# Quiver[1] representations and moduli spaces of sheaves on $\mathbf{P}^2$

**S. Kuleshov.**

## 1. Introduction.

Representations of the quivers with two vertices are used for geometrical description of moduli spaces of stable sheaves on the projective plane. These representations are called Kronecker moduli ([1],[2]). Unfortunately, application of this fine construction is bounded by the condition: stable sheaves belong to a subcategory generated by an exceptional pair.

Papers [3],[4],[5] are some generalization of the Kronecker moduli method. In these papers moduli spaces of sheaves on $\mathbb{P}^2$ are realized as a closed subset in relative Grassman's variety over the Kronecker moduli space. But these works do not describe the general situation.

On the other hand, it was proved in ([6]), that the derived category of coherent sheaves on $\mathbb{P}^2$ is equivalent to the derived category of representations of 3 vertices quiver with relations. This gives a hope that there exists a universal construction of geometrical description of stable sheaves moduli on $\mathbb{P}^2$. This preprint deals with this general construction.

The idea is the following. Let $(E_0, E_1, E_2)$ be a full exceptional collection on $\mathbb{P}^2$ (for example, $(\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}))$ and $X$ be a sheaf such that

$$\mu(E_0) < \mu(X) < \mu(E_2), \tag{1.1}$$

then $X$ is the cohomology of the monad

$$M(X): \qquad 0 \longrightarrow U_0 \otimes E_0 \xrightarrow{\alpha} U_1 \otimes E_1 \xrightarrow{\beta} U_2 \otimes E_2 \longrightarrow 0, \tag{1.2}$$

where $U_0, U_1, U_2$ are finite dimension vector spaces such that dimensions of $U_i$'s depend on the topological type of $X$ and the bundles $E_j$'s only (see proposition 3.1).

Moreover, the correspondence between the the stable sheaves $X$'s and monads $M(X)$ is functorial,. i.e., any morphism of stable sheaves $\varphi : X \longrightarrow X'$, where the slopes of $X$ and $X'$ satisfy condition (1.1), induces linear maps $\varphi_i : U_i \longrightarrow U_i'$ such that the following diagram is commutative[2].

$$\begin{array}{lccccccccc}
M(X): & 0 & \longrightarrow & U_0 \otimes E_0 & \xrightarrow{\alpha} & U_1 \otimes E_1 & \xrightarrow{\beta} & U_2 \otimes E_2 & \longrightarrow & 0 \\
 & & & \downarrow {\scriptstyle \varphi_0 \otimes id} & & \downarrow {\scriptstyle \varphi_1 \otimes id} & & \downarrow {\scriptstyle \varphi_2 \otimes id} & & \\
M(X'): & 0 & \longrightarrow & U_0' \otimes E_0 & \xrightarrow{\alpha'} & U_1' \otimes E_1 & \xrightarrow{\beta'} & U_2' \otimes E_2 & \longrightarrow & 0
\end{array}$$

And vise verse, any morphism of monads induces a morphism of their cohomologies.

The morphism $\alpha$ from (1.2) is determined by a tensor $a \in U_0^* \otimes U_1 \otimes H_{01}$, and $\beta$ is determined by $b \in U_1^* \otimes U_2 \otimes H_{12}$, where $H_{ij}$ is the vector space $\mathrm{Hom}(E_i, E_j)$. In such a way, an element $(a, b) \in R = U_0^* \otimes U_1 \otimes H_{01} \bigoplus U_1^* \otimes U_2 \otimes H_{12}$ corresponds to monad (1.2). We see that elements $(a, b)$ and $(a'b')$ correspond to isomorphic monads iff they belong to one orbit the group $G = \mathrm{GL}(U_0) \times \mathrm{GL}(U_1) \times \mathrm{GL}(U_2)$ action.

---

[1]This work was started at the Independent University of Moscow and was done during the author‘s stay at the Max Plank Institute für Mathematik in 1998.

[2]The collection of morphisms $\varphi_i$'s, inducing the commutative diagram is called a morphism of monads.

Further, a vector $(a,b) \in R$ determines a pair of linear maps

$$U_0 \xrightarrow{a} U_1 \otimes H_{01}, \qquad U_1 \xrightarrow{b} U_2 \otimes H_{12}.$$

These maps are a representation of a quiver $Q$ with 3 vertices and without relations. The quiver $Q$ has $h_{01}$ arrows from the zero vertex to the first and $h_{12}$ arrows from the first one to the second.

The moduli spaces of quiver representations was studded by A. King in [7]. By this work, to construct quotient $R/G$ we need to extend the group $G$ action to the trivial bundle $R \times \mathbb{C}$ with help of a character $\chi_\theta$ of the group. After this operation the moduli space of semistable representations (w.r.t $\chi_\theta$ ) $\mathfrak{N}(h_{01}, h_{12}; u_0, u_1, u_2)$ is defined as the projective spectrum of relative invariants ring.

In the preprint we show that a stable quiver representation corresponds to a stable sheaf $X$ on $\mathbb{P}^2$. Moreover, the description of moduli space of stable sheaves has the form.

**1.1. Theorem.** *Let $(E_0, E_1, E_2)$ be a full exceptional collection on $\mathbb{P}^2$ such that $-2 \le \mu(E_0) < \mu(E_2) \le 0$. Suppose that integers $r,\ c_1,\ c_2$ satisfy the conditions:*

$$\mu = \frac{c_1}{r} \in (\mu(E_0), \mu(E_2)), \qquad (r, c_1) = 1.$$

*Then the moduli space $\mathfrak{M}^s(r, c_1, c_2)$ of stable sheaves $X$ on $\mathbb{P}^2$ with* $\mathrm{rk}\,(X) = r$*, $c_1(X) = c_1$, and $c_2(X) = c_2$ is isomorphic to closed subveriety $\mathfrak{Z}$ of the moduli space*

$$\mathfrak{N} = \mathfrak{N}^s(h_{01}, h_{12}; u_0, u_1, u_2)$$

*of stable representations of the quiver $Q$ with 3 vertices. The subveriety $\mathfrak{Z}$ is defined as the zero set of the section of the universal sheaf $\mathcal{U}_0^* \otimes \mathcal{U}_2 \otimes H_{02}$ on $\mathfrak{N}$.*

The author is grateful to A. Gorodentsev for useful discussions. I am thankful to Max-Plank-Institut für Mathematik for hospitality.

## 2. Notations and preliminaries.

Here we use the following notations.
$\mathcal{O}(n)$ is a line bundle on $\mathbb{P}^2$, in particular,
$\mathcal{O}$ is the structure sheaf of projective plane;
$F(n)$ is the tensor product $F \otimes \mathcal{O}(n)$.
For an exceptional collection $(E_0, E_1, E_2)$ on $\mathbb{P}^2$
$r_i$ is rank of bundle $E_i$;
$d_i$ is the first Chern class of $E_i$;
$\mu_i = \dfrac{d_i}{r_i}$ is the slope of $E_i$;
$H_{ij}$ is the vector space $\mathrm{Hom}(E_i, E_j)$;
$h_{ij} = \dim H_{ij}$.
We identify a bundle with the locally free sheaf of its local sections. The stability of sheaves we understand in the sense of Mumford – Takemoto.

**Exceptional bundles.**[3]

[3]The details about the exceptional bundles on $\mathbb{P}^2$ can be found in [8] and [9].

Recall that a sheaf $E$ on $\mathbb{P}^2$ is called *exceptional* if it is *simple and superrigid,* i.e.,

$$\mathrm{Hom}(E,E)=\mathbb{C} \qquad \text{and} \qquad \mathrm{Ext}^q(E,E)=0 \text{ for } q>0.$$

It is known that the exceptional sheaves on $\mathbb{P}^2$ are locally free and stable.

An ordered collection $(E_0,E_1,E_2)$ of exceptional bundles on $\mathbb{P}^2$ is called *exceptional collection* or exceptional triple if $\mathrm{Ext}^q(E_i,E_j)=0\ \forall q$ whenever $i>j$. It is known that such the collections on $\mathbb{P}^2$ are full, in other words, the minimal triangulated subcategory in the derived category of coherent sheaves on $\mathbb{P}^2$ $(D^b\mathbb{P}^2)$ containing the bundles of the collection coincide with $D^b\mathbb{P}^2$.

The braid group acts on the set of all exceptional triples on $\mathbb{P}^2$ by mutations of neighboring exceptional pairs. If $(E,F)$ is an exceptional pair, then its the left mutation is defined as $(L_EF,E)$, where $L_EF$ is the kernel of the canonical epimorphism

$$0\longrightarrow L_EF\longrightarrow E\otimes \mathrm{Hom}(E,F)\xrightarrow{\mathrm{lcan}} F\longrightarrow 0. \tag{2.1}$$

The canonical morphism lcan corresponds to the tensor $can\in \mathrm{Hom}(E,F)^*\otimes \mathrm{Hom}(E,F)$ such that the map $can\in \mathrm{End}(\mathrm{Hom}(E,F))$ is identical. The right mutation of the pair $(E,F)$ is $(F,R_FE)$, where $R_FE$ the cokernel of the canonical morphism

$$0\longrightarrow E\xrightarrow{\mathrm{rcan}} \mathrm{Hom}(E,F)^*\otimes F\longrightarrow R_FE\longrightarrow 0. \tag{2.2}$$

The morphism rcan corresponds the identical map $can\in \mathrm{End}(\mathrm{Hom}(E,F))$ too.

If $(E_0,E_1,E_2)$ is a full exceptional collection on $\mathbb{P}^2$, then

$$R_{E_2}R_{E_1}E_0=E_0(3) \qquad \text{and} \qquad L_{E_0}L_{E_1}E_2=E_2(-3).$$

Besides ranks $r_i$ of the bundles from the exceptional collection satisfy Markov's equation

$$r_0^2+r_1^2+r_2^2=3r_0r_1r_2. \tag{2.3}$$

Dimensions of the spaces $H_{ij}=\mathrm{Hom}(E_i,E_j)$ can be calculated by the following formulae

$$h_{01}=3r_2, \qquad h_{12}=3r_0, \qquad h_{02}=3\bar{r}_1=3(3r_0r_2-r_1). \tag{2.4}$$

Finally, the most important fact from the theory of exceptional bundles is the generalization of Beilinson spectral sequence to the case of arbitrary exceptional collection. For the projective plane this generalization has the form:

**2.1. Theorem.** *(Gorodentcev) Let $(E_0,E_1,E_2)$ be a full exceptional collection on $\mathbb{P}^2$. Let us define its the left dual one as $(F_0,F_1,F_2)$, where*

$$F_0=E_2, \qquad F_1=R_{E_2}E_1, \qquad F_2=E_0(3).$$

*Then for any coherent sheaf $X$ on $\mathbb{P}^2$ there exists a spectral sequence, converging to $X$ on the principal diagonal with the $\mathbf{E}_1$-part in the form*

$$\mathbf{E}_1^{pq}=\mathrm{Ext}^q(F_{-p},X)\otimes E_{2-p}.$$

**Quiver representations.** Hear we formulate some results of paper [7] for the 3 vertices quiver $Q$ with $h_{01}$ arrows from the zero vertex to the first one and $h_{12}$ arrows from the first vertex to the second one.

**Definition.** A pair of linear maps $(a, b)$

$$a : U_0 \longrightarrow U_1 \otimes H_{01} \qquad \text{and} \qquad b : U_1 \longrightarrow U_2 \otimes H_{12}$$

is called *the representation of the quiver* $Q$ with the dimension vector $(u_0, u_1, u_2)$, where

$$\dim H_{ij} = h_{ij}, \qquad u_i = \dim U_i.$$

We see that the representation is an element of the vector space

$$R = U_0^* \otimes U_1 \otimes H_{01} \bigoplus U_1^* \otimes U_2 \otimes H_{12}.$$

The representation $(a, b)$ is isomorphic to another one $(a', b')$ if there exists isomorphisms $g_i$'s of the vector spaces $U_i$'s such that the following diagrams are commutative

$$\begin{array}{ccc} U_0 & \xrightarrow{a} & U_1 \otimes H_{01} \\ g_0 \downarrow & & \downarrow g_1 \otimes id \\ U_0 & \xrightarrow{a'} & U_1 \otimes H_{01} \end{array} \qquad \begin{array}{ccc} U_1 & \xrightarrow{b} & U_2 \otimes H_{12} \\ g_1 \downarrow & & \downarrow g_2 \otimes id \\ U_1 & \xrightarrow{b'} & U_2 \otimes H_{12} \end{array}.$$

Thus the group

$$G = \mathrm{GL}(U_0) \times \mathrm{GL}(U_1) \times \mathrm{GL}(U_2).$$

acts on the vector space $R$.

To construct moduli space of the quiver representations let us choose a character $\chi$ of $G$ ($\chi : G \longrightarrow \mathbb{C}^*$) such that the restriction of $\chi$ to the kernel of the action $\Delta$ is trivial. Let us extend the action of $G$ to the trivial bundle $R \times \mathbb{C}$ by the following rule $g \cdot (v, z) = (g \cdot v, \chi^{-1}(g) z)$. Recall that a function $f \in \mathbb{C}[R]$ is called a relative invariant of weight $\chi$ if $f(g \cdot v) = \chi(g) f(v)$ $\forall g \in G$ and $v \in R$. The set of all such invariants is denoted by $\mathbb{C}[R]^{G,\chi}$.

**Definition.** i) A vector $v \in R$ is called $\chi$-semistable, if there is a relative invariant $f \in \mathbb{C}[R]^{G,\chi^n}$ with $n \geq 1$ such that $f(v) \neq 0$.

ii) A vector $v \in R$ is $\chi$-stable if there is a relative invariant $f \in \mathbb{C}[R]^{G,\chi^n}$ with $n \geq 1$ such that $f(v) \neq 0$ and, further, $\dim G \cdot v = \dim G/\Delta$ and the $G$-action on $\{v \in R | fv \neq 0\}$ is closed.

Hence the moduli space of the quiver representations can be described as

$$\mathrm{Proj}\Big(\bigoplus_{n \geq 0} \mathbb{C}[R]^{G,\chi^n}\Big).$$

It is easily shown, that in our case the kernel of $G$-action $\Delta$ consists of homotheties of the spaces $U_i$ with a common coefficient. A character of the group $G$ has the form

$$\chi(g_0, g_1, g_2) = \Big(\det g_0\Big)^{\theta_0} \Big(\det g_1\Big)^{\theta_1} \Big(\det g_2\Big)^{\theta_2},$$

where $\theta_i \in \mathbb{Z}$. Therefore a choice of a character $\chi$ such that $\chi|_\Delta = 1$ is equivalent to a choice of on additive function $\theta$ on $\mathbb{Z}^3$ satisfying the condition

$$\theta(u_0, u_1, u_2) = \sum_{i=0}^{2} \theta_i u_i = 0, \tag{2.5}$$

where $u_i = \dim U_i$. Such the functions are called polarizations. Furthermore, we assume that the notion $\chi_\theta$-stable is the same as $\theta$-stable.

The following numerical criterion was proved in [7].

**2.2. Theorem.** *A representation $(a, b)$ of the quiver is $\theta$-semistable, if for any its subrepresentation $a' : U'_0 \longrightarrow U'_1 \otimes H_{01}, \quad b' : U'_1 \longrightarrow U'_2 \otimes H_{12}$ the inequality $\sum \theta_i u'_i \geq 0$ holds (the strong inequality for $\theta$-stable representation).*

Below we will fix the polarization and thus will speak of stable representations w.r.t. this polarization, without specifying it each time. We denote by $\mathfrak{N}(h_{01}, h_{12}; u_0, u_1, u_2)$ the moduli space of the quiver representations with the dimension vector $(u_0, u_1, u_2)$.

It was noted in the paper [7] that the quasi-projective subveriety $\mathfrak{N}^s(h_{01}, h_{12}; u_0, u_1, u_2)$ consisting of only stable representation is *fine* whenever the dimension vector $(u_0, u_1, u_2)$ is primitive, i.e. the integers $u_0, u_1, u_2$ are coprime.

Moreover, if the system

$$\begin{cases} \sum \theta_i v_i = 0 \\ 0 < v_i \leq u_i \end{cases}$$

has no solutions except $(u_0, u_1, u_2)$, then $\mathfrak{N}^s(h_{01}, h_{12}; u_0, u_1, u_2) = \mathfrak{N}(h_{01}, h_{12}; u_0, u_1, u_2)$.

## 3. A map from sheaves moduli space to representations moduli space.

**3.1. Proposition.** *Let $(E_0, E_1, E_2)$ be a full exceptional collection on $\mathbb{P}^2$ and $X$ be a stable sheaf such that $\mu_0 = \mu(E_0) < \mu(X) < \mu(E_2) = \mu_2$. Then $X$ is the cohomology of the monad*

$$M(X): \qquad 0 \longrightarrow U_0 \otimes E_0 \xrightarrow{\alpha} U_1 \otimes E_1 \xrightarrow{\beta} U_2 \otimes E_2 \longrightarrow 0, \tag{3.1}$$

*where $U_0 = \mathrm{Ext}^1(E_0(3), X)$, $U_1 = \mathrm{Ext}^1(R_{E_2}E_1, X)$, $U_2 = \mathrm{Ext}^1(E_2, X)$.*

**Proof.** By theorem 2.1, there exists a spectral sequence converging to $X$ on the principal diagonal with $\mathbf{E_1}$-part in the form

$$\mathbf{E_1^{-2,q}} = \mathrm{Ext}^q(E_0(3), X) \otimes E_0, \ \mathbf{E_1^{-1,q}} = \mathrm{Ext}^q(R_{E_2}E_1, X) \otimes E_1, \ \mathbf{E_1^{0,q}} = \mathrm{Ext}^q(E_2, X) \otimes E_2.$$

It follows from the stability of exceptional bundles and the condition

$$\mu(F) - 3 < \mu(X) < \mu(F) \quad \text{for } F \in \{E_2, R_{E_2}E_1, E_0(3)\}$$

that $\mathrm{Ext}^q(F, X) = 0$ if $q \neq 1$. Hence the diagram of the $\mathbf{E_1}$-part has the form

| | | |
|---|---|---|
| $0$ | $0$ | $0$ |
| $\mathrm{Ext}^1(E_0(3), X) \otimes E_0$ | $\mathrm{Ext}^1(R_{E_2}E_1, X) \otimes E_1$ | $\mathrm{Ext}^1(E_2, X) \otimes E_2$ |
| $0$ | $0$ | $\mathbf{E_1^{0,0}} = 0$ |

We see that $\mathbf{E_2^{p,q}} = \mathbf{E_\infty^{p,q}}$. This completes the proof.

Taking into account the functorial properties of the spectral sequence, we obtain

**3.2. Corollary.** *A morphism of stable sheaves induces a morphism of monads.*

*Moreover, two stable sheaves are isomorphic iff corresponding monads are isomorphic.*

Now we note, that any sequence of morphisms $\alpha$ and $\beta$ from (3.1) determines a representation $(a, b)$ of the quiver $Q$ with dimension vector $(u_0, u_1, u_2)$:

$$U_0 \xrightarrow{a} U_1 \otimes H_{01} \qquad U_1 \xrightarrow{b} U_2 \otimes H_{12}.$$

Whence, for a stable sheaf $X$ satisfying the assumptions of proposition 3.1 we can construct a representation $(a_X, b_X)$ of the quiver $Q$.

Consider the following composition

$$\gamma : U_0 \xrightarrow{a} U_1 \otimes H_{01} \xrightarrow{b\otimes id} U_2 \otimes H_{12} \otimes H_{01} \xrightarrow{id\otimes\delta} U_2 \otimes H_{02},$$

where $\delta : H_{12} \otimes H_{01} \longrightarrow H_{02}$ is the composition map, which takes each tensor $f \otimes g$ to $f \circ g \in H_{02} = \mathrm{Hom}(E_0, E_2)$. We say that a representation $(a, b)$ has zero composition (or $(a, b)$ is representation with zero composition) whenever the composition $\gamma$ is trivial.

**3.3. Lemma.** *The sequence of sheaves morphisms (3.1) is a complex iff the corresponding representation $(a, b)$ has zero composition.*

**Proof.** We fix basses $\{e_i^j\}_{i=1,\dots,u_j}$ in the spaces $U_j$ $(j = 0, 1, 2)$. Then the morphisms $\alpha$ and $\beta$ can be wrote as matrices

$$A = (g_{kj})_{\substack{k=1,\dots,u_1 \\ j=1,\dots,u_0}}, \qquad B = (f_{ik})_{\substack{i=1,\dots,u_2 \\ k=1,\dots,u_1}},$$

where $g_{kj} \in H_{01}$, $f_{ik} \in H_{12}$. Therefore the morphisms composition $\beta \circ \alpha : U_0 \otimes E_0 \longrightarrow U_2 \otimes E_2$ has the form

$$B \cdot A = \left( \sum_{k=1}^{u_1} f_{ik} \circ g_{kj} \right)_{\substack{i=1,\dots,u_2 \\ j=1,\dots,u_0}}$$

Thus sequence (3.1) is a complex iff for each pair of induces $i$, $j$ the following relation $\sum_{k=1}^{u_1} f_{ik} \circ g_{kj} = 0$ holds.

On the other hand, the linear map $a$ takes the basis vector $e_j^0$ of $U_0$ to the tensor $\sum_{k=1}^{u_1} e_k^1 \otimes g_{kj}$. The linear map $b$ takes the basis vector $e_k^1$ of $U_1$ to $\sum_{i=1}^{u_2} e_i^2 \otimes f_{ik}$. Therefore the action of $\gamma$ to the vector $e_j^0$ has the form

$$e_j^0 \xrightarrow{a} \sum_{k=1}^{u_1} e_k^1 \otimes g_{kj} \xrightarrow{b\otimes id} \sum_{k,i} e_i^2 \otimes f_{ik} \otimes g_{kj} \xrightarrow{id\otimes\delta} \sum_{i=1}^{u_2} e_i^2 \otimes \sum_{k=1}^{u_1} f_{ik} \circ g_{kj}.$$

In other words, $\gamma = 0$ iff for any pair of $i$, $j$ the condition $\sum_{k=1}^{u_1} f_{ik} \circ g_{kj} = 0$ holds.

With this moment we fix the topological type of stable sheaf $X$ from proposition 3.1. That is we fix the vector $x = (r, c_1, c_1^2 - 2c_2)$ in $K_0(\mathbb{P}^2)$ (here $r$ is rank of sheaf, $c_1$, $c_2$ are Chern classes of it) and dimensions $u_i$ of the spaces $U_i$. Note that

$$r = u_1 r_1 - u_0 r_0 - u_2 r_2 > 0.$$

To construct the moduli space of the quiver representations, let us choose a polarization $\theta$, it is let us fix integers $\theta_i$ $(i = 0, 1, 2)$ satisfying condition (2.5). By definition put

$$\theta_0 = u_2 \bar{r}_1 - u_1 r_2 \quad \theta_1 = u_0 r_2 - u_2 r_0 \quad \theta_2 = u_1 r_0 - u_0 \bar{r}_1, \tag{3.2}$$

where $r_i = \mathrm{rk}\, E_i$ and $\bar{r}_1 = \mathrm{rk}\, R_{E_2} E_1$. Bellow we understand stability of representations as the stability w.r.t the polarization $\theta$ with $\theta_i$ fixed above.

By given $v = (v_0, v_1, v_2) \in \mathbb{Z}^3$ we define an element of $K_0(\mathbb{P}^2)$ by the rule

$$x(v) = x(v_0, v_1, v_2) = v_1 e_1 - v_0 e_0 - v_2 e_2,$$

where the vectors of $K_0(\mathbb{P}^2)$ $e_i$'s correspond to the exceptional bundles $E_i$'s. Besides we define a slope function $\mu$ on $K_0(\mathbb{P}^2)$, $\mu = \dfrac{r(x)}{c_1(x)}$. In these notations the following lemma holds

**3.4. Lemma.** *Suppose that $v = (v_0, v_1, v_2) \in \mathbb{Z}^3$ has nonzero coordinates and the inequality $v_1 r_1 - v_0 r_0 - v_2 r_2 > 0$ holds, where $r_i = \operatorname{rk} E_i$. Then*

$$\mu(x(v)) \geq \mu(x(u_0, u_1, u_2)) \qquad \Longleftrightarrow \qquad \sum_i \theta_i v_i \leq 0.$$

**Proof.** Since the functions $r$ and $c_1$ are additive on $K_0(\mathbb{P}^2)$, the inequality $\mu(x(v)) \geq \mu(x(u_0, u_1, u_2))$ is equivalent to

$$\frac{v_1 d_1 - v_0 d_0 - v_2 d_2}{v_1 r_1 - v_0 r_0 - v_2 r_2} \geq \frac{u_1 d_1 - u_0 d_0 - u_2 d_2}{u_1 r_1 - u_0 r_0 - u_2 r_2},$$

where $d_i = c_1(E_i)$. Because of the denominators of these fractions are positive, the direct calculations show that this inequality is equivalent to

$$v_0\Big(u_1 r_0 r_1 (\mu_1 - \mu_0) - u_2 r_0 r_2 (\mu_2 - \mu_0)\Big) +$$

$$v_1\Big(u_2 r_1 r_2 (\mu_2 - \mu_1) - u_0 r_0 r_1 (\mu_1 - \mu_0)\Big) +$$

$$v_2\Big(u_0 r_0 r_2 (\mu_2 - \mu_0) - u_1 r_1 r_2 (\mu_2 - \mu_1)\Big) \geq 0$$

(here $\mu_i = \mu(E_i)$). Recall that if $i < j$, the pair $(E_i, E_j)$ is exceptional. In particular, $\chi(E_j, E_i) = 0$ and

$$h_{ij} = \chi(E_i, E_j) - \chi(E_j, E_i).$$

We use Riemann – Roch formula for the projective plane,

$$\chi(E, F) = \operatorname{rk} E \cdot \operatorname{rk} F \left(1 + \frac{3}{2}(\mu(F) - \mu(E)) + q(E) + q(F) - \frac{c_1(E) c_1(F)}{\operatorname{rk} E \operatorname{rk} F}\right), \tag{3.3}$$

where $q(E) = \dfrac{c_1^2 - 2c_2}{2 \operatorname{rk} E}$ to obtain

$$\chi(E_i, E_j) - \chi(E_j, E_i) = 3 r_i r_j (\mu_j - \mu_i).$$

On the other hand, dimensions of the spaces $\operatorname{Hom}(E_i, E_j)$ for exceptional pairs satisfy the relations (2.4). Therefore,

$$3 r_0 r_1 (\mu_1 - \mu_0) = h_{01} = 3 r_2, \;\; 3 r_0 r_2 (\mu_2 - \mu_0) = h_{02} = \bar{r}_1, \;\; 3 r_1 r_2 (\mu_2 - \mu_1) = h_{12} = 3 r_0.$$

By the choice of the polarizations

$$u_1 r_0 r_1 (\mu_1 - \mu_0) - u_2 r_0 r_2 (\mu_2 - \mu_0) = u_1 r_2 - u_2 \bar{r}_1 = -\theta_0,$$

$$u_2 r_1 r_2 (\mu_2 - \mu_1) - u_0 r_0 r_1 (\mu_1 - \mu_0) = u_2 r_0 - u_0 r_2 = -\theta_1,$$

$$u_0 r_0 r_2 (\mu_2 - \mu_0) - u_1 r_1 r_2 (\mu_2 - \mu_1) = u_0 \bar{r}_1 - u_1 r_0 = -\theta_2.$$

This implies our lemma.

We need the following remark.

**3.5. Remark.** *The integers $\theta_i$, from (3.2), can be rewrote in the form*

$$\theta_0 = r_0 r(\mu_0 - \mu), \quad \theta_1 = r_1 r(\mu - \mu_1), \quad \theta_2 = r_2 r(\mu_2 - \mu),$$

*where $r$ and $\mu$ are the values of the rank function and the slope function on the vector $x(u_0, u_1, u_2)$ respectively.*

**Proof.** Using the formulae, obtaining in the previous proof, we get

$$\theta_0 = u_2 r_0 r_2(\mu_2 - \mu_0) - u_1 r_0 r_1(\mu_1 - \mu_0).$$

Consequently,

$$\theta_0 = u_2(d_2 r_0 - d_0 r_2) - u_1(d_1 r_0 - d_0 r_1) = d_0(u_1 r_1 - u_0 r_0 - u_2 r_2) - r_0(u_1 d_1 - u_2 d_2 - u_0 d_0).$$

But $u_1 r_1 - u_0 r_0 - u_2 r_2 = r$, and $\dfrac{u_1 d_1 - u_2 d_2 - u_0 d_0}{u_1 r_1 - u_0 r_0 - u_2 r_2} = \mu$. Therefore, $\theta_0 = r r_0(\mu_0 - \mu)$. Similarly we can obtain the formulae for $\theta_1$ and $\theta_2$.

**3.6. Proposition.** *Let $X$ be a stable sheaf on $\mathbb{P}^2$ with slop $\mu$ satisfying the condition $\mu_0 < \mu < \mu_2$. Then the corresponding quiver representation $(a_X, b_X)$ is stable too.*

**Proof.** Recall that $X$ is the cohomology of monad (3.1) (proposition 3.1). Therefore $(a_X, b_X)$ is the representation with zero composition (see 3.3), i.e. the maps composition

$$U_0 \xrightarrow{a_X} U_1 \otimes H_{01} \xrightarrow{b_X \otimes id} U_2 \otimes H_{12} \otimes H_{01} \xrightarrow{id \otimes \delta} U_2 \otimes H_{02}$$

is trivial. Suppose that the representation $(a_X, b_X)$ is no stable. Taking into account numerical criterion 2.2, we see that there exists a subrepresentation

$$a' : U_0' \longrightarrow U_1' \otimes H_{01} \qquad b' : U_1' \longrightarrow U_2' \otimes H_{12}$$

with dimension vector $(u_0', u_1', u_2')$ such that

$$\sum_i \theta_i u_i' \leq 0. \tag{3.4}$$

Consider the following commutative diagram

$$\begin{array}{ccccccc}
U_0 & \xrightarrow{a_X} & U_1 \otimes H_{01} & \xrightarrow{b_X \otimes id} & U_2 \otimes H_{12} \otimes H_{01} & \xrightarrow{id \otimes \delta} & U_2 \otimes H_{02} \\
\uparrow \varphi_0 & & \uparrow \varphi_1 \otimes id & & \uparrow \varphi_2 \otimes id & & \uparrow \varphi_2 \otimes id \\
U_0' & \xrightarrow{a'} & U_1' \otimes H_{01} & \xrightarrow{b' \otimes id} & U_2' \otimes H_{12} \otimes H_{01} & \xrightarrow{id \otimes \delta} & U_2' \otimes H_{02} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0 & & 0
\end{array}.$$

The diagram yields that $(a', b')$ has zero composition, i.e., the corresponding sequence of sheaves morphisms is a complex. Besides, the following diagram is commutative too.

$$\begin{array}{lccccccc}
M(X): & 0 \longrightarrow & U_0 \otimes E_0 & \xrightarrow{\alpha} & U_1 \otimes E_1 & \xrightarrow{\beta} & U_2 \otimes E_2 & \longrightarrow 0 \\
& & \uparrow \varphi_0 \otimes id & & \uparrow \varphi_1 \otimes id & & \uparrow \varphi_2 \otimes id & \\
& & U_0' \otimes E_0 & \xrightarrow{\alpha'} & U_1' \otimes E_1 & \xrightarrow{\beta'} & U_2' \otimes E_2 & \\
& & \uparrow & & \uparrow & & \uparrow & \\
& & 0 & & 0 & & 0 &
\end{array}.$$

Whence we have, $\alpha'$ is a monomorphism and the restriction of $\varphi_2 \otimes id$ to $B = \operatorname{im}\beta'$ is a monomorphism too, i.e., we get a monomorphism of monads

$$
\begin{array}{lccccccc}
M(X): & 0 \longrightarrow & U_0 \otimes E_0 & \xrightarrow{\alpha} & U_1 \otimes E_1 & \xrightarrow{\beta} & U_2 \otimes E_2 & \longrightarrow 0 \\
 & & \uparrow {\scriptstyle \varphi_0 \otimes id} & & \uparrow {\scriptstyle \varphi_1 \otimes id} & & \uparrow {\scriptstyle \varphi_2 \otimes id} & \\
M(X'): & 0 \longrightarrow & U_0' \otimes E_0 & \xrightarrow{\alpha'} & U_1' \otimes E_1 & \xrightarrow{\beta'} & B & \longrightarrow 0\ \cdot \\
 & & \uparrow & & \uparrow & & \uparrow & \\
 & & 0 & & 0 & & 0 &
\end{array}
\tag{3.5}
$$

Denote by $X'$ the cohomology of the monad $M(X')$. At first we show that inequality (3.4) implies the following

$$\mu(X') \geq \mu(X), \tag{3.6}$$

i.e.,

$$\frac{u_1'd_1 - d(B) - u_0'd_0}{u_1'r_1 - r(B) - u_0'r_0} \geq \frac{d}{r},$$

where $r(B)$ and $d(B)$ are rank and the first Chern class of the sheaf $B$, respectively.

It is easy to prove that this inequality is equivalent to

$$u_0'rr_0(\mu_0 - \mu) + u_1'rr_1(\mu - \mu_1) + rr(B)(\mu(B) - \mu) \leq 0.$$

Taking into account remark 3.5 and inequality (3.4), we see that to prove the last inequality it remains to check that

$$rr(B)(\mu(B) - \mu) \leq u_2'rr_2(\mu_2 - \mu).$$

Since $B$ is a subsheaf of the semistable bundle $U_2' \otimes E_2$, then $\mu(B) \leq \mu_2$, i.e.,

$$rr(B)(\mu(B) - \mu) \leq rr(B)(\mu_2 - \mu).$$

Furthermore, by the assumptions of the lemma, $\mu_2 - \mu > 0$, and $r(B) \leq u_2'r_2$. Hence,

$$rr(B)(\mu_2 - \mu) \leq u_2'rr_2(\mu_2 - \mu).$$

Thus inequality (3.6) is proved.

Commutative diagram (3.5) is divided into two diagrams with exact columns and rows

$$
\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \longrightarrow & \ker\beta/\ker\beta' & \longrightarrow & (U_1/U_1') \otimes E_1 & \longrightarrow & (U_2 \otimes E_2)/B & \longrightarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \longrightarrow & \ker\beta & \longrightarrow & U_1 \otimes E_1 & \xrightarrow{\beta} & U_2 \otimes E_2 & \longrightarrow 0 \\
 & \uparrow {\scriptstyle \varphi'} & & \uparrow {\scriptstyle \varphi_1 \otimes id} & & \uparrow {\scriptstyle \varphi_2 \otimes id} & \\
0 \longrightarrow & \ker\beta' & \longrightarrow & U_1' \otimes E_1 & \xrightarrow{\beta'} & B & \longrightarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & 0 & & 0 & & 0 &
\end{array}
$$

$$
\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \longrightarrow & (U_0/U_0') \otimes E_0 & \longrightarrow & \ker\beta/\ker\beta' & \longrightarrow & \operatorname{coker}\varphi & \longrightarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \longrightarrow & U_0 \otimes E_0 & \xrightarrow{\alpha} & \ker\beta & \longrightarrow & X & \longrightarrow 0\ . \\
 & \uparrow {\scriptstyle \varphi_0 \otimes id} & & \uparrow {\scriptstyle \varphi'} & & \uparrow {\scriptstyle \varphi} & \\
0 \longrightarrow & U_0' \otimes E_0 & \xrightarrow{\alpha'} & \ker\beta' & \longrightarrow & X' & \longrightarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & 0 & & 0 & & \ker\varphi &
\end{array}
$$

Besides, the sequence

$$0 \longrightarrow \ker\varphi \longrightarrow (U_0/U_0') \otimes E_0 \longrightarrow (\ker\beta/\ker\beta') \longrightarrow \operatorname{coker}\varphi \longrightarrow 0 \tag{3.7}$$

is exact too.

Suppose $\ker\varphi = 0$, then $X' \subset X$ and the inequality (3.6) contradicts the stability of the sheaf $X$. This contradiction proves the proposition in this case.

If $\ker\varphi \neq 0$, it follows from (3.7) and the stability of exceptional bundles that $\mu(\ker\varphi) \leq \mu_0$. By the assumption of the proposition, $\mu_0 < \mu(X)$. Combining these inequalities with (3.6), we get $\mu(\ker\varphi) < \mu(X')$. Hence, $\mu(\operatorname{coker}\varphi) > \mu(X') \geq \mu(X)$. On the other hand, $\operatorname{coker}\varphi$ is a subsheaf of the stable sheaf $X$. This contradiction proves the proposition.

Thus we have the set theoretical inclusion of the set of stable sheaves corresponding to the vector $x(u_0, u_1, u_2) \in K_0(\mathbb{P}^2)$ to the moduli space $\mathfrak{N} = \mathfrak{N}^s(h_{01}, h_{12}; u_0, u_1, u_2)$ of stable representations of the quiver $Q$. Note that there exists a compact coarse moduli space $\mathfrak{M}(x)$ of semistable in Gieseker sense sheaves on $\mathbb{P}^2$ with the vector $x$. To construct algebraic morphism $\mathfrak{M}(x) \longrightarrow \mathfrak{N}$ we need to recall the definition of coarse moduli space.

Denote by $\Sigma_x(S)$ the set of equivalence classes of families of semistable by Gieseker sheaves on $\mathbb{P}^2$ with the vector $x$ and base $S$, i.e.,

$$\Sigma_x(S) = \left\{ [\mathcal{X}] \mid \mathcal{X} \text{ is a sheaf on } S \times \mathbb{P}^2 : \begin{array}{l} \mathcal{X}|_{s\times\mathbb{P}^2} \text{ is a stable sheaf on } \mathbb{P}^2 \text{ with} \\ \text{the vector } x \in K_0(\mathbb{P}^2) \end{array} \right\}$$

In such a way, the contravariant functor

$$\Sigma_x : \mathcal{A}lg \longrightarrow \mathcal{E}ns$$

from the category of algebraic varieties to the set category is defined. An algebraic variety $\mathfrak{M}(x)$ is called the coarse moduli space for $\Sigma_x$, if the following conditions hold

1) there exists a natural transformation of contravariant functors

$$\Sigma_x \longrightarrow \operatorname{Hom}(\cdot, \mathfrak{M}(x)),$$

which is bijective for any (reduced) point;

2) for every algebraic variety $\mathfrak{N}$ and every natural transformation $\Sigma_x \longrightarrow \operatorname{Hom}(\cdot, \mathfrak{N})$ there is a unique algebraic mapping $f : \mathfrak{M}(x) \longrightarrow \mathfrak{N}$ such that the diagram

$$\begin{tikzcd} \Sigma_x \arrow[rr] \arrow[dr] & & \operatorname{Hom}(\cdot, \mathfrak{M}(x)) \arrow[dl, "f_*"] \\ & \operatorname{Hom}(\cdot, \mathfrak{N}) & \end{tikzcd}$$

is commutative.

**3.7. Proposition.** *Let $x$ be a vector from $K_0(\mathbb{P}^2)$ such that $r(x)$ and $c_1(x)$ are coprime, $\mu(x) \in (\mu_0, \mu_2)$, and the moduli space $\mathfrak{M}(x)$ is not empty. Then there exists algebraic monomorphism $m : \mathfrak{M}(x) \longrightarrow \mathfrak{N}$.*

**Proof.** It follows from the definition that it is sufficient to prove that there exists a natural transformation of functors $\Sigma_x \longrightarrow \operatorname{Hom}(\cdot, \mathfrak{N})$. Note that the condition $(r(x), c_1(x)) = 1$ implies that any semistable in Gieseker sense sheaf from $\mathfrak{M}(x)$ is stable in the sense of Mumford – Takemoto.

Let $\mathcal{X}$ be a family of stable sheaves on $\mathbb{P}^2$ with vector $x$ and base $S$, i.e. $\mathcal{X}$ is a sheaf on the direct product $S \times \mathbb{P}^2$ such that the restriction of it to $s \times \mathbb{P}^2$ is a stable sheaf on $\mathbb{P}^2$ with topological invariants vector $x$.

By the relative variant of theorem 2.1, there exists a spectral sequence on $S \times \mathbb{P}^2$ converging to $\mathcal{X}$ with $\mathbf{E_1}$-part in the form

$$\mathbf{E_1^{p,q}} = \pi^* R^q_{\pi_*} \mathcal{H}om(g^* F_{-p}, \mathcal{X}) \otimes g^* E_{2-p},$$

where $(F_0, F_1, F_2)$ is the left dual collection to the collection $(E_0, E_1, E_2)$ and $\pi$ and $g$ are projections of the direct product $S \times \mathbb{P}^2$ to the first factor and the second one respectively. By the same reasons as in 3.1, we obtain that the sheaf $\mathcal{X}$ is the cohomology of the monad

$$0 \longrightarrow \pi^* \mathcal{F}_0 \otimes g^* E_0 \xrightarrow{\mathcal{A}} \pi^* \mathcal{F}_1 \otimes g^* E_1 \xrightarrow{\mathcal{B}} \pi^* \mathcal{F}_2 \otimes g^* E_2 \longrightarrow 0,$$

where $\mathcal{F}_i = R^1_{\pi_*} \mathcal{H}om(g^* F_{-p}, \mathcal{X})$. Let us tensoring by $g^* E_0^*$ the morphism $\mathcal{A}$ of the monad, and the morphism $\mathcal{B}$ tensoring by $g^* E_1^*$:

$$\pi^* \mathcal{F}_0 \otimes g^*(E_0^* \otimes E_0) \xrightarrow{\mathcal{A} \otimes id} \pi^* \mathcal{F}_1 \otimes g^*(E_0^* \otimes E_1)$$

$$\pi^* \mathcal{F}_1 \otimes g^*(E_1^* \otimes E_1) \xrightarrow{\mathcal{B} \otimes id} \pi^* \mathcal{F}_2 \otimes g^*(E_1^* \otimes E_2)$$

Applying the functor $R_{\pi_*}$ to these sequences, we get morphisms of bundles on $S$:

$$\mathcal{F}_0 \xrightarrow{A} \mathcal{F}_1 \otimes H_{01}, \qquad \mathcal{F}_1 \xrightarrow{B} \mathcal{F}_2 \otimes H_{12}.$$

By construction, the fibers $A$ and $B$ over a point $X$ of the base $S$ determine the representation $(a_X, b_X)$ of the quiver $Q$ corresponding to the sheaf $X$. Proposition 3.6 yields that the representation $(a_X, b_X)$ is stable. Hence the family of stable sheaves with the base $S$ determines the family of stable representations of the quiver with the same base.

Since $(r(x), c_1(x)) = 1$, the dimension vector $(u_0, u_1, u_2)$ is primitive. Therefore the moduli space of stable representations $\mathfrak{N}$ is fine (see the end of section 2). Thus there exists an algebraic map $s : S \longrightarrow \mathfrak{N}$ such that $\mathcal{F}_i = s^* \mathcal{U}_i$, where $\mathcal{U}_i$'s are the universal bundles over $\mathfrak{N}$. In other words, there is a natural transformation of functors. This completes the proof.

## 4. The image of $\mathfrak{M}(x)$ under the map $m$.

In the previous section we constructed the algebraic map $m$ from the moduli space of sheaves $\mathfrak{M}(x)$ to the moduli space of the stable representations $\mathfrak{N}$. Now we describe the image of $\mathfrak{M}(x)$ under this map. At first we note that dimension of tangent space of $\mathfrak{M}(x)$ in a point $X$ equals $\dim \operatorname{Ext}^1(X, X)$. Since stable sheaves are simple and the anticanonical class on $\mathbb{P}^2$ is ample, we have, $\dim \operatorname{Ext}^1(X, X) = 1 - \chi(X, X)$.

On the other hand, $x = u_1 e_1 - u_0 e_0 - u_2 e_2$, where $e_i$'s are vectors from $K_0(\mathbb{P}^2)$ corresponding to the bundles of the exceptional collections. Therefore, $\chi(X, X) = \sum u_i^2 + u_0 u_2 h_{02} - u_0 u_1 h_{01} - u_1 u_2 h_{12}$, i.e., the variety $\mathfrak{M}(x)$ is smooth and

$$\dim \mathfrak{M}(x) = u_0 u_1 h_{01} + u_1 u_2 h_{12} - u_0 u_2 h_{02} - \sum u_i^2 + 1.$$

Dimension of moduli space $\mathfrak{N}$ can by calculated as the difference of dimension of the vector space $R$ and dimension of the group $G/\Delta$, i.e.,

$$\dim \mathfrak{N} = u_0 u_1 h_{01} + u_1 u_2 h_{12} - \sum u_i^2 + 1.$$

Besides, by lemma 3.3 we get that the image of the variety $\mathfrak{M}(x)$ under the map $m$ is a subset of the zero set $\mathfrak{Z}$ of morphism $\Gamma : \mathcal{U}_0 \longrightarrow \mathcal{U}_2 \otimes H_{02}$. We see that if the map $\Gamma$ is general, then dimension of $\mathfrak{Z}$ coincide with dimension of $\mathfrak{M}(x)$. We will prove that the the image coincides with $\mathfrak{Z}$ too. This will imply theorem 1.1.

We start with a simple remark.

**4.1. Remark.** *Let $(a, b)$ be a stable representation with zero composition such that the corresponding complex of sheaves (3.1) is a monad. Then the cohomology of this monad $X$ is stable sheaf.*

**Proof.** Suppose that $X$ is non-stable. Then there exists a subsheaf $X'$ such that $\operatorname{rk} X' < \operatorname{rk} X$ and $\mu(X') \geq \mu(X)$. Without loss of generality it can be assumed that $X'$ is stable.

Since $\mu(X') \geq \mu(X) > \mu(F_i(-3))$, we have $\operatorname{Ext}^2(F_i, X') = 0$ for $i = 0, 1, 2$ (hear $F_i$'s are the bundles of the left dual collection for $(E_0, E_1, E_2)$ as before).

On the other hand, $X' \subset X$ and $\operatorname{Hom}(F_i, X) = 0$. This yields, $\operatorname{Hom}(F_i, X') = 0$. Therefore, the spectral sequence constructed by the collection $(E_0, E_1, E_2)$ and the sheaf $X'$ is degenerated into the monad

$$M(X') \quad 0 \longrightarrow U_0' \otimes E_0 \xrightarrow{\alpha'} U_1' \otimes E_1 \xrightarrow{\beta'} U_2' \otimes E_2 \longrightarrow 0$$

with cohomology $X'$. The monomorphism of sheaves $X' \longrightarrow X$ induces a monomorphism of the monads $M(X') \longrightarrow M(X)$ ( 3.2). Hence the representation $(a_{X'}, b_{X'})$ corresponding to the subsheaf $X'$ is a subrepresentation of $(a_X, b_X)$.

Now using lemma 3.4 and numerical criterion 2.2, we get that the subrepresentation $(a_{X'}, b_{X'})$ contradicts to the stability of $(a_X, b_X)$. This contradiction proves the remark.

Thus we see that, for theorem 1.1 to be proved it is necessary and sufficient to check that any stable representation with zero composition induces a monad namely. We start with the case of exceptional collection $\big(\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}\big)$.

**4.2. Proposition.** *Suppose that $(E_0, E_1, E_2) = \big(\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}\big)$ and*

$$a : U_0 \longrightarrow U_1 \otimes H_{01}, \qquad b : U_1 \longrightarrow U_2 \otimes H_{12}$$

*is a representation with zero composition. In addition we assume that slope $\mu$ of the vector $x(u_0, u_1, u_2)$ satisfy the inequality $\mu_0 < \mu < \mu_2$. Then the corresponding morphisms of sheaves $\alpha : U_0 \otimes \mathcal{O}(-2) \longrightarrow U_1 \otimes \mathcal{O}(-1)$, $\beta : U_1 \otimes \mathcal{O}(-1) \longrightarrow U_2 \otimes \mathcal{O}$ are a monomorphism and an epimorphism (respectively).*

**Proof.** Taking into account the duality of these morphisms, it is sufficient to check that $\alpha$ is a monomorphism of bundles. Namely, we show that the fiber of $\alpha$ over any point $x \in \mathbb{P}^2$ $\alpha_x : U_0 \otimes \mathbb{C} \longrightarrow U_1 \otimes \mathbb{C}$ is injective.

Recall that if we fixed basses for the spaces $U_i$'s, the linear map $a$ and a morphism $\alpha$ can be wrote as a matrix $A = (g_{kj})_{\substack{k=1,\dots,u_1 \\ j=1,\dots,u_0}}$ with $g_{kj} \in \mathrm{H}^0(\mathcal{O}(1))$. Assume that the morphism $\alpha$ fiber over a point $x$ $\alpha_x$ has nonzero kernel, i.e., there is a vector $u \in U_0$ such that $\alpha_x(u) = 0$. Without loss of generality it can be assumed that $u$ is the first basis vector of the space $U_0$. Then the morphisms $g_{k1} \in H_{01} = \mathrm{H}^0(\mathcal{O}(1))$ vanish at the point $x$ when $k = 1, ..., u_1$. Because of the zero set of a morphism $g \in \mathrm{H}^0(\mathcal{O}(1))$ is a line on the projective plane, dimension of the linear span $H = \langle g_{11}, ..., g_{u_0 1}\rangle$ less or equal to 2. This implies that the matrix $A$ in suitable basses $\{e_i^j\}$ of the spaces $U_j$ $(j = 0, 1)$ has the form

$$A = \begin{pmatrix} g_{11} & * & \cdots & * \\ g_{21} & * & \cdots & * \\ 0 & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & * & \cdots & * \end{pmatrix}$$

Holding fixed a basis for $U_2$ we can write the linear map $b$ as a matrix $B = (f_{ik})_{\substack{i=1,\dots,u_2 \\ k=1,\dots,u_1}}$ with $f_{ik} \in \mathrm{H}^0(\mathcal{O}(1))$. By assumption, the representation $(a, b)$ has zero composition, i.e., $B \cdot A = 0$. In particular,

$$f_{i1} \circ g_{11} + f_{i2} \circ g_{21} = 0 \qquad \forall i = 1, ..., u_2. \tag{4.1}$$

Let us consider 3 cases.

CASE 1. Suppose that $\dim H = 2$, that is the morphisms $g_{11}$ and $g_{21}$ are not proportional. Then the exact sequence takes place.

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{g_{11} \oplus g_{21}} \mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow J_x \longrightarrow 0,$$

where $J_x$ is the ideal sheaf of the point $x$. Applying the functor $\mathrm{Hom}(\cdot, \mathcal{O})$ we have,

$$0 \longrightarrow \mathrm{Hom}(J_x, \mathcal{O}) \longrightarrow \mathrm{Hom}(\mathcal{O}(-1), \mathcal{O})^{\oplus 2} \xrightarrow{\delta} \mathrm{Hom}(\mathcal{O}(-2), \mathcal{O}) \longrightarrow 0.$$

Here the morphism $\delta$ takes a pair $(f_1, f_2) \in \mathrm{Hom}(\mathcal{O}(-1), \mathcal{O})^{\oplus 2}$ to $f_1 \circ g_{11} + f_2 \circ g_{21}$. Since $\dim \mathrm{Hom}(J_x, \mathcal{O}) = 1$, then $\dim\langle (f_{i1}, f_{i2}) \rangle \leq 1$ (see (4.1)). Therefore, changing basis of the space $U_2$, the matrix $B$ can be rewrote in the form

$$B = \begin{pmatrix} f_{11} & f_{12} & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & * & \cdots & * \end{pmatrix}$$

In such a way we constructed a subrepresentation $(a', b')$ of the representation $(a, b)$ with dimension vector $(1, 2, 1)$. Let us compute value of the polarization $\theta$ at this vector. In the case when exceptional collection is $\big(\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}\big)$ the polarization has the form

$$\theta_0 = 2u_2 - u_1 = r(\mu_0 - \mu), \quad \theta_1 = u_0 - u_2 = r(\mu - \mu_1), \quad \theta_2 = u_1 - 2u_0 = r(\mu_2 - \mu) \tag{4.2}$$

(see (3.2), 3.5). Hence, $\theta_0 + 2\theta_1 + \theta_2 = 0$,. This contradicts the numerical criterion 2.2 and the assumption of stability of the representation $(a, b)$.

CASE 2. Suppose that $\dim H = 1$. We can assume that $g_{12} = 0$, i.e., the condition $D \cdot A = 0$ implies the inequality $f_{i1} \circ g_{11} = 0\ \forall i = 1, ..., u_2$. We see that this is possible in the case $f_{i,1} = 0$ only. Therefore the representation $(a, b)$ has a subrepresentation with dimension vector $(1, 1, 0)$. We have

$$\theta(1, 1, 0) = u_2 + u_0 - u_1 = \frac{1}{2}(2u_2 - u_1 + 2u_0 - u_1).$$

By the assumption of the proposition, $2u_2 - u_1 = r(\mu_0 - \mu) < 0, \quad u_1 - 2u_0 = r(\mu_2 - \mu) > 0$, i.e., $\theta(1, 1, 0) < 0$. By the stability of $(a, b)$, it is impossible.

CASE 3. Now suppose that $\dim H = 0$. Then the representation $(a, b)$ has a subrepresentation with dimension vector $(1, 0, 0)$. This contradicts stability of $(a, b)$ too.

This completes the proof.

To check that $\alpha$ is a monomorphism and $\beta$ is an epimorphism in the general case we need a mutations of representations. Further we deal with the mutations.

**4.3. Lemma.** *Let a representation* $U_0 \otimes H_{01}^* \xrightarrow{a} U_1 \xrightarrow{b} U_2 \otimes H_{12}$ *with zero composition corresponds to a complex of sheaves* $U_0 \otimes E_0 \xrightarrow{\alpha} U_1 \otimes E_1 \xrightarrow{\beta} U_2 \otimes E_2$*, where* $(E_0, E_1, E_2)$ *is a full exceptional collection. Denote by* $L_0 U_1$ *the kernel of the map* $a$*, i.e.,*

$$0 \longrightarrow L_0 U_1 \xrightarrow{L_0 a} U_0 \otimes H_{01}^* \xrightarrow{a} U_1,$$

*and by $L_0H_{12}$ the linear space* $\mathrm{Hom}(RE_0, E_2)$. *Then there exits a linear map* $L_0b : U_0 \longrightarrow U_2 \otimes L_0H_{12}$ *such that the representation* $L_0U_1 \otimes H_{01} \xrightarrow{L_0a} U_0 \xrightarrow{L_0b} U_2 \otimes L_0H_{12}$ *corresponds to the sheaves complex* $L_0U_1 \otimes E_1 \xrightarrow{L_0\alpha} U_0 \otimes RE_0 \xrightarrow{L_0\beta} U_2 \otimes E_2$, *where, as usually,* $RE_0 = R_{E_1}E_0$ *is the right mutation of the exceptional pair* [4] $(E_0, E_1)$. *Moreover it follows from the fact that $\alpha$ is monomorphism that $L_0\alpha$ is a monomorphism too. If $\beta$ and $a$ are epimorphisms, then $L_0\beta$ is epimorphism too.*

**Proof.** At first we check that the diagram is commutative

$$\begin{array}{ccc} U_0 \otimes E_0 & \xrightarrow{\alpha} & U_1 \otimes E_1 \\ \uparrow {\scriptstyle id} & & \uparrow {\scriptstyle a\otimes id} \\ U_0 \otimes E_0 & \xrightarrow{id\otimes \mathrm{rcan}} & U_0 \otimes H_{01}^* \otimes E_1 \end{array} \tag{4.3}$$

The sheaf morphism $\alpha$ and the linear map $a$ are determined by one tensor of $U_0^* \otimes U_1 \otimes H_{01}$, which in a fixed basis $f_1, ..., f_n$ for $H_{01}$ has the form $\sum\limits_i \varphi_i \otimes f_i$, where $\varphi_i \in \mathrm{Hom}(U_0, U_1)$. The fiber of $\alpha$ over a point $x \in \mathbb{P}^2$ acts by the rule $\alpha_x(v \otimes e) = \sum\limits_i \varphi_i(v) \otimes f_{i,x}(e)$ The linear map $a$ takes each tensor $v \otimes g \in U_0 \otimes H_{01}^*$ to the vector $a(v \otimes g) = \sum\limits_j \varphi_j(v)\langle g, f_j\rangle$, where $\langle\cdot,\cdot\rangle$ is the natural pairing on the tensor product $H_{01}^* \otimes H_{01}$.

Besides, the canonical map rcan : $E_0 \longrightarrow H_{01}^* \otimes E_1$ is determined by $\sum\limits_i f_i \otimes f^i$, where $f^1, ..., f^n$ is the dual basis for $H_{01}^*$. Therefore the action of this map on a fiber vector $e \in E_{0,x}$ has the form $e \longrightarrow \sum\limits_i f_{i,x}(e) \otimes f^i$.

Combining these formulae together, we obtain

$$v\otimes e \xrightarrow{id\otimes\mathrm{rcan}} \sum_i v\otimes f_{i,x}(e)\otimes f^i \xrightarrow{a\otimes id} \sum_i\Big(\sum_j \varphi_j(v)\langle f^i, f_j\rangle\Big)\otimes f_{i,x}(e) = \sum_i \varphi_i(v)\otimes f_{i,x}(e) = \alpha_x(v\otimes e).$$

This proves the commutativity of the diagram.

Since diagram (4.3) is commutative, we have one more commutative diagram with exact columns and rows

$$\begin{array}{ccccccccc} & & U_0 \otimes E_0 & \xrightarrow{\alpha} & U_1 \otimes E_1 & \longrightarrow & \mathrm{coker}\,\alpha & \longrightarrow & 0 \\ & & \uparrow {\scriptstyle id} & & \uparrow {\scriptstyle a\otimes id} & & \uparrow {\scriptstyle \exists\beta'} & & \\ 0 & \longrightarrow & U_0 \otimes E_0 & \xrightarrow{id\otimes\mathrm{rcan}} & U_0 \otimes H_{01}^* \otimes E_1 & \xrightarrow{id\otimes\mathrm{lcan}} & U_0 \otimes RE_0 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow {\scriptstyle L_0a\otimes id} & & \uparrow {\scriptstyle L_0\alpha} & & , \\ & & 0 & \longrightarrow & L_0U_1 \otimes E_1 & \xrightarrow{id} & L_0U_1 \otimes E_1 & \longrightarrow & 0 \\ & & & & \uparrow & & & & \\ & & & & 0 & & & & \end{array}$$

where $L_0\alpha = (id \otimes \mathrm{lcan}) \circ (L_0a \otimes id)$. As before we can check that the morphism $L_0\alpha$ and the linear map $a$ are determined by one tensor from $(L_0U_1)^* \otimes U_0 \otimes H_{01}^*$. The commutative diagram implies

(i) $\beta' \circ L_0\alpha = 0$,

(ii) if $a$ is an epimorphism, then $\beta'$ is an epimorphism too,

(iii) if $\alpha$ is an inclusion of bundles, then $L_0\alpha$ is also an inclusion of bundles.

[4] Recall that $\mathrm{Hom}(E_0, E_1)^* \cong \mathrm{Hom}(E_1, RE_0)$.

By assumption the following commutative diagram takes place

$$\begin{array}{ccccccc}
 & & U_2 \otimes E_2 & \xrightarrow{id} & U_2 \otimes E_2 & & \\
 & & \uparrow \beta & & \uparrow \exists \beta'' & & \\
U_0 \otimes E_0 & \xrightarrow{\alpha} & U_1 \otimes E_1 & \longrightarrow & \operatorname{coker} \alpha & \longrightarrow & 0 \\
\uparrow id & & \uparrow & & & & \\
U_0 \otimes E_0 & \xrightarrow{\alpha} & \ker \beta & & & & \\
 & & \uparrow & & & & \\
 & & 0 & & & &
\end{array} .$$

In particular, we see that the fact that $\beta$ is an epimorphism gives that $\beta''$ is an epimorphism too. Let us define the left mutation $L_0\beta$ as the composition $L_0\beta = \beta'' \circ \beta'$ and let $L_0 b$ be the linear map $U_0 \longrightarrow U_2 \otimes L_0 H_{12}$, corresponding to the same tensor form $U_0^* \otimes U_2 \otimes L_0 H_{12}$, as $L_0\beta$ corresponds.

**4.4. Remark.** *We say that the representation $(L_0 a, L_0 b)$ constructed in the previous lemma is the left mutation of the representation $(a, b)$. By construction the left mutation $(L_0 a', L_0 b')$ of a subrepresentation $(a', b')$ of $(a, b)$ is a subrepresentation of $(L_0 a, L_0 b)$.*

**4.5. Lemma.** *Let a representation with zero composition*

$$U_0 \otimes H_{01}^* \xrightarrow{a} U_1 \xrightarrow{b} U_2 \otimes H_{12}$$

*corresponds to the complex of sheaves $U_0 \otimes E_0 \xrightarrow{\alpha} U_1 \otimes E_1 \xrightarrow{\beta} U_2 \otimes E_2$, where $(E_0, E_1, E_2)$ is a full exceptional collection. Denote by $R_1 U_0$ the cokernel of the map $a$, i.e,*

$$U_0 \xrightarrow{a} U_1 \otimes H_{01} \xrightarrow{R_1 a} R_1 U_0 \longrightarrow 0.$$

*Then there exists a linear map $R_1 b : R_1 U_0 \longrightarrow U_2 \otimes H_{02}$ such that the representation $U_1 \otimes H_{01} \xrightarrow{R_1 a} R_1 U_0 \xrightarrow{R_1 b} U_2 \otimes H_{02}$ corresponds to the complex of sheaves*

$$U_1 \otimes LE_1 \xrightarrow{R_1 \alpha} R_1 U_0 \otimes E_0 \xrightarrow{R_1 \beta} U_2 \otimes E_2,$$

*where, as usually, $LE_1 = L_{E_0} E_1$ is the left mutation of the bundle $E_1$. Moreover, if $\alpha$ is a monomorphism of bundles and $a$ is injective, then $R_1\alpha$ is a monomorphism of bundles. If $\beta$ is an epimorphism, then $R_1\beta$ is too.*

**Proof.** Application of the functor $\operatorname{Hom}(E_0, \cdot)$ to the sequence

$$U_0 \otimes E_0 \xrightarrow{\alpha} U_1 \otimes E_1 \longrightarrow \operatorname{coker} \alpha \longrightarrow 0$$

gives

$$U_0 \xrightarrow{a} U_1 \otimes H_{01} \xrightarrow{R_1 a} \operatorname{Hom}(E_0, \operatorname{coker} \alpha).$$

By the assumption of the lemma $R_1 U_0$ is the image of the map $R_1 a$. Hence there exists the commutative diagram

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\uparrow & & \uparrow & & \uparrow & & \\
U_0 \otimes E_0 & \xrightarrow{\alpha} & U_1 \otimes E_1 & \longrightarrow & \operatorname{coker} \alpha & \longrightarrow & 0 \\
\uparrow & & \uparrow id \otimes \mathrm{lcan} & & \uparrow \beta' & & \\
U_0 \otimes E_0 & \xrightarrow{a \otimes id} & U_1 \otimes H_{01} \otimes E_0 & \xrightarrow{R_1 a \otimes id} & R_1 U_0 \otimes E_0 & \longrightarrow & 0 \\
\uparrow & & \uparrow id \otimes \mathrm{rcan} & & \uparrow R_1 \alpha & & \\
0 & \longrightarrow & U_1 \otimes LE_1 & \longrightarrow & U_1 \otimes LE_1 & & \\
 & & \uparrow & & & & \\
 & & 0 & & & &
\end{array} ,$$

where $R_1\alpha = (R_1 a \otimes id) \circ (id \otimes \mathrm{rcan})$, and $\beta'$ is the restriction of the canonical map $\mathrm{lcan} : \mathrm{Hom}(E_0, \operatorname{coker}\alpha) \otimes E_0 \longrightarrow \operatorname{coker}\alpha$ to the subspace $R_1U_0 \subset \mathrm{Hom}(E_0, \operatorname{coker}\alpha)$. We see that $R_1\alpha$ and $R_1a$ are determined by one tensor from $U_1^* \otimes R_1U_0 \otimes H_{01}^*$. This diagram implies

(i) $\beta'$ is epimorphism,

(ii) $\beta' \circ R_1\alpha = 0$,

(iii) if $\alpha$ is an inclusion of bundles, then $R_1\alpha$ is inclusion of bundles too.

Further, as before, the following commutative diagram takes place

$$\begin{array}{ccccccc}
 & & U_2 \otimes E_2 & \xrightarrow{id} & U_2 \otimes E_2 & & \\
 & & \uparrow \beta & & \uparrow \exists\beta'' & & \\
U_0 \otimes E_0 & \xrightarrow{\alpha} & U_1 \otimes E_1 & \longrightarrow & \operatorname{coker}\alpha & \longrightarrow & 0 \\
\uparrow id & & \uparrow & & & & \\
U_0 \otimes E_0 & \xrightarrow{\alpha} & \ker\beta & & & & \\
 & & \uparrow & & & & \\
 & & 0 & & & &
\end{array} .$$

Moreover, if $\beta$ is an epimorphism, we see that $\beta''$ is an epimorphism too. To prove the lemma it is remains to define $R_1\beta$ as the composition $\beta'' \circ \beta'$, and we define $R_1b$ as the corresponding linear map.

**4.6. Remark.** *The representation $(R_1a, R_1b)$ constructed in the previous lemma is called the right mutation of the representation $(a, b)$. By construction, the right mutation $(R_1a', R_1b')$ of a subrepresentation $(a', b')$ of $(a, b)$ is a subrepresentation of $(R_1a, R_1b)$.*

The next lemma follows from the proves of the lemmas 4.3 and 4.5.

**4.7. Lemma** *(i) If the map $a : U_0 \otimes H_{01}^* \longrightarrow U_1$ of a representation $(a, b)$ is an epimorphism, then $(R_1L_0a, R_1L_0b) = (a, b)$.*

*(ii) If the map $a : U_0 \longrightarrow U_1 \otimes H_{01}$ of a representation $(a, b)$ is a monomorphism, then $(L_0R_1a, L_0R_1b) = (a, b)$.*

Below we show when the mutations of stable representations are stable.

**4.8. Lemma.** *Suppose that $\mu = \dfrac{u_1d_1 - u_0d_0 - u_2d_2}{u_1r_1 - u_0r_0 - u_2r_2} = \frac{d}{r} \in (\mu_0, \mu_2)$ and a representation with zero composition $U_0 \otimes H_{01}^* \xrightarrow{a} U_1 \xrightarrow{b} U_2 \otimes H_{12}$ is stable w.r.t. the polarization $\theta$ with*

$\theta_0 = u_2\bar{r}_1 - u_1r_2 = rr_0(\mu_0 - \mu) < 0,$

$\theta_1 = u_0r_2 - u_2r_0 = rr_1(\mu - \mu_1),$

$\theta_2 = u_1r_0 - u_0\bar{r}_1 = rr_2(\mu_2 - \mu) > 0.$

*Then the right mutation of the representation $U_1 \otimes H_{01} \xrightarrow{R_1a} R_1U_0 \xrightarrow{R_1b} U_2 \otimes H_{02}$ is stable w.r.t. the polarization $R_1\theta$ with*

$R_1\theta_0 = u_2(3r_2\bar{r}_1 - r_0) - (3r_2u_1 - u_0)r_2 = rr_0(\bar{\mu}_1 - \mu) < 0,$

$R_1\theta_1 = u_1r_2 - u_2\bar{r}_1 = rr_0(\mu - \mu_0) > 0,$

$R_1\theta_2 = (3r_2u_1 - u_0)\bar{r}_1 - u_1(3r_2\bar{r}_1 - r_0) = u_1r_0 - u_0\bar{r}_1 = rr_2(\mu_2 - \mu) > 0,$

*where $\bar{\mu}_1 = \mu(L_{E_0}E_1)$.*

**Proof.** At first we show that the map $a : U_0 \longrightarrow U_1 \otimes H_{01}$ is a monomorphism. Assume that it is not true, i.e., $a$ has a nonzero kernel $U_0' \subset U_0$. Then the pair of trivial maps $U_0' \longrightarrow \{0\} \otimes H_{01}$, $\{0\} \longrightarrow \{0\} \otimes H_{12}$ determines a subrepresentation of $(a, b)$ with dimension vector $(u_0', 0, 0)$. Since $\theta_0 < 0$, the values of the polarization $\theta$ at this vector is negative. This contradicts the stability of $(a, b)$.

Thus, $a$ is an monomorphism and by the previous lemma we have

$$(L_0R_1a, L_0R_1b) = (a, b). \tag{4.4}$$

Besides, $\dim R_1U_0 = h_{01}u_1 - u_0$. Further, since $h_{01} = 3r_2$, then $\dim R_1U_0 = 3r_2u_1 - u_0$. Using this fact we can prove that $R_1\theta$ is a polarization, i.e., $R_1\theta_0u_1 + R_1\theta_1 \dim R_1U_0 + R_1\theta_2u_2 = 0$.

Now we assume that the representation $(R_1a, R_1b)$ is no stable. By criterion 2.2, it has a subrepresentation $U_0' \otimes H_{01} \xrightarrow{a'} U_1' \xrightarrow{b'} U_2' \otimes H_{02}$ with dimension vector $(u_0', u_1', u_2')$ such that

$$\sum R_1\theta_iu_i' \leq 0. \tag{4.5}$$

If $a'$ is not epimorphism, then taking $\operatorname{im} a'$ in place of $U_1'$ we get a subrepresentation $(a'', b'')$ of $(R_1a, R_1b)$ such that $a''$ is an epimorphism and the value of the polarization $R_1\theta$ at dimension vector of $(a'', b'')$ is non-positive too (by assumption $R_1\theta_1 > 0$). Therefore we can assume that $a' : U_0' \otimes H_{01} \longrightarrow U_1'$ is an epimorphism.

Taking into account 4.4 and (4.4) we get that the left mutation $(L_0a', L_0b')$ of the subrepresentation $(a', b')$ is a subrepresentation of $(a, b)$ with dimension vector $(3r_2u_0' - u_1', u_0', u_2')$. We have

$$\theta(3r_2u_0' - u_1', u_0', u_2') = (u_2\bar{r}_1 - u_1r_2)(3r_2u_0' - u_1') + (u_0r_2 - u_2r_0)u_0' + (u_1r_0 - u_0\bar{r}_1)u_2'.$$

Substituting the values of $R_1\theta_i$ to inequality (4.5), we obtain

$$u_0'(u_2(3r_2\bar{r}_1 - r_0) - r_2(3r_2u_1 - u_0)) + u_1'(u_1r_2 - u_2\bar{r}_1) + u_2'(u_1r_0 - u_0\bar{r}_1) \leq 0.$$

This is equivalent to

$$\theta(3r_2u_0' - u_1', u_0', u_2') \leq 0.$$

We get the contradiction with the stability of the representation $(a, b)$. This concludes the proof.

**4.9. Lemma.** *Suppose that* $\mu = \dfrac{u_1d_1 - u_0d_0 - u_2d_2}{u_1r_1 - u_0r_0 - u_2r_2} = \dfrac{d}{r} \in (\mu_1, \mu_2)$ *and a representation with zero composition* $U_0 \otimes H_{01}^* \xrightarrow{a} U_1 \xrightarrow{b} U_2 \otimes H_{12}$ *is stable w.r.t. the polarization* $\theta$ *from the previous lemma. Then the left mutation of the representation*

$$L_0U_1 \otimes H_{01} \xrightarrow{L_0a} U_0 \xrightarrow{L_0b} U_2 \otimes L_0H_{12}$$

*is stable w.r.t. the polarization* $L_0\theta$ *with*
$L_0\theta_0 = u_2r_0 - u_0r_2 = rr_0(\mu_1 - \mu) < 0,$
$L_0\theta_1 = (3r_2u_0 - u_1)r_2 - u_2r_1 = r(3r_2r_1 - r_0)(\mu - \mu(RE_0)),$
$L_0\theta_2 = u_0r_1 - (3r_2u_0 - u_1)r_0 = rr_2(\mu_2 - \mu) > 0.$

**Proof.** If the image of the map $a$ does not coincide with $U_1$, then the representation $(a, b)$ has a subrepresentation with dimension vector $(u_0, \dim(\operatorname{im} a), u_2)$. By assumption $\mu > \mu_1$, hence $\theta_1 = rr_1(\mu - \mu_1) > 0$ and $\theta(u_0, \dim(\operatorname{im} a), u_2) < 0$. This contradicts the stability of $(a, b)$. Therefore the map $a$ is an epimorphism and from lemma 4.7 it follows that $(R_1L_0a, R_1L_0b) = (a, b)$. Besides, $\dim L_0U_1 = 3r_2u_0 - u_1$. Using this equality, it is easy to show that $L_0\theta$ is a polarization for representations with dimension vector $(3r_2u_0 - u_1, u_0, u_2)$.

Suppose that the representation $(L_0a, L_0b)$ has a destabilizing subrepresentation $(a', b')$ with dimension vector $(u_0', u_1', u_2')$, i.e.,

$$L_0\theta(u_0', u_1', u_2') \leq 0. \tag{4.6}$$

As before, we can assume that $a' : U_0' \longrightarrow U_1' \otimes H_{01}^*$ is an inclusion. In other words, $(R_1a', R_1b')$ is a subrepresentation of $(a, b)$ with dimension vector $(u_0', 3r_2u_1' - u_0', u_2')$. It can by proved that the subrepresentation $(R_1a', R_1b')$ destabilizes the representation $(a, b)$. This contradiction proves the lemma.

By the dual reasons we can prove the following lemmas.

**4.10. Lemma.** *Let* $U_0 \otimes H_{01}^* \xrightarrow{a} U_1 \xrightarrow{b} U_2 \otimes H_{12}$ — *be a representation with zero composition associated with the complex of sheaves* $U_0 \otimes E_0 \xrightarrow{\alpha} U_1 \otimes E_1 \xrightarrow{\beta} U_2 \otimes E_2$. *Denote by* $R_2U_1$ *the cokernel of the map* $b$, *i.e.,*

$$U_1 \xrightarrow{b} U_2 \otimes H_{12} \xrightarrow{R_2 b} R_2U_1 \longrightarrow 0.$$

*Then there exists a linear map* $R_2a : U_0 \otimes \mathrm{Hom}(E_0, LE_2)^* \longrightarrow U_2$ *such that the representation* $(R_2a, R_2b)$ *corresponds to the complex of sheaves* $U_0 \otimes E_0 \xrightarrow{R_2\alpha} U_2 \otimes LE_2 \xrightarrow{R_2\beta} R_2U_1 \otimes E_1$. *Moreover, if* $\beta$ *is an epimorphism, then* $R_2\beta$ *is an epimorphism too. If* $\alpha$ *is a monomorphism of bundles,* $R_2\alpha$ *is a monomorphism of bundles too.*

**4.11. Lemma.** *Let* $U_0 \otimes H_{01}^* \xrightarrow{a} U_1 \xrightarrow{b} U_2 \otimes H_{12}$ *be a representation with zero composition associated with the complex of sheaves* $U_0 \otimes E_0 \xrightarrow{\alpha} U_1 \otimes E_1 \xrightarrow{\beta} U_2 \otimes E_2$. *Denote by* $L_1U_2$ *the kernel of the map* $b$, *i.e.*

$$0 \longrightarrow L_1U_2 \xrightarrow{L_1 b} U_1 \otimes H_{12}^* \xrightarrow{b} U_2.$$

*Then there exists a linear map* $L_1a : U_0 \longrightarrow L_1U_2 \otimes H_{02}$ *such that the representation* $(L_1a, L_1b)$ *corresponds to the complex of sheaves* $U_0 \otimes E_0 \xrightarrow{L_1\alpha} L_1U_2 \otimes E_2 \xrightarrow{L_1\beta} U_1 \otimes RE_1$. *Moreover, if* $\beta$ *and* $b$ *are epimorphisms, then* $L_1\beta$ *is epimorphism too. If* $\alpha$ *is a monomorphism of bundles,* $L_1\alpha$ *is a monomorphism of bundles too.*

**4.12. Remark.** *1. If* $(a', b')$ *is a subrepresentation of* $(a, b)$ *then* $(L_1a', L_1b')$ *is subrepresentation of* $(L_1a, L_1b)$ *and* $(R_2a', R_2b')$ *is subrepresentation of* $(R_2a, R_2b)$.

*2. If the map* $b$ *of* $(a, b)$ *is inclusion, then* $(L_1R_2a, L_1R_2b) = (a, b)$. *If it is an epimorphism, then* $(R_2L_1a, R_2L_1b) = (a, b)$.

**4.13. Lemma.** *Suppose that a representation* $U_0 \otimes H_{01}^* \xrightarrow{a} U_1 \xrightarrow{b} U_2 \otimes H_{12}$ *with zero composition is stable w.r.t. the polarization* $\theta$ *with*

$$\theta_0 = u_2\bar{r}_1 - u_1r_2, \quad \theta_1 = u_0r_2 - u_2r_0, \quad \theta_2 = u_1r_0 - u_0\bar{r}_1.$$

*If, an additional,* $\mu = \frac{d}{r} \in (\mu_0, \mu_1)$, *then the representation* $(R_2a, R_2b)$ *is stable w.r.t.* $R_2\theta$ *with*

$R_2\theta_0 = (3r_0u_2 - u_1)r_2 - u_2r_1,$
$R_2\theta_1 = u_0r_1 - (3r_0u_2 - u_1)r_0,$
$R_2\theta_2 = u_2r_0 - u_0r_2.$

*If* $\mu \in (\mu_0, \mu_2)$, *then the representation* $(L_1a, L_1b)$ *is stable w.r.t.* $L_1\theta$ *with*

$L_1\theta_0 = u_1(3\bar{r}_1r_0 - r_2) - (3r_0u_1 - u_2)\bar{r}_1 = u_2\bar{r}_1 - u_1r_2,$
$L_1\theta_1 = u_0\bar{r}_1 - u_1r_0,$
$L_1\theta_2 = (3r_0u_1 - u_2)r_0 - u_0(3\bar{r}_1r_0 - r_2).$

Now we prove the following proposition, which proves theorem 1.1.

**4.14. Proposition.** *Let* $\sigma = (E_0, E_1, E_2)$ *be a full exceptional collection with* $-2 \le \mu_0 < \mu_2 \le 0$ *and* $u = (u_0, u_1, u_2)$ *be a collection of nonnegative integer such that*[5] $\mu(x(u)) \in (\mu_0, \mu_2)$. *Suppose that a representation with zero composition* $U_0 \otimes H_{01}^* \xrightarrow{a} U_1 \xrightarrow{b} U_2 \otimes H_{12}$ *is stable w.r.t. the polarization* $\theta$ *with* $\theta_i$ *from (3.2), and dimension vector of* $(a, b)$ *is equal to*

[5] See the definition of $x(u)$ on p. 6

*$u = (u_0, u_1, u_2)$. Then the corresponding complex of sheaves $U_0 \otimes E_0 \xrightarrow{\alpha} U_1 \otimes E_1 \xrightarrow{\beta} U_2 \otimes E_2$ is a monad.*

**Proof.** Real we need to prove that $\alpha$ is a monomorphism and $\beta$ is an epimorphism. It is known that all exceptional collection on $\mathbb{P}^2$, with slopes of bundles satisfying the condition of the proposition can be obtained by the "interior" mutations from the collection $\big(\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}\big)$, i.e., by the mutations $L_0$ and $R_2$. Because of this we study the properties of the morphisms $\alpha$ and $\beta$ with help of the mutation induction. The base of the induction (the case of the collection $\big(\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}\big)$) was done in proposition 4.2.

Assume that the proposition is true for a collection $\tau = (F_0, F_1, F_2)$, and the collection $\sigma$ was obtained from $\tau$ by one of the mutations

$$L_1\tau = (E_0, LE_2, E_1), \qquad R_1\tau = (E_1, RE_0, E_2).$$

We assume that $\sigma = R_1\tau$, i.e., $E_0 = F_1,\ E_1 = RF_0,\ E_2 = F_2$. the second case can be proved by the same method. Thus, $\tau = L_0\sigma$.

The mutations $(R_1a, R_1b)$ of the representation $(a, b)$ is stable w.r.t. $R_1\theta$ (4.5) and it is associated with the sheaves complex $U_1 \otimes F_0 \xrightarrow{R_1\alpha} R_1U_0 \otimes F_1 \xrightarrow{R_1\beta} U_2 \otimes F_2$. By the inductive hypothesis, $R_1\alpha$ is a monomorphism and $R_1\beta$ is an epimorphism. Since the representation $(a, b)$ is stable, the map $a : U_0 \longrightarrow U_1 \otimes H_{01}$ is inclusion. Therefore, by remark 4.7, $(L_0R_1a, L_0R_1b) = (a, b)$. By 4.3, we get that $\alpha = L_0R_1\alpha$ is a monomorphism and $\beta = L_0R_1\beta$ is an epimorphism.


[1] Drezet J.-M., Le Potier J.: Fibrés stables et fibrés exceptionnels sur $\mathbf{P}_2$. Ann. scient. Éc. Norm. Sup. 4$^e$ série. **18** (1985). 193–244.

[2] Karpov B.V. Semistable sheaves on two-dimensional quadric and Kronecker modules. transl. in Russian Acad. Sci. Izv. Math. **40** (1993), no. 1, 33–66.

[3] Ellingsrud G., Stromme S.A., Towards the Chow ring of the moduli space for stable sheaves on $\mathbb{P}_2$ with $c_1 = 1$. Preprint of the Oslo University, 1991.

[4] Gorodentsev A.L., Leenson M.I. How to calculate the correlation function in twisted SYM N=2, $N_f = 4$ QFT on projective plane. alg-geom / 9604011 v2.

[5] Kuleshov S. Moduli spaces of sheaves necessary for testing of S-duality conjecture. Preprint of Max–Planck–Institut, 97–32. 1997.

[6] Bondal A.I. Representations of associative algebras and coherent sheaves. transl. in Math. USSR - Izv. **34** (1990) no. 1, 23–42.

[7] King A. Moduli of Representations of Finite Dimensional Algebras J. Math. Oxford 45, (1994) P. 515-530.

[8] Drezet J.-M. Fibrés stables et fibrés exceptionnels sur $\mathbf{P_2}$. Ann. Ec. N. Sup. **18** (1985), 193–244.

[9] Gorodentsev A.L., Rudakov A.N. Exceptional vector bundles on projective spaces. Duke Math. J. **54** (1987) no. 1, 115–130.