + \beta_1 q$ or $m = \alpha_1 p + \beta_1 q - pq$ with $0 \le \alpha_1 \le q-1$ the unique integer such that $\alpha_1 p \equiv m (\mathrm{mod}\ q)$ and $0 \le \beta_1 \le p-1$ the unique integer such that $\beta_1 q \equiv m (\mathrm{mod}\ p)$. The cyclotomic coefficient $a_{pq}(m)$ equals*

$$\begin{cases} 1 & \text{if } m = \alpha_1 p + \beta_1 q \text{ with } 0 \le \alpha_1 \le \rho,\ 0 \le \beta_1 \le \sigma; \\ -1 & \text{if } m = \alpha_1 p + \beta_1 q - pq \text{ with } \rho + 1 \le \alpha_1 \le q-1,\ \sigma + 1 \le \beta_1 \le p-1; \\ 0 & \text{otherwise.} \end{cases}$$

The following result of Kaplan [8] together with Dirichlet's theorem shows that given one counter-example (p, q, r) infinitely many counter-examples to Beiter's conjecture exist with the same values of p and q.

Theorem 8 (Nathan Kaplan, 2007). *For any prime $s > q$ such that $s \equiv \pm r (\mathrm{mod}\ pq)$ we have $A(pqr) = A(pqs)$.*

`Example`. Put $p = 17$ and $q = 29$. By computation one finds that $A(pq \cdot 1931) = 10$. On applying Kaplan's result one then finds from this that also $A(pq \cdot 2917) = A(pq \cdot 2999) = 10$.

Implicit in Kaplan's proof of Theorem 8 is the following result using which we immediately infer that Theorem 7 holds true.

Proposition 2 *Suppose that $a_{pqr}(n) = m$. Write $n = [\frac{n}{r}]r + n_0$ with $0 \le n_0 < r$.*
1) Let $s > r$ be a prime satisfying $s \equiv r (\mathrm{mod}\ pq)$. Then

$$a_{pqs}\Big(\Big[\frac{n}{r}\Big]s + n_0\Big) = m.$$

2) Let $t > pq$ be a prime satisfying $t \equiv -r (\mathrm{mod}\ pq)$. Let $0 \le n_1 < pq$ be the unique integer such that $n_1 \equiv q + p - 1 - n_0 (\mathrm{mod}\ pq)$. Then

$$a_{pqt}\Big(\Big[\frac{n}{r}\Big]t + n_1\Big) = -m.$$

2 A counter-example construction for $p = 11$

Using Lemma 1 and Lemma 2 (we leave this as an exercise to the reader) one finds that for the p, q and r in the Möller construction we have $b_{f(m)} = 1$ for $0 \le m \le (p-1)/2$ and $b_{f(m)} = 0$ for the remaining m in (3), giving $a_{pqr}(n) = (p+1)/2$. Likewise, for the Lehmer example we find $b_{f(m)} = 1$ for $0 \le m \le (p-3)/2$ and $b_{f(m)} = 0$ for the remaining m, giving $a_{pqr}(n) = (p-1)/2$.

For the counter-examples to Beiter's conjecture we find in general rather more complicated vectors $(b_{f(0)}, \ldots, b_{f(p-1)})$ and $(b_{f(q)}, \ldots, b_{f(q+p-1)})$. However, some of them are regular enough as to build a general construction on. We give an example which is intended as an appetizer that should help the reader digest more easily the general construction given in Theorem 10. Notice that Theorem 10 implies Theorem 9. The first few counter-examples produced by Theorem 9 are given in Table 3.

Table 3: Some counter-examples produced by Theorem 9

p	q	α	r	n	$a_{pqr}(n)$
11	59	2	877	175410	-7
	103	4	1229	381000	-7
	191	6	4639	3173086	-7
		7	16937	10280769	-7
	257	8	3011	2788196	-7
		9	1163	987397	-7
		10	8731	6740342	-7
	367	12	56999	72844732	-7
		13	811	974021	-7
		14	39157	44012478	-7

Theorem 9 *Let $q < r$ be primes such that $q \equiv 4 (\mathrm{mod}\ 11)$ and $r \equiv -3 (\mathrm{mod}\ 11)$. Let $1 \leq \alpha \leq q-1$ be the unique integer such that $11r\alpha \equiv 1 (\mathrm{mod}\ q)$. Suppose that*

$$\frac{q}{33} < \alpha \leq \frac{3q-1}{77}.$$

Then $a_{11qr}(10 + (6q - 77\alpha)r) = -7$.

Proof. Put $p = 11$ and $n = 10 + (6q - 77\alpha)r$. Note that $n \geq 10$. We will compute $a_{pqr}(n)$ using Lemma 1. Since this will have -7 as outcome, it follows that $n \leq \varphi(pqr)$. Observe that

$$0 \leq \frac{n-10}{r} = (q - 7\alpha)p + 6q - pq \leq 6q < pq$$

and so $f(10) = (q-7\alpha)p+6q-pq$. This expresses $f(10)$ as a linear combination in p and q. Let $0 \leq r_1 < pq$ be the unique integer with $r_1 \equiv -\frac{1}{r} (\mathrm{mod}\ pq)$. It is easy to see that $r_1 = q-\alpha p$. On noting that $f(m) \equiv f(10)+(m-10)r_1 \equiv f(10)+(m-10)(q-\alpha p)$, we infer that $f(m)$ is congruent modulo pq to the corresponding entry in Table 4 (and likewise for $f(m+q)$ on noting that $-q/r \equiv 4q (\mathrm{mod}\ pq)$ and $f(m+q) \equiv f(m) - q/r \equiv f(m) + 4q (\mathrm{mod}\ pq)$). In the $f(m+q)$ column we trivially have

$$0 \leq f(q) < \cdots < f(q+p-1) = 10q - 7\alpha p \leq 10q < pq.$$

Using this we infer that $f(m+q)$ is actually equal to the corresponding entry in Table 4. In the $f(m)$ column we have $0 \leq f(0) < \cdots < f(4) < pq$ and on using that $\alpha \leq q/(2p)$ we find $0 \leq f(5) < \cdots < f(10) < 6q < pq$. Again we see that $f(m)$ is actually equal to the corresponding entry in Table 4.

Now we are ready to invoke Lemma 2. One computes that $\sigma = 2$ and $\rho = (8q-10)/11$. The conditions on α ensure that $q - 7\alpha \geq \rho + 1$ and $3\alpha \leq \rho$. On applying Lemma 2 we then infer that $c_m := a_{pq}(f(m))$ and $c_{m+q} := a_{pq}(f(m+q))$ are as given in Table 4. Since $[n/r] = f(10)$ it follows, by Lemma 1 that $b_m = c_m$ if $f(m) \leq f(10)$ and $b_m = 0$ otherwise. Thus to compute say the b_m column we have $b_m = 0$ if $c_m = 0$. If $c_m \neq 0$ we have

$$b_m = \begin{cases} c_m & \text{if } f(m) \leq f(10); \\ 0 & \text{otherwise.} \end{cases}$$

Note that clearly $f(7) < .... < f(10)$. It then follows that the b_m column equals the c_m column. Next let us determine the c_{m+q} column. We claim that $f(q) < f(q+1) < f(q+2) < f(10)$. To establish this we have to check that $\alpha p + 2q < (q - 7\alpha)p - 5q$. Note that $f(q+4) < \cdots < f(q+10)$. The conditions on α ensure that $f(q+4) > f(10)$ and we see that $f(10) < f(q+4) < \cdots < f(q+10)$ and thus $b_{q+4} = \ldots = b_{q+10} = 0$. Finally, on applying Lemma 1 we infer that

$$a_{pqr}(n) = \sum_{m=0}^{p-1} b_{f(m)} - \sum_{m=0}^{p-1} b_{f(m+q)} = -4 - 3 = -7.$$

This completes the proof. □

`Remark`. Note that the conditions imposed on α are such that $\alpha \leq q/(2p)$, $3\alpha \leq \rho$, $q - 7\alpha \geq \rho + 1$, $f(q+4) > f(10)$ and $f(q+2) \leq f(10)$.

Table 4: `A counter-example construction for` $p = 11$

m	$f(m)$	c_m	b_m	$m+q$	$f(m+q)$	c_{m+q}	b_{m+q}
0	$3\alpha p + q(p-4)$	0	0	q	$3\alpha p$	1	1
1	$2\alpha p + q(p-3)$	0	0	$q+1$	$2\alpha p + q$	1	1
2	$\alpha p + q(p-2)$	0	0	$q+2$	$\alpha p + 2q$	1	1
3	$q(p-1)$	0	0	$q+3$	$3q$	0	0
4	$(q-\alpha)p$	0	0	$q+4$	$(q-\alpha)p + 4q - pq$	-1	0
5	$(q-2\alpha)p + q - pq$	0	0	$q+5$	$(q-2\alpha)p + 5q - pq$	-1	0
6	$(q-3\alpha)p + 2q - pq$	0	0	$q+6$	$(q-3\alpha)p + 6q - pq$	-1	0
7	$(q-4\alpha)p + 3q - pq$	-1	-1	$q+7$	$(q-4\alpha)p + 7q - pq$	-1	0
8	$(q-5\alpha)p + 4q - pq$	-1	-1	$q+8$	$(q-5\alpha)p + 8q - pq$	-1	0
9	$(q-6\alpha)p + 5q - pq$	-1	-1	$q+9$	$(q-6\alpha)p + 9q - pq$	-1	0
10	$(q-7\alpha)p + 6q - pq$	-1	-1	$q+10$	$(q-7\alpha)p + 10q - pq$	-1	0

3 General counter-example construction

3.1 The negative coefficient case

We now establish a more general counter-example construction. The approach will be similar to that of the previous section. For reasons of space the analogue of Table 4, Table 5, is split into two tables, for $f(m)$, respectively $f(m+q)$.

Table 5A: `General negative coefficient construction,` $f(m)$ `case`

m	$f(m)$	c_m	b_m
0	$(\sigma+1)\alpha p+(p-\beta)q$	0	0
1	$\sigma\alpha p+(p-\beta+1)q$	0	0
$\cdots$	$\cdots$	0	0
$\sigma+1$	$0\cdot p+(\sigma+1+p-\beta)q$	0	0
$\sigma+2$	$(q-\alpha)p+(\sigma+2+p-\beta)q-pq$	-1	0
$\cdots$	$\cdots$	-1	0
$\beta-1$	$(q-(\beta-\sigma-2)\alpha)p+(p-1)q-pq$	-1	0
β	$(q-(\beta-\sigma-1)\alpha)p+0\cdot q$	0	0
$\beta+1$	$(q-(\beta-\sigma)\alpha)p+1\cdot q-\delta_1 pq$	0	0
$\cdots$	$\cdots$	0	0
$\beta+\sigma$	$(q-(\beta-1)\alpha)p+\sigma q-\delta_\sigma pq$	0	0
$\beta+\sigma+1$	$(q-\beta\alpha)p+(\sigma+1)q-pq$	-1	-1
$\cdots$	$\cdots$	-1	-1
$\beta+k$	$(q-(\beta-\sigma-1+k)\alpha)p+kq-pq$	-1	-1
$\cdots$	$\cdots$	-1	-1
$p-1$	$(q-(p-\sigma-2)\alpha)p+(p-\beta-1)q-pq$	-1	-1

In Table 5A δ_j is the unique integer such that the corresponding entry is in the interval $[0, pq)$.

Table 5B: `General negative coefficient construction,` $f(m+q)$ `case`

$m+q$	$f(m+q)$	c_{m+q}	b_{m+q}
q	$(\sigma+1)\alpha p$	1	1
$q+1$	$\sigma\alpha p+q$	1	1
$\cdots$	$\cdots$	1	1
$q+\sigma$	$\alpha p+\sigma q$	1	1
$q+\sigma+1$	$0\cdot p+(\sigma+1)q$	0	0
$q+\sigma+2$	$(q-\alpha)p+(\sigma+2)q-pq$	-1	0
$\cdots$	$\cdots$	-1	0
$q+p-1$	$(q-(p-\sigma-2)\alpha)p+(p-1)q-pq$	-1	0

Lemma 3 *Let p be a prime. Let $1\le\beta\le(p-3)/2$. Let $q>p$ be a prime satisfying $q\equiv\beta(\mathrm{mod}\ p)$ and $r>q$ be a prime satisfying $qr\equiv-1(\mathrm{mod}\ p)$. Let $1\le\alpha\le q-1$ be the unique integer such that $pr\alpha\equiv1(\mathrm{mod}\ q)$. Put*

$$w_-=(p-\beta-1)q-(p-\sigma-2)\alpha p,$$

where ρ and σ are uniquely determined by $(p-1)(q-1)=\rho p+\sigma q,\ \rho,\sigma\ge0$. Suppose that

$$p\ge\beta+\sigma+2,\ \beta\ge\sigma+2$$

and

$$\alpha\le\frac{q(\sigma+1)}{p\beta},\tag{4}$$

$$\alpha \le \frac{q(p-1-\sigma)-(p-1)}{p(\sigma+1)}, \tag{5}$$

$$\alpha \le \frac{q(\sigma+1)-1}{p(p-\sigma-2)}, \tag{6}$$

$$\alpha \le \frac{q(p-\sigma-1-\beta)}{p(p-\sigma-1)}, \tag{7}$$

and

$$\alpha > \frac{q(p-3-\sigma-\beta)}{p(p-\sigma-3)}, \tag{8}$$

then $a_{pqr}(p-1+rw) = \beta - p$.

`Remark`. The conditions (4), (5), (6), (7) and (8) are used to ensure that respectively, $f(\beta+\sigma+1) \ge 0$, $(\sigma+1)\alpha \le \rho$, $q-(p-\sigma-2)\alpha \ge \rho+1$, $f(q+\sigma) \le f(p-1)$ and $f(q+\sigma+2) > f(p-1)$.

Proof of Lemma 3. Reasoning as in the proof of Theorem 9 we find that $f(m)$ and $f(m+q)$ are congruent modulo pq to the numbers given in Table 5A, respectively 5B.

In Table 5A we distinguish 3 ranges: $0 \le m \le \beta$, $\beta+1 \le m \le \beta+\sigma$ and $\beta+\sigma+1 \le m \le p-1$. In the first range the entries 0 up to β are non-negative and in ascending order. Since the entry for β is $< pq$ it follows that for $0 \le m \le \beta$, $f(m)$ is actually equal to the corresponding entry given in Table 5A. Since $(\sigma+1)\alpha \le \rho$ and $p-\beta \ge \sigma+1$ it follows by Lemma 1 that $c_m = 0$ (and hence $b_m = 0$) for $m = 0, \ldots, \sigma+1$. For $\sigma+2 \le m \le \beta-1$ one finds that $c_m = -1$. Since

$$f(\sigma+2) = f(q+\sigma+2) + (p-\beta)q > f(q+\sigma+2) > f(p-1),$$

we infer that $b_m = 0$ for $\sigma+2 \le m \le \beta-1$. Clearly $b_\beta = c_\beta = 0$.

In the second range we have, for $1 \le k \le \sigma$,

$$(q-(\beta-\sigma+k-1)\alpha)p + kq - \delta_k pq$$

as entry in row $\beta+k$, where a priori δ_k is an integer. Since

$$q-1 \ge q-(\beta-\sigma+k-1)\alpha \ge q-(p-\sigma-2)\alpha \ge \rho+1 > 0,$$

we find that $0 \le (q-\beta-\sigma+k-1)\alpha)p + kq < 2pq$ and hence $\delta_k \in \{0,1\}$. By Lemma 1 again we now find that $c_m = b_m = 0$ for $\beta+1 \le m \le \beta+\sigma$.

In the range $\beta+\sigma+1 \le m \le p-1$ the entries in Table 5A are in ascending order and the final entry is less than pq. Since $\alpha \le (\sigma+1)q/(\beta p)$, it follows that the $\beta+\sigma+1$ entry in the table is ≥ 0. It follows that $f(m)$ is actually equal to the corresponding entry in Table 5A. By Lemma 1 it now follows that $c_m = -1$ for $\beta+\sigma+1 \le m \le p-1$. Since $f(m) \le f(p-1)$ for $\beta+\sigma+1 \le m \le p-1$ we infer that also $b_m = -1$ in this range.

Establishing the correctness of the $f(m+q)$ and c_{m+q} column is straightforward and left to the reader. Note that once we have $f(q+\sigma) \le f(p-1)$ and

$f(q+\sigma+2) > f(p-1)$, the b_{m+q} column is as given in the table. That these inequalities hold is ensured by conditions (7), respectively (8). Finally on applying Lemma 1 we infer that

$$a_{pqr}(p-1+rw_-) = -\sum_{j=\beta+\sigma+1}^{p-1} 1 - \sum_{j=q}^{q+\sigma} 1 = -(p-1-\beta-\sigma) - (\sigma+1) = \beta - p.$$

This concludes the proof. □

In the next lemma the set of *real* numbers α satisfying (4), (5), (6), (7) and (8) is determined.

Lemma 4 *Let $\mathcal{I}$ be the set of real numbers satisfying* (4), (5), (6), (7) and (8) *and suppose the conditions of Lemma* 3 *are satisfied. Then the set $\mathcal{I}$ is non-empty iff $p \le \beta + 2\beta^* + 1$. In that case*

$$\mathcal{I} = \begin{cases} \left(\frac{q}{p}\left(1 - \frac{\beta}{p-\beta^*-2}\right), \frac{q}{p}\left(1 - \frac{\beta}{p-\beta^*}\right)\right] & \text{if } p < \beta + 2\beta^* + 1; \\ \left(\frac{q}{p}\left(1 - \frac{\beta}{p-\beta^*-2}\right), \frac{q\beta^*-1}{p(p-\beta^*-1)}\right] & \text{if } p = \beta + 2\beta^* + 1. \end{cases}$$

If $\mathcal{I}$ is non-empty then it consists of positive reals only.

Proof. From $(\rho+1)p + (\sigma+1)q = pq + 1$ we infer that $(\sigma+1)q \equiv 1 (\mathrm{mod}\ p)$ and hence $\sigma \equiv 1/q - 1 (\mathrm{mod}\ p)$. Since $q \equiv \beta (\mathrm{mod}\ p)$, we infer that $\sigma = \beta^* - 1$. Note that if (6) is satisfied then, since $p - \sigma - 2 \ge \beta$, automatically (4) is satisfied. Note that if α satisfies (6), then

$$\alpha \le \frac{q(\sigma+1)}{p(p-\sigma-3)}.$$

Now if α is also to satisfy (8), then we must have $\sigma + 1 > p - 3 - \sigma - \beta$ and so $p \le 2\sigma + 3 + \beta$. Thus if $p > \beta + 2\beta^* + 1$, then $\mathcal{I}$ is empty and hence we may assume that $p \le \beta + 2\beta^* + 1$. By (7) and since $p \ge 2\sigma + 3 = 2\beta^* + 1$ we infer that

$$\alpha \le \frac{q}{p} \le \frac{q(p-2-\sigma)}{p(\sigma+1)} \le \frac{q(p-1-\sigma)-(p-1)}{p(\sigma+1)},$$

and hence the condition (5) is also superfluous. Note that the inequality

$$\frac{q(p-\sigma-1-\beta)}{p(p-\sigma-1)} \le \frac{q(\sigma+1)-1}{p(p-\sigma-2)}$$

can be rewritten as

$$p \le \beta + 3 + 2\sigma - \frac{1}{q} - \frac{\beta}{p-\sigma-1}.$$

We observe that

$$0 < \frac{1}{q} + \frac{\beta}{p-\sigma-1} \le \frac{1}{q} + \frac{\beta}{\beta+1} \le \frac{1}{q} + 1 - \frac{1}{p} < 1.$$

It follows that if $p = \beta + 3 + 2\sigma = \beta + 2\beta^* + 1$, then (7) is redundant and if $p < \beta + 2\beta^* + 1$, then (6) is redundant. In the latter case we obtain that

$$\mathcal{I} = \Big(\frac{q}{p}\Big(1 - \frac{\beta}{p - \beta^* - 2}\Big), \frac{q}{p}\Big(1 - \frac{\beta}{p - \beta^*}\Big)\Big],$$

a clearly non-empty interval. In the former case we obtain

$$\mathcal{I} = \Big(\frac{q}{p}\Big(1 - \frac{\beta}{p - \beta^* - 2}\Big), \frac{q\beta^* - 1}{p(p - \beta^* - 1)}\Big)\Big],$$

in which case an easy calculation shows that it is a non-empty interval. Since $\beta + \beta^* \leq 2\beta - 1 \leq p - 4$ it follows that $p - \beta^* - 2 > \beta$ and thus if $\mathcal{I}$ is non-empty, it contains positive reals only. □

Theorem 10 *Suppose that $\mathcal{B}_-(p)$ is non-empty and $\beta \in \mathcal{B}_-(p)$. Suppose also that $p < \beta + 2\beta^* + 1$. Let $q > p$ be a prime satisfying $q \equiv \beta (\mathrm{mod}\ p)$ and $q \geq q_-(p)$ with $q_-(p) = p(p - \beta^*)(p - \beta^* - 2)/(2\beta)$. Then the interval*

$$\mathcal{I} = \Big(\frac{q}{p}\Big(1 - \frac{\beta}{p - \beta^* - 2}\Big), \frac{q}{p}\Big(1 - \frac{\beta}{p - \beta^*}\Big)\Big]$$

contains at least one integer a. Let $r > q$ be a prime with $r(q - pa) \equiv -1 (\mathrm{mod}\ pq)$, then

$$a_{pqr}(p - 1 + [(p - \beta - 1)q - (p - \beta^* - 1)ap]r) = \beta - p < -\frac{(p+1)}{2}$$

is a counter-example to Beiter's conjecture.

In case $p = \beta + 2\beta^* + 1$ the same conclusion holds, but with $q_-(p)$ replaced by $(\beta + \beta^* - 1)(p(\beta + \beta^*) + 1)/\beta$ and $\mathcal{I}$ by

$$\mathcal{I} = \Big(\frac{q}{p}\Big(1 - \frac{\beta}{p - \beta^* - 2}\Big), \frac{q\beta^* - 1}{p(p - \beta^* - 1)}\Big].$$

Proof. Note that as a function of q the length of $\mathcal{I}$ is increasing. If the length of $\mathcal{I}$ is at least one, then it contains at least one positive integer. Now $q_+(p)$ is obtained on solving the equation $|\mathcal{I}| = 1$ for q. The proof is completed on invoking Lemma 3 and Lemma 4. □

Small counter-examples produced by this result are given in Table 8.

3.2 The positive coefficient construction

Since the method of proof in this section is similar to that in the previous section, some of the details will be suppressed.

Lemma 5 *Let p be a prime. Let $1 \leq \beta \leq (p - 3)/2$. Let $q > p$ be a prime satisfying $q \equiv \beta (\mathrm{mod}\ p)$ and $r > q$ be a prime satisfying $qr \equiv -1 (\mathrm{mod}\ p)$. Let $1 \leq \alpha \leq q - 1$ be the unique integer such that $pr\alpha \equiv 1 (\mathrm{mod}\ q)$. Put*

$$w_+ = 1 + (p - \beta - 1)q - (p - \beta)\alpha p,$$

where ρ and σ are uniquely determined by $(p-1)(q-1) = \rho p + \sigma q,\ \rho, \sigma \geq 0$. Suppose that

$$\beta + \sigma \geq p - 1$$

and

$$\alpha < \frac{(\sigma+1)q - 1}{p(\beta - 1)}, \tag{9}$$

$$\alpha \leq \frac{q(p - \sigma - 1)}{p(p - \beta)}, \tag{10}$$

$$\alpha \leq \frac{q(\sigma + 1 - \beta)}{p(\sigma + 1)}, \tag{11}$$

and

$$\alpha > \frac{q(p - 2\beta - 1)}{p(p - \beta - 1)}, \tag{12}$$

then $a_{pqr}(p - 1 + rw_+) = p - \beta$.

`Remark`. The conditions (9), (10), (11) and (12) are used to ensure that respectively, $\rho + 1 + (\beta - 1)\alpha \leq q - 1$, $\rho + 1 - (p - \beta)\alpha \geq 0$, $f(q + p - \sigma - 2) \leq f(p - 1)$ and $f(q + \beta) > f(p - 1)$.

Proof of Lemma 5. The proof is a more general variant of the proof of Theorem 9. Again we make use of tables in the proof.

Table 6A: `General positive coefficient construction:` $f(m)$ `case`

m	$f(m)$	c_m	b_m
0	$(\rho + 1 + (\beta - 1)\alpha)p + (\sigma - \beta + 1)q - \tau_0 pq$	0	0
$\cdots$	$\cdots$	0	0
$\beta - 2$	$(\rho + 1 + \alpha)p + (\sigma - 1)q - \tau_{\beta-2} pq$	0	0
$\beta - 1$	$(\rho + 1)p + \sigma q - \tau_{\beta-1} pq$	0	0
β	$(\rho + 1 - \alpha)p + (\sigma + 1)q - \tau_\beta pq$	0	0
$\cdots$	$\cdots$	0	0
$p - \sigma + \beta - 2$	$(\rho + 1 - (p - \sigma - 1)\alpha)p + (p - 1)q - \tau_{p-\sigma+\beta-2} pq$	0	0
$p - \sigma + \beta - 1$	$(\rho + 1 - (p - \sigma)\alpha)p$	1	1
$p - \sigma + \beta$	$(\rho + 1 - (p - \sigma + 1)\alpha)p + q$	1	1
$\cdots$	$\cdots$	1	1
$p - 1$	$(\rho + 1 - (p - \beta)\alpha)p + (\sigma - \beta)q$	1	1

In Table 5A τ_j is the unique integer in $\{0, 1\}$ such that the corresponding entry for $f(m)$ is in the interval $[0, pq)$.

Table 6B: General positive coefficient construction: $f(m+q)$ case

$m+q$	$f(m+q)$	c_{m+q}	b_{m+q}
q	$(\rho+1+(\beta-1)\alpha)p+(\sigma+1)q-pq$	-1	-1
$\cdots$	$\cdots$	-1	-1
$q+p-\sigma-2$	$(\rho+1+(\beta+\sigma+1-p)\alpha)p+(p-1)q-pq$	-1	-1
$q+p-\sigma-1$	$(\rho+1+(\beta+\sigma-p)\alpha)p$	0	0
$\cdots$	$\cdots$	0	0
$q+\beta-1$	$(\rho+1)p+(\beta+\sigma-p)q$	0	0
$q+\beta$	$(\rho+1-\alpha)p+(\beta+\sigma+1-p)q$	1	0
$\cdots$	$\cdots$	1	0
$q+p-1$	$(\rho+1-(p-\beta)\alpha)p+\sigma q$	1	0

Since $(\rho+1)p+(\sigma+1)q = qp+1$ we can rewrite w_+ as

$$w_+ = (\rho+1-(p-\beta)\alpha)p+(\sigma-\beta)q.$$

The condition (10) ensures that $\rho+1-(p-\beta)\alpha \geq 0$. From $\beta \leq (p-3)/2$ and $\beta+\sigma \geq p-1$ we infer that $\sigma \geq \beta+2$. It follows that

$$0 \leq w_+ = (p-1)q+1-(p-\beta)\alpha p-\beta q < pq.$$

Thus $f(p-1) = w_+$. The condition (9) ensures that $\rho+1+(\beta-1)\alpha \leq q-1$. Let $0 \leq r_1 < pq$ be the unique integer with $r_1 \equiv -\frac{1}{r} (\mathrm{mod}\ pq)$. It is easy to see that $r_1 = q-\alpha p$. We have $f(m) \equiv w_+ + (m-p+1)r_1 (\mathrm{mod}\ pq)$. Using these observations one arrives at Table 6A. For $m \leq p-\sigma+\beta-2$ we do not care about whether $\tau_m = 0$ or $\tau_m = 1$; in either case we find $b_m = 0$ and hence $c_m = 0$.

On noting that $f(m+q) \equiv f(m)+\beta q (\mathrm{mod}\ pq)$ (cf. the proof of Theorem 9), we easily infer that the $f(m+q)$ are as given in Table 6B, with the caveat that the entries from $q+p-\sigma-1$ to $q+\beta-1$ do not arise if $\beta+\sigma = p-1$. Using that $f(q+p-\sigma-2) \leq f(p-1)$ and $f(q+\beta) > f(p-1)$ (a consequence of α satisfying (11), respectively (12))), we deduce that the b_{m+q} and c_{m+q} columns are as given in Table 6B. Finally on applying Lemma 1 we infer that

$$a_{pqr}(p-1+rw_+) = \sum_{j=p-\sigma+\beta-1}^{p-1} 1 + \sum_{j=q}^{q+p-\sigma-2} 1 = (\sigma-\beta+1)+(p-\sigma-1) = p-\beta.$$

This concludes the proof. □

Lemma 6 *Let $\mathcal{I}$ be the set of real numbers satisfying* (9), (10), (11), (12) *and suppose the conditions of Lemma* 5 *are satisfied. Put*

$$\gamma = \min\Big\{\frac{p-\beta^*}{p-\beta}, \frac{\beta^*-\beta}{\beta^*}\Big\}.$$

The set $\mathcal{I}$ is non-empty iff $\beta^* \leq 2\beta$. In that case

$$\mathcal{I} = \Big(\frac{q(p-1-2\beta)}{p(p-1-\beta)}, \frac{q\gamma}{p}\Big].$$

If $\mathcal{I}$ is non-empty then it consists of positive reals only.

Proof. Left to the reader. □

`Remark`. Note that

$$\gamma = \begin{cases} \frac{p-\beta^*}{p-\beta} & \text{if } p < \beta + \frac{\beta^*}{\beta}(\beta^* - \beta); \\ \frac{\beta^*-\beta}{\beta^*} & \text{otherwise.} \end{cases}$$

On combining the latter two lemmas one obtains an explicit counter-example construction in the positive case.

Theorem 11 *Suppose that $\mathcal{B}_+(p)$ is non-empty and $\beta \in \mathcal{B}_+(p)$. Let $q > p$ be a prime satisfying $q \equiv \beta (\mathrm{mod}\ p)$ and $q \geq q_+(p)$ with*

$$q_+(p) = \frac{p(p-1-\beta)}{\gamma(p-1-\beta) - p + 1 + 2\beta}.$$

Then the interval

$$\mathcal{I} = \Big[\frac{q(p-1-2\beta)}{p(p-1-\beta)}, \frac{q\gamma}{p}\Big]$$

contains at least one integer a. Let $r > q$ be a prime with $r(q-pa) \equiv -1 (\mathrm{mod}\ pq)$, then

$$a_{pqr}(p-1+[(p-\beta-1)q - (p-\beta)ap]r) = p - \beta > \frac{(p+1)}{2}$$

is a counter-example to Beiter's conjecture.

Proof. Note that as a function of q the length of $\mathcal{I}$ is increasing. If the length of $\mathcal{I}$ is at least one, then it contains at least one positive integer. Now $q_+(p)$ is obtained on solving the equation $|\mathcal{I}| = 1$ for q. The proof is completed on combining Lemma 5 and Lemma 6. □

Small counter-examples produced by this result are given in Table 7.

4 The proofs of Theorem 4 and Theorem 6

As is well-known the distribution of inverses modulo p can be studied by connecting this problem to Kloosterman sums and estimates for those. For us, the following typical lemma, see e.g. Cobeli [6, Lemma 4, Chapter 3.2], will do.

Let $\mathcal{I} = \{a, a+r, \ldots, a+(M-1)r\} \subset [1,p]$ and put

$$N(\mathcal{I}_1, \mathcal{I}_2; p) = \#\{(x,y) : x \in \mathcal{I}_1,\ y \in \mathcal{I}_2,\ xy \equiv 1 (\mathrm{mod}\ p)\}.$$

Lemma 7 *Let p be a prime number. Suppose $|\mathcal{I}_1| \leq p$ and $|\mathcal{I}_2| \leq p$. Then*

$$\Big|N(\mathcal{I}_1, \mathcal{I}_2; p) - \frac{|\mathcal{I}_1| \cdot |\mathcal{I}_2|}{p}\Big| \leq \sqrt{p}\ (2 + \log p)^2.$$

The set of points (x,y) with $xy \equiv d (\mathrm{mod}\ p)$ is called a *modular hyperbola*. For a survey of this area of study see e.g. Shparlinski [14].

Proof of Theorem 6. Bachman's upper bound $p-\lceil p/4\rceil$ shows that $M(p)\le 3p/4$.

Suppose that for $p\ge 29$ and some $0<\epsilon<1/6$ we have

$$\frac{p}{3}(1+\epsilon)\le\beta\le\frac{p}{3}(1+2\epsilon),\ \frac{2p}{3}(1-\epsilon)\le\beta^*\le\frac{2p}{3}(1+\epsilon). \tag{13}$$

Then one checks that $\beta\in\mathcal{B}_+(p)$. It then follows by Theorem 4 that $M(p)\ge 2p(1-\epsilon)/3$. It only remains to show that for every p sufficiently large there is a β satisfying (13). This follows on invoking Lemma 7 with $\mathcal{I}_1$ the integers in the interval $[\frac{p}{3}(1+\epsilon),\frac{p}{3}(1+2\epsilon)]$ and $\mathcal{I}_2$ the integers in the range $[\frac{2p}{3}(1-\epsilon),\frac{2p}{3}(1+\epsilon)]$. This completes the proof. □

Proposition 3 *Let $\epsilon>0$. There are infinitely many primes p such that there exist primes q and r so that*

$$A(pqr)=\min(\frac{p-1}{2}+a,p-a)\ge(\frac{2}{3}-\epsilon)p,$$

with a as in Theorem 2.

Proof. Suppose that

$$p\ge 29,\ 0<\epsilon<\frac{1}{9},\ \beta\in\mathcal{B}_+(p),\ \beta<\frac{p}{3}(1+3\epsilon),\ \beta+\beta^*=p, \tag{14}$$

then by Theorem 4 we obtain $a_{pqr_+}(n_+)=p-\beta\ge(\frac{2}{3}-\epsilon)p$. In this case also Bachman's upper bound given in Theorem 2 gives $A(pqr_+)=\beta^*=p-\beta$. To see this note that $r^*=p-\beta$ and $q^*=\beta^*$. Since $\beta<\beta^*$ we infer that $a=p-\beta^*$. By Theorem 2 it then follows that $A(pqr_+)=\min(\frac{p-1}{2}+p-\beta^*,\beta^*)\le\beta^*$. Since $a_{pqr_+}(n_+)=\beta^*$, it follows that $A(pqr_+)=\beta^*$ and thus Bachman's upper bound is assumed.

Duke et al. [7] proved that if f is a quadratic polynomial with complex roots, $0\le a<b\le 1$, then $\#\{(p,\nu):p\le x,\ f(\nu)\equiv 0(\mathrm{mod}\ p),\ a\le\nu/p\}<b\}\sim(b-a)\pi(x)$, where $\pi(x)$ denotes the number of primes $p\le x$. In particular it follows that there are asymptotically $\epsilon\pi(x)/3$ primes p for which there exist x satisfying $x+x^*=p$ (that is $x^2+1\equiv 0(\mathrm{mod}\ p)$), and $p/3<x<p(1+3\epsilon)/3$. On putting $x=\beta$ we then see that $\beta\in\mathcal{B}_+(p)$ and thus it follows that there exists infinitely many primes p for which there is a β satisfying (14). □

The final result in this section shows that by elementary methods one can easily prove that $M(p)\ge(\frac{2}{3}-\epsilon)p$ for infinitely many primes p.

Proposition 4 *Let $\epsilon>0$, $e\ge 1$ be an integer, $N=2^{2e+1}$ and p a prime satisfying $p\equiv N-9(\mathrm{mod}\ 3N)$. If $p>\frac{N^2}{2}-9$, then*

$$M(p)\ge\frac{(2N-1)p-9}{3N}>\Big(\frac{2}{3}-\frac{N}{3(N^2-18)}\Big)p.$$

If $p>\frac{N^2}{2}-9$ and $N>\frac{1}{3\epsilon}+3$, then $M(p)>(\frac{2}{3}-\epsilon)p$.

Proof. Consider $\beta=((N+1)p+9)/(3N)$. Then $\beta^*=(2p+N)/3$. One checks that if $p>N^2/2-9$, then $\beta^*/2<\beta$ and finally that $\beta\in\mathcal{B}_+(p)$. Then invoke Theorem 4. □

Examples.
If $p \equiv 23 (\mathrm{mod}\ 24)$ and $p > 23$ then $M(p) \geq (5p-3)/8 > 0.608p$.
If $p \equiv 23 (\mathrm{mod}\ 96)$ and $p > 503$ then $M(p) \geq (21p-3)/32 > 0.656p$.
If $p \equiv 119 (\mathrm{mod}\ 384)$ and $p > 8183$ then $M(p) \geq (85p-3)/128 > 0.664p$.
If $p \equiv 503 (\mathrm{mod}\ 1536)$ and $p > 131063$ then $M(p) \geq (341p-3)/512 > 0.666p$.

Proof of Theorem 4. Follows on combining Theorem 10 and Theorem 11. Theorem 11 together with Theorem 2 yields that $A(pqr_+) = p - \beta$ if $\beta \in \mathcal{B}_+(p)$ and $\beta + \beta^* = p$ (cf. the proof of Proposition 3). □

5 Reciprocal cyclotomic polynomials

We point out that for the so called reciprocal cyclotomic polynomials the analogue of Beiter's conjecture is known. Let

$$\frac{1}{\Phi_n(x)} = \sum_{k=0}^{\infty} c_n(k) x^k$$

be the Taylor series of $1/\Phi_n(x)$ around $x = 0$. The coefficients turn out to be periodic with period dividing n. Moree [13] established the following result concerning the height, $H(n)$, of $1/\Phi_n(x)$ (thus $\max_{k \geq 0} |c_n(k| = H(n)$).

Theorem 12 *Let $p < q < r$ be odd primes. Then $H(pqr) = p - 1$ iff*

$$q \equiv r \equiv \pm 1 (\mathrm{mod}\ p) \text{ and } r < \frac{(p-1)}{(p-2)}(q-1).$$

In the remaining cases $H(pqr) < p - 1$.

This result in combination with Dirichlet's theorem on arithmetic progressions shows that for every odd prime p there are infinitely many pairs (q, r) such that $H(pqr) = p - 1$.

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Table 7: The set $\mathcal{B}_+(p)$, all β and β^* included, smallest q for which α exists in the interval $\mathcal{I}$ of Theorem 11 and smallest r.

p	β	β^*	q	α	r	n	$a_{pqr}(n)$
13	5	8	83	2	2423	906214	8
19	8	12	179	2	32611	44774921	11
23	9	18	331	5	161741	435730276	14
29	12	17	331	3	5783	22079522	17
31	14	20	107	1	5281	6263296	17
37	17	24	239	1	61507	233849650	20
41	17	29	263	2	35149	143478258	24
43	19	34	191	1	4051	13619504	24
	20	28	751	2	144323	2099178077	23
47	18	34	347	3	444539	2501865538	29
	20	40	1289	7	1015207	25006578870	27
53	22	41	181	1	5471	20724200	31
	23	30	977	4	208207	4575140670	30
	24	42	607	2	89899	1251663829	29
59	27	35	263	1	89527	560976240	32
61	27	52	271	1	141773	973980570	34
	29	40	761	1	45161	977284100	32
67	26	49	1031	6	188891	4676752335	41
	30	38	1303	4	4930693	182401126215	37
	32	44	367	1	279413	2831571408	35
71	27	50	311	2	119159	849127104	44
	29	49	739	4	376393	6915092266	42
	30	45	811	3	157433	3732421634	41
	31	55	457	2	146063	1773789142	40
73	27	46	3823	21	3233051	328212871490	46
	30	56	1709	7	3510587	174848296194	43
	34	58	691	1	711181	16650169644	39
	35	48	619	1	2069	41649042	38

Table 8: The set $\mathcal{B}_{-}(p)$, all β and β^* included, smallest q for which a exists in the interval $\mathcal{I}$ of Theorem 10 and smallest r.

p	β	β^*	q	a	r	n	$a_{pqr}(n)$
11	4	3	59	2	877	175410	-7
17	7	5	41	1	29	5294	-10
23	10	7	79	1	5743	3463051	-13
29	13	9	419	5	39779	140419898	-16
31	13	12	199	2	72497	164350729	-18
41	18	16	797	5	79319	1000529906	-23
	19	13	593	4	359311	2883470815	-22
43	18	12	1093	10	924101	12320114574	-25
47	22	15	163	1	136379	334810491	-25
53	25	17	1297	7	994369	21909926598	-28
59	23	18	967	7	697303	12080774533	-36
	26	25	557	2	900551	12544675488	-33
	28	19	1031	5	28183	547454833	-31
61	25	22	1123	6	1021301	25937981557	-36
	28	24	577	2	200971	2828063972	-33
71	32	20	1381	7	4008001	110733051698	-39
	33	28	317	1	5581	48817077	-38
	34	23	1951	8	296473	12908434490	-37
73	33	31	1201	3	1668019	63151199412	-40

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