Geometry of the trilogarithm

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July 7, 1997

Contents

1 Introduction

The classical n-logarithm is defined by induction as an integral

$$Li_n(z) := \int_0^z Li_{n-1}(t) d\log t, \quad Li_1(z) = -\log(1-z)$$

So it can be written as an n-dimensional integral

$$Li_n(z) = \int_{0 \leq 1-t_1 \leq \ldots \leq t_n} \frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_n}{t_n}$$

Aomoto considered [A] more general integrals where the differential form $\frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_n}{t_n}$ is integrated over an arbitrary n-dimensional real simplex in $\mathbb{C}^n$. Let me recall this construction in a more formal setting.

Let F be a field. An n-simplex in $P^n(F)$ is a collection of hyperplanes $L = (L_0, \ldots, L_n)$. It is nondegenerate if the intersection of the hyperplanes L_i is empty. A face of a simplex is any nonempty intersection of hyperplanes from L. A pair of simplices is admissible if L and M have no common faces of the same dimension. Now let $F = \mathbb{C}$. Then there is a canonical n-form ω_L in $\mathbb{C}P^n$ with logarithmic poles on the hyperplanes L_i. If $z_i = 0$ are homogeneous equations of L_i then

$$\omega_L = d\log(z_1/z_0) \wedge \ldots \wedge d\log(z_n/z_0)$$

Let Δ_M be an n-cycle representing a generator of $H_n(\mathbb{C}P^n, M)$.

The Aomoto n-logarithm is a multivalued function on configurations of admissible pairs of simplices $(L; M)$ in $\mathbb{C}P^n$ defined as follows:

$$\Lambda_n(L; M) := \int_{\Delta_M} \omega_L$$

The classical n-logarithm corresponds to a very special pair of simplices in $\mathbb{P}^n$, see fig.1, so it is a very special case of the Aomoto n-logarithm.

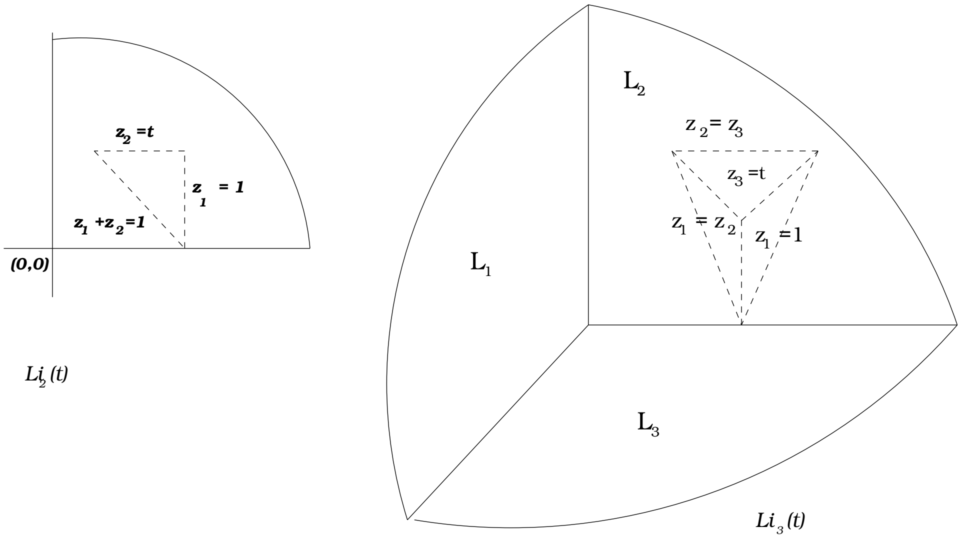


figure 1

In the present paper, which is a continuation of [BGSV1-2] and [G0-3], we express explicitly the Aomoto trilogarithm by classical trilogarithms and investigate the algebric-geometric structures staying behind: different realisations of the weight three motivic complexes. For the Aomoto dilogarithm a similar problem was solved in [BGSV1-2].

The function $\Lambda_3(L; M)$ is defined on configurations, i.e. projective equivalence classes, of $4 + 4$ points in $\mathbb{P}^3$ (vertices of a pair of tetrahedra), while the classical trilogarithm lives on $\mathbb{P}^1$. To build a bridge between these functions we relate each of them with the Grassmannian trilogarithm $\mathcal{P}_3^G$ defined on configurations six points in $\mathbb{P}^2$ (see section 4.1 for a definition of Grassmannian polylogarithms). Our main geometric construction, the map a_3 defined in s. 3.3, (see its different versions on fig. 3, 8 and 9), permits to go from a configuration of $4 + 4$ points in $\mathbb{P}^3$ to configurations of 6 points in $\mathbb{P}^2$. Then we apply the generalized cross-ratio map r_3 from [G0-3] to get to $\mathbb{P}^1$. The map a_3 sheds a new light on the key ansatz from [G0-3] leading to the functional equation for the classical trilogarithm, see fig. 10 and the discussion there.

The Aomoto n-logarithm for $n > 3$ **can not** be expressed by the classical n-logarithm. However the *explicit* relation between the Aomoto n-logarithm, which is defined on configurations of $2 \times (n+1)$ points in $\mathbb{P}^n$, and the Grassmannian n-logarithm, which lives on configurations of $2n$ points in $\mathbb{P}^{n-1}$, should exist for all n. In section 4 we explain how such a relation would give an explicit construction of a certain graded co-Lie algebra $G(F)_\bullet$ over $\mathbb{Q}$, which should be isomorphic to the Lie algebra of the Galois group of the category of mixed Tate motives over a field F. It would be very interesting to relate our approach with the work of M. Hanamura and R. MacPherson [HM]. The cohomology of this Lie algebra should give the appropriate pieces of Quillen's K-theory of the field F modulo torsion.

There are several other candidates for the motivic co-Lie algebra, see [BK], [BMS-BGSV], [G5]. However all of them are constructed as Hopf algebras, so we get the co-Lie algebras as the quotient by the decomposable elements. Our approach should lead directly to a co-Lie algebra. The degree 2 and 3 parts of its standard cochain complex are precisely the Bloch-Suslin complex and the weight three motivic complex defined in [G0-3], so $G(F)_\bullet$ should be the smallest possible realization of the motivic Lie algebra.

Our results partially generalize the work [BGSV] from $\mathbb{P}^2$ to $\mathbb{P}^3$. We discuss in s. 4 what remains to be done. An application to an explicit construction of the weight four motivic complexes will be discussed elsewhere.

I am extremely grateful to Herbert Gangl who wrote a proof of lemma (3.7), helped me to check coefficients in the formulas and and pointed out a lot of errors in the preliminary version of the paper.

The results of this paper were obtained in May 1992 during my stay in the Max-Planck-Institute (Bonn). The paper was written in MPI later on (when I learned how to draw pictures using computer). I am very grateful to the MPI for hospitality and support.

The work was partially supported by the NSF grant DMS-9500010.

2 The scissors congruence groups of pairs of simplices in $P^n(F)$

1.The scissors congruence groups $A_n(F)$. Let me recall some definitions from [BMS], [BGSV]. It is handy to work with configurations of points than with hyperplanes. Let us apply the projective duality $P^n \longrightarrow \hat{P}^n$ which transforms the configuration of $2(n+1)$ hyperplanes $(L_0, ..., L_n; M_0, ..., M_n)$ to a configuration of points $(l_0, ..., l_n; m_0, ..., m_n)$. Abusing notations we will denote it also $(L; M)$, where now $L = (l_0, ..., l_n)$ and $M = (m_0, ..., m_n)$.

The group $A_n(F)$ is generated by configurations of $2(n+1)$ points $(l_0, ..., l_n; m_0, ..., m_n)$ in $P^n(F)$ which are vertices of admissible pairs of simplices subject to the following relations:

1)*Nondegeneracy.* $(L; M) = 0$ if $(l_0, ..., l_n)$ or $(m_0, ..., m_n)$ belong to a hyperplane.

2) *Skew symmetry.* $(\sigma L; M) = (L; \sigma M) = (-1)^{|\sigma|}(L; M)$ for any permutation $\sigma \in S_{n+1}$.

3)*Additivity.* For any configuration $(l_0, ..., l_{n+1})$

$$\sum_{i=0}^{n+1}(-1)^i(l_0, ..., \hat{l}_i, ..., l_{n+1}; m_0, ..., m_n) = 0$$

if all the terms are admissible (additivity in L). A similar condition is imposed for $(m_0, ..., m_{n+1})$ (additivity in M).

$\hat{3}$)*Dual additivity.* For any configuration $(l_0, ..., l_{n+1})$

$$\sum_{i=0}^{n+1}(-1)^i(l_i|l_0, ..., \hat{l}_i, ..., l_{n+1}; m_0, ..., m_n) = 0$$

if all the terms are admissible, and a similar condition is imposed for $(m_0, ..., m_{n+1})$.

4) *Projective invariance.* $(gL; gM) = (L; M)$ for any $g \in PGL_{n+1}(F)$.

These relations reflect properties of Aomoto polylogarithms.

The cross-ratio provides a canonical isomorphism

$$a_1 : A_1(F) \longrightarrow F^*, \quad a_1 : (l_0, l_1; m_0, m_1) \longmapsto r(l_0, l_1, m_0, m_1)$$

2. A coproduct on the generic part of $A_n(F)$ ([BMS], [BGSV]). Set $A_0 = \mathbb{Z}$. Let $A_n^0(F) \subset A_n(F)$ be the subgroup generated by pairs of simplices in generic position. Let us define a coproduct $\nu : A_n^0 \longrightarrow \oplus_k A_k^0 \otimes A_{n-k}^0$. Set for $k, n-k > 0$

$$\nu = \oplus \nu_{n-k,k}, \qquad \nu_{n-k,k} : A_n^0 :\longrightarrow A_{n-k}^0 \otimes A_k^0,$$

$$\nu_{n-k,k} : (l_0, ..., l_n; m_0, ..., m_n) \longmapsto$$

$$\sum_{I,J}(-1)^{\sigma(I,J)}(l_{i_1}...l_{i_k}|l_0, ..., \hat{l}_{i_1}, ..., \hat{l}_{i_k}, ..., l_n; m_0, m_{j_1}, ..., m_{j_{n-k}})$$

$$\otimes(m_{j_1}...m_{j_{n-k}}|l_0, l_{i_1}, ..., l_{i_k}; m_0, ..., \hat{m}_{j_1}, ..., \hat{m}_{j_{n-k}}, ..., m_n)$$

where $I := \{0 < i_1 < ... < i_k\}$, $J := \{0 < j_1 < ... < j_{n-k}\}$. Here $\sigma(I, J) = sign(I, \bar{I}) \cdot sign(J, \bar{J})$, where $sign(I, \bar{I})$ is the sign of the permutation $(1, ..., n) \to (I, \bar{I})$ (similarly $sign(J, \bar{J})$). For example

$$\nu_{2,1} : (l_0, ..., l_3; m_0, ..., m_3) \longmapsto$$

$$(l_3|l_0, l_1, l_2; m_0, m_2, m_3) \otimes (m_2, m_3|l_0, l_3; m_0, m_1) + ...$$

Proposition 2.1 *$(A(F)_\bullet, \nu)$ is a graded coalgebra.*

I will need this statement only in the case of degree 3. In this case it is not hard to deduce it from proposition (2.3) below.

Proposition 2.2 *$P : (L; M) \longrightarrow (M; L)$ defines an antiautomorphism of the graded coalgebra $(A(F)_\bullet, \nu)$.*

Proof follows from the definitions.

3. Another formula for $\nu_{1,n-1}$ and $\nu_{n-1,1}$. Unfortunately in the definition of the coproduct ν we have to choose vertices l_0 in L and m_0 in M first, so the skew-symmetry is not obvious. In the next proposition we give another formula for $\nu_{1,n-1}$ and $\nu_{n-1,1}$ which is skew-symmetric from the beginning and much more convenient.

Proposition 2.3 *(See figure 2)*

$$\nu_{1,n-1} : (l_0, ..., l_n; m_0, ..., m_n) \longmapsto$$

$$- \sum_{i,j=0}^{n} (-1)^{i+j} \Delta(l_i, m_0, ..., \hat{m}_j, ..., m_n) \otimes (l_i | l_0, ..., \hat{l}_i, ..., l_n; m_0, ..., \hat{m}_j, ..., m_n)$$

$$\nu_{n-1,1} : (l_0, ..., l_n; m_0, ..., m_n) \longmapsto$$

$$- \sum_{i,j=0}^{n} (-1)^{i+j} (m_j | l_0, ..., \hat{l}_i, ..., l_n; m_0, ..., \hat{m}_j, ..., m_n) \otimes \Delta(m_j, l_0, ..., \hat{l}_i, ..., l_n)$$

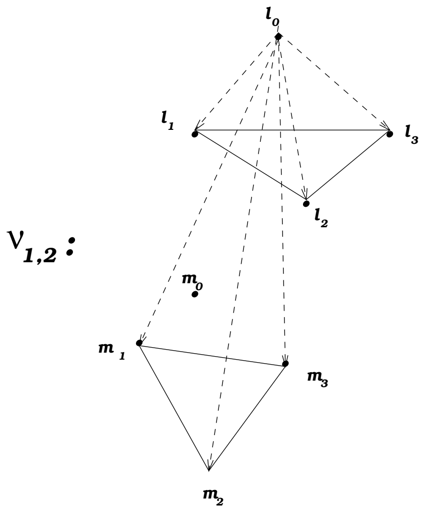


figure 2

Proof. Let us compute

$$\nu_{n-1,1}(l_0,...,l_n;m_0,...,m_n) = \tag{1}$$

$$\sum_{i,j=1}^{n}(-1)^{i+j}(l_i|l_0,...,\hat{l}_i,...,l_n;m_0,...,\hat{m}_j,...,m_n)\otimes\Delta(m_1...\hat{m}_j...m_n|l_0,l_i;m_0,m_j)$$

using the formula

$$r(l_1,l_2,l_3,l_4)=\frac{\Delta(l_1,l_3)\Delta(l_2,l_4)}{\Delta(l_1,l_4)\Delta(l_2,l_3)}$$

for the cross-ratio. We will get

$$\sum_{i,j=1}^{n}(-1)^{i+j}(l_i|l_0,...,\hat{l}_i,...,l_n;m_0,...,\hat{m}_j,...,m_n)\otimes\Delta(l_0,m_0,...,\hat{m}_j,...,m_n) \tag{2}$$

$$-\sum_{i,j=1}^{n}(-1)^{i+j}(l_i|l_0,...,\hat{l}_i,...,l_n;m_0,...,\hat{m}_j,...,m_n)\otimes\Delta(l_i,m_0,...,\hat{m}_j,...,m_n) \tag{3}$$

$$-\sum_{i,j=1}^{n}(-1)^{i+j}(l_i|l_0,...,\hat{l}_i,...,l_n;m_0,...,\hat{m}_j,...,m_n)\otimes\Delta(l_0,m_1,...,m_n) \tag{4}$$

$$+\sum_{i,j=1}^{n}(-1)^{i+j}(l_i|l_0,...,\hat{l}_i,...,l_n;m_0,...,\hat{m}_j,...,m_n)\otimes\Delta(l_i,m_1,...,m_n) \tag{5}$$

Applying to (2) dual additivity in L we can rewrite it as

$$-\sum_{j=1}^{n}(-1)^{j}(l_0|l_1,...,l_n;m_0,...,\hat{m}_j,...,m_n)\otimes\Delta(l_0,m_0,...\hat{m}_j,...,m_n)$$

(3) is already in the desired shape. Applying to (4) first additivity in M and then dual additivity in L we get

$$-(l_0|l_1,...,l_n;m_1,...,m_n)\otimes\Delta(l_0,m_1,...,m_n)$$

Finally applying additivity in M to (5) we get

$$-\sum_{i=1}^{n}(-1)^{i}(l_i|l_0,...,\hat{l}_i,...,l_n;m_1,...,m_n)\otimes\Delta(l_i,m_1,...,m_n)$$

So we conclude that (1) is equal to

$$-\sum_{i,j=0}^{n}(-1)^{i+j}(l_i|l_0,...,\hat{l}_i,...,l_n;m_0,...,\hat{m}_j,...,m_n)\otimes\Delta(l_i,m_0,...,\hat{m}_j,...,m_n)$$

The considerations for $\nu_{1,n-1}$ are similar. The proposition is proved.

3 Main construction

1. The weight two case. Let $B_2(F)$ be the quotient of the free abelian group $\mathbb{Z}[F^*\backslash 1]$ generated by the symbols $\{x\}$ where $x \in F^*\backslash 1$ modulo the subgroup $R_2(F)$ generated by the "five term relations", i.e. by the elements

$$\sum_{i=1}^{5}(-1)^i\{r(x_1, ..., \hat{x}_i, ..., x_5)\}, \qquad x_i \in \mathbb{P}^1(F), \quad x_i \neq x_j$$

Denote by $\{x\}_2$ the image of the generator $\{x\}$ in $B_2(F)$. One can prove (see for example [G1]) that there is a well defined homomorphism

$$\delta_2 : B_2(F) \longrightarrow \Lambda^2 F^*, \quad \{x\}_2 \longmapsto (1-x) \wedge x$$

The complex we get is called the Bloch complex.

In [BGSV1-2] there was defined a homomorphism of complexes

$$\begin{array}{ccc} A_2 & \xrightarrow{\nu} & A_1 \otimes A_1 \\ \downarrow a_2 & & \downarrow a_1 \wedge a_1 \\ B_2 & \xrightarrow{\delta_2} & \Lambda^2 F^* \end{array}$$

Namely

$$a_2(l_0, l_1, l_2; m_0, m_1, m_2) :=< l_0, l_1, l_2; m_0, m_1, m_2 >_2:= \tag{6}$$

$$\sum_{i,j=0}^{2}(-1)^{i+j}\{r(l_i|l_0, ..., \hat{l}_i, ..., l_2, m_0, ..., \hat{m}_j, ..., m_2)\}_2 \in B_2(F)$$

2. The weight three motivic complex related to the classical trilogarithm. Let V_3 be a three dimensional vector space over a field F. Choose a volume form $\omega_3 \in detV_3^*$. For any three vectors l_1, l_2, l_3 in V_3 set

$$\Delta(l_1, l_2, l_3) :=< l_1 \wedge l_2 \wedge l_3, \omega_3 >$$

Let us define the generalized cross-ratio of six generic points $x_1, ..., x_6$ on the plane $\mathbb{P}^2$ by setting

$$r_3(x_1, ..., x_6) := \frac{1}{15}\text{Alt}_6\{\frac{\Delta(\tilde{x}_1, \tilde{x}_2, \tilde{x}_4)\Delta(\tilde{x}_2, \tilde{x}_3, \tilde{x}_5)\Delta(\tilde{x}_3, \tilde{x}_1, \tilde{x}_6)}{\Delta(\tilde{x}_1, \tilde{x}_2, \tilde{x}_5)\Delta(\tilde{x}_2, \tilde{x}_3, \tilde{x}_6)\Delta(\tilde{x}_3, \tilde{x}_1, \tilde{x}_4)}\} \in \mathbb{Z}[F^*\backslash 1] \tag{7}$$

Here $\tilde{x}$ is a vector projecting to the point x. The ratio does not depend on the choise of these vectors.

The group $R_3(F)$ of functional equations for the trilogarithm. I will use the following definition, which is a bit ad hoc. $R_3(F)$ is the group generated by $\{x\}_3 - \{x^{-1}\}_3$, the "seven term" relations (containing actually 7! terms)

$$\sum_{i=1}^{7}(-1)^i r_3(x_1, ..., \hat{x}_i, ..., x_7), \qquad x_i \in \mathbb{P}^2(F) \tag{8}$$

where the points $x_1, ..., x_7$ are in *generic* position in the plane, and Kummer's functional equation for the trilogarithm:

$$K(x,y) := -\{\frac{x(1-y)^2}{y(1-x)^2}\} - \{xy\} - \{\frac{x}{y}\} - 2\{1\}+$$

$$2\Big(\{\frac{-x(1-y)}{(1-x)}\} + \{\frac{x(1-y)}{y(1-x)}\} + \{\frac{-y(1-x)}{1-y}\} + \{\frac{1-x}{1-y}\} + \{y\} + \{x\}\Big)$$

It is may be true that Kummer's relation follows from the generic seven term relations and the one $\{x\}_3 - \{x^{-1}\}_3$. But I just add them to the list.

Remark. A more natural way to define the group $R_3(F)$ is this. We extend the generalized cross-ratio to arbitrary configurations of 6 points on the plane. Then $R_3(F)$ is given by the seven term relations for arbitrary configurations of seven points in the plane. Using the main results of [G1] one can show that both definitions lead to the same group. In particular we get Kummer's relations for a certain degenerate configuration of seven points. To extend the definition of r_3 we take the "limit value" of the definition (7) using $\{0\}_3 = \{\infty\}_3 = 0$. If two of the points x_i coincide or four of them are on a line then $r_3(x_1, ..., x_6) = 0$. For the remaining cases see lemma 3.7 below. To make the exposition shorter I will not use this approach.

Set

$$B_3(F) := \frac{\mathbb{Z}[F^* \backslash 1]}{R_3(F)}$$

Denote by $\{x\}_3$ the image in $B_3(F)$ of the generator $\{x\}$. One can prove ([G1-3]) that there is a well defined homomorphism

$$\delta_3 : B_3(F) \longrightarrow B_2(F) \otimes F^*, \quad \{x\}_3 \longmapsto \{x\}_2 \otimes x$$

We get the weight three motivic complex related to the classical trilogarithm

$$B_3(F) \longrightarrow B_2(F) \otimes F^* \longrightarrow \Lambda^3 F^*$$

where $\{x\}_2 \otimes y \longmapsto (1-x) \wedge x \wedge y$.

Lemma 3.1 *Using the notations introduced in (6)*

$$\delta_3 \circ r_3(l_1, l_2, l_3, l_4, l_5, l_6) = -\frac{1}{18}\mathrm{Alt}_6\Big(< l_1, l_2, l_3; l_4, l_5, l_6 >_2 \otimes \Delta(l_1, l_2, l_3)\Big) =$$
$$-2 < l_1, l_2, l_3; l_4, l_5, l_6 >_2 \otimes \Delta(l_1, l_2, l_3) + \quad 19 \textit{ other terms} \tag{9}$$

Proof. Let $C_n(V_3)$ be the free abelian group generated by the configurations of n generic vectors (i.e. n-tuples of vectors in generic position modulo the action of $GL(V_3)$) in V_3. Let

$$d : C_n(V_3) \to C_{n-1}(V_3), \quad d(l_1, ..., l_n) := \sum_{i=1}^{n} (-1)^{i-1} (l_1, ..., \hat{l}_i, ..., l_n)$$

Define a homomorphism $C_5(V_3) \to B_2(F) \otimes F^*$ by setting

$$f_5(3)(l_1, ..., l_5) := \frac{1}{2}\mathrm{Alt}_5\Big(\{r(l_1|l_2, ..., l_5)\}_2 \otimes \Delta(l_1, l_2, l_3)\Big)$$

According to theorem 2.3 in [G3] the following diagram is commutative:

$$\begin{array}{ccc} C_6(V_3) & \xrightarrow{d} & C_5(V_3) \\ r_3 \downarrow & & \downarrow f_5(3) \\ B_3(F) & \xrightarrow{\delta_3} & B_2(F) \otimes F^* \end{array}$$

Therefore

$$\delta_3 \circ r_3(l_1, l_2, l_3, l_4, l_5, l_6) = -\frac{1}{2}\mathrm{Alt}_6\Big((l_1|l_2, l_3, l_4, l_5)_2 \otimes \Delta(l_1, l_2, l_3)\Big)$$

which equals to (9).

3. A homomorphism between the weight three motivic complexes. We have constructed in chapter 2 a complex

$$A_3^0 \xrightarrow{\nu_{2,1} \oplus \nu_{1,2}} A_2 \otimes A_1 \oplus A_1 \otimes A_2 \xrightarrow{\nu_{1,1} \otimes Id - Id \otimes \nu_{1,1}} A_1 \otimes A_1 \otimes A_1$$

Let us define a homomorphism of complexes

$$\begin{array}{ccccc} A_3^0 & \xrightarrow{\nu} & A_2 \otimes A_1 \oplus A_1 \otimes A_2 & \longrightarrow & A_1 \otimes A_1 \otimes A_1 \\ \downarrow a_3 & & \downarrow a_2 \wedge a_1 & & \downarrow \wedge^3 a_1 \\ B_3 & \xrightarrow{\delta_3} & B_2(F) \otimes F^* & \longrightarrow & \Lambda^3 F^* \end{array}$$

where $a_2 \wedge a_1(x_2 \otimes x_1 + y_1 \otimes y_2) := x_2 \otimes x_1 - y_2 \otimes y_1$ and $\wedge^3 a_1(x_1 \otimes x_2 \otimes x_3) = x_1 \wedge x_2 \wedge x_3$.

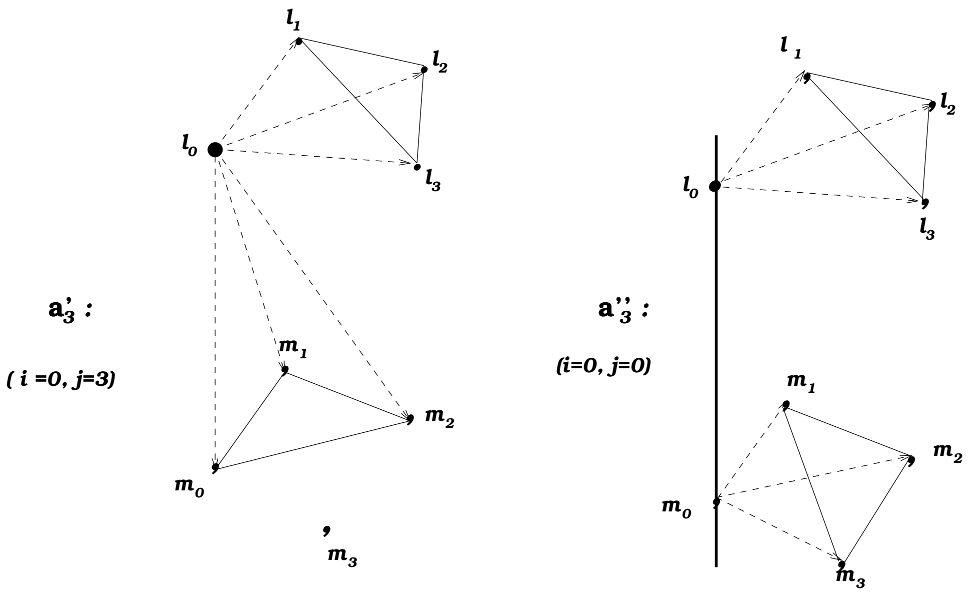


figure 3

Let $x_1, x_2, x_3; y_1, y_2, y_3$ be six different points on a line. Set

$$\mu_3(x_1, x_2, x_3; y_1, y_2, y_3) =$$
$$\frac{1}{4} \cdot \mathrm{Alt}_{(x_1,x_2,x_3)(y_1,y_2,y_3)} \Big(\{r(x_1, y_2, x_2, y_1)\}_3 - \{r(x_1, y_1, x_2, y_2)\}_3 \Big)$$

This formula, which contains 18 terms, means simply that $m_3(x_1, x_2, x_3; y_1, y_2, y_3)$ is skewsymmetric with respect to x_1, x_2, x_3 and y_1, y_2, y_3 and a typical term is $\{r(x_1, y_2, x_2, y_1)\}_3$.

We set

$$a_3 := \frac{1}{6} a_3' - \frac{1}{3} a_3''$$

where a_3' and a_3'' are defined on generators by the following formula (see fig. 2):

$$a_3'(l_0, l_1, l_2, l_3; m_0, m_1, m_2, m_3) := \sum_{i,j=0}^{3} (-1)^{i+j} r_3(l_i | l_0, ..., \hat{l}_i, ..., l_3, m_0, ..., \hat{m}_j, ..., m_3)$$

$$a_3''(l_0, ..., l_3; m_0, ..., m_3) := \sum_{i,j=0}^{3} (-1)^{i+j} \cdot \mu_3(l_i, m_j | l_0, ..., \hat{l}_i, ..., l_3; m_0, ..., \hat{m}_j, ..., m_3)$$

Denote by $\tilde{A}_3^0$ the free abelian group generated by the generators of the group A_3^0, i.e. by pairs of tetrahedra in generic position.

Theorem 3.2 *The diagram*

$$\begin{array}{ccc}
\tilde{A}_3^0 & \stackrel{\nu}{\longrightarrow} & A_2 \otimes A_1 \oplus A_1 \otimes A_2 \\
\downarrow a_3 & & \downarrow a_2 \wedge a_1 \\
B_3 & \stackrel{\delta_3}{\longrightarrow} & B_2(F) \otimes F^*
\end{array}$$

is commutative.

Remark. The proof of this theorem is the crucial point of the paper. The commutativity of the diagram is what we really wanted for the homomorphism a_3. Surprisingly neither homomorphism a_3' nor a_3'' make the diagram commutative, even up to a scalar. Only their sum does the job. Moreover, theorem (3.2) "morally" implies that a_3 should be a homomorphism of groups. We will prove this later on.

Proof. Because of the skew-symmetry of the formulas for ν (see proposition 2.3) it is sufficient to compute for

$$(a_2 \wedge a_1) \circ \nu(l_0, ..., l_3; m_0, ..., m_3) \quad \text{and} \quad \delta_3 \circ a_3(l_0, ..., l_3; m_0, ..., m_3)$$

the $B_2(F)$-factors of the following elements of F^*:

$$\Delta(m_0, m_1, m_2, m_3), \quad \Delta(l_0, m_0, m_1, m_2), \quad \Delta(l_0, l_1, m_0, m_1),$$

$$\Delta(l_0, l_1, l_2, m_0), \quad \Delta(l_0, l_1, l_2, l_3)$$

Since $\nu(L; M) = \nu(M, L)$ by proposition (2.2), and $a_3(L; M) = a_3(M; L)$ by lemma (3.5), we see that it is sufficient to consider only the first three of them.

It follows from proposition (2.3) that in $(a_2 \wedge a_1) \circ \nu(l_0, ..., l_3; m_0, ..., m_3)$ appears only

$$< l_0|l_1, l_2, l_3; m_1, m_2, m_3 >_2 \otimes \Delta(l_0, m_1, m_2, m_3) \tag{10}$$

Step 1. Let us do the computations for $\delta_3 \circ a_3(l_0, ..., l_3; m_0, ..., m_3)$. The crucial and most nontrivial case is the term with $\Delta(l_0, l_1, m_0, m_1)$. If the diagram is commutative it must be zero because of the observation we just made. The only summands in $a_3'(l_0, ..., l_3; m_0, ..., m_3)$ that give a contribution to this term are:

$$-r_3(l_0|l_1, l_2, l_3, m_0, m_1, m_2) + r_3(l_0|l_1, l_2, l_3, m_0, m_1, m_3) +$$

$$r_3(l_1|l_0, l_2, l_3, m_0, m_1, m_2) - r_3(l_1|l_0, l_2, l_3, m_0, m_1, m_3)$$

Lemma (3.1) shows using the symmetry $1 <-> 4, 2 <-> 5$ that the term with $\Delta(l_0, l_1, m_0, m_1)$ in $\delta_3 \circ a_3(l_0, ..., l_3; m_0, ..., m_3)$ is

$$-2\Big(- < l_0|l_1, m_0, m_1; l_2, l_3, m_2 >_2 + < l_0|l_1, m_0, m_1; l_2, l_3, m_3 >_2 +$$

$$< l_1|l_0,m_0,m_1;l_2,l_3,m_2 >_2 - < l_1|l_0,m_0,m_1;l_2,l_3,m_3 >_2\Big)\otimes\Delta(l_0,l_1,m_0,m_1) =$$

$$2\cdot \mathrm{Alt}_{l_0l_1}\mathrm{Alt}_{m_2m_3}\Big(< l_0|l_1,m_0,m_1;l_2,l_3,m_2 >_2\Big)\otimes\Delta(l_0,l_1,m_0,m_1) \qquad (11)$$

Step 2. The contribution for $\Delta(l_0,l_1,m_0,m_1)$ in $\delta_3\circ a_3''(l_0,...,l_3;m_0,...,m_3)$ is coming from

$$\delta_3\circ\mu_3\Big((l_0m_0|l_1,l_2,l_3;m_1,m_2,m_3) - (l_0m_1|l_1,l_2,l_3;m_0,m_2,m_3)$$

$$-(l_1m_0|l_0,l_2,l_3;m_1,m_2,m_3) + (l_1m_1|l_0,l_2,l_3;m_0,m_2,m_3)\Big)$$

And the contribution is equal to

$$\mathrm{Alt}_{(l_0,l_1);(m_2,m_3);(m_0,m_1);(l_2,l_3)}\Big((l_0m_0|l_1,m_1,l_2,m_2)_2\Big)\otimes\Delta(l_0,l_1,m_0,m_1) =$$

$$\mathrm{Alt}_{(l_0,l_1);(m_2,m_3)}\Big(< l_0|m_0,l_1,m_1;l_2,m_2,l_3 >_2\Big)\otimes\Delta(l_0,l_1,m_0,m_1) =$$

$$\mathrm{Alt}_{(l_0,l_1);(m_2,m_3)}\Big(< l_0|l_1,m_0,m_1;l_2,l_3,m_2 >_2\Big)\otimes\Delta(l_0,l_1,m_0,m_1)$$

Comparing the last formula with (11) we see that

$$\tfrac{1}{2}a_3' - a_3'' \textit{ has contribution } 0\otimes\Delta(l_0,l_1,m_0,m_1)$$

Step 3. Now look at the terms of $\delta_3\circ a_3(l_0,...,l_3;m_0,...,m_3)$ with $\Delta(m_0,m_1,m_2,m_3)$ and $\Delta(l_0,m_1,m_2,m_3)$.

1) $\Delta(m_0,m_1,m_2,m_3)$ does not appear in $a_3(l_0,...,l_3;m_0,...,m_3)$.

2i)Using proposition (2.3) we get that $\Delta(l_0,m_1,m_2,m_3)$ appears in $\delta_3\circ a_3'(l_0,...,l_3;m_0,...,m_3)$ as

$$2 < l_0|l_1,l_2,l_3;m_1,m_2,m_3 >_2 \otimes\Delta(l_0,m_1,m_2,m_3)$$

2ii) Let us compute $\delta_3\circ a_3''(l_0,...,l_3;m_0,...,m_3)$. The nontrivial nontrivial contribution of $\Delta(l_0,m_1,m_2,m_3)$ is given by

$$-1/2\mathrm{Alt}_{(m_1,m_2,m_3)}\mu_3(l_0m_1|l_1,l_2,l_3;m_0,m_2,m_3)$$

and it is equal to

$$1/2\mathrm{Alt}_{(l_1,l_2,l_3)(m_1,m_2,m_3)}(l_0m_1|l_1,m_2,l_2,m_3)_2\otimes\Delta(l_0,m_1,m_2,m_3) =$$

$$-2 < (l_0|m_1,m_2,m_3;l_1,l_2,l_3) > \otimes\Delta(l_0,m_1,m_2,m_3)$$

For the last step use $(l_0m_1|l_1,m_2,l_2,m_3)_2 = -(l_0m_1|m_2,m_3,l_1,l_2)_2$. So the contribution of $\frac{1}{2}a_3' - a_3''$ is

$$3\cdot < l_0|l_1,l_2,l_3;m_1,m_2,m_3 >_2 \otimes\Delta(l_0,m_1,m_2,m_3)$$

It remains to compare this answer with (10). Theorem (3.2) is proved.

Theorem 3.3 *a_3 is a homomorphism of groups.*

We start the proof with

Proposition 3.4 *Both a_3' and a_3'' send relations 2), 3), 4) to zero.*

Proof. This is clear for relations 2) and 4). It is also clear that a_3'' is additive in L and in M. To check that a_3'' sends the dual additivity in L to zero notice that the typical term for $a_3''(\sum_{i=0}^{3}(-1)^i \cdot (l_i|l_0,...,\hat{l}_i,...,l_4;m_0,...,m_3))$ is

$$\pm\hat{m}_3(l_il_jm_k|l_0,...,\hat{l}_i,...,\hat{l}_j,...,l_4;m_0,...,\hat{m}_k,...,m_3)$$

It is symmetric in l_i, l_j. So we get 0 after alternation in $(l_0,...,l_4)$.

Similarly using the skewsymmetry of r_3 we immediately see that the map a_3' sends dual additivity in L to zero. The additivity in M of a_3' is obvious. Therefore the proposition follows immediately from the following lemma.

Lemma 3.5 $a_3'(L;M) = a_3'(M;L); \qquad a_3''(L;M) = a_3''(M;L)$

Proof. The statement about a_3'' is clear from the definition. Let us prove it for a_3'. Applying the 7-term relation for the configuration of 7 points in $\mathbb{P}^2$ $(l_i|l_0,...,\hat{l}_i,...,l_3,m_0,...,m_3)$ we get

$$a_3'(l_0,,...,l_3;m_0,...,m_3) = -\sum_{0\leq i\neq j\leq 3}\gamma(i,j)r_3(l_i|l_0,...,\hat{l}_i,...,\hat{l}_j,...,l_3,m_0,...,m_3) \tag{12}$$

where $\gamma(i,j) = (-1)^{i+j}$ if $i < j$ and $-(-1)^{i+j}$ otherwise. Applying the dual 7-term relation to the configuration $(l_0,...,\hat{l}_i,...,l_3,m_0,...,m_3)$ of 7 points in $\mathbb{P}^3$ we can rewrite (12) as

$$-\sum_{i,j=0}^{3}(-1)^{i+j}r_3(m_i|l_0,...,\hat{l}_j,...,l_3,m_0,...,\hat{m}_i,...,m_3)$$

It remains to interchange the l- and m- tetrahedra using the skewsymmetry. The lemma is proved.

Neither a_3' nor a_3'' send the relation 1) to zero. Only their weighted sum a_3 does the job. To show this we have to prove the proposition 3.6 below.

Let $L_0,...,L_3$ be 4 lines and $m_0,...,m_3$ 4 points in P^2. Let

$$(L_0,...,\hat{L}_i,...,L_3;m_0,...,\hat{m}_j,...,m_3)$$

be a pair of triangles where the first triangle is given by its sides and the second one by its vertices (see fig. 4). For example

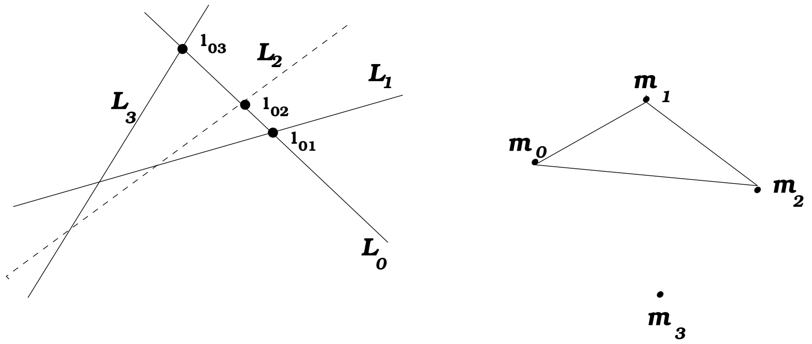
$$(L_1,L_2,L_3;m_1,m_2,m_3) = (l_{23},l_{13},l_{12},m_1,m_2,m_3)$$

where $l_{ij} := L_i \cap L_j$, so the first three points are the vertices of the triangle (L_1,L_2,L_3).

Proposition 3.6 *Let $L_0,...,L_3$ be 4 lines and $m_0,...,m_3$ 4 points in P^2. Then*

$$\sum_{i,j=0}^{3}(-1)^{i+j}r_3(L_0,...,\hat{L}_i,...,L_3;m_0,...,\hat{m}_j,...,m_3) = \tag{13}$$

$$2\sum_{i,j=0}^{3}(-1)^{i+j}r_3\cdot\mu_3(m_i|l_{j0},...,\hat{l}_{jj},...,l_{j3};m_0,...,\hat{m}_i,...,m_3) \tag{14}$$



the pair of triangles $(L_0,L_1,L_3;m_0,m_1m_2)$

figure 4

Proof. Applying the 7-term relation to the configuration of 7 points formed by the vertices of the triangle $L_0,...,\hat{L}_j,...,L_3$ and $m_0,...,m_3$ we rewrite (13) as

$$-\frac{1}{2}\mathrm{Alt}_{(L_0,...,L_3)}r_3(l_{13},l_{23},m_0,m_1,m_2,m_3)$$

(We alternate the indices $0,1,2,3$ in the l-variables only). It is equal to

$$-r_3(l_{13},l_{23},m_0,m_1,m_2,m_3)+r_3(l_{03},l_{23},m_0,m_1,m_2,m_3)-r_3(l_{03},l_{13},m_0,m_1,m_2,m_3)$$

$$+\ 9 \text{ other terms}$$

Consider the three intersection points of the line L_i with the other lines and add to them the points m_0,m_1,m_2,m_3. For instance $(l_{03},l_{13},l_{23},m_0,m_1,m_2,m_3)$ is the configuration related to the line L_3. Applying the 7-term relation to these configurations we rewrite the previous formula as

$$-r_3\Big(\sum_{i=0}^{3}(-1)^i(l_{03},l_{13},l_{23},m_0,...,\hat{m}_i,...,m_3)-\sum_{i=0}^{3}(-1)^i(l_{10},l_{20},l_{30},m_0,...,\hat{m}_i,...,m_3) \tag{15}$$

$$+\sum_{i=0}^{3}(-1)^i(l_{21},l_{31},l_{01},m_0,...,\hat{m}_i,...,m_3)-\sum_{i=0}^{3}(-1)^i(l_{32},l_{02},l_{12},m_0,...,\hat{m}_i,...,m_3)\Big) \tag{16}$$

Now comes a little trick: we will use the fact that one can extend the generalized cross ratio to certain degenerate configurations of six points such that the seven term relation holds.

Lemma 3.7 *Suppose the points l_1,l_2,l_3 are on the same line. Then*

$$r_3(l_1,l_2,l_3,m_1,m_2,m_3) = -2\mu_3(l_1,l_2,l_3;m_1,m_2,m_3)$$

Remark. It is easy to check that

$$\delta_3 \circ (r_3 + 2\mu_3)(l_1,l_2,l_3,m_1,m_2,m_3) = 0$$

This implies that $(r_3 + 2\mu_3)(l_1,l_2,l_3,m_1,m_2,m_3)$ is a functional equation for the trilogarithm. The following proof verifies that it is already in the group R_3 defined above (so we do not have to enlarge this group).

Proof. The following proof based on direct computation using $\{0\}_3 = \{\infty\}_3 = 0$ was provided to me by Herbert Gangl. Consider the following special configuration of 6 points on the plane (fig. 5)

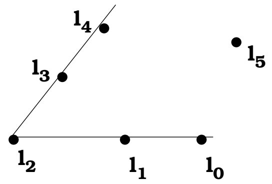


figure 5

given by the columns of the matrix

$$\mathcal{C}(b,c) := \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & c & 1 & b & 1 & 0 \end{pmatrix}$$

It is sufficient to prove the lemma for this configuration, since the general case will follow by the seven term relation.

Set

$$\tilde{l}_3\{x\} := \{1-x^{-1}\}_3 - \{1-x\}_3$$

$$M_3(\mathcal{C}(b,c)) := \tilde{l}_3\Big(\{\frac{b-c}{1-c}\}_3 - \{b\}_3 + \{\frac{b(1-c)}{b-c}\}_3 - \{1-c\}_3 + \{\frac{b-c}{b}\}_3\Big)$$

A configuration of six points x_1,x_2,x_3,l_1,l_2,l_3 on the plane, three of which, x_1,x_2,x_3, are on a line L, is determined completely by the configuration $(x_1,x_2,x_3,n_1,n_2,n_3)$ of 3+3 points on the line L, where the point $n_1 := l_2l_3 \cap L$ and so on, see fig. 6.

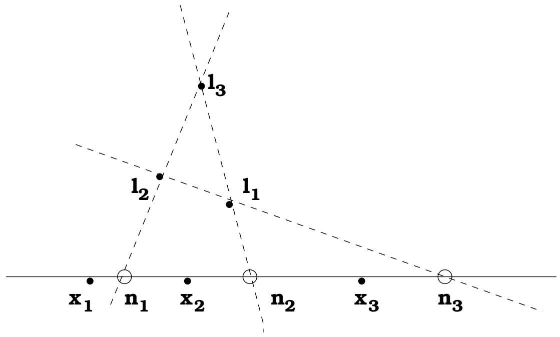


figure 6

(Recall that by configuration we always mean its projective equivalence class.)

Let $\tilde{\mathcal{C}}(b,c)$ be the configuration of six points on the line corresponding to the configuration $\mathcal{C}(b,c)$ by this rule. Then

$$M_3(\mathcal{C}(b,c)) = -\mu_3(\tilde{\mathcal{C}}(b,c))$$

One has

$$15 \cdot r_3(\mathcal{C}(b,c)) = -18\{1-b\}_3 - 12\{\frac{b}{1-c}\}_3 - 18\{\frac{b-c}{b-1}\}_3 - 18\{c\}_3$$

$$-12\{\frac{1-b}{c}\}_3 - 24\{\frac{b(c-1)}{c(b-1)}\}_3 - 6\{\frac{c}{(c-b)(1-b)}\}_3 - 6\{\frac{c(b-c)}{b-1}\}_3 - 18\{\frac{c-b}{c}\}_3$$

$$+18\{1-b^{-1}\}_3 + 12\{b-c\}_3 + 18\{\frac{b-1}{c-1}\}_3 + 12\{\frac{b-1}{c}\}_3 + 18\{\frac{b}{c}\}_3$$

$$+24\{\frac{b-c}{c(b-1)}\}_3 + 6\{\frac{b(b-1)}{c(c-1)}\}_3 + 18\{1-c^{-1}\}_3 + 6\{\frac{(1-b)(1-c)}{b\cdot c}\}_3$$

Recall two Kummer relations:

$$6K(\frac{b}{1-c},\frac{c}{1-b}) = 12\{\frac{b}{1-c}\}_3+12\{\frac{c}{1-b}\}_3+12\{\frac{1-b}{1-c}\}_3+12\{\frac{c}{c-1}\}_3+12\{\frac{b}{b-1}\}_3$$

$$+12\{\frac{c}{b}\}_3 - 6\{\frac{bc}{(1-b)(1-c)}\}_3 - 6\{\frac{b(b-1)}{c(c-1)}\}_3 - 6\{\frac{c(1-b)}{b(1-c)}\}_3 - 12\{1\}_3$$

and

$$-6K(b-c,\frac{b-1}{c}) = -12\{b-c\}_3-12\{\frac{b-1}{c}\}_3-12\{c\}_3-12\{1-b\}_3-12\{\frac{c}{c-b}\}_3$$

$$-12\{\frac{b-1}{b-c}\}_3 + 6\{\frac{(c-b)(1-b)}{c}\}_3 + 6\{\frac{c(b-c)}{b-1}\}_3 + 6\{\frac{c(b-1)}{b-c}\}_3 + 12\{1\}_3$$

Then one computes, adding the three expressions above,

$$15r_3(\mathcal{C}(b,c)) + 6K(\frac{b}{1-c}, \frac{c}{1-b}) - 6K(b-c, \frac{b-1}{c}) =$$

$$30\Big(-\{1-b\}_3 + \{\frac{b}{b-1}\}_3 - \{\frac{b-c}{b-1}\}_3 + \{\frac{1-b}{1-c}\}_3 - \{\frac{c(1-b)}{b(1-c)}\}_3 + \{\frac{c-b}{c(1-b)}\}_3$$

$$-\{c\}_3 + \{1-c^{-1}\}_3 - \{\frac{c-b}{c}\}_3 + \{\frac{c}{b}\}_3\Big) = -30\mu_3(\mathcal{C}(b,c))$$

which is exactly what we wanted. The lemma, and hence theorem (3.3) is proved.

Comparing this lemma with (15), (16) we get the proposition.

Another way to proceed. The right hand side of (13) considered as a function of 4 lines and 4 points on the plane satisfies the 5 term relation with respect to lines as well as points. Using this observation it is sufficient to check the proposition for the degenerate configuration of lines and points when 3 of the lines L_i pass through a point and 3 points among the m_i are on a line (see fig. 7).

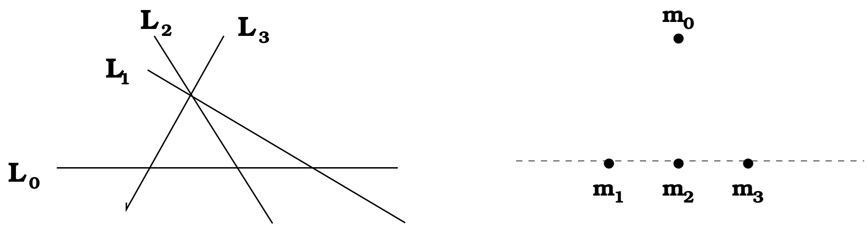


figure 7

4. Another way to define the homomorphism a_3. Let us associate to 8 points $(l_0,...,l_3; m_0,...,m_3)$ in $\mathbb{P}^3$ a degenerate configuration of 6 points on the plane, denoted

$$d(l_i, m_j || l_0, ..., \hat{l}_i, ..., l_3; m_0, ..., \hat{m}_j, ..., m_3)$$

as follows. Consider the configuration

$$(l_i | l_0, ..., \hat{l}_i, ..., l_3; m_0, ..., m_3)$$

of 7 points in $\mathbb{P}^2$. We construct out of them a degenerate configuration of 6 lines in the plane. Let L_{ij} (resp (M_{ij}) be the line through the points l_i and

l_j (resp. m_i and m_j). Take the three lines formed by the sides of the triangle $(l_i|l_0, ..., \hat{l}_i, ..., l_3)$ and add to them the three lines $M_{j0}, ...\hat{M}_{jj}, ..., M_{j3}$ connecting m_j with the other three m-points. For example

$$d(l_0, m_0||l_1, l_2, l_3; m_1, m_2, m_3) = (L_{23}, L_{13}, L_{12}, M_{01}, M_{02}, M_{03})$$

Let $\bar{C}_{2n}(F)$ be the free abelian group generated by the configurations of *arbitrary* $2n$ F-points in $\mathbb{P}^{n-1}$. One can define a_3 as a composition

$$A_3^0(F) \xrightarrow{p_3} \bar{C}_6(P^2)/\{7 \text{ term relations}\} \xrightarrow{r_3} B_3(F)$$

where (see fig. 8)

$$p_3(l_0, l_1, l_2, l_3; m_0, m_1, m_2, m_3) := \sum_{i,j=0}^{3} \Big((-1)^{i+j}(l_i|l_0, ..., \hat{l}_i, ..., l_3, m_0, ..., \hat{m}_j, ..., m_3)$$

$$+2 \cdot d(l_i, m_j||l_0, ..., \hat{l}_i, ..., l_3; m_0, ..., \hat{m}_j, ..., m_3)\Big) \qquad (17)$$

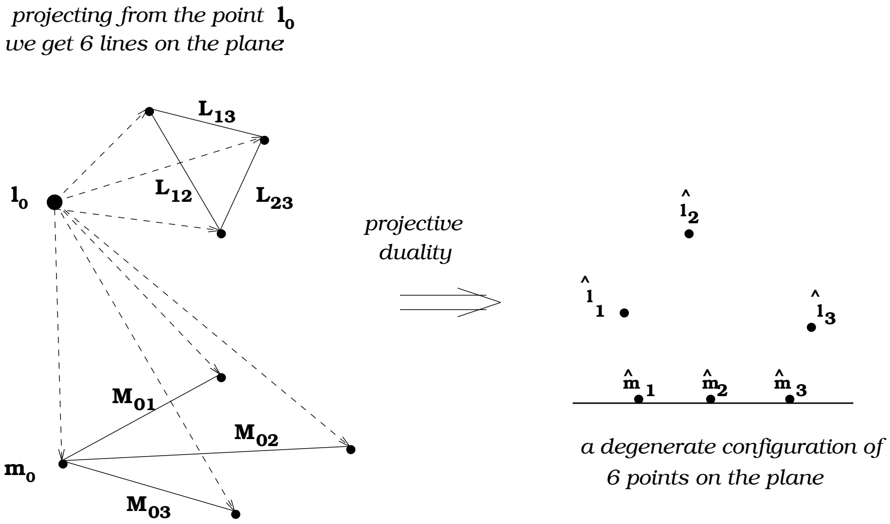


figure 8

There is a different candidate for the term (17). We define a configuration

$$\hat{d}(l_i, m_j||l_0, ..., \hat{l}_i, ..., l_3; m_0, ..., \hat{m}_j, ..., m_3) \qquad (18)$$

of $3+3$ points on a line as follows. Consider the planes L_i and M_j in $\mathbb{P}^3$ (see fig. 9 for $i = 0, j = 0$.) Their intersection is a line $L_i \cap M_j$. The sides of the triangles $(l_0, ..., \hat{l}_i, ..., l_3)$ and $(m_0, ..., \hat{m}_j, ..., m_3)$ cut this line in $3+3$ points. This is the configuration (18) we promised to define.

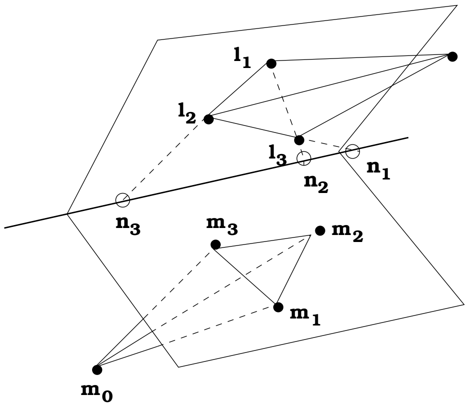


figure 9

Recall that we can think of a configuration of $3+3$ points on a line as of a configuration of six points on a plane (see fig. 6). So we can describe configuration (18) by the configuration of points $(n_0, ..., \hat{n}_i, ..., n_3, m_0, ..., \hat{m}_j, ..., m_3)$ on the plane M_j, or by a similar configuration on the plane L_i.

The definition of (18) given on fig. 9 is projectively dual to the one on fig. 8. This means that if we consider the points l_i, m_j as planes in the dual space $\hat{\mathbb{P}}^3$ then (17) corresponds to (18).

Remark. One has (mainly thanks to lemma (3.7))

$$\sum_{i,j=0}^{3} (-1)^{i+j} \Big(d(l_i, m_j || l_0, ..., \hat{l}_i, ..., l_3; m_0, ..., \hat{m}_j, ..., m_3) - \hat{d}(l_i, m_j || l_0, ..., \hat{l}_i, ..., l_3; m_0, ..., \hat{m}_j, ..., m_3) \Big) = 0 \tag{19}$$

Let us imagine that we would be able to prove this using only the (possibly degenerate) seven term relations but not lemma (3.7). Then it is straightforward to show that p_3 sends to zero all the defining relations for the group A_3 *except* the degeneracy relation. So we can define the group R_3 by adding to the (possibly degenerate) seven term relations the image of the degeneracy relations, i.e. the relations from proposition (3.6). Moreover we get a nice free gift: now we can skip lemma (3.7) together with its computational proof. So instead of the mysterious relation (8) for the classical trilogarithm which we think of as a relation

$$\text{generic configuration of 6 points on } \mathbb{P}^2 \quad = \sum \quad \text{of degenerate configurations}$$

(and which *does not* follow from the seven term relations between the configurations of six points on the plane), we would have the geometrically natural relations from proposition (3.6). However I was not able to prove (19) without using lemma (3.7). I wish somebody will try.

5. The key ansatz from [G0-3]. Consider the following admissible pair of tetrahedra which represents 0 on A_3 (see fig 10).

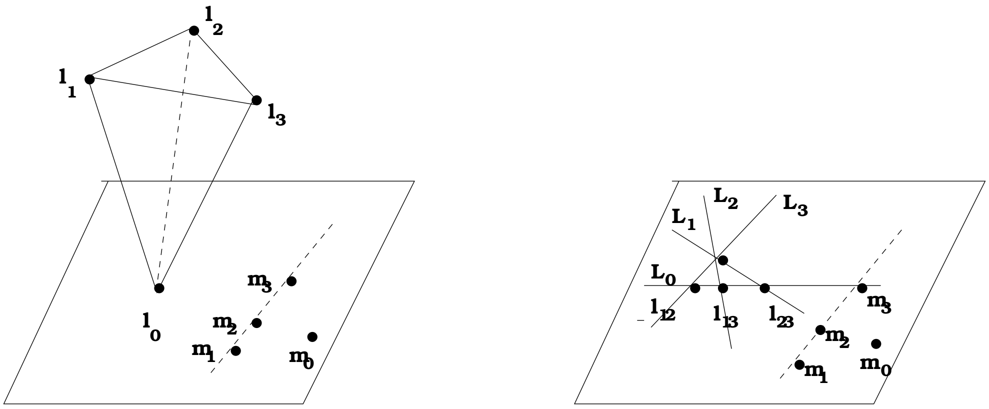


figure 10

Here l_0, m_0, m_1, m_2, m_3 are on the same plane. The right hand side of the picture illustrates the computation of the homomorphism a_3.

Then

$$a_3'(L, M) = (l_{12}, l_{13}, l_{23}, m_1, m_2, m_3)$$

Indeed, all the other terms in (15)-(16) vanish since by definition a degenerate configuration of six points on the plane represents zero in B_3 if two of them coincide or four are on the same line. The condition that for such (L, M) one has $a_3(L, M) = (a_3' + 2a_2'')(L, M) = 0$ just means that we get a formula expressing the configuration $l_{12}, l_{13}, l_{23}, m_1, m_2, m_3$ as a sum of configurations like on fig. 11 (each of them corresponds to a generator $\{x\}_3$ of the group B_3).

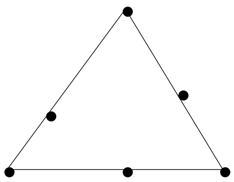


figure 11

6. A formula for $\Lambda_3(L, M)$. One can define (see [BGSV]) a commutative product map

$$\mu : A_k \otimes A_l \to A_{k+l}$$

Then one should have a structure of the Hopf algebra on $A_\bullet$ given by the coproduct ν and the product μ (at the present moment there are still some problems with proving that $\nu \circ \mu = \mu \otimes \mu(\nu)$). Let

$$P_n := \oplus_{k+l=n}\mu(A_k \otimes A_l) \qquad (k > 0, l > 0)$$

Consider the homomorphism $\pi_3 : A_3 \to A_3$ given by

$$\pi_3(x) := x - \frac{1}{2}\mu \circ \nu(x) + \frac{1}{3}\mu \circ \nu_{1,1,1}(x)$$

where $\nu_{1,1,1} : A_3 \to A_1 \otimes A_1 \otimes A_1$ is the composition

$$A_3 \xrightarrow{\nu_{2,1}} A_1 \otimes A_1 \xrightarrow{\nu_{1,1}\otimes id} A_1 \otimes A_1 \otimes A_1$$

Lemma 3.8 *π_3 is a projector and $Ker\pi_3 = P_3$.*

See also section 4.3 for a generalization.

Proof. Let us check that $P_3 \subset Ker\pi_3$. This implies $\pi_3^2 = \pi_3$. Since $\pi_3 = id$ on A_3/P_3 we have $Ker\pi_3 = P_3$.

Suppose $\nu_{1,1}(y_2) = \sum_i y_i' \otimes y_i'' \subset A_1 \otimes A_1$. Let $x_1 \in A_1$. Denote by $\cdot$ the product in $A_\bullet$. Then

$$\nu \circ \mu(y_2 \cdot x_1) = \sum_i (x_1 \cdot y_i' \otimes y_i'' + y_i' \otimes y_i'' \cdot x_1) + y_2 \otimes x_1 + x_1 \otimes y_2)$$

So a simple calculation proves the assertion.

One has

$$\pi_3(l_3(x)) = l_3(x) - \frac{1}{2}\mu(l_2(x) \otimes x) + \frac{1}{12}\mu(l_1(x) \otimes x \otimes x) \tag{20}$$

where $l_i(x)$ is the generator of A_n corresponding to the classical n-logarithm. For $n = 2$ and $n = 3$ it is given by the picture on fig.1. Set

$$L_3(x) := Li_3(x) - \frac{1}{2}Li_2(x)\log x + \frac{1}{12}Li_1(x)(\log x)^2$$

Theorem 3.9

$$\Lambda_3(\pi_3(L, M)) = L_3(a_3(L, M)) \tag{21}$$

Remark. To get a local coincidence of two multivalued analytic functions in this formula we should choose appropriate cycles of integration. The theorem claims in particular that it is possible to do this.

Proof. The main result of this paper implies that

$$\nu(\pi_3(L, M)) = \nu(l_3 \circ a_3(L, M))$$

Since the differential of $\Lambda_3(L, M)$ is determined by the coproduct of (L, M) (see the lemma below) this implies that the differentials of both sides of (21) coincide. So the difference between the left and right hand sides is a constant. Considering the additivity relation in L (which has odd number of terms) we deduce that this constant is zero.

Lemma 3.10 *Let $(L, M) \in A_n(\mathbb{C})$ and the $A_{n-1} \otimes A_1$ component of $\nu(L, M)$ is $\sum x_i \otimes y_i$. Then*

$$d\Lambda_3(L, M) = \sum d\Lambda_3(x) d\log y_i$$

This is a particular case of a general fact about the differential of the period of n-framed mixed Tate motives. A direct proof can be given by an explicit calculation using proposition 2.3.

4 Configurations of $2n$ points in $\mathbb{P}^{n-1}$, Grassmannian n-logarithms and motivic co-Lie algebra of a field

1. Grassmannian polylogarithms. Let me recall the construction of the Grassmannian polylogarithm function $\mathcal{P}_n^G(h_1, ..., h_{2n})$ given in [G]. It is a function on configurations of *arbitrary* $2n$ hyperplanes in $\mathbb{CP}^{n-1}$.

Let $f_1, ..., f_m$ be m complex-valued functions on a manifold X. We attach to them the following $(m-1)$-form. Let $c_{j,m} := \frac{1}{(2j+1)!(n-2j-1)!}$. Set

$$r_{m-1}(f_1, ..., f_m) := \tag{22}$$

$$\frac{1}{(2\pi i)^m} \text{Alt}_m \sum_{j \geq 0} c_{j,m} \log |f_1| d\log |f_2| \wedge ... \wedge d\log |f_{2j+1}| \wedge di \arg f_{2j+2} \wedge ... \wedge di \arg f_m$$

Now let f_i be a rational function on $\mathbb{CP}^{n-1}$ with the divisor $(f_i) = h_i - h_{2n}$, $i = 1, ..., 2n-1$. Then

$$\mathcal{P}_n^G(h_1, ..., h_{2n}) := \int_{\mathbb{CP}^{n-1}} r_{2n-2}(f_1, ..., f_{2n-1})$$

Theorem 4.1 *$\mathcal{P}_n^G(h_1, ..., h_{2n})$ satisfies the following functional equations:*

a) For any $2n+1$ hyperplanes $h_1, ..., h_{2n+1}$ in $\mathbb{CP}^{n-1}$ one has

$$\sum_{i=1}^{2n+1} (-1)^i \mathcal{P}_n^G(h_1, ..., \hat{h}_i, ..., h_{2n+1}) = 0$$

b) For any $2n+1$ hyperplanes $p_1, ..., p_{2n+1}$ in $\mathbb{CP}^n$ one has

$$\sum_{i=1}^{2n+1} (-1)^i \mathcal{P}_n^G(p_1 \cap p_i, ..., p_{2n+1} \cap p_i) = 0$$

c) The function $\mathcal{P}_n^G(h_1, ..., h_{2n})$ is skewsymmetric with respect to $h_1, ..., h_{2n}$.

d) $\mathcal{P}_n^G(h_1, ..., h_{2n}) = 0$ if the intersection of certain $2k$ hyperplanes among $h_1, ..., h_{2n}$ has codimension $\leq k$.

Proof. a), b) are proved in [G4] and c) is clear.

d). One can show that up to a (computable) rational number d_n one has

$$\int_{\mathbb{CP}^{n-1}} r_{2n-2}(f_1, ..., f_{2n-1}) = d_n \cdot \int_{\mathbb{CP}^{n-1}} \log|f_1| d\log|f_2| \wedge ... \wedge d\log|f_{2n-1}| \quad (23)$$

If $h_2 \cap ... \cap h_{2k+1}$ has codimension k then $d\log|f_2| \wedge ... \wedge d\log|f_{2k+1}| = 0$, so the integral is zero.

On the other hand since for $(L, M) \in A_n(\mathbb{C})$ the function $\Lambda_n(L, M)$ is a period of the mixed Hodge structure

$$H^n(\mathbb{CP}^n \backslash L, M)$$

one can define a single valued function $\mathcal{L}_n(L, M) \in \mathbb{R}$ as the $\mathbb{R}$-period of this mixed Hodge structure. Then $\mathcal{L}_n : A_n(\mathbb{C}) \rightarrow \mathbb{R}$ is a homomorphism of groups.

Problems. 1. Find a complete list of functional equations for the function $\mathcal{P}_n^G$.

2. Find the relationship between the Aomoto and Grassmannian polylogarithms.

In the next section we will formulate these problems more precisely and explain their importance.

2 Conjectures. I conjecture that one can describe *explicitly* a certain subgroup

$$\mathcal{R}_n^G \subset \bar{C}_{2n}(\mathbb{P}^{n-1})$$

such that $\mathcal{R}_n^G(\mathbb{C})$ is the group of all functional equations for the Grassmannian n-logarithm function $\mathcal{P}_n^G$. The subgroup should be given by an explicit "universally defined" finite list of "relations". More precisely, one should have a finite set of varieties $R_i(n)$ over $\mathbb{Z}$ and morphisms $s_i(n) : \mathbb{Z}[R_i(n)] \rightarrow \mathbb{Z}[\bar{C}_{2n}(\mathbb{P}^{n-1})]$ given by finite correspondences over $\mathbb{Z}$ such that

$$\mathcal{R}_n^G(F) = \sum s_i(n) \Big(\mathbb{Z}[R_i(n)] \Big)$$

The properties of the subgroups $\mathcal{R}_n^G(F)$ are formulated in the conjecture below. Set

$$G_n(F) := \quad \frac{\mathbb{Q}[\text{ configurations of any } 2n \ F\text{-points in } \mathbb{P}^{n-1}]}{\mathcal{R}_n^G(F)}$$

Then $\mathcal{P}_n^G : G_n(\mathbb{C}) \rightarrow \mathbb{R}, \quad (l_1, ..., l_{2n}) \longmapsto \mathcal{P}_n^G(l_1, ..., l_{2n})$.

Let $G_\bullet := \oplus_{n=1}^{\infty} G_n$. Recall that a structure of a Lie coalgebra on $G_\bullet$ is given by homomorphisms

$$G_n \xrightarrow{\delta_n} \oplus_{i \leq n/2} G_i \wedge G_{n-i}$$

such that if $\delta := \oplus\delta_n$ then

$$G_\bullet \xrightarrow{\delta} \Lambda^2 G_\bullet \overset{\delta\otimes id - id\otimes\delta}{\longrightarrow} \Lambda^3 G_\bullet \longrightarrow ...$$

is a complex.

Conjecture 4.2 *a) $G_\bullet$ has a natural structure of a graded Lie coalgebra over $\mathbb{Q}$.*

b) The category of graded finite dimensional modules over $G_\bullet(F)$ is equivalent to the category of mixed Tate motives over $SpecF$. In particular one should have

$$H^i_{(n)}\Big(G_\bullet(F)\Big) = gr^\gamma_n K_{2n-i}(F)\otimes\mathbb{Q}$$

Here $H^i_{(n)}$ is the degree n part of H^i. Set

$$G_1(F) := F^*\otimes\mathbb{Q}, \quad G_2(F) := B_2(F)\otimes\mathbb{Q}, \quad G_3(F) := B_3(F)\otimes\mathbb{Q}$$

Then the first components of the cobracket δ are given by

$$\delta_2 : B_2(F)\to\Lambda^2 F^*_{\mathbb{Q}}, \qquad \delta_3 : B_3(F)\to G_2(F)\otimes F^*_{\mathbb{Q}}$$

where the homomorphisms δ_2 and δ_3 were defined in s.3.1 and 3.2.

The 7-term relation for the generalized cross-ratio r_3 is rather mysterious. Its analog needed to define the group $G_4(F)$ is unknown. However the results of this paper suggest the following strategy. Denote by $\tilde{A}_n$ the free abelian group generated by the generators of the group A_n, i.e. by admissible pairs of simplices. Let us assume that we have defined already the subgroups $\mathcal{R}^G_m$ for $m < n$.

Conjecture 4.3 *There exists a homomorphism $a_n : \tilde{A}_n \to \bar{C}_{2n}$ such that*

a) The following diagram is commutative

$$\begin{array}{ccc}
\tilde{A}_n & \stackrel{\nu}{\longrightarrow} & \oplus_{1\le i\le n} A_i\otimes A_{n-i} \\
\downarrow a_n & & \downarrow a_i\wedge a_{n-i} \\
\bar{C}_{2n} & \stackrel{\delta_n}{\longrightarrow} & \oplus_{i\le n/2} G_i(F)\wedge G_{n-i}(F)
\end{array}$$

b) $\mathcal{L}_n(L;M) = c_n\mathcal{P}^G_n(a_n(L,M))$ for any $(L;M)\in A_n(\mathbb{C})$, where c_n is a normalization constant.

(See also conjecture 1.42 in [G1]).

Assuming this we *introduce* $\mathcal{R}^G_n(F)$ as the image under the map a_n of the defining relations for the group A_n. Thus $\delta(\mathcal{R}^G_n) = 0$ and we are getting a commutative diagram

$$\begin{array}{ccc} A_n & \stackrel{\nu}{\longrightarrow} & \oplus_{1\leq i\leq n} A_i \otimes A_{n-i} \\ \downarrow a_n & & \downarrow a_i \wedge a_{n-i} \\ G_n(F) & \stackrel{\delta_n}{\longrightarrow} & \oplus_{i\leq n/2} G_i(F) \wedge G_{n-i}(F) \end{array}$$

The results of s. 3 show how this program works for G_3.

Conjecture 4.4 *The map a_n induces an isomorphism*

$$\bar{a}_n : A_n/P_n \to G_n(F)$$

For $n = 2$ this was proved in [BGSV1-2].

For $n = 3$ one can prove that $a_3(P_3) = 0$. To show that $\bar{a}_3$ is an isomorphism one should construct a homomorphism $L_3 : B_3(F) \to A_3(F)$ splitting the map a_3. The homomorphism $L_3 : \mathbb{Z}[F^*] \to A_n, \quad \{x\}_3 \longmapsto L_3\{x\}_3$ is given by the right hand side of formula (20). Then one should prove that the map $L_3 : \mathbb{Z}[F^*] \to A_3/P_3$ is surjective and $L_3(R_3) = 0$.

Conjecture (4.4) just means that the dual to the Hopf algebra $A_\bullet := \oplus A_n$ is isomorphic to the universal enveloping of the Lie algebra $G_\bullet^\vee$. (Here $A \to A^\vee$ is the duality between the ind and pro $\mathbb{Q}$-vector spaces). It seems quite remarkable that the universal enveloping algebra of $G_\bullet^\vee$ admits a completely different description (the only similar situation which comes to mind is Lusztig's construction of $U(\mathcal{N})$). So we get two different descriptions of the motivic Lie algebra (reflecting the properties of the Aomoto and Grassmannian polylogarithms). It is even more interesting that there are two more ways of thinking about the same Lie algebra (!): the wonderful "cycle" construction of Bloch and Kriz [BK] (so far the only one which is completely done), and the construction reflecting the properties of multiple polylogarithms ([G5]). This definitely shows the richness of the subject.

All constructions of different models of the motivic Hopf algebra of the category of mixed Tate motives are based on the following idea. The set of appropriately defined equivalence classes of n-framed mixed Tate motives over F form an abelian group $\mathcal{A}_n$ and $\mathcal{A}_\bullet := \oplus_{n\geq 1}\mathcal{A}_n$ is a commutative Hopf algebra. It is isomorphic to the fundamental Hopf algebra of the category of mixed Tate motives over F (see [BGSV1-2], [BMS] for the definition of n-framed mixed motives). Consider a universal variation of n-framed mixed Tate motives over a base X_n. For any F-point of $x \in X_n$ we get an element $m_n(x) \subset \mathcal{A}_n$. Universality of the variation means that the map $\mathbb{Z}[X_n(F)] \to \mathcal{A}_n$ is surjective. The kernel of this map is supposed to be described explicitly. So

$$\oplus_{n>0} \frac{\mathbb{Z}[X_n(F)]}{\text{Ker} \quad m_n}$$

should have a natural structure of a Hopf algebra, and one needs to determine it. (The variation of mixed Tate motives for the "cycle" Hopf algebra of [BK] can be found in the last section of [G4]). So it is quite interesting that we could get a construction of a co-Lie algebra directly, without constructing first its universal enveloping algebra.

3. A canonical map $\mathbb{Z}[F^*] \to A_n(F)$. Let A be a commutative graded Hopf algebra with a product μ and coproduct ν, A_+ the kernel of the augmentation homomorphism and

$$\tilde{\nu} := \nu - (id \otimes 1 + 1 \otimes id) : A \longmapsto A_+^{\otimes 2}$$

the restricted coproduct. We define a map of graded vector spaces $\tilde{\nu}_{[k]} : A \longrightarrow A_+^{\otimes k}$ as a composition

$$A \xrightarrow{\tilde{\nu}} A_+ \otimes A_+ \xrightarrow{\tilde{\nu}\otimes id} A_+ \otimes A_+ \otimes A_+ \xrightarrow{\tilde{\nu}\otimes id} ... \xrightarrow{\tilde{\nu}\otimes id} A_+^{\otimes k}$$

Let $\mu_k : A^{\otimes k} \to A$ be the product map. Set

$$\pi := \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \mu_k \circ \tilde{\nu}_{[k]} = Id - \frac{1}{2}\mu \circ \tilde{\nu} + ...$$

Proposition 4.5 $\pi^2 = \pi$ *and* $Ker\pi = P$.

Proof. Since $\pi = id$ (mod P) one needs to show only that $P \subset Ker\pi$.

Now return to the Hopf algebra $A_\bullet(F)$. Let B_k be the Bernoulli numbers.

Proposition 4.6

$$\pi(l_n(x)) = \sum_{k \geq 0} \frac{B_k}{k!} l_{n-k}(x) \cdot x^k$$

Here $x^m := x \cdot ... \cdot x$ (m times) is the product in $A_\bullet(F)$ of the element of A_1 corresponding to x under the canonical isomorphism $A_1(F) \to F^*$. Notice that this formula coincides with the formula for the function $\Lambda_n(z)$ in s. 4.1 of [BD]. However we get it in a quite different way.

Proof. Let

$$l(x,t) := \sum_{k \geq 1} l_k(x) t^k$$

Lemma 4.7 $\tilde{\nu} : l(x,t) \longmapsto l(x,t) \otimes (e^{x \cdot t} - 1)$

This is the generating function for the standard formula

$$\tilde{\nu}(l_k(x)) = \sum_{1 \leq i \leq k-1} l_{k-i}(x) \otimes \frac{x^i}{i!}$$

for the coproduct of the classical polylogarithm. Therefore

$$\mu \circ \nu_{[k]} : \quad l(x,t) \longmapsto l(x,t) \cdot (e^{x \cdot t} - 1)^{k-1}$$

So

$$\pi : l(x,t) \longmapsto \quad - \sum_{k \geq 1} \frac{(-1)^k}{k} l(x,t) \cdot (e^{x \cdot t} - 1)^{k-1} =$$

$$= l(x,t) \sum_{k \geq 1} \frac{(1 - e^{x \cdot t})^k}{k} \frac{1}{e^{x \cdot t} - 1} \quad = \quad l(x,t) \cdot \frac{xt}{e^{x \cdot t} - 1} \quad = \quad \sum_{n \geq 1} \Big(\sum_{k \geq 0} \frac{B_k}{k!} l_{n-k}(x) \cdot x^k \Big) t^n$$

The proposition is proved.

The kernel of the map

$$L_n : \mathbb{Z}[F^*] \longmapsto A_n(F), \quad \{x\} \longmapsto \pi(l_n(x))$$

should coincide with the subgroup of all functional equations for the n-logarithm.

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