Zeros of Abelian Integrals and Hilbert's XVIth problem

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ABSTRACT *We survey recent developments towards the solution of the (weak version of) Hilbert's XVIth problem. The main theme is the study of number of limit cycles appearing in a perturbed Hamiltonian system which can be estimated by the number of zeros of Abelian integrals. Some methods which use Picard-Lefschetz theorem, Wronskian and Fuchsian differential equations are discussed.*

0. **Introduction to limit cycles**

In an autonomous dynamical system,

$$\frac{d}{dt}x_i = v_i(x) \qquad 1 \leq i \leq n, x = (x_1, \dots, x_n). \tag{0.1}$$

whose right hand side consists of smooth functions in $x \in \mathbb{R}^n$ (i.e. terms independent of time t), periodic régime may observed. A mathematical description of this phenomenon is furnished by the theory of 'cycles' (closed orbits of system (0.1)) initiated by Henri Poincaré (Mémoires sur les courbes définies par les équations différentielles I-IV, Oeuvres T.1, Gauthier-Villars, Paris). The following theorem assures us a simple geometric intuition in the study of orbits of the system (0.1) for $n = 2$.

The orbit (or phase curve) means the curve issued from a point $(x_1, x_2) = (x_1(0), x_2(0)$ as a solution to the differential equation (0.1).

Theorem 0.1. *([5]) An orbit (phase curve) of the dynamical system (0.1) for $n = 2$ i.e.*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}, \qquad x = (x_1, x_2) \tag{0.1$'$}$$

is either a single point or is diffeomorphic to a circle or a straight line.

The following example illustrates the theorem.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{pmatrix}$$

$\gamma_0 = (0,0), \gamma_1, \gamma_3 :$ open orbits diffeomorphic to a straight line

$\gamma_2 :$ closed orbit diffeomorphic to a circle

The orbit which is diffeomorphic to a circle is called 'cycle'.

Suppose that a smooth vector field has a cycle γ. Choose a point 'p' on γ and draw a straight line 'l' passing through it and transversal to γ. An orbit starting from a point 'q' of the line 'l', sufficiently close to p shall return to a point $\Delta(q)$ on l. We call $\Delta(q)$ the monodromy transformation of $q \in l$.

It is evident that $\Delta(p) = p$ for p on a cycle γ. Especially we introduce the following notion.

Definition 1. *A limit cycle of the system (0,1)' is an isolated orbit (diffeomorphic to a circle) of the system (0,1)'. In other words, a closed orbit is a limit cycle if the fixed point of the monodromy transformation (i.e. $q \in l$ such that $\Delta(q) = q$) is isolated.*

In 1900, in the occasion of the International Congress of Mathematicians, David Hilbert posed the following question among other 22 problems that shall stimulate mathematical investigations in different domains all along the XXth century (Mathematische Probleme).

Question No. 16

1. *Let $X(x,y)$ be a polynomial of degree n with real coefficients in two variables. Then how many closed ovals the level set $X_0 = \{(x,y) \in \mathbb{R}^2 : H(x,y) = 0\}$ may have? Decide all possible dispositions of these ovals on $\mathbb{R}^2$ (or its projectivization $\mathbb{PR}^2$).*
2. *Let $X_1(x,y), X_2(x,y)$ be two polynomials of degree n with real coefficients. Then how many limit cycles the dynamical system*
$$\frac{dy}{dx} = \frac{X_1(x,y)}{X_2(x,y)}$$
may have?

Let us explain the connexion between two sub-questions 1. and 2. that may seem to be, at a first glance, alien one another.

Since more than 60 years ago [21], [10],[11], [3],[4] it has been known that the problem of the number of limit cycles of a perturbed Hamiltonian dynamical system that emerge from the original Hamiltonian system can be reduced to that of number of zeros of Abelian integrals associated to the Hamiltonian. As far as it concerns only the number of limit cycles of a perturbed Hamiltonian system (i.e. less general than the problem of Hilbert himself) , we may call it a **weak version of Hilbert's XVIth problem**.

To see this let us look at a non-perturbed Hamiltonian system on the plane, $(x,y) \in \mathbf{R}^2$:
$$dH(x,y) = 0. \tag{0.2}$$
Or in other words
$$\dot{x} = \frac{\partial}{\partial y} H(x,y)$$
$$\dot{y} = -\frac{\partial}{\partial x} H(x,y), \tag{0.2$'$}$$
where the Hamiltonian $H(x,y)$ is a polynomial of degree n. The problem to estimate the number of limit cycles for the system (0.2) is a purely algebraic question i.e. the number of closed component of real energy surface $X_t = \{(x,y) \in \mathbf{R}^2 : H(x,y) + t = 0\}$.

Further we consider a perturbed dynamical system of (0.2) :
$$dH(x,y) + \epsilon(P(x,y)dx + Q(x,y)dy) = 0, \tag{0.3}$$
or
$$\dot{x} = \frac{\partial}{\partial y} H(x,y) + \epsilon Q(x,y)$$
$$\dot{y} = -\frac{\partial}{\partial x} H(x,y) - \epsilon P(x,y), \tag{0.3$'$}$$
with polynomial perturbation of degree at most m, $P(x,y) = \sum_{0 \leq i+j \leq m} P_{i,j} x^i y^j$, $Q(x,y) = \sum_{0 \leq i+j \leq m} Q_{i,j}\, x^i y^j$. Let us try to write down the monodromy transform for the system (0.3) in the neighbourhood of a closed cycle γ_t of (0.2). We will be interested in the limit cycles that may appear near γ_t. Denote such a cycle by $\gamma_t + \epsilon\delta$, $|\epsilon| << 1$, δ a curve homotopically equivalent to a closed cycle. Then the integral
$$T(t) = \int_{\gamma_t + \epsilon\delta} \dot{x} dy - \dot{y} dx = \int_{\gamma_t + \epsilon\delta} dH + \epsilon(P(x,y)dx + Q(x,y)dy)$$

$$= \int_{\gamma_t + \epsilon\delta} dH + \epsilon \int_{\gamma_t} P(x,y)dx + Q(x,y)dy + \epsilon^2 \int_{\delta} P(x,y)dx + Q(x,y)dy,$$

gives the monodromy transform. This can be interpreted as an asymptotic expansion in ϵ whose first term corresponding to ϵ^0 equals to

$$\oint_{\gamma_t+\epsilon\delta} H_y dy + H_x dx = H(\text{begin}) - H(\text{end}).$$

Thus if one choses $\gamma_t + \epsilon\delta$ a closed curve, then the above expression must be equal to 0. On the other hand, the term corresponding to ϵ^1 is expressed by

$$I_{P,Q}(t) = \int_{\gamma_t} Pdx + Qdy, \tag{0.4}$$

which does not always vanish. If (0.4) possesses an isolated zero at $t = \tilde{t}$, then $T(\tilde{t}) = O(\epsilon^2)$. Hence we conclude that $T(t) = \epsilon I_{P,Q}(t) + O(\epsilon^2)$ must have so much zeros as $I_{P,Q}(t)$ if $\mid \epsilon \mid << 1$. If the Hamiltonian system (0.3) has a closed cycle at $t = \tilde{t}$, $I_{P,Q}(t)$ vanishes at point $t = \tilde{t}$. That is to say, the vanishing of the integral $I_{P,Q}(t)$ is a necessary condition for the existence of a limit cycle. As a matter of fact we have the following theorem.

Theorem 0.2. *If $\gamma_{\tilde{t}}$ generates a limit cycle of system (0.3) then $I_{P,Q}(t)$ has simple zero at $t = \tilde{t}$. Conversely if $I_{P,Q}(t)$ has an isolated zero at $t = \tilde{t}$, then $\gamma_{\tilde{t}}$ generates a limit cycle of system (0.3).*

Definition 2. *The integral $I_{P,Q}(t)$ defined in (0.4) is called the Abelian integral associated to the Hamiltonian $H(x,y)$.*

This reduction permits us to apply the techniques of algebraic geometry to the study of limit cycles.

In 1984, A.N. Varchenko and A.G. Khovanski proved independently the finiteness of the number of zeros in the following form.

Theorem 0.3. *For any positive integers n, m , there exists an integer $c(n,m)$ such that the number of isolated zeros of $I_{P,Q}(t)$ defined for $\deg H \leq n, \deg P, Q \leq m$ (taking the multiplicity into account) must be less than $c(n,m)$.*

Remark 1. Here the term "isolated zeros" has been underlined not by chance. As a matter of fact, if we set $Pdx + Qdy = gdH + dR$ for $g, R \in \mathbb{R}[x,y]$ then $I_{P,Q}(t) \equiv 0$. One proves this property in view of the Stokes' theorem:

$$\begin{aligned} I_{P,Q}(t) = \textstyle\int_{\gamma_t} gdH + dR &= \textstyle\int_{\gamma_t} d(gH+R) + \int_{\gamma_t} H\, dg \\ &= (-t)\textstyle\int_{\gamma_t} dg = 0 \end{aligned}$$

It is to be remarked that the above theorem states nothing (except its existence) on the numbers $c(n,m)$. To answer what kind of behaviour of $c(n,m)$ can we expect, more detailed studies are demanded. Quite recently (1995), Yu. S. Ilyashenko [13], D. Novikov and S. Yakovenko [17] established the following estimate for the Hamiltonian $H(x,y)$ with Morse type singularities. Morse type singularity means that in a neighbourhood of a critical point (x^*, y^*) where $\operatorname{grad} H(x,y)$ degenerates (i.e. $(\frac{\partial}{\partial x}H(x^*,y^*), \frac{\partial}{\partial y}H(x^*,y^*)) = (0,0)$), smooth coordinate change $(x,y) \mapsto (X(x,y), Y(x,y))$ (also $(X,Y) \mapsto (x(X,Y), y(X,Y))$ is smooth) brings the Hamiltonian into the following canonical form: $H(x,y) = X^2 + Y^2$. For details on the Morse type singularity see S. Pnevmatikos "Anazetontas tén taxi sti perioxi ton krisimwn simeiwn", Taxi Kai Xaos, Tomos III.

Theorem 0.4. *Suppose $\deg H(x,y), P(x,y), Q(x,y) \leq n$ in (0.3). If $H(x,y) + t = 0$ has only Morse type singularity for any $t \in \mathbb{R}$, then $\sharp\{$ zeros of $I_{P,Q}(t)\} \leq \exp(c_0 n + c_1)$ for some constants $c_0, c_1 > 0$ independent of n.*

Theorem 0.4 is certainly a progress in comparison with Theorem 0.3, however it can not be a final version of the answer to Hilbert's XVIth problem. As we will see later in the following sections, more precise estimates of zeros of $I_{P,Q}(t)$ are known for lower degree Hamiltonians. The aim of this article is to give rough sketches of possible methods that must be also useful to study the Abelian integrals associated to wider class of Hamiltonians.

1. **Picard-Lefschetz theorem**

1.1 **Vanishing cycles and an algebraic lemma.**

In a series of works, G.S. Petrov proposed a powerful machinery to estimate zeros of elliptic and hyperelliptic integrals. His method consists mainly in the study of ramification properties of the (hyper-) elliptic integrals by means of the Picard-Lefschetz theorem. Shortly speaking, this theorem describes the monodromy of the homology group $H_1(X_s, \mathbb{Z})$ in terms of the intersection number of base elements of $H_1(X_s, \mathbb{Z})$.

We will trace the argument of Petrov in chasing his central work [19].

Let us look at the following Hamiltonian

$$H(x, y; s_0, s_1) = y^2 + x^4 - x^2 - s_1 x - s_0. \tag{1.1}$$

Assume that it has 3 distinct real critical values $s_0^{(1)} < s_0^{(2)} < s_0^{(3)}$. (this is a condition on s_1). We fix s_1 in our further discussions. We say that $H(x^*, y^*; 0, s_1)$ is a critical value of $H(x, y; s)$ (we will denote simply $H(x, y; s)$ instead of $H(x, y; s_0, s_1)$) if the gradient of H at a point $(x, y) = (x^*, y^*)$ degenerates i.e.

$$\operatorname{grad} H(x^*, y^*; s) = \left(\frac{\partial}{\partial x} H(x^*, y^*; s), \frac{\partial}{\partial x} H(x^*, y^*; s) \right) = (0, 0).$$

You can imagine it as a bottom of potential valley. Let us define a family of so called "vanishing cycles" on a family of Riemann surfaces $X_s = \{(x, y) \in \mathbb{C}^2 : H(x, y; s) = 0\}$. The vanishing cycles form a basis in the 1st homology group of X_s.

Denote by x_i^* $(i = 1, 2, 3)$ critical points of $H(x, 0, 0, s_1)$ so that $s_0^{(i)} = -H(x_i^*, 0, 0, s_1)$. Let $x_0(s) < x_1(s) < x_2(s) < x_3(s)$ be the consecutive roots of the equation $H(x, 0; s) = 0$ for $s_0^{(1)} < s_0 < s_0^{(3)}$. We denote by $\gamma_1(s)$ (resp. $\gamma_2(s)$) an 1-dimensional real cycle on X_s whose projection on $\mathbb{C}_x^1$-plane goes around roots $x_0(s), x_1(s)$ (resp. $x_2(s), x_3(s)$).

If one observes the deformation of cycles that happens in approaching s_0 to $s_0^{(1)}$, then it is visually clear that $\gamma_1(s)$ tends to a point as $x_2(s) \to x_3(s)$ in this limit process. Under this situation we call $\gamma_1(s)$ the vanishing cycle associated with the critical value $s_0^{(1)}$. Similarly $\gamma_2(s)$ is the vanishing cycle associated with the critical value $s_0^{(2)}$. As a matter of fact, the closed curve γ_t of (0.4) can be regarded as one of these cycles $\gamma_i(s)$, we define the integral

$$I_{\omega,\gamma_i}(s) = \int_{\gamma_i(s)} \omega \tag{1.2}$$

for a form

$$\omega = P dx + Q dy$$

under the condition $\deg P, Q \leq n$. With these notions, Petrov's theorem can be formulated as follows:

Theorem 1.1. *The elliptic integral $I_{\omega,\gamma_i(s)}$ $(i = 1, 2)$ for ω of (1.2) has at most $n + \left[\frac{n-1}{2}\right]$ zeros on the interval $s_0 \in (s_0^{(1)}, s_0^{(3)})$.*

We begin the proof of this theorem with the following lemma. Further we write simply $I_\omega(s)$ if no specification of integration cycle is demanded.

Lemma 1.2. 1. *For every ω of (1.2), there exist three polynomials $P_0(s), P_1(s), P_2(s) \in \mathbb{R}[s_0, s_0]$ such that*

$$I_\omega(s) = P_0(s)I_{ydx}(s) + P_1(s)I_{xydx}(s) + P_2(s)I_{x^2ydx}(s) \tag{1.3}$$

$$\deg_{s_0} P_0(s), \deg_{s_0} P_1(s) \leq p-1, \deg_{s_0} P_2(s) \leq p-2 \quad \textit{if} \quad n = 2p \tag{1.4}$$

$$\deg_{s_0} P_0(s) \leq p, \deg_{s_0} P_1(s), \deg_{s_0} P_2(s) \leq p-1 \quad \textit{if} \quad n = 2p+1.$$

Here $\deg_{s_0} P_i(s)$ denotes a positive integer d such that

$$P_i(s) = P_{id}s_0^d + P_{id-1}(s_i)s_0^{d-1} + \ldots + P_{i0}(s_1)$$

with $P_{id} \neq 0$.

2. *The following estimates hold for $\mid s_0 \mid >> 1$ and fixed s_1:*

$$\mid I_{x^iydx}(s) \mid \leq c_0 \mid s_0 \mid^{\frac{3+i}{4}}$$

for certain positive constant $c_0 > 0$.

Proof We consider the following space of 1-forms

$$H' = \frac{A^1_{\mathbb{R}^2}}{A^0_{\mathbb{R}^2}dH + dA^0_{\mathbb{R}^2} + HA^1_{\mathbb{R}^2}}, \tag{1.5}$$

where $A^0_{\mathbb{R}^2} = \mathbb{R}[x,y]$ and $A^1_{\mathbb{R}^2}$ is a space of 1-forms with real polynomial coefficients. It means that to every 1-form $Pdx + Qdy \in A^1_{\mathbb{R}^2}$ we associate a representative $\psi \in H'$ in such a way that

$$Pdx + Qdy = \psi + gdH + dh + H\varphi \tag{1.6}$$

for some $g, h \in A^0_{\mathbb{R}^2}, \varphi \in A^1_{\mathbb{R}^2}$. This decomposition is important because of the following equalities derived from Stoke's theorem:

$$\begin{aligned} &\int_{\gamma_i(s)} dh = 0 \\ &\int_{\gamma_i(s)} gdH = \int_{\gamma_i(s)} d(gH) - \int_{\gamma_i(s)} Hdg = 0 \\ &\int_{\gamma_i(s)} H\varphi = (-s_0)\int_{\gamma_i(s)} \varphi \end{aligned} \tag{1.7}$$

for a vanishing cycle $\gamma_i(s) \in H_1(X_s, \mathbb{Z})$. In other words, if $\psi = 0 \in H'$ then $\int_{\gamma_i(s)} \psi = 0$

After direct calculus of the quotient space H', we see that

$$H' \cong \langle ydx, xydx, x^2ydx \rangle_{\mathbb{R}} \tag{1.8}$$

From this one obtains (1.3) in applying iteratively (1.7) to (1.6) (cf. Weierstrass division theorem [6]). Before proving (1.4) we shall remark the quasihomogeneous property of $H(x,y;s)$ i.e.

$$H(\lambda x, \lambda^2 y; \lambda^4 s_0, \lambda^3 s_1) = \lambda^4 H(x,y;s_0,s_1)$$

for every $\lambda > 0$. We prescribe so called quasihomogeneous weight $w(\cdot)$ to each variable i.e.

$$w(x) = 1, w(y) = 2, w(s_1) = 3, w(s_0) = 4.$$

Consequently we can supply the form $x^\alpha y^\beta dx$ with the quasihomogeneous weight $w(x^\alpha y^\beta dx) = \alpha+2\beta+1$. The statement on $\deg_{s_0} P_i(s)$ can be deduced from an evident formula:

$$\deg_{s_0} P_i(s) \leq \left[\frac{w(\omega) - w(x^i y dx)}{w(s_0)}\right].$$

The estimate 2) is derived from the calculus of quasihomogeneous weights of the base forms of H' like (1.8)

q.e.d.

<u>1.2</u> **Picard-Lefschetz theorem**
In this section we review the essence of the Picard-Lefschetz theorem. Let us imagine what kind of changing happens with the Riemann surface X_s (and with the cycle $\gamma_i(s)$) if you turn the value $s_0 \in \mathbb{C}$ around $s_0^{(3)}$ on the complex plane. Here we consider a turn around $s_0^{(3)}$ in the following manner:

$$\lambda(t) = e^{2\pi it} \cdot \alpha + s^{(3)}, 0 \leq t < 1, s_0^{(2)} - s_0^{(3)} < \alpha < 0.$$

Thus it is possible to construct a family of diffeomorphism Γ_t depending continuously on t,

$$\Gamma_t : X_{(\lambda(0),s_1)} \to X_{(\lambda(t),s_1)}.$$

Though the Riemann surface $X_{(\lambda(1),s_1)}$ coming at the end of this process is identical to $X_{(\lambda(0),s_1)}$ the automorphism of $H_1(X_s,\mathbb{Z})$ induced by Γ_1 is not an identical mapping.

Especially the cycle γ_1 is transformed into another cycle $\Gamma_1(\gamma_1)$ such that

$$\Gamma_1(\gamma_1) - \gamma_1 \sim -\gamma_3$$

here '$\sim$' denotes homotopy equivalence. In the same way,

$$\Gamma_1(\gamma_2) - \gamma_2 \sim -\gamma_3.$$

The transformation

$$\gamma_i \to \Gamma_1(\gamma_i) \sim \gamma_i - \gamma_3 \qquad i = 1,2 \tag{1.9}$$

is called the monodromy transform of vanishing cycles around $s_0 = s_0^{(3)}$. (Not to confuse this termonology with $\Delta(q)$ appeared in § 0). More generally we have the following theorem.

Theorem 1.3. **<u>Picard-Lefschetz Theorem</u>**
Let $H(x,y;s)$ parametrized by $s \in \mathbf{R}^\mu$ be a Hamiltonian with the property

$$\left(\frac{\partial}{\partial x}H(x,y;0), \frac{\partial}{\partial y}H(x,y;0)\right) = (0,0) \quad \textit{if and only if} \quad (x,y) = (0,0).$$

Let $\gamma_i(s)$ be the vanishing cycle on $X_s = \{(x,y) \in \mathbb{C}^2; H(x,y;s) = 0\}$ that vanishes at $s = s^{(i)} \in \mathbb{R}^\mu$. Then the monodromy transform around $s^{(i)}$ is given by the following formula for any $\delta \in H_1(X_s,\mathbb{Z})$:

$$\Gamma_1(\delta) = \delta + (\gamma_i, \delta) \cdot \gamma_i \tag{1.10}$$

where $(\gamma_i \cdot \delta)$ denotes the intersection index between γ_i and δ.

The Picard-Lefschetz theorem reduces the study of monodromy of cycles to the calculus of intersection index (number) between base elements of the homology group. See for the detail [6].

In his famous investigations on the Landau singularities appeared in quantum physics, F. Pham [20] made use of Picard-Lefschetz theorem to describe the monodromy of integrals

$$I_{\omega,\gamma_i}(s) = \int_{\gamma_i(s)} \omega$$

in taking into account that its variation around its ramfication point $s^{(j)}$ is given by

$$\int_{\gamma_i} \omega \longmapsto \int_{\gamma_i} \omega + \int_{\langle\gamma_j,\gamma_i\rangle\gamma_j} \omega = \int_{\gamma_i} \omega + \langle\gamma_j,\gamma_i\rangle \int_{\gamma_j} \omega, \tag{1.11}$$

according to the Picard-Lefschetz theorem.

1.3 **Petrov's proof of the theorem 1.1.**

In the sequel let us denote $\frac{d}{ds_0} I_\omega(s) = I'_\omega(s)$.

Lemma 1.4. 1. *The vector of integrals* $\mathbf{I}(s) = (I_{ydx}(s), I_{xydx}(s), I_{x^2ydx}(s))$ *satisfy the following system of differential equations.*

$$S(s)\frac{\partial}{\partial s_0}\mathbf{I} = (L+V)\mathbf{I},$$

where

$$S(s) = \begin{bmatrix} s_0 & \frac{3}{4}s_1 & \frac{1}{2} \\ -\frac{1}{8}s_1 & s_0-\frac{3}{4} & \frac{3}{4}s_1 \\ -\frac{3}{16}s_1^2 & -\frac{1}{2}s_1 & s_0-\frac{3}{4} \end{bmatrix},$$

$$L+V = \begin{bmatrix} \frac{3}{4} & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{8} & 0 & \frac{5}{4} \end{bmatrix}.$$

2. *The differentiated integral* $I'_\omega(s)$ *defined for* ω *of (1.2) admits the following decomposition:*

$$I'_\omega(s) = Q_0(s)I_{y^{-1}dx}(s) + Q_1(s)I_{xy^{-1}dx}(s) + Q_2(s)I_{x^2y^{-1}dx}(s)$$

where $Q_i(s)$ *are polynomials with* $\deg_{s_0} Q_i(s)$ *satisfying the same estimates for* $\deg_{s_0} P_i(s)$ *as in (1.5).*

3. *The integral* $I_{y^{-1}dx,\gamma_i}(s)$ $(i=1,2,3)$ *has no zeros on the ray* $s_0 < s_0^{(2)}$.

Proof

As for 1) see § 3.

The statement 2) is a direct consequence of 1) and lemma 1.2, 1). One shows 3) for the case $\gamma_i = \gamma_2$. On the interval $(s_0^{(1)}, s_0^{(2)})$ $I_{y^{-1}dx,\gamma_2}(s)$ is negative. On the ray $\{s_0 \le s_0^{(2)}\}$ the roots of the equation $H(x,0;s_0,s_1) = 0$ form an equilateral trapezeum, hence $I_{y^{-1}dx,\gamma_2}(s) = -I_{y^{-1}dx,\gamma_1}(s)$ which is equal to the integrals along remaining two sides. The sides contract to the narrowest base. The sum of integrals along these sides is of sign definite hence $I_{y^{-1}dx,\gamma_2}(s)$ must be of sign definite. For $\gamma_i = \gamma_1, \gamma_3$ similar argument holds. **q.e.d.**

Proof of the Theorem 1.1 consists of several steps.

Step 1

After the lemma 1.4 we pursue the following argument. By Rolle's theorem, it is enough to estimate the number of zeros of $I'_{\omega,\gamma_i}(s)$ to estimate that of $I_{\omega,\gamma_i}(s)$. Let us investigate the number of zeros of $\left(\frac{I'_{\omega,\gamma_i}(s)}{I_{y^{-1}dx,\gamma_i}(s)}\right)$ in the domain $\mathcal{D} = \{s_0 \in \mathbb{C}_{s_0} \setminus \{s_0 \ge s_0^{(3)}\}; |s_0| \le R\}$ with R large enough, that is equal to

$$\sharp\{\text{zeros} \quad (Q_0(s) + Q_1(s)\left(\frac{I_{xy^{-1}dx,\gamma_i}(s)}{I_{y^{-1}dx,\gamma_i}(s)}\right) + Q_2(s)\left(\frac{I_{x^2y^{-1}dx,\gamma_i}(s)}{I_{y^{-1}dx,\gamma_i}(s)}\right))\} \quad \text{in the domain} \quad \mathcal{D},$$

which makes sense by virtue of the Lemma 1.4, 3) This number is in its turn estimated by

$$\frac{1}{2}\sharp\{\text{zeros} \quad Im(Q_0(s) + Q_1(s)\frac{I_{xy^{-1},\gamma_i}(s)}{I_{y^{-1}dx,\gamma_i}(s)} + Q_2(s)\left(\frac{I_{x^2y^{-1}dx,\gamma_i}(s)}{I_{y^{-1}dx,\gamma_i}(s)}\right))\} \quad \text{along} \quad \partial\mathcal{D}.$$

This is seen from the fact that a complex valued function $f(s)$ has a zero in an domain if its imaginary part $Im\, f(s)$ has two zeros on the boundary of the domain. If $Im\, f(s)$ makes $2N$ turns around zero on $\partial\mathcal{D}$, it implies that $f(s)$ has N zeros in $\mathcal{D}$.

To apply the argument principle of complex functions we estimate the zeros of $Im\left(\frac{I'_{\omega,\gamma_i}(s)}{I_{y^{-1}dx,\gamma_i}(s)}\right)$ along a circle of large radius $\{s_0 \in \mathbf{C}; |s_0| = R\}$. The lemma 1.2, 2) and lemma 1.4, 2) give the estimate

$$\frac{1}{2}\sharp\{\text{zeros of} \quad Im\left(\frac{I'_{\omega,\gamma_i}}{I_{y^{-1}dx,\gamma_i}}(s)\right) \quad \text{along} \quad \{s_0 \in \mathbf{C}; |s_0| = R\}\} \leq \left[\frac{n+1}{2}\right]. \tag{1.12}$$

It remains to estimate zeros of $Im\left(\frac{I'_{\omega,\gamma_i}(s)}{I_{y^{-1}dx,\gamma_i}(s)}\right)$ along the cut $\{s_0 \geq s_0^{(3)}\}$.

Step 2

The following equality follows easily from the fact $Im\, Q_0(s) \equiv 0$ on $\{s_0 \geq s_0^{(3)}\}$:

$$\begin{aligned}
&\frac{1}{2}\sharp\{\text{zeros} \quad Im(Q_0(s) + Q_1(s)\left(\frac{I_{xy^{-1}dx}(s)}{I_{y^{-1}dx}(s)}\right) + Q_2(s)\left(\frac{I_{x^2y^{-1}dx}(s)}{I_{y^{-1}dx}(s)}\right))\} \quad \text{along} \quad \{s_0 \geq s^{(3)}\} \\
&= \frac{1}{2}\sharp\{\text{zeros} \quad Im(Q_1(s)\left(\frac{I_{xy^{-1}dx}(s)}{I_{y^{-1}dx}(s)}\right) + Q_2(s)\left(\frac{I_{x^2y^{-1}dx}(s)}{I_{y^{-1}dx}(s)}\right))\} \quad \text{along} \quad \{s_0 \geq s^{(3)}\} \\
&= \frac{1}{2}\sharp\{\text{zeros of}\{\frac{1}{|I_{y^{-1}dx,\gamma_i(s)}|^2}((Im\, I_{xy^{-1}dx} Re\, I_{y^{-1}dx} - Im\, I_{y^{-1}dx} Re\, I_{xy^{-1}dx})\, Q_1(s) \\
&\quad + (Im\, I_{x^2y^{-1}dx} Re\, I_{y^{-1}dx} - Im\, I_{y^{-1}dx} Re\, I_{x^2y^{-1}dx})\, Q_2(s))\} \quad \text{along} \quad \{s_0 \geq s^{(3)}\}.
\end{aligned} \tag{1.13}$$

Here we have fixed $\gamma_i = \gamma_1$ and we omitted suffices indicating the cycle of integration.

Step 3

Let us show that on the cut $\{s_0 \geq s_0^{(3)}\}, Im\, I_{\omega,\gamma_1}(s) = \pm\frac{i}{2} I_{\omega,\gamma_3}(s)$ for any form $\omega \in A^1_{\mathbb{R}^2}$. As $I_{\omega,\gamma_1}(s)$ is a complex conjugate function,

$$I_{\omega,\gamma_1}(s) - I_{\omega,\gamma_1}(\bar{s}) = 2i\, Im\, I_{\omega,\gamma_1}(s) \tag{1.14}$$

On the other hand this difference is nothing but the variation of the integral $I_{\omega,\gamma_1}(s)$ after its turn around the point $s_0 = s_0^{(3)}$ i.e. the monodromy transform around $s_0 = s_0^{(3)}$ minus identity. Here apply Picard-Lefschetz theorem (cf. (1.9) or (1.11)) in order to express the monodromy transformation by means of cycles:

$$I_{\omega,\gamma_1}(\bar{s}) = I_{\omega,\gamma_1 \pm \gamma_3}(s). \tag{1.15}$$

In comparing (1.14) and (1.15) we get

$$Im\, I_{\omega,\gamma_1}(s) = \pm\frac{i}{2} I_{\omega,\gamma_3}(s). \tag{1.16}$$

Step 4

Let us examine the term

$$Im\, I_{xy^{-1}dx,\gamma_1}(s) Re\, I_{y^{-1}dx,\gamma_1}(s) - Im\, I_{y^{-1}dx,\gamma_1}(s)\, Re\, I_{xy^{-1}dx,\gamma_1}(s) \tag{1.17}$$

that appeared in (1.13). By using (1.16) we see

$$(1.17) = \pm \frac{i}{2} \det \begin{pmatrix} I_{y^{-1}dx,\gamma_1}(s) & I_{y^{-1}dx,\gamma_3}(s) \\ I_{xy^{-1}dx,\gamma_1}(s) & I_{xy^{-1}dx,\gamma_3}(s) \end{pmatrix}.$$

Let us denote this determinant $i\omega(\gamma_1,\gamma_3)$. Similarly

$$Im\, I_{x^2y^{-1}dx,\gamma_1} Re\, I_{y^{-1}dx,\gamma_1} - Im\, I_{y^{-1}dx,\gamma_1}\, Re\, I_{x^2y^{-1}dx,\gamma_1} \tag{1.18}$$
$$= \pm \frac{i}{2} \det \begin{pmatrix} I_{y^{-1}dx,\gamma_1}(s) & I_{y^{-1}dx,\gamma_3}(s) \\ I_{x^2y^{-1}dx,\gamma_1}(s) & I_{x^2y^{-1}dx,\gamma_3}(s) \end{pmatrix}.$$

The Wronskian method explained in the following §2 and the lemma 1.2.2) yield that (1.15) is a holomorphic function bounded at ∞. That is to say (1.18) is a real constant.

Step 5

In combining (1.17) and (1.18), we see that the number (1.13) is nothing but

$$\frac{1}{2}\sharp \quad \text{zeros of} \quad \{i\omega(\gamma_1,\gamma_3)Q_1(s) + \text{real} \quad \text{const.}\, Q_2(s)\}\,. \tag{1.19}$$

By the Wronskian method we see that $\omega(\gamma_1,\gamma_3)$ is also holomorphic at $s_0 = s_0^{(3)}, s_0^{(1)}$. Thus the fuction $i\omega(\gamma_1,\gamma_3)Q_1(s) + \text{real} \quad \text{const.}\, Q_2(s)$ is holomorphically extended to $D_1 = \mathbb{C}\setminus\{s_0 < s_0^{(2)}\}$. Let us estimate (1.19) near $s_0 = +\infty$. After the argument principle like in Step 1 and lemma 1.2. 2), the function $i\omega(\gamma_1,\gamma_3)Q_1(s) + \text{real const.}\, Q_2(s)$ goes around the origin less than $\left[\frac{n-1}{2}\right]$ times if s_0 turns around a large circle $\{|s_0| = R, R >> 1\}$. The cut along $\{s_0 < s_0^{(2)}\}$ gives the contribution to (1.19) estimated by

$$\frac{1}{2}\sharp\{\text{zeros of} \quad Im\, i\omega(\gamma_1,\gamma_3)Q_1(s)\}. \tag{1.20}$$

The lemma 1.4. 3), and the following lemma entail that (1.20) must be less than $\deg Q_1(s) \le \frac{n}{2} - 1$.

Lemma 1.5. *The real valued function $i\omega(\gamma_1,\gamma_3)$ admits a holomorphic extension to D_1 and its imaginary part is proportional to $I_{y^{-1}dx,\gamma_1(s)}$ on the ray $s_0 < s_0^{(2)}$.*

Proof The holomorphicity follows from the Wronskian argument. As for the second we observe

$$Im\, i\omega(\gamma_1,\gamma_3) = \frac{1}{2}\left(\omega(\gamma_1,\gamma_3) + \bar{\omega}(\gamma_1,\gamma_3)\right) = \frac{1}{2}\omega(\gamma_1,\gamma_3+\bar{\gamma}_3), \tag{1.21}$$

because γ_1 is a real cycle on the ray $s_0^{(1)} < s_0 < s_0^{(3)}$.The sum of cycles $\gamma_3+\bar{\gamma}_3$ is homotopically equivalent to $\pm\gamma_2$ on the interval $(s_0^{(1)}, s_0^{(2)})$ and to the cycle $\pm(\gamma_2-\gamma_1)$ on the ray $\{s_0 > s_0^{(1)}\}$. Thus (1.21) is proportional to

$$\det \begin{pmatrix} I_{y^{-1}dx,\gamma_1} & I_{y^{-1}dx,\gamma_1+\gamma_2} \\ I_{xy^{-1}dx,\gamma_1} & I_{xy^{-1}dx,\gamma_1+\gamma_2} \end{pmatrix} = \det \begin{pmatrix} I_{y^{-1}dx,\gamma_1} & 0 \\ I_{xy^{-1}dx,\gamma_1} & \text{const.} \end{pmatrix} = \quad \text{const.}\, I_{y^{-1}dx,\gamma_1}$$

The last equality results from the fact that $\gamma_1+\gamma_2$ is homotopic to a cycle encircling the infinity and the elementary calculus of residues around $x=\infty$.

q.e.d.

At last the contribution to (1.19) from the little circle around $s_0 = s_0^{(2)}$ can be estimated as zero, as the following asymptotic expansion holds,

$$i\omega(\gamma_1,\gamma_3) = a_0 \log(s_0 - s_0^{(2)}) + a_1(s_0 - s_0^{(2)})\log(s_0 - s_0^{(2)}) + \dots$$

after the Wronskian argument and Picard-Lefschetz theorem. In summary, estimates (1.12), (1.19), (1.20) plus Rolle's theorem give the desired estimate of the Theorem 1.1. **q.e.d.**

2. Wronskian method

In a work appeared in 1978 Yu.S. Ilyashenko [12] proposed a new method to study the multiplicity of zeros of Abelian integrals by means of their Wronskians. His main result obtained after this method is the following.

Theorem 2.1. *Let us look at the Hamiltonian*

$$H(x,y) = x^3 - 3x - y^2$$

and a perturbed Hamiltonian system:

$$dH + \epsilon(P(x,y)dx + Q(x,y)dy) = 0 \tag{2.1}$$

with $\deg P(x,y) \leq 2, \deg Q(x,y) \leq 1$. *Then the multiplicity of limit cycles that can appear in (2.1) is less than 2.*

We remark that the above theorem covers results established before and by F. Takens and R.I. Bogdanov (Versal deformations of a singularity of a vector field on the plane in the case of zero eigenvalues. Petrovski Seminar Proceedings (1976) vol. 2. English translation: Seclecta Math. Sovietica 1. (1981) 389-421).

To prove the Theorem 2.1, Ilyashenko studied the Abelian integrals

$$I_{ydx,\gamma}(s_0) = \int_{\gamma(s_0)} ydx, I_{xydx,\gamma}(s_0) = \int_{\gamma(s_0)} xydx \tag{2.2}$$

where $\gamma(s_0)$ a vanishing cycle of the Riemann surface $X_{s_0} = \{(x,y) \in \mathbb{C}^2 : H(x,y) + s_0 = 0\}$. It is well known that X_{s_0} is topologically equivalent to a torus in this case and $\dim H_1(X_{s_0}, \mathbb{Z}) = 2$. So we choose a basis $\{\gamma_1, \gamma_2\}$ of $H_1(X_{s_0}, \mathbb{Z})$ such that γ_1 is the vanishing cycle associated with the critical value $s_0 = 2$. To $I_{\omega,\gamma_i}(s_0)$ defined in the same manner as in § 1 (see (1.2)), we associate the Wronskian as follows:

$$W(s_0, \omega) := det \begin{pmatrix} I'_{\omega,\gamma_1}(s_0) & I'_{\omega,\gamma_2}(s_0) \\ I''_{\omega,\gamma_1}(s_0) & I''_{\omega,\gamma_2}(s_0) \end{pmatrix} \tag{2.3}$$

where $I'_{\omega,\gamma}(s_0) = \frac{d}{ds_0} I_{\omega,\gamma}(s_0)$. We shall be interested in the case $\omega = (a+bx)ydx$. Remark that

$$\frac{d}{ds_0} \int_{\gamma_i} (a+bx)ydx = \int_{\gamma_i} (a+bx)\frac{dx}{y}.$$

In principle the integrals $I_{\omega,\gamma_i}(s_0)$ ramify around the critical values of the Hamiltonian $H(x,y)$ (in this case $s_0 = \pm 2, \infty$). The great advantage to make use of the Wronskian $W(s_0,\omega)$ is, however the univalence of $W(s_0,\omega)$ (not a multivalued function). In fact we will see that $W(s_0,\omega)$ is a rational function in s_0.

Let us observe the action induced by monodromy transformation on $W(s_0,\omega)$. Let us denote a turn of $s_0 \in \mathbb{C}$ around $s_0 = -2$ by l. Then by the aid of the Picard-Lefschetz formula (1.10), (1.11):

$$l_* W(s_0,\omega) = \det \begin{pmatrix} I'_{\gamma_1} + \langle\gamma_1,\gamma_2\rangle I'_{\gamma_2} & I'_{\gamma_2} + \langle\gamma_2,\gamma_2\rangle I'_{\gamma_2} \\ I''_{\gamma_1} + \langle\gamma_1,\gamma_2\rangle I''_{\gamma_2} & I''_{\gamma_2} + \langle\gamma_2,\gamma_2\rangle I''_{\gamma_2} \end{pmatrix}$$

It is known by E. Cartan's formula that $\langle\gamma_2,\gamma_2\rangle = 0$ ([20]). Therefore the above expression is equal to

$$\begin{aligned} &\det \begin{pmatrix} I'_{\gamma_1} + \langle\gamma_1,\gamma_2\rangle I'_{\gamma_2} & I'_{\gamma_2} \\ I''_{\gamma_1} + \langle\gamma_1,\gamma_2\rangle I''_{\gamma_2} & I''_{\gamma_2} \end{pmatrix} \\ &= \det \begin{pmatrix} I'_{\gamma_1} & I'_{\gamma_2} \\ I''_{\gamma_1} & I''_{\gamma_2} \end{pmatrix} \cdot \det \begin{pmatrix} 1 & 0 \\ \langle\gamma_1,\gamma_2\rangle & 1 \end{pmatrix} \\ &= \det \begin{pmatrix} I'_{\gamma_1} & I'_{\gamma_2} \\ I''_{\gamma_1} & I''_{\gamma_2} \end{pmatrix} = W(s_0,\omega). \end{aligned}$$

The same argument works for a turn around $s_0 = 2$. Hence $W(s_0, \omega)$ does not ramify neither at $s_0 = 2$ nor $s_0 = -2$, it is an univalent function on $\mathbb{C}_{s_0}$. On the other hand, by computing the growth by the size $|\gamma_i(s_0)|$ and the integrand ω as s_0 approaches to a critical value of $H(x,y)$, one sees

$$W(s_0, w) = O(s_0^{-2/3}) \quad \text{as} \quad s_0 \to \infty \tag{2.4}$$

$$W(s_0, \omega) = O((s_0 \pm 2)^{-1}) \quad \text{as} \quad s_0 \to \pm 2. \tag{2.5}$$

This is verified by the same argument as that of lemma 1.2, 2). From the above estimates (2.4) and (2.5), one concludes that $W(s_0, \omega)$ admits the following simple expression:

$$W(s_0, \omega) = \frac{A}{s_0 - 2} + \frac{B}{s_0 + 2} = \frac{(A+B)s_0 + 2(A-B)}{(s_0^2 - 4)} \tag{2.6}$$

for $A, B \in \mathbb{C}$. All rational functions satisfying (2.4) and (2.5) can be written down like this. On proving $A + B \neq 0$, Ilyashenko asserts that the multiplicity of zeros of $I_{\omega,\gamma_2}(s_0)$ must be less than 2 because

$$\{\text{the multiplicity of zeros} \quad I'_{\omega,\gamma_1}(s_0)\} + \{\text{the multipliciy of zeros of} \quad I''_{\omega,\gamma_2}(s_0)\} \leq 1,$$

This is an evident conclusion from (2.3) and (2.6). **q.e.d.**

Several authors exploited the above method to obtain estimates on the muliplicities of zeros of abelian integrals defined for other Hamiltonians. For example, P. Mardešic obtained the following "best estimate" ever known till now for the Hamiltonians with minimally restrictive conditions. Namely he considered the situation below. Let $H(x,y) = \sum_{i=0}^{n+1} H_i(x,y) \in \mathbb{R}[x,y]$ be a polynomial of degree $n+1$. Here $H_i(x,y)$ is a polynomial of homogeneous degree i. Assume that every critical value (singularity) of $H(x,y)$ is of Morse type (see the commentary before Theorem 0.1). Secondly suppose that the algebraic variety $\{(x,y,z) \in \mathbb{CP}^2 : H_{n+1}(x,y) - z^{n+1} = 0\}$ is a smooth submanifold of $\mathbb{CP}^2$. Then the following estimate holds.

Theorem 2.2. *Let $\gamma(s_0)$ be a vanishing cycle of $X_{s_0} = \{(x,y) \in \mathbb{C}^2 : H(x,y) + s_0 = 0\}$.*

Suppose $\deg P(x,y), Q(x,y) \leq n$ *and the conditions above are satisfied then the multiplicity of isolated zeros of an Abelian integral $I_{Pdx+Qdy,\gamma}(s_0)$ is less than $\frac{n^4+n^2-2}{2}$.*

To prove this theorem, Madešic studied the Wronskian defined as the determinant of a $n^2 \times n^2$ matrix

$$W(s_0, \omega) = \det \left(\left(\frac{d}{ds_0} \right)^j I_{\gamma_i}(s_0) \right)_{\substack{1 \leq i \leq n^2 \\ 0 \leq j \leq n^2 - 1}},$$

where $\{\gamma_1, \dots, \gamma_{n^2}\}$ a basis of the homology group $H_1(X_{s_0}, \mathbb{Z})$. He shows the univalence hence the rationality of $W(s_0, \omega)$ by an similar argument to the above one, on showing

$$W(s_0, \omega) = O\left(s_0^{\frac{3n^2 - n^4}{2}}\right) \quad \text{as} \quad s_0 \to \infty$$

$$W(s_0, \omega) = O\left(\left(s_0 - s_0^{(i)}\right)^{2-n^2}\right) \quad \text{as} \quad s_0 \to s_0^{(i)} \quad i = 1, \dots, n^2$$

where $s_0^{(i)}$ is a critical value of the Hamiltonian $H(x,y)$. This implies that the Wronskian is a rational function of the form,

$$W(s_0, \omega) = \frac{A(s_0)}{B(s_0)},$$

where *deg* $A(s_0) \leq \frac{n^4 - n^2}{2}$, *deg* $B(s_0) \leq (n^2 - 1)n^2$. Hence we see the multiplicity of isolated zeros of an Abelian integral $(\frac{\partial}{\partial s_0})^{n^2-1} I_{Pdx+Qdy,\gamma_i}(s_0)$ is less than $\frac{n^4-n^2}{2}$ which gives the estimate of the theorem **q.e.d.**

3. Fuchsian differential equation

3.1 Statement of the result

We remember that in the lemma 1.2, 1) we observed a system of differential equations satisfied by $I_{x^iy^\ell dx}(s), 0 \leq i \leq 2$ defined for the Hamiltonian (1.1). As a matter of fact analogous system of differential equations appears in the investigation of Abelian integrals defined for more general type of Hamiltonians. Namely this method allows one to obtain estimates on the multiplicity of zeros of the Abelian integrals.

Let us try to get a system for the case of a Hamiltonian of the following form:

$$H(x, y) = F(x, s') - y^\nu, \tag{3.1}$$

where the polynomial $F(x, s')$ is assumed to be as follows:

$$F(x, s') = x^{\mu+1} + s_{\mu-1}x^{\mu-1} + \ldots + s_1 x. \tag{3.2}$$

that depends on $\mu - 1$ real parameters $(s_1, \ldots, s_{\mu-1})$.

Remark that every polynomial $G(X)$ of degree $\mu + 1$ in one variable can be reduced to the form $F(x, s') + s_0$ after a simple change of coordinate $X \to x = X + c$, for certain $c \in \mathbf{R}$. If you put $\mu = 2$ and $\nu = 2$ you reproduce the situation of §1.

In effect, every notion that has been introduced in §1 can be defined and studied in a similar manner for the Hamiltonian (3.1). That is to say we shall look at a family of Riemann surface

$$X_s^{(\mu,\nu)} = \{(x, y) \in P^2\mathbf{C}; F(x, s') - y^\nu + s_0 = 0\} \tag{3.3}$$

and the vanishing cycles of the homology group $H_1(X_s^{(\mu,\nu)})$. It is known that the rank of this group is equal to $\mu(\nu - 1)$. Further we shall look at the quotient space like (1.5) for our Hamiltonian (3.1) and obtain from it a basis of the cohomology group $H^1(X_s^{(\mu,\nu)})$ in the following manner:

$$x^k y^\ell dx, (0 \leq k \leq \mu - 1, 1 \leq \ell \leq \nu - 1). \tag{3.4}$$

It justifies the duality between $H_1(X_s^{(\mu,\nu)})$ and $H^1(X_s^{(\mu,\nu)})$. That is to say, the ranks of two groups are equal.

In order to introduce the Abelian integrals, we fix a vanishing cycle $\gamma_s \in H_1(X_s^{(\mu,\nu)})$. We define the Abelian integral

$$K_i^\lambda(s) = \int_{\gamma_s} x^i (F(x, s') + s_0)^\lambda dx, \qquad i = 0, \ldots, \mu + 1. \tag{3.5}$$

for $\lambda = \frac{\ell}{\nu}$. Further we shall use the notation λ in this sense for the sake of simpicity. You imagine γ_s as a sum of cycles which turn around two roots of the equation $F(z, s') + s_0 = 0$.

This is a (multivalued) function in μ variables. Leray's residue theorem ([7], [6]) assures that one needs not to fix concretely a cycle γ_s in (3.5) during the search for differential equations satisfied by (3.5). Namely Leray's residue theorem tells us,

$$\frac{\partial}{\partial s_0} K_\ell^\lambda = \int_{\gamma_s} z^i \frac{\partial}{\partial s_0} (F(z, s') + s_0)^\lambda dz. \tag{3.6}$$

Every integral defined by (3.5) satisfies the same differential equation independent of the cyle γ_s.

Further we shall impose a condition on the Hamiltonian (3.1) that its critical values $s_0^{(i)}$ never coincides (i.e. $s_0^{(i)} \neq s_0^{(j)}$ if $i \neq j$) and it has singularity of Morse type i.e. the Hessian is non-degenerate at each critical point (cf. §1).

As we intend to establish an estimate of the multiplicity of zeros of a ramifying function, we will precise the definition of this notion:

Definition 3. *If a convergent series in the neighbourhood of $t = t_0$ defines a multivalued function in a sector $\Delta_\theta = \{t : |t - t_0| < \theta\} \subset \mathbf{C}$,*

$$f(t) = \sum_{\rho \in \mathbf{Q}, k \in \mathbf{N}} f_{\rho,k}(t - t_0)^\rho (log(t - t_0))^k$$

this series is called a Dulac function.

If the above series has the following characteristic exponents:

$$\rho_1 = \{min\ \rho; f_{\rho,k} \neq 0\},$$

$$k_1 = \{max\ k; f_{\rho_1,k} \neq 0\},$$

we say that a Dulac function has t_0 as a zero of multiplicity $(k_1 + 1)([\rho_1] + 1)$.

Dulac [8] remarked that the monodromy transform $\Delta(q)$ of a dynamical system defined in §0 is a Dulac function in q.

Further we shall understand the zero of the Abelian integral as that of a Dulac function in variable s_0.

After Picard-Lefschetz theorem, the integral $K_k^{\frac{m}{\nu}}(s)$ has the critical values $\{s_0^{(1)}, \cdots, s_0^{(\mu)}\}$ of $F(x, s') + y^\nu$ as its ramifiation points.

Theorem 3.1. *Under the above situation let us consider the Abelian integral $K_k^{\frac{m}{\nu}}(s)$ defined along a cycle $\gamma_s = \{(x, y) \in \mathbf{R}^2 : H(x, y) + s_0 = 0\}$,*

$$K_k^{\frac{m}{\nu}}(s) = \int_{\gamma_s} x^k y^m(x, y) dx,$$

with $K \in \mathbf{N}, m \in \mathbf{Z}$. Suppose $K_k^{\frac{m}{\nu}}(s) \not\equiv 0$. Then we have the following estimates on the multiplicity N of zeros of the Abelian integral $K_k^{\frac{m}{\nu}}(s)$.

i) At one of its ramification points $\tilde{s}_0 \in \{s_0^{(1)}, \cdots, s_0^{(\mu)}\}$, N satisfies:

$$N \leq 2[\frac{m}{\nu} + \frac{3}{2}].$$

ii) At a point where it is smooth i.e. $s_0 = \tilde{s}_0 \notin \{0, s_0^{(1)}, \cdots, s_0^{(\mu)}\}$, N does not surpass μ.

iii) At the orign $s_0 = 0$, N satisfies:

$$N \leq max\{\mu - 1, [\frac{m}{\nu}] + k + 2\}.$$

3.2 Differential operator annihilating the Abelian integral.

To get concrete expression of the Abelian integrals in the neighbourhood of ramification points, we start from the following proposition.

Proposition 3.2 ([15], [18]). *The Abelian integrals $K_0^\lambda(s), \ldots, K_{\mu+1}^\lambda(s)$ satisfy the following overdetermined system of differential equations:*

$$(3.7)_i \qquad \sum_{\ell=0}^{\mu-1} s_\ell \frac{\partial}{\partial s_0} K_{\ell+i}^\lambda + \frac{\partial}{\partial s_0} K_{\mu+1+i}^\lambda = \lambda K_i^\lambda, \quad 0 \leq i \leq \mu - 1,$$

$$(3.8)_j \qquad \sum_{\ell=1}^{\mu-1} \ell s_\ell \frac{\partial}{\partial s_0} K_{\ell+j}^\lambda + (\mu + 1) \frac{\partial}{\partial s_0} K_{\mu+1+j}^\lambda = -(j + 1) K_j^\lambda, \quad -1 \leq j \leq \mu - 1.$$

We get $(3.7)_i$ in applying directly (3.6) to the definition of the integral (3.5). The left hand side of $(3.8)_j$ turns out to be equal to

$$\int_{\Gamma_s} z^{j+1}\frac{\partial}{\partial z}(F(z,s')+s_0)^\lambda dz,$$

which in its turn must be equal to the right hand side of $(3.8)_j$ by integration by parts.

The relations $(3.7)_i$ $(3.8)_j$ can be rewritten in a matrix representation as follows.

$$\Sigma\cdot\vec{b}=\vec{a},$$

where

$$\Sigma=\begin{bmatrix}
s_0 & s_1 & \cdots & s_{\mu-2} & s_{\mu-1} & 0 & 1 & \cdots & 0 & 0 & 0\\
0 & s_0 & \cdots & s_{\mu-3} & s_{\mu-2} & s_{\mu-1} & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & \cdots & s_{\mu-4} & s_{\mu-3} & s_{\mu-2} & s_{\mu-1} & \ddots & 0 & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & s_1 & s_2 & s_3 & s_4 & \cdots & 1 & 0 & 0\\
0 & 0 & \cdots & s_0 & s_1 & s_2 & s_3 & \cdots & 0 & 1 & 0\\
0 & 0 & \cdots & 0 & s_0 & s_1 & s_2 & \cdots & s_\mu & 0 & 1\\
s_1 & 2s_2 & \cdots & (\mu-1)s_{\mu-1} & 0 & \mu+1 & 0 & \cdots & 0 & 0 & 0\\
0 & s_1 & \cdots & (\mu-2)s_{\mu-2} & (\mu-1)s_{\mu-1} & 0 & \mu+1 & \ddots & 0 & 0 & 0\\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & s_1 & 2s_2 & 3s_3 & \cdots & \cdots & \mu+1 & 0 & 0\\
0 & 0 & \cdots & 0 & s_1 & 2s_2 & \cdots & \cdots & 0 & \mu+1 & 0\\
0 & 0 & \cdots & 0 & 0 & s_1 & \cdots & \cdots & (\mu-1)s_{\mu-1} & 0 & \mu+1
\end{bmatrix},$$

$$\vec{a}={}^t(\lambda K_0^\lambda,\lambda K_1^\lambda,\cdots,\lambda K_{\mu-1}^\lambda,0,-K_0^\lambda,-2K_1^\lambda,\cdots,-\mu K_{\mu-1}^\lambda)$$

$$\vec{b}={}^t(\frac{\partial}{\partial s_0}K_0^\lambda,\frac{\partial}{\partial s_0}K_1^\lambda,\cdots,\frac{\partial}{\partial s_0}K_{2\mu}^\lambda).$$

Here one shall remark the following simple relation, $\frac{\partial}{\partial s_i}K_j^\lambda(s)=\frac{\partial}{\partial s_j}K_i^\lambda(s)$.

In fact 2μ Abelian integrals take part in the equations $(3.7)_i,(3.8)_j$ instead of μ integrals. You may wonder why I do not speak about $\mu(\nu-1)$ integrals that can be obtained from the froms (3.4). The reason is that for fixed $\lambda=\frac{\ell}{\nu}$ with $1\leq\ell\leq\nu-1$ the integrals have non-trivial relations only among μ integrals $K_j^\lambda(s)$ $0\leq j\leq\mu-1$ after Proposition 3.2.

Proposition 3.3. *The integrals* $\mathbf{K}(s)=(K_0(s)^\lambda,\cdots,K_{\mu-1}^\lambda(s)),s=(s_0,\cdots,s_{\mu-1})$ *satisfy the following system of differential equations:*

$$S\frac{\partial}{\partial s_0}\mathbf{K}=(L+V(s_2,\cdots,s_{\mu-1}))\mathbf{K},\tag{3.9}$$

where $V(s_2,\cdots,s_{\mu-1})$ *is a rank* $\mu-2$ *matrix with polynomial entries depending only on* $s_2,\cdots,s_{\mu-1}$.

The matrix S admits the following representation

$$S=s_0\mathrm{id}_\mu+C(s_1,\ldots,s_{\mu-1})$$

with a polynomial matrix $C(s_1,\ldots,s_{\mu-1})$ *and*

$$L=\mathrm{diag}(\lambda+\frac{1}{\mu+1},\ldots,\lambda+\frac{\mu}{\mu+1}).$$

We will make use of the notation $\Delta_\mu(s)$ designating a monic polynomial in variable s_0 that is the discriminant of the polynomial $z^{\mu+1}+s_{\mu-1}z^{\mu-1}+\ldots+s_1z+s_0$. It means that for $s\in\mathbf{R}^\mu$ the equation $F(x,s')+s_0=0$ has multiple (double, triple etc.) roots. The discriminant can be calculated as determinant of the matrix $S(s)$ of the Proposition 3.3,

$$\Delta_\mu(s_0,s')=\det\ S(s)=\det\Sigma.$$

Specifically the matrix Σ is often called Sylvester's resultant (cf. van der Waerden, Modern Algebra).

Definition 4. One says that $s' \in \mathbf{R}^{\mu-1}$ belongs to the bifurcation set of the Hamiltonian (3.1) $B \subset \mathbf{R}^{\mu-1}$ if and only if there exists $s_0 \in \mathbf{C}$ such that

$$\Delta_\mu(s_0, s') = \frac{\partial}{\partial s_0}\Delta_\mu(s_0, s') = 0.$$

Remark 2. The bifurcation set classifies the Hamiltonians (3.2). The Hamiltonians outside of B are those in generic position i.e. the case that you encounter most often in the nature as they compose a subset of codimension 0 in the space of all Hamiltonians. And our condition that the Hamiltonian has only Morse type singularities and different critical values is equivalent to say that the Hamiltonian is outside of B. The next typical case will be those on the regular part of B. After that comes those on codimension 1 subset of B, etc.

After Proposition 3.3, our strategy shall be developed in the following manner. We seek for a differential operator that annihilates $K_i^\lambda(s)$ (Proposition 3.4). It turns out that these operators are Fuchsian differential operators with its singular locus along $D = \{s \in \mathbf{C}^\mu; \Delta_\mu(s) = 0\}$ possibly plus an affine plane. We shall determine the characteristic exponents (i.e. how fast the function annihilated by these operators can vanish at its singular locus) by reducing the general situation to a special example by means of isomonodromy technique.The maximum of the characteristic exponents shall estimate the multiplicity of the zeros of the integrals $K_i^\lambda(s)$.

In order to get the Fuchsian differential operator annihilating $K_i^\lambda(s)$, one shall introduce determinant of a matrix with operator elements like the following.

$$\mathbf{P}(s, \frac{\partial}{\partial s_0}) = \begin{bmatrix} p_{0,0} & p_{0,1} & \cdots & p_{0,\mu-1} \\ p_{1,0} & \vdots & \cdots & p_{1,\mu-1} \\ \vdots & \vdots & \cdots & \vdots \\ p_{\mu-1,0} & p_{\mu-1,1} & \cdots & p_{\mu-1,\mu-1} \end{bmatrix},$$

with operator elements like the following,

$$det\mathbf{P}(s, \frac{\partial}{\partial s_0}) :=$$

$$= \sum_{i_0, i_1, \cdots, i_{\mu-1}} sign(i_0, i_1, \cdots, i_{\mu-1}) p_{i_{\mu-1},\mu-1} \cdots p_{i_1,1} p_{i_0,0}, \tag{3.10}$$

for $p_{i,j} = p^1_{i,j}(s)\frac{\partial}{\partial s_0} + p^0_{i,j}(s)$, with $p^1_{i,j}(s), p^0_{i,j}(s) \in \mathbf{R}[s]$. In (3.10), the indices $(i_0, i_1, \cdots, i_{\mu-1})$ run through all the permutations of $(0, 1, \cdots, \mu-1)$.

Proposition 3.4. *Differential operator $\mathcal{P}(s, \frac{\partial}{\partial s})$ of degree μ defined below annihilates the integral $K_0^\lambda(s)$,*

$$\mathcal{P}(s, \frac{\partial}{\partial s}) K_0^\lambda(s) = 0,$$

where

$$\mathcal{P}(s, \frac{\partial}{\partial s}) := det(S\frac{\partial}{\partial s_0} - L - V(s')),$$

the matrices $S, L, V(s')$ are those which are defined in the Proposition 3.3.

We can get by a similar method a differential operator $\mathcal{P}^{(k)}_{x^0}(s, \frac{\partial}{\partial s})$ of degree $\mu+1$ annihilating K_k^λ :

$$\mathcal{P}^{(k)}_0(s, \frac{\partial}{\partial s_0}) K_k^\lambda(s) = 0.$$

These operators have the following "principal terms" (the terms with highest order differentials):

$$\mathcal{P}(s, \frac{\partial}{\partial s}) = \Delta_\mu(s)(\frac{\partial}{\partial s_0})^\mu + \sum_{1 \le j \le \mu} P_{\mu-j}(s)(\frac{\partial}{\partial s_0})^{\mu-j}, \tag{3.11}$$

and

$$\mathcal{P}_0^{(k)}(s, \frac{\partial}{\partial s})$$

$$= s_0 \Delta_\mu(s)(\frac{\partial}{\partial s_0})^{\mu+1} + \sum_{1 \le j \le \mu+1} P^{(k)}_{\mu-j+1}(s)(\frac{\partial}{\partial s_0})^{\mu+1-j}, \tag{3.12}$$

for $k \geq 1$. And these operators are Fuchsian differential operators, that is to say, $deg_{s_0} P_{\mu-j}(s) = \mu - j$, $deg_{s_0} P^{(k)}_{\mu+1-j}(s) = \mu + 1 - j$. The notation deg_{s_0} is that of (1.4). Remark that this Proposition already proves the statement $ii)$ of Theorem 3.1, because as it is easy to see, a solution to a differential equation of order $\mu + 1$ cannot have a zero of multiplicity strictly larger than μ. Now the proof of the statement $i), iii)$ remained.

3.3 Calculus of the characteristic exponents

In order to explain the notion of the characteristic exponents, let us remember Frobenius method for a second order Fuchsian operator. Let $P(t, \frac{\partial}{\partial t})$ be an ordinary differential operator defined as follows:

$$P(t, \frac{d}{dt}) = t^2 b_2(t)(\frac{d}{dt})^2 + t b_1(t) \frac{d}{dt} + b_0(t),$$

where $b_2(0)b_1(0)b_0(0) \neq 0$. Then the solutions of the equation $P(t, \frac{d}{dt})u(t) = 0$ are known to be linear combination of functions of the form $v(t) = t^\lambda (\log t)^\kappa \alpha(t)$ with $\alpha(t)$ a holomorphic function at $t = 0$ i.e. a Dulac function. To determine numbers λ and permissible values of κ, we shall solve so called indicial equation (see (3.13) below) for $P(t, \frac{d}{dt})$. Substituting $u(t) = t^\lambda$ one sees that a function like $v(t)$ can be solution of the equation if and only if

$$b_2(0)\lambda(\lambda - 1) + b_1(0)\lambda + b_0(0) = 0. \tag{3.13}$$

If the equation (3.13) has distinct roots λ_1, λ_2 called characteristic exponents, then we have two linearly independent solutions:

$$u(t) = t^{\lambda_i} \alpha_i(t), \quad i = 1, 2$$

i.e. $\kappa = 0$. While if the (3.13) has a double root λ_0, then we have two linearly independent solutions:

$$u(t) = t^{\lambda_0} \alpha_0(t) \ \text{ or } \ = t^{\lambda_0} (\log t) \tilde{\alpha}_0(t),$$

i.e. $\kappa = 0, 1$. See the book by Coddington-Levinson, Ince or Whittaker-Watson "Modern Analysis" to review more detailed and generalized discussion.

We can apply the same method to a higher order Fuchsian differential operator. As a matter of fact, the indicial equation associated to the equation (3.11) (resp. (3.12)) has always a set of μ (resp. $\mu + 1$) rational numbers independent of the choice of $s' \in \mathbf{R}^{\mu-1} \setminus B$. One can see it from the fact that the characteristic exponents of (3.12) must be rational numbers $\lambda_1(s'), \cdots, \lambda_{\mu+1}(s')$, while the indicial equation itself changes continuously depending on $s' \in \mathbf{R}^{\mu-1} \setminus B$. On B the multiplicity of the indicial equation can change and we cannot extend it beyond B.

After Picard-Lefschetz theorem explained in §1, the characteristic exponents of (3.12) must satisfy

$$e^{2\pi i \lambda_1(s')} = -e^{2\pi i \frac{m}{\nu}}, e^{2\pi i \lambda_2(s')} = \cdots = e^{2\pi i \lambda_{\mu+1}(s')} = 1.$$

This means that

$$\lambda_i(s'_1) - \lambda_i(s'_2) \in \mathbf{Z}, 1 \leq i \leq \mu + 1,$$

for $s'_1 \neq s'_2 \in \mathbf{R}^{\mu-1} \setminus B$. In taking into account the continuous dependence of the characteristic exponents on $s' \in \mathbf{R}^{\mu-1}$ we conclude that $\lambda_i(s')$ must be constant on $s' \in \mathbf{R}^{\mu-1} \setminus B$ (isomonodromy technique). Now we can choose any point $s' \in \mathbf{R}^{\mu-1} \setminus B$ to calculate concretely the characteristic exponents of (3.11), (3.12). Let us choose for the sake of simplicity $s' = (s_1, 0, \cdots, 0)$. Then the equations under question will be as follows:

$$(3.13). \qquad [\prod_{j=0}^{\mu-1}(s_0\frac{\partial}{\partial s_0} - \frac{j+1}{\mu+1} + j - \frac{m}{\nu}) + \frac{s_1}{\mu+1}(-\frac{\mu}{\mu+1}s_1\frac{\partial}{\partial s_0})^{\mu}]K_0^{\frac{m}{\nu}}(s_0, s_1) = 0$$

On the other hand for $K_k^{\frac{m}{\nu}}(s_0, s_1)$, we have

$$[(s_0\frac{\partial}{\partial s_0} - \frac{k}{\mu+1} + \mu - 1 - \frac{m}{\nu})\prod_{j=0}^{\mu-1}(s_0\frac{\partial}{\partial s_0} +$$

$$(3.14) \qquad + \frac{\mu j - k}{\mu+1} - \frac{m}{\nu}) + \frac{s_1}{\mu+1}(-\frac{\mu}{\mu+1}s_1\frac{\partial}{\partial s_0})^{\mu}(s_0\frac{\partial}{\partial s_0} - k - 1 - \frac{m}{\nu})]K_k^{\frac{m}{\nu}}(s_0, s_1) = 0.$$

If you calculate the characteristic exponents at singular points of the equations (3.13), (3.14) we get the following result.

Proposition 3.5. *The characteristic exponents at a singular point* $s_0 = \mu(-\frac{s_1}{\mu+1})^{\frac{\mu+1}{\mu}}$ *of the equations (3.13) are the following:*

$$\rho = 0, 1, \cdots, \mu - 2, \frac{1}{2} + \frac{m}{\nu}.$$

At the same point, the characteristic exponents of (3.14) are the follwoing:

$$\rho = 0, 1, \cdots, \mu - 1, \frac{1}{2} + \frac{m}{\nu}.$$

At the point $s_0 = 0$ *its characteristic exponents are as follows:*

$$\rho = 0, 1, \cdots, \mu - 1, \frac{m}{\nu} + k + 1.$$

As it has been explained on the Frobenius method, if the indicial equation has double roots ρ then there exists a solution to (3.14) in the form

$$K_k^{\frac{m}{\nu}}(s_0, s_1) = (s_0 - s_0(s_1))^{\rho}\log(s_0 - s_0(s'))\alpha_0'(s_0 - s_0(s')),$$

with a holomorphic function $\alpha_0'(\bullet)$ in the neighbourhood of $s_0 = s_0(s')$. This situation can happen only with $\rho = \frac{1}{2} + \frac{m}{\nu}$. After definition 3, the multiplicity of zeros N of $K_k^{\frac{m}{\nu}}(s_0, s_1)$ must satisfy the inequality of the statement Theorem 3.1, i).

The statement iii) follows from the fact that $\frac{m}{\nu} \notin \mathbf{N}$. If $\frac{m}{\nu} = \ell \in \mathbf{N}$ then the integral $K_k^{\ell}(s)$ has an exact form as its integrand. Hence $K_k^{\frac{m}{\nu}}(s) \equiv 0$. **Q.E.D.**

The detailed proof of Theorem 3.1 will appear in [22]

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