On the Invariants of Base Changes of Pencils of Curves, II

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Introduction

Semistable reduction of pencils of curves has been studied by many authors in various ways. (cf. [AW], [De], [DM], [X3]). In this part of the series, we shall investigate semistable reduction from the point of view of numerical invariants. As an application, we obtain two numerical criterions for a base change to be stablizing, and for a fibration to be isotrivial. We also obtain a canonical class inequality for any fibrations. Some other applications are presented.

Let $f : S \longrightarrow C$ be a fibration of a smooth complex projective surface S over a curve C, and denote by g the genus of a general fiber of f. We assume that $g > 0$ and S is relatively minimal with respect to f, i.e., S has no (-1)-curves contained in a fiber of f. The basic relative numerical invariants of f are defined as follows,

$$\begin{aligned} \chi_f &= \chi(\mathcal{O}_S) - (g-1)(g(C)-1), \\ K_f^2 &= K_S^2 - 8(g-1)(g(C)-1), \\ e_f &= \chi_{\rm top}(S) - 4(g-1)(g(C)-1). \end{aligned}$$

These invariants are nonnegative integers satisfying the Noether equality $12\chi_f = K_f^2 + e_f$. We denote by $\omega_{S/C} = \omega_S \otimes f^*\omega_C$ the relative canonical sheaf of f, and $K_{S/C}$ the relative canonical divisor corresponding to $\omega_{S/C}$. Then $\chi_f = \deg f_*\omega_{S/C}$ and $K_f^2 = K_{S/C}^2$. If $g > 1$ and f is not locally trivial, then χ_f and K_f^2 are positive ([Ar], [Be2], [Pa], or [BPV], Theorem 18.2), in this case, we define the slope of f as

$$\lambda_f = K_f^2/\chi_f.$$

$e_f = \sum_F e_F = \sum_F(\chi_{\rm top}(F) - (2-2g))$ is zero iff f is smooth.

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

A fiber of f is called semistable if it consists of simple components meeting normally. f is said to be semistable if every fiber of it is semistable.

Let $\pi : \widetilde{C} \longrightarrow C$ be a base change of degree d. Then the pull-back fibration $\widetilde{f} : \widetilde{S} \longrightarrow \widetilde{C}$ of f with respect to π is defined as the relative minimal model of the desingularization of $S \times_C \widetilde{C} \longrightarrow \widetilde{C}$. (cf. Sect. 1.3). Since $g > 0$, so the relative minimal model is unique, hence $\widetilde{f}$ is determined uniquely by f and π. Due to Kodaira's classification of singular fibers, the semistable reduction of an elliptic fibration is quite clear, so we always assume that $g \geq 2$.

We define

$$\chi_\pi = \chi_f - \frac{1}{d}\chi_{\widetilde{f}}, \quad K_\pi^2 = K_f^2 - \frac{1}{d}K_{\widetilde{f}}^2, \quad e_\pi = e_f - \frac{1}{d}e_{\widetilde{f}}$$

as the basic numerical invariants of π with respect to f. Obviously, they are rational numbers satisfying $12\chi_\pi = K_\pi^2 + e_\pi$. Xiao [X4] and I [Ta] proved that these invariants are nonnegative, and one of them vanishes if and only if π is an *invariant base change.* (See Definition 1.7).

Definition I. We shall call π a *stablizing* (resp. *trivial*) base change if all of the fibers of $\widetilde{f}$ (resp. f) over the ramification locus R_π (resp. the branch locus B_π) of π are semistable. We shall also call π the *semistable reduction of the fibers over* B_π.

The well-known semistable reduction theorem says that for any fibration f, there exists a base change π such that $\widetilde{f}$ is semistable. In particular, let π be a base change totally ramified over F (i.e., over $f(F)$) and some other semistable fibers, and let F' be the minimal embedded resolution of F. If the degree of π is exactly the greatest common divisor of the multiplicities of the components in F', then it is well-known that π is stablizing. We shall call π the *canonical semistable reduction* of F, and denote it by ϕ_F.

Definition II. For any fiber F of f, we define its basic invariants to be the basic invariants of $\phi = \phi_F$, and denote them respectively by

$$c_1^2(F) = K_\phi^2, \quad c_2(F) = e_\phi, \quad \chi_F = \chi_\phi.$$

We shall show that these invariants are independent of the choice of the base changes (Lemma 2.3). They are nonnegative rational numbers satisfying the Noether equality

$$12\chi_F = c_1^2(F) + c_2(F).$$

We can see also that one of them vanishes iff F is semistable. In fact, these invariants can be computed directly from the embedded resolution of F (see Proposition 3.1 for the formulas). For simplicity, if $B = F_1 + \cdots + F_s$, then we define $c_1^2(B) = c_1^2(F_1) + \cdots + c_1^2(F_s)$. Similarly, we can define $c_2(B)$ and χ_B.

Definition III. A fibration $f : S \longrightarrow C$ is *trivial* if S is isomorphic to $F \times C$ over C. It is *isotrivial* if it becomes trivial after a finite base change.

If $\widetilde{f}$ is a semistable model of f under a semistable reduction π, then a natural problem is:

What is the effect of a non-semistable fiber on the invariants of $\widetilde{f}$?

([X2], Problem 7). In this paper the effect is completely determined.

In what follows, we denote by $\mathcal{B}_\pi = f^*(B_\pi)$ the locus of branched fibers, and by $\mathcal{R}_\pi = \widetilde{f}^*(R_\pi)$ the locus of ramified fibers.

The main results of this paper are the following.

Theorem A. *Let $f : S \longrightarrow C$ be a fibration, and let $\pi : \widetilde{C} \longrightarrow C$ be a base change of degree d. Then*

$$K_\pi^2 = c_1^2(\mathcal{B}_\pi) - \frac{1}{d}c_1^2(\mathcal{R}_\pi), \quad e_\pi = c_2(\mathcal{B}_\pi) - \frac{1}{d}c_2(\mathcal{R}_\pi), \quad \chi_\pi = \chi_{\mathcal{B}_\pi} - \frac{1}{d}\chi_{\mathcal{R}_\pi}.$$

Corollary. *For any fibration $f : S \longrightarrow C$ and any base change $\pi : \widetilde{C} \longrightarrow C$, we have*

1)

$$K_\pi^2 \le c_1^2(\mathcal{B}_\pi), \qquad e_\pi \le c_2(\mathcal{B}_\pi), \qquad \chi_\pi \le \chi_{\mathcal{B}_\pi},$$

and one of the equalities holds iff π is stablizing.

2)

$$\sum_F c_1^2(F) \le K_f^2, \quad \sum_F \chi_F \le \chi_f, \quad \sum_F c_2(F) \le e_f,$$

where F runs over all of the non-semistable fibers of f. Furthermore, one of the first two equalities holds iff f is isotrivial, and the last equality holds iff the semistable model of f is smooth.

3) If f is non-isotrivial, then we have

$$\lambda_{\widetilde{f}} = \frac{K_f^2 - c_1^2(\mathcal{B}_\pi)}{\chi_f - \chi_{\mathcal{B}_\pi}}.$$

Hence the slope of $\widetilde{f}$ is completely determined by the branched non-semistable fibers.

Due to this theorem, the study of the invariants of stablizing base changes can be reduced to the local study of $c_1^2(F)$ and $c_2(F)$. First of all, from definition, it is trivial to see that

$$c_2(F) \le e_F \ (=: \chi_{\mathrm{top}}(F) - (2-2g)),$$

with equality iff the semistable model of F is a smooth fiber. In Sect. 3.3, we obtain

Theorem B.

$$c_1^2(F) \le 2c_2(F),$$

with equality iff $F = nF_{\mathrm{red}}$ and F_{red} has at worst ordinary double points as its singularities. Hence for any stablizing base change π, we have

$$K_\pi^2 \le 8\chi_\pi.$$

We show that $c_1^2(F)$ is in fact bounded by the genus g, i.e.,

Theorem C.

$$c_1^2(F) \le 4g - 4.$$

As an application of this inequality, we obtain the following *canonical class inequality.*

Theorem D. *If f is a fibration of genus $g \ge 2$, then*

$$K_{S/C}^2 \le (2g-2)(2g(C) - 2 + 3s),$$

where s is the number of singular fibers of f.

Note that other canonical class inequalities are already known for semistable fibrations:

$$K_{S/C}^2 \le (2g-2)(2g(C) - 2 + s);$$
$$K_{S/C}^2 < 4g(g-1)(2g(C) - 2 + s);$$
$$K_{S/C}^2 \le 8(g-1)^2(2g(C) - 2 + s).$$

These inequalities are due respectively to Vojta [Vo], Szpiro [Sz] and Esnault and Viehweg [EV]. In a later paper, by using the results of this paper we shall give a linear (in g) and effective height inequality for algebraic points on a curve over functional fields.

As another application, we find some new phenomena for fibrations. (Sect. 4.1). For example, from the corollary above, we can see that every stable model $\widetilde{f}$ of f has the same slope λ determined by

$$K_f^2 - \lambda \chi_f = \sum_F c_1^2(F) - \lambda \sum_F \chi_F,$$

where F runs over all of the non-semistable fibers of f. From Theorem B we know that if $\lambda_f > 8$, then any non-trivial stablizing base change π makes the slope increase. We have also found some relationships between non-semistable fibers and the slope of a fibration.

Finally, in Sect. 4.3, we consider the computation of the Horikawa number of a genus 3 non-semistable fiber F through semistable reductions. We reduce it to the computation for its semistable models $\widetilde{F}$.

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Notations. If D is a local curve and $p \in D$, then we denote by ν_p the multiplicity of D at p, and denote respectively by μ_p, δ_p, k_p the Milnor number, geometric genus and the number of local branches of (D_{red}, p). Hence $\mu_p = 2\delta_p - k_p + 1$. If F is a curve on a smooth surface, then we denote by μ_F the total Milnor number of the singularities of F.

If a, b are two natural numbers, then we denote by (a, b) the greatest common divisor of a and b, and let $[a, b] = \frac{(a,b)^2}{ab}$. $[x]$ is the greatest integer $\le x$

1 Preliminaries and technical lemmas

1.1 Embedded resolution of curve singularities

Let $(B,p) \subset \mathbb{C}^2$ be a local curve (not necessarily reduced) in a neighborhood U_0 of $p=(0,0)$. Assume that (B_{red},p) is a singular point, we say also that p is a singular point of B.

Definition 1.1. The embedded resolution of curve singularity $(B,p)=(B_0,p_0)$ is a sequence

$$(U_0,B_0) \xleftarrow{\sigma_1} (U_1,B_1) \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_r} (U_r,B_r)$$

satisfying the following conditions.

(1) σ_i is the blowing-up of U_{i-1} at a singular point $p_{i-1} \in B_{i-1}$ with $\mu_{p_{i-1}} > 1$.

(2) $B_{r,\mathrm{red}}$ has at worst ordinary double points as its singularities.

(3) B_i is the total transformation of B_{i-1}.

It is well-known that embedded resolution exists and is unique for any curve singularity $(B,p) \subset \mathbb{C}^2$.

We denote by m_i the the multiplicity of $(B_{i,\mathrm{red}},p_i)$. Let

$$\alpha_p = \sum_{i=0}^{r-1}(m_i-2)^2. \tag{1}$$

If $q \in B_r$ is a double point, and a_q, b_q are the multiplicities of the two components of (B_r,q), then we let

$$\beta_p = \sum_{q\in B_r}[a_q,b_q]. \tag{2}$$

Lemma 1.2.

$$\mu_p = \sum_{i=0}^{r-1}(m_i-1)(m_i-2)+k_p-1, \tag{3}$$

$$\delta_p = \frac{1}{2}\sum_{i=0}^{r-1}(m_i-1)(m_i-2)+k_p-1. \tag{4}$$

Proof. In the embedded resolution, we let $E_1 \cap (B_1-E_1) = p_1,\cdots,p_s$. Then by ([Ta], Lemma 1.3) we have

$$\mu_p = (m_p-1)(m_p-2)-1+\sum_{i=1}^{s}\mu_{p_i}. \tag{5}$$

On the other hand, it is obvious that

$$k_p = \sum_{i=1}^{s}(k_{p_i}-1), \tag{6}$$

hence (3) can be obtained easily by using induction on r, and (4) follows from (3) and $\mu_p = 2\delta_p-(k_p-1)$. **Q.E.D.**

Lemma 1.3. *For any singular point (B,p), we have*

$$\alpha_p + \beta_p \le \mu_p. \tag{7}$$

Proof. First we prove (7) for the case $m_p = 2$, i.e., $(B_{\rm red}, p)$ is a double point. Assume that (B,p) is defined by $f(x,y) = 0$ at 0.

If $f = x^a(x+y^k)^b$ and $k = 1$, then $\alpha_p = 0$, $\mu_p = 1$ and $\beta_p = [a,b]$, (7) is obvious. If $k > 1$, then by the computation of the embedded resolution, we have

$$\alpha_p = k-1,\ \mu_p = 2k-1,\ \beta_p = 1 - \frac{1}{k} + [a, k(a+b)] + [b, k(a+b)] \le 1,$$

hence (7) holds strictly.

If $f = (x^2 + y^{2k+1})^n$, then

$$\alpha_p = k,\ \mu_p = 2k,\ \beta_p = \frac{3}{2}(1 - \frac{1}{2k+1}),$$

thus we can see that $\alpha_p + \beta_p \le \mu_p$.

Now we assume that $m_p \ge 3$. In this case, we shall prove (7) by using induction on μ_p. From (5) we know $\mu_{p_i} < \mu_p$, by induction hypothesis, we have $\alpha_{p_i} + \beta_{p_i} \le \mu_{p_i}$. On the other hand, we know

$$\beta_p = \sum_{i=1}^{s} \beta_{p_i}, \quad \alpha_p = (m_p - 2)^2 + \sum_{i=1}^{s} \alpha_{p_i},$$

from (5), (7) follows immediately. **Q.E.D.**

1.2 On the resolution of the singularity of $z^d = f(x,y)$

Now we assume that (B,p) is defined by $f(x,y) = 0$ at $p = (0,0)$. Let $\Sigma \subset \mathbb{C}^3$ be a local surface defined by $z^d = f(x,y)$, and let V_0 be the normalization of Σ. Then, V_0 is a d-cyclic cover $\pi_0 : V_0 \longrightarrow U_0$, the singular points of V_0 (lying over p) can be resolved by the embedded resolution of (B,p), it goes as follows.

Let V_r be the normalization of $U_r \times_{U_0} V_0$, and let $\eta : M \longrightarrow V_r$ be the minimal resolution of the singularities of V_r.

$$\begin{array}{ccccc} V_0 & \xleftarrow{\ \tau\ } & V_r & \xleftarrow{\ \eta\ } & M \\ {\scriptstyle \pi_0}\downarrow & & {\scriptstyle \pi_r}\downarrow & & \downarrow{\scriptstyle \pi_r \eta} \\ U_0 & \xleftarrow[\sigma]{} & U_r & = & U_r \end{array}$$

Then π_r is a cyclic covering branched along B_r. If near $q \in B_r$, B_r is defined by $x^a y^b = 0$, then V_r is locally the normalization of $z^d = x^a y^b$, which are cyclic quotient singularities, hence can be resolved by June-Hirzebruch method (cf. [BPV], p.83). Hence $\phi = \tau\eta : M \longrightarrow V_0$ is the resolution of V_0, we shall call ϕ the *embedded resolution of* V_0.

Denote by $E_p = \sum_{i=1}^{s} E_i$ the exceptional curves of ϕ, and let $K_\phi = \sum_{i=1}^{s} r_i E_i$ be the rational canonical divisor of E_p, which is determined uniquely by the adjunction formula $K_\phi E_i + E_i^2 = 2p_a(E_i) - 2$. Then K_ϕ^2 is an invariant of the resolution ϕ. If ϕ is minimal, then $K_\phi^2 = K_p^2 \leq 0$ is an invariant of the singularities of V_0, which is independent of the resolution. $K_p^2 = 0$ iff V_0 has at worst rational double points as its singularities. We denote by $b_2(E_p)$ the number of components of E_p. The following Lemma can be obtained by a direct computation of the normalization. (cf. Sect. 5 or [X3])

Lemma 1.4. *If (B, p) is defined by $x^a y^b = 0$, and d is divided by a and b, then E_p is $d_p = (a, b)$ curves of type A_n, where*

$$d_p n = b_2(E_p) = [a, b]d - (a, b). \tag{8}$$

Lemma 1.5. *Assume that d is divided by all of the multiplicities of the components in the embedded resolution B_r. Then*

$$-\frac{1}{d} K_\phi^2 = \alpha_p. \tag{9}$$

The proof of this lemma will be given in Sect. 5.

Now we recall the normalization of Σ. (cf. [Ta], Lemma 2.1).

Lemma 1.6. *For any point $p \in B$, $\pi_0^{-1}(p)$ consists of $d_p = \gcd(d, n_1, \cdots, n_s)$ points if there are exactly s components $\Gamma_1, \cdots, \Gamma_s$ passing through p.*

1.3 The construction of base changes

In this section, we recall the construction of the pullback fibration $\widetilde{f}$ of $f : S \longrightarrow C$ under a base change.

Let $\pi : \widetilde{C} \longrightarrow C$ be a base change of degree d. Then the pull-back fibration $\widetilde{f} : \widetilde{S} \longrightarrow \widetilde{C}$ of f with respect to π is defined as the relative minimal model of the desingularization of $S \times_C \widetilde{C} \longrightarrow \widetilde{C}$. In fact, the pull-back fibration $\widetilde{f} : \widetilde{S} \longrightarrow \widetilde{C}$ can be constructed as follows.

Let $\rho_1 : S_1 \longrightarrow S \times_C \widetilde{C}$ be the normalization of $S \times_C \widetilde{C}$, let $\rho_2 : S_2 \longrightarrow S_1$ be the minimal desingularization of S_1. Then we have a fibration $f_2 : S_2 \longrightarrow \widetilde{C}$. Let $\widetilde{\rho} : S_2 \longrightarrow \widetilde{S}$ be the contraction of (-1)-curves such that $\widetilde{f} : \widetilde{S} \longrightarrow \widetilde{C}$ is a relative minimal model. Since we have assumed that $g > 1$, so $\widetilde{\rho}$ is unique. Hence $\widetilde{f} : \widetilde{S} \longrightarrow \widetilde{C}$ is determined uniquely by f and π.

$$\begin{tikzcd}
\widetilde{S} \arrow[d, "\widetilde{f}"] & S_2 \arrow[l, "\widetilde{\rho}"'] \arrow[r, "\rho_2"] \arrow[d, "f_2"] & S_1 \arrow[r, "\rho_1"] \arrow[d, "f_1"] & S \times_C \widetilde{C} \arrow[r, "\Pi'"] \arrow[d] & S \arrow[d, "f"] \\
\widetilde{C} \arrow[r, equal] & \widetilde{C} \arrow[r, equal] & \widetilde{C} \arrow[r, equal] & \widetilde{C} \arrow[r, "\pi"'] & C
\end{tikzcd}$$

Let $\Pi_2 = \Pi' \circ \rho_1 \circ \rho_2 : S_2 \longrightarrow S$.

Definition 1.7. If $\pi : \widetilde{C} \longrightarrow C$ is a base change satisfying

$$\widetilde{\rho}^* K_{\widetilde{S}/\widetilde{C}} \equiv \Pi_2^* K_{S/C},$$

then we shall call it *an invariant base change.*

In fact if $g \geq 2$, then f is invariant iff the fibers F in the branch locus are reduced and F has at worst d_F-simple singularities, where d_F is the greatest ramification index of π over $f(F)$. A d-simple singularity is a simple curve singularity $f(x, y) = 0$ such that $z^d = f(x, y)$ is a simple surface singularity. Hence 2-simple is ADE, 3-simple is $A_1, \cdots, A_4$, 4 and 5-simple are A_1, A_2, d-simple is A_1 if $d > 5$.

Let F be a singular fiber. We always denote by F' the embedded resolution of F, and denote by M_F the greatest common divisor of the multiplicities of the components in F'.

2 On the invariants of a base change

2.1 Local computations of K_π^2

In this section, we first consider the computation of the invariant K_π^2 for a base change $\pi : \widetilde{C} \longrightarrow C$. Without loss of generality, we assume that π is totally ramified over $p_1, \cdots, p_s$. Let ρ_2 be the embedded resolution of singularities, let $F_1, \cdots, F_s$ be the fibers of f corresponding to $p_1, \cdots, p_s$, and let $\mathcal{B}_\pi = \sum_{i=1}^s F_i = \sum_\Gamma n_\Gamma \Gamma$. From Lemma 1.6, it is easy to see that

$$K_{S_2} \equiv \Pi_2^* \left(K_S + \sum_{\Gamma \subset \mathcal{B}_\pi} \left(1 - \frac{(d, n_\Gamma)}{d} \right) \Gamma \right) + K_{\rho_2}. \tag{10}$$

where K_{ρ_2} is the rational canonical divisor of the exceptional set of ρ_2. On the other hand, we have

$$K_{\widetilde{C}} = \pi^* \left(K_C + \sum_{i=1}^s \left(1 - \frac{1}{d} \right) p_i \right). \tag{11}$$

Note that $f_2^* \pi^* = \Pi_2^* f^*$, hence from (10) and (11) we can obtain

$$K_{S_2/\widetilde{C}} = \Pi_2^* \left(K_{S/C} - \sum_{i=1}^s H_{F_i} \right) + K_{\rho_2},$$

where $H_i = \sum_{\Gamma \subset F_i} h_\Gamma \Gamma$, $h_\Gamma = n_\Gamma - 1 - \frac{1}{d}(n_\Gamma - (d, n_\Gamma))$. Hence

$$dK_f^2 - K_{f_2}^2 = d \sum_{i=1}^s (2H_{F_i} K_S - H_{F_i}^2) - K_{\rho_2}^2.$$

If we let $K_\pi^2(f_2) = K_f^2 - \frac{1}{d} K_{f_2}^2$, then

$$K_\pi^2 = K_\pi^2(f_2) - \frac{1}{d} \#\{ (-1)\text{-curves contracted by } \widetilde{\rho} \}.$$

Proposition 2.1. *With the notations above, we have*

$$K_\pi^2(f_2) = \sum_{i=1}^{s}(2H_{F_i}K_S - H_{F_i}^2) - \sum_{i=1}^{s}\sum_{p\in F_i}\frac{1}{d}K_p^2, \tag{12}$$

In the case when π is the base change of $F_1, \cdots, F_s$, if d is divided by M_{F_i}, $i = 1, \cdots s$, we can see that $H_{F_i} = F_i - F_{i,\mathrm{red}}$. Note that we have (cf. [Ta], (7))

$$e_F = \chi_{\mathrm{top}}(F) - (2-2g) = 2N_F + \mu_F,$$

where $N_F = g - p_a(F_{\mathrm{red}}) = \frac{1}{2}((F - F_{\mathrm{red}})K_S - F_{\mathrm{red}}^2)$ is an invariant of F. From Lemma 1.5 we have

Proposition 2.2. *If d is divided by M_{F_i} for all i, then*

$$K_\pi^2(f_2) = \sum_{i=1}^{s}(4N_{F_i} + F_{i,\mathrm{red}}^2) + \sum_{i=1}^{s}\sum_{p\in F_i}\alpha_p. \tag{13}$$

Note that the right hand side of (13) is independent of d.

2.2 Proof of Theorem A

We consider first the composition of base changes.

Let $\pi_1 : C_1 \longrightarrow C$ and $\pi_2 : \widetilde{C} \longrightarrow C_1$ be two base changes, let f_1 be the pullback fibration of f under π_1, and let f_2 be that of f_1 under π_2. By the universal property of fiber product and the uniqueness of the relative canonical model (when $g > 0$), we know f_2 is nothing but the pullback fibration $\widetilde{f}$ of f under $\pi = \pi_1 \circ \pi_2$. Hence we have the *basic equalities*:

$$\begin{aligned} K_\pi^2 &= K_{\pi_1}^2 + \frac{1}{\deg \pi_1}K_{\pi_2}^2, \\ e_\pi &= e_{\pi_1} + \frac{1}{\deg \pi_1}e_{\pi_2}, \\ \chi_\pi &= \chi_{\pi_1} + \frac{1}{\deg \pi_1}\chi_{\pi_2}. \end{aligned} \tag{14}$$

Lemma 2.3. *Let $f : S \longrightarrow C$ be a fibration, and let $F_1, \cdots, F_s$ be fibers of f. Considering all of the semistable reduction π of $F_1, \cdots, F_s$, we have that K_π^2, e_π and χ_π are independent of π.*

Proof. Let $\pi_1 : C_1 \longrightarrow C$ and $\pi_2 : C_2 \longrightarrow C$ be two semistable reduction of $F_1, \cdots, F_s$, let $\deg \pi_i = d_i$, $i = 1, 2$, and let f_i be the pullback fibration of f under π_i. We shall prove that

$$K_{\pi_1}^2 = K_{\pi_2}^2.$$

For this, we consider the pullback of π_1 and π_2,

$$\pi = \pi_1 \times_C \pi_2 : \widetilde{C} = C_1 \times_C C_2 \longrightarrow C.$$

Note that if necessary, we can choose $\widetilde{C}$ to be the normalization of a component of $C_1 \times_C C_2$. Let $p_i : \widetilde{C} \longrightarrow C_i$ be the i-th projection, it is obvious that $\deg p_1 = d_2$, $\deg p_2 = d_1$. Then we have $\pi = p_1 \circ \pi_1 = p_2 \circ \pi_2$ (composition of base changes). Since π_1 and π_2 are semistable reductions, so the fibers of f_i over $F_1, \cdots, F_s$ are semistable, and thus p_i is an invariant base change. It implies that $K_{p_i}^2 = 0$ for $i = 1, 2$. Then by using the basic equalities (14), we have

$$K_\pi^2 = K_{\pi_1}^2 = K_{\pi_2}^2,$$

hence, we have $K_{\pi_1}^2 = K_{\pi_2}^2$.

The proof for χ_π, e_π is the same as above. **Q.E.D.**

Lemma 2.4. *In the situation of Lemma 2.3, we have*

$$K_\pi^2 = \sum_{i=1}^{s} c_1^2(F_i), \quad e_\pi = \sum_{i=1}^{s} c_2(F_i), \quad \chi_\pi = \sum_{i=1}^{s} \chi_{F_i}.$$

Proof. By Lemma 2.3, we can assume that π is the pullback of the canonical semistable reductions $\pi_i = \phi_{F_i} : C_i \longrightarrow C$, $i = 1, \cdots, s$. We can assume that π_i is unramified over the fibers F_j for $j \neq i$. Without loss of generality, we assume also that $s = 2$. As in the proof of Lemma 2.3, we have

$$K_\pi^2 = K_{\pi_1}^2 + \frac{1}{d_1} K_{p_1}^2.$$

Since p_1 is totally ramified semistable reduction, hence $K_{p_2}^2$ can be computed locally from the branched non-semistable fibers, which are the pullback of F_2 under π_1. Hence we know

$$K_{p_1}^2 = d_1 K_{\pi_2}^2.$$

By definition, $K_{\pi_i}^2 = c_1^2(F_i)$. Hence we have obtained the desired equality.

Note that the local property used above holds for e_π and χ_π. **Q.E.D.**

Proof of Theorem A. Let $\hat{\pi} : \hat{C} \longrightarrow \widetilde{C}$ be the semistable reduction of the ramified fibers $\mathcal{R}_\pi$. Then we know that $\pi \circ \hat{\pi}$ is also the semistable reduction of the branched fibers $\mathcal{B}_\pi$. By Lemma 2.4 and the basic equalities we can obtain the equalities in this theorem. **Q.E.D.**

3 On the invariants of non-semistable fibers

3.1 The computations of the invariants c_1^2, c_2 and χ

In what follows, we shall consider the computation of the invariants $c_1^2(F)$, $c_2(F)$ and χ_F. By Noether equality, we only need to compute c_1^2 and c_2.

First note that if we use embedded resolution to resolve the singularities of F, then the number

$$c_{-1}(F) = \frac{1}{d} \#\{(-1)\text{-curves in } F' \text{ contracted by } \widetilde{\rho}\}$$

is also independent of the stablizing base change if d is divided by M_F.

Theorem 3.1.

$$\begin{aligned} c_1^2(F) &= 4N_F + F_{\mathrm{red}}^2 + \sum_{p\in F} \alpha_p - c_{-1}(F), \\ c_2(F) &= 2N_F + \mu_F - \sum_{p\in F} \beta_p + c_{-1}(F). \end{aligned} \tag{15}$$

Proof. The first formula has been proved in Proposition 2.2. In order to prove the second formula, we consider the stablizing base change π of F whose degree is divided by M_F. By definition, if $\widetilde{F}$ and F_2 are respectively the pullback fibers of F in $\widetilde{S}$ and S_2, (note that S_2 is the embedded resolution, not the minimal resolution), then we have

$$e_\pi = \frac{1}{d}(de_F - e_{\widetilde{F}}) = e_F - \frac{1}{d}e_{F_2} + c_{-1}(F).$$

Since F_2 is semistable, so e_{F_2} is the number of singular points of F_2, which is exactly the number $d\sum_{p\in F}\beta_p$ (Lemma 1.4). We have known that $e_F = 2N_F + \mu_F$, hence the second formula has been obtained. **Q.E.D.**

Remark. From the formulas above and the Noether formula, we can see that χ_F is independent of $c_{-1}(F)$, hence it can be computed directly from embedded resolution. In fact, if we consider the canonical semistable reduction of F, then we can prove that

$$c_{-1}(F) = \sum_{q'\in F'} \beta_{q'},$$

where F' is the embedded resolution of F, and q' runs over the singular points of F' such that the (-2)-curves coming from p' are contracted to points of the semistable model of F.

Example. Note that the discussion above holds for elliptic fibrations. In this case, $K_\pi^2 = 0$ for all base changes, so we have $c_1^2(F) = 0$. By a direct computation we have

$$c_2(F) = 12\chi_F = \begin{cases} 0, & \text{if } F \text{ is of type } {}_m\mathrm{I}_b, \\ 6, & \text{if } F \text{ is of type } \mathrm{I}_b^* \ (b > 0), \\ e_F, & \text{otherwise.} \end{cases}$$

The result above shows the well-known fact that the semistable model of an elliptic fiber is smooth except for type ${}_m\mathrm{I}_b$ $(b > 0)$ and type I_b^* $(b > 0)$.

3.2 Proof of Theorem C

Lemma 3.2.

$$\sum_{p\in F} \alpha_p \le 2p_a(F_{\mathrm{red}}), \tag{16}$$

the equality holds iff $p_a(F_{\mathrm{red}}) = 0$, hence F is a tree of nonsingular rational curves.

Proof. We shall use the notations of Sect. 1.1. By (1) and (4), we have

$$\begin{aligned}\alpha_p &= 2\delta_p - \sum_{i=0}^{r-1}(m_i - 2) - 2(k_p - 1) \\ &\leq 2\delta_p - 2(k_p - 1).\end{aligned} \tag{17}$$

On the other hand, if $F_{\mathrm{red}} = \sum_{i=1}^{l_F} \Gamma_i$, then we have

$$p_a(F_{\mathrm{red}}) = \sum_{i=1}^{l_F} p_a(\widetilde{\Gamma}_i) + \sum_{p\in F} \delta_p - l_F + 1. \tag{18}$$

Hence we only need to prove that

$$\sum_{p\in F}(k_p - 1) \geq l_F - 1. \tag{19}$$

But this inequality is an immediate consequence of the connectedness of F. So (16) holds.

If the equality in (16) holds, then from (17) we know that $\alpha_p = 0$ for any $p \in F$, hence $p_a(F_{\mathrm{red}}) = 0$. Then from (18) and (19), we can see F is a tree of smooth rational curves. **Q.E.D.**

Theorem 3.3.

$$c_1^2(F) \leq 4g - 4. \tag{20}$$

Proof. From Lemma 3.2 we have

$$c_1^2(F) \leq 4N_F + F_{\mathrm{red}}^2 + 2p_a(F_{\mathrm{red}}) - c_{-1}(F),$$

and the equality holds iff $p_a(F_{\mathrm{red}}) = 0$. Hence it is easy to prove that $c_1^2(F) \leq 4g - 3 - c_{-1}(F)$, and the equality holds iff F satisfies

$$p_a(F_{\mathrm{red}}) = 0, \quad F_{\mathrm{red}}K = 1. \tag{21}$$

So it is enough to prove that for the fibers F satisfying (21) we have $c_{-1}(F) \geq 1$.

Now we consider the canonical semistable reduction ϕ_F of F, we know the degree of $\phi_F = M_F$. We can see that the fiber F satisfying (21) is a tree of a (-3)-curve Γ and some (-2)-curves E. We note first that if F contains a (-2)-curve E such that F_{red} has only one singular point p on E, then p is an ordinary double point of type $(n, 2n)$, where n is the multiplicity of E in F. Since the pullback fiber $\widetilde{F}$ of F is semistable, so for any component Γ in $\widetilde{F}$, $-\Gamma^2$ is the intersection number of Γ with the other components. Thus we can see easily that the inverse image of E in the minimal resolution surface consists of n (-1)-curves, hence the exceptional curves of p can be contracted to a point. That is to say we contracted $n + [n, 2n]d - (n, 2n) = \frac{1}{2}d$ (-1)-curves. Thus the contribution of (E, p) to $c_{-1}(F)$ is $\frac{1}{2}$. On the other hand, we know easily that there are at least two such (-2)-curves in F, hence $c_{-1}(F) \geq 1$. This completes the proof. **Q.E.D.**

3.3 Proof of Theorem B

Theorem 3.4. *For any singular fiber F, we have*

$$c_1^2(F) \leq 2c_2(F), \tag{22}$$

with equality iff $F = nF_{\text{red}}$ and F has at worst ordinary double points as its singularities.

Proof. From (15), we have

$$2c_2(F) - c_1^2(F) = 3c_{-1}(F) - F_{\text{red}}^2 + \sum_p (2\mu_p - 2\beta_p - \alpha_p).$$

Then by Lemma 1.3, $\mu_p - \beta_p \geq \alpha_p$, hence we have

$$2c_2(F) - c_1^2(F) \geq -F_{\text{red}}^2 + \sum_{p\in F} \alpha_p \geq 0.$$

If $c_1^2(F) = 2c_2(F)$, then $F_{\text{red}}^2 = 0$, and $\alpha_p = 0$ for all $p \in F$. By the well-known Zariski's lemma ([BPV], p.90), we have $F = nF_{\text{red}}$. Since $\alpha_p = 0$ implies p is an ordinary double point, so F_{red} has at worst nodes as its singularities. The converse is obvious. **Q.E.D.**

Proposition 3.5. *If all of the multiple components of F are (-2)-curves, then*

$$c_1^2(F) \leq c_2(F). \tag{23}$$

Proof. The proof is similar to that of Theorem 3.4. **Q.E.D.**

4 Applications

4.1 On the slopes of fibrations

From the corollary to Theorem A, Theorem 3.4 and the Noether equality, we have

Theorem 4.1. *For any stablizing base change π, we have*

$$K_\pi^2 \leq 8\chi_\pi. \tag{24}$$

As in the case of fibrations, we have the following definition of slopes.

Definition 4.2. If F is a non-semistable fiber, $\chi_F \neq 0$, and so we can define the slope of F as

$$\lambda_F = c_1^2(F)/\chi_F.$$

From Theorem 3.4, we know $0 < \lambda_F \leq 8$.

If π is a non-invariant base change, then we define the slope of π as

$$\lambda_\pi = K_\pi^2/\chi_\pi.$$

Note that a non-trivial stablizing base change π satisfies $\chi_\pi > 0$, so Theorem 4.1 says that its slope $\lambda_\pi \leq 8$.

We have known in the Introduction that for a stablizing base change π,

$$K_f^2 - \lambda_{\tilde{f}}\chi_f = c_1^2(\mathcal{B}_\pi) - \lambda_{\tilde{f}}\chi_{\mathcal{B}_\pi}. \tag{25}$$

Corollary 4.3. *If $f : S \longrightarrow C$ is a non-semistable fibration with $\lambda_f > 8$, then through any non-trivial stablizing base change, we have*

$$\lambda_{\widetilde{f}} > \lambda_f. \tag{26}$$

In what follows, we shall consider a set of fibrations Σ which is invariant under base changes, i.e., if $f \in \Sigma$, then $\widetilde{f} \in \Sigma$.

Corollary 4.4. *Let f (resp. f') be a fibration in Σ with maximal (resp. minimal) slope.*

1) For any non-semistable fiber F of f (resp. f'), we have

$$\lambda_F \geq \lambda_f, \qquad (resp.\ \lambda_F \leq \lambda_{f'}).$$

2) If $\lambda_f > 8$, then f is semistable.

3) If $\lambda_f > 6$, then any non-semistable fiber of f has at least one multiple component which is not a (-2)-curve.

Proof. Considering the canonical stablizing base change of F and using (25), we can prove 1). 2) and 3) are immediate consequences of (22)–(25) and the assumption. **Q.E.D.**

Remark. This corollary can be used to classify singular fibers of a fibration with minimal slope in the sense above. For example,

I) Xiao ([X1], [X4]) has proved that for any relatively minimal fibration f of genus g,

$$\lambda_f \geq 4 - 4/g.$$

Furthermore, if f is a hyperelliptic fibration, then

$$\lambda_f \leq 12 - \frac{4g+2}{[g^2/2]}.$$

II) If f is non-hyperelliptic, then the lower bounds λ_g of the slope are $\lambda_3 = 3$, $\lambda_4 = 24/7$, $\lambda_5 = 40/11$. (cf. [Ch], [Ho], [Ko], [Re]).

If we consider fibrations over $\mathbb{P}^1$, and we only consider base changes with two ramification points, then the above results can also be used.

4.2 Canonical class inequality for general fibrations

First we recall Miyaoka's inequality and refer to [Hi] for the details.

Lemma 4.5. [Mi] *If S is a smooth surface of general type, and $E_1, \cdots, E_n$ are disjoint ADE curves on S, then we have*

$$\sum_{i=1}^{n} m(E_i) \leq 3c_2(S) - c_1^2(S),$$

where $m(E)$ is defined as follows,

$$m(A_r) = 3(r+1) - \frac{3}{r+1}; \quad m(D_r) = 3(r+1) - \frac{3}{4(r-2)}, \quad for\ r \geq 4;$$

$$m(E_6) = 21 - \frac{1}{8}; \quad m(E_7) = 24 - \frac{1}{16}; \quad m(E_8) = 27 - \frac{1}{40}.$$

The condition "of general type" can be replaced by some other conditions. (cf. [Mi]).

Theorem 4.6. *If f is a fibration of genus $g>1$ over a curve C of genus b, then*

$$K_{S/C}^2 \leq 3\sum_{y\in C}\delta_y^{\#} + (2g-2)\max(2b-2,0), \tag{27}$$

where $\delta_y^{\#} = e_{F_y} - \frac{1}{3}\sum_{E\subset F_y} m(E) \leq 4g-3$, and the sum is taken over all of the disjoint ADE curves E in F_y.

Proof. The inequality (27) is an immediate consequence of Miyaoka-Yau inequality (cf. [Vo], Vojta's proof). So we only need to prove that $\delta_y^{\#} \leq 4g-3$.

We denote by l_D the number of components of a curve D. Let $F_{\mathrm{red}} = D + \sum_{E\subset F} E$ be the reduced part of a fiber F. Then

$$\begin{aligned}\chi_{\mathrm{top}}(F) &= \chi_{\mathrm{top}}(D) + \sum_{E\subset F}(\chi_{\mathrm{top}}(E) - \#(D\cap E)) \\ &= \chi_{\mathrm{top}}(\tilde{D}) - \sum_{p\in D}(k_p(D)-1) + \sum_{E\subset F}(l_E+1) - \#(D\cap\sum_{E\subset F}E) \\ &\leq 2l_D + \sum_{E\subset F}(l_E+1) - \sum_{p\in D}(k_p(D)-1) - \sum_{E\subset F}\#(D\cap E).\end{aligned}$$

From the definition of $m(E)$, we know

$$m(E) \geq 3(l_E+1) - 1,$$

hence

$$\begin{aligned}\delta_F^{\#} &= 2g-2+\chi_{\mathrm{top}}(F) - \sum_{E\subset F} m(E) \\ &\leq 2g-2+2l_D - \sum_{p\in D}(k_p(D)-1) - \#(D\cap\sum_{E\subset F}E) + \sum_{E\subset F}1.\end{aligned}$$

So it is enough to prove that

$$\sum_{p\in D}(k_p(D)-1) + \sum_{E\subset F}(\#(D\cap E)-1) \geq l_D - 1. \tag{28}$$

Indeed, if D is connected, then we have

$$\sum_{p\in D}(k_p(D)-1) \geq l_D - 1,$$

hence (28) holds. If D has r connected components $D_1, \cdots, D_r$, then

$$\sum_{p\in D}(k_p(D)-1) \geq \sum_{i=1}^{r}(l_{D_i}-1) = l_D - r.$$

On the other hand, from the connectedness of F, we can prove easily that

$$\sum_{E\subset F}(\#(D\cap E)-1) \geq r-1.$$

Hence (28) also holds. **Q.E.D.**

In [Vo], Vojta proved that if f is a semistable fibration with s singular fibers, then

$$K_{S/C}^2 \leq (2g-2)(2b-2+s).$$

By using base changes, we can obtain a similar inequality for general fibrations.

Theorem 4.7. *If f has s singular fibers, then*

$$K_{S/C}^2 \leq (2g-2)(2b-2+3s) \tag{29}$$

Proof. If $b > 0$, by Kodaira-Parshin's construction, modulo an étale base change, there exists a stablizing base change totally ramified over the singular fibers. Note that (29) is unchanged under an étale base change. If π is stablizing base change, then from Theorem A, we have

$$K_f^2 = c_1^2(\mathcal{B}_\pi) + \frac{1}{d}K_{\widetilde{f}}^2.$$

Combining Theorem 3.3 and Vojta's canonical class inequality for semistable fibrations, we can obtain immediately (29).

If $C = \mathbb{P}^1$, then $s \geq 3$ ([Be1]), hence there exist base changes $\pi : \widetilde{C} \longrightarrow C$ totally ramified over the s singular fibers, whose degrees can be arbitrarily large. Note that $g(\widetilde{C}) > 0$, hence (29) holds for $\widetilde{f}$. By Lemma 2.6 and and Theorem 3.3, we have

$$\begin{aligned} K_f^2 &= \frac{1}{d}K_{\widetilde{f}}^2 + K_\pi^2 \leq \frac{1}{d}K_{\widetilde{f}}^2 + \sum_{i=1}^{s} c_1^2(F_i) \\ &\leq 4(g-1)(g(\widetilde{C})-1)/d + (6g-6)s/d + (4g-4)s \\ &= 4(g-1)(b-1) + \frac{d-1}{d}2s + (4g-4)s + (6g-6)s/d, \end{aligned}$$

then let $d \to \infty$, we obtain (29). **Q.E.D.**

4.3 On Horikawa number of a non-semistable fiber of genus 3

Let $f : S \longrightarrow C$ be a relatively minimal non-hyperelliptic fibration of genus 3, and let F be a fiber of f. The Horikawa number of F is defined as (cf. [Re])

$$H_F = \text{length coker}\left(S^2 f_*\omega_{S/C} \hookrightarrow f_*(\omega_{S/C}^{\otimes 2})\right)_{f(F)}.$$

The global invariants of f depend on this number. In fact, Reid [Re] shows that

$$K_f^2 - 3\chi_f = \sum_F H_F. \tag{30}$$

In general, it is quite difficult to compute H_F. The aim here is to try to reduce the computation of a non-semistable fiber to the computation of its semistable models, by using semistable reduction.

Theorem 4.8. *Let $\widetilde{F}$ be the semistable model of F under a stablizing base change of degree d. Then*

$$\frac{1}{d}H_{\widetilde{F}} = H_F + \frac{1}{4}(c_2(F) - 3c_1^2(F)). \tag{31}$$

Proof. We can assume that the branch locus of the base change consists of generic smooth fibers and F, hence

$$\begin{aligned}(c_1^2(F)-3\chi_F) &= K_\pi^2 - 3\chi_\pi \\ &= (K_f^2-3\chi_f) - \frac{1}{d}(K_{\widetilde{f}}-3\chi_{\widetilde{f}}) \\ &= H_F - \frac{1}{d}H_{\widetilde{F}}.\end{aligned}$$

By using Noether formula, we can obtain (31). **Q.E.D.**

Examples. If F is a genus 2 curve with an ordinary cusp, then we can take $d=6$. Then the semistable model $\widetilde{F}$ consists of a nonsingular elliptic curve E and a nonsingular curve C of genus 2, with $EC=1$. Since $c_1^2(F)=\frac{1}{6}$ and $c_2(F)=\frac{11}{6}$, hence we have

$$H_{\widetilde{F}} = 6H_F + 2.$$

If $F=2C$, C is a smooth curve of genus 2, then we can take $d=2$, hence $\widetilde{F}$ is a nonsingular hyperelliptic curve of genus 3. We have $c_1^2(F)=4$, $c_2(F)=2$. Hence

$$H_{\widetilde{F}} = 2H_F - 5.$$

So we can compute directly the Horikawa numbers of some special singular fibers, e.g., if their semistable models are non-hyperelliptic curves of genus 3.

5 The proof of Lemma 1.5

In this section, we shall use freely the notations of Sect. 1.2. Note first that Lemma 1.5 is a special case of the following theorem.

Theorem 5.1. *For the embedded resolution given in Sect. 1.2, we have*

$$-K_p^2 = d\sum_{i=0}^{r-1}\left(m_i - 2 + \frac{1}{d}\left((m_i^*,d)-m_i(d)\right)\right)^2 - \sum_{q\in B_r} K_q^2$$

where $m_i, m_i^*, m_i(d)$ are the multiplicities of $B_{i,\mathrm{red}}$, B_i, $B_i(d)$ respectively, and $B_i(d)=\sum_\Gamma (d,n_\Gamma)\Gamma$ if $B_i=\sum_\Gamma n_\Gamma\Gamma$.

Proof. Since we only need to find $K_p = K_\phi$, without loss of generality, we may assume that U_0 is a compact smooth surface, and the reduced curve of $B = B_0 = \sum_\Gamma n_\Gamma\Gamma$ has only one singular point p, (otherwise we can resolve the other singularities of B by using embedded resolution). So we have a formula similar to (11):

$$K_M = \phi^*\pi_0^*\left(K_{U_0} + \sum_{\Gamma\subset B}(1-\frac{(d,n_\Gamma)}{d})\Gamma\right) + K_\phi,$$

i.e.,

$$K_M = \eta^*\pi^*\sigma^*\left(K_{U_0} + B_{0,\mathrm{red}} - \frac{1}{d}B_0(d)\right) + K_\phi. \tag{32}$$

On the other hand, π_r is determined by $B_r = \sigma^*(B)$, hence we have also

$$K_M = \eta^*\pi_r^*\left(K_{U_r} + B_{r,\mathrm{red}} - \frac{1}{d}B_r(d)\right) + K_\eta. \tag{33}$$

From Definition 1.1 it is easy to prove that

$$\begin{aligned} K_{U_i} + B_{i,\mathrm{red}} - \frac{1}{d}B_i(d) =&\sigma_i^*\left(K_{U_{i-1}} + B_{i-1,\mathrm{red}} - \frac{1}{d}B_{i-1}(d)\right) \\ &- \left(m_{i-1} - 2 + \frac{1}{d}((m_{i-1}^*, d) - m_{i-1}(d))\right)E_i. \end{aligned} \tag{34}$$

If we denote by $\mathcal{E}_i$ the total inverse image of E_i in U_r, then from (32)-(34), we have

$$K_\phi = -\eta^*\pi_r^*\left(\sum_{i=1}^{r}\left(m_{i-1} - 2 + \frac{1}{d}((m_{i-1}^*, d) - m_{i-1}(d))\right)\mathcal{E}_i\right) + K_\eta.$$

Hence we obtain the desired equality. **Q.E.D.**

Remark. In order to resolve the singularities of V_0, we can used the *d-resolution of* (B, p), i.e., replace the condition (3) in Definition 1.1 by

(3′) If $E_i = \sigma_i^{-1}(p_{i-1})$ is the exceptional curve, then we have

$$B_i = \sigma_i^*(B_{i-1}) - d\left[\frac{m_{i-1}^*}{d}\right]E_i.$$

Then we have also the formula in Theorem 5.1.

Finally, we consider the computation of K_η^2. If (B_r, q) is defined by $x^a y^b = 0$, then $-K_q^2$ can be computed as follows.

I. If $(d, a) = (d, b) = (a, b) = 1$, then we let q, q' be two integers with $0 < q, q' < d$, $aq + b \equiv 0 \pmod{d}$, $qq' \equiv 1 \pmod{d}$. If

$$\frac{d}{q} = [e_1, \cdots, e_r] = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ddots - \cfrac{1}{e_r}}},$$

then

$$-K_q^2 = \sum_{i=1}^{r}(e_i - 2) + \frac{q + q' + 2}{d} - 2.$$

II. If $(d, a, b) = 1$, then the singularity of V_r over q is isomorphic to the normalization of $z^{d'} = x^{a'}y^{b'}$, where $a = a'(d, a)$, $b = b'(d, b)$ and $d = d'(d, a)(d, b)$, hence the computation is reduced to (I).

III. If $d_0 = (d, a, b) > 1$, then we have

$$z^d - x^a y^b = \prod_{i=1}^{d_0}\left(z^{d/d_0} - x^{a/d_0}y^{b/d_0}\exp\left(2\pi i\sqrt{-1}/d_0\right)\right).$$

Hence the singularity decomposes into d_0 singularities of type II.

References

[Ar] Arakelov, S. Ju., *Families of algebraic curves with fixed degeneracy*, Math. USSR Izv. **5** (1971), 1277–1302.

[AW] Artin, M., Winters, G., *Degenerate fibres and stable reduction of curves*, Topology **10** (1971), 373–383.

[Be1] Beauville, A., *Le nombre minimum de fibres singulières d'un courbe stable sur* $\mathbb{P}^1$, in Séminaire sur les pinceaux de courbes de genre au moins deux, ed. L. Szpiro, Astérisque **86** (1981), 97–108.

[Be2] Beauville, A., *L'inégalité* $p_g \geq 2q-4$ *pour les surfaces de type général*, Bull. Soc. Math. France **110** (1982), no. 3, 319–346.

[BPV] Barth, W., Peters, C., Van de Ven, A., *Compact Complex Surfaces*, Berlin, Heidelberg, New York: Springer, 1984.

[Ch] Chen, Z., *On the lower bound of the slope of a non-hyperelliptic fibration of genus 4*, Intern. J. Math. **4** (1993), no. 3, 367–378.

[De] Deschamps, M., *Réduction semi-stable*, in Séminaire sur les pinceaux de courbes de genre au moins deux, ed. L. Szpiro, Astérisque **86** (1981), 1–34.

[DM] Deligne, P., Mumford, D., *The irreducibility of the space of curves of given genus*, Publ. IHES **36** (1969), 75–109.

[EV] Esnault, H., Viehweg, E., *Effective bounds for semipositive sheaves and the height of points on curves over complex functional fields*, Compositio Mathematica **76** (1990), 69–85.

[Hi] Hirzebruch, F., *Singularities of algebraic surfaces and characteristic numbers*, The Lefschetz Centennial Conference, Part I (Mexico City), Contemp. Math., **58** (1986), Amer. Math. Soc. Providence, R.I., 141–155.

[Ho] Horikawa, E., *Notes on canonical surfaces*, Tohoku Math. J. **43** (1991), 141–148.

[Ko] Konno, K., *Non-hyperelliptic fibrations of small genus and certain irregular canonical surfaces*, Preprint (1992).

[Mi] Miyaoka, Y., *The maximal number of quotient singularities on surfaces with given numerical invariants*, Math. Ann. **268**, 154–171.

[Pa] Parshin, A. N., *Algebraic curves over function fields* I, Math. USSR Izv. **2** (1968), 1145–1170.

[Re] Reid, M., *Problems on pencils of small genus*, Preprint.

[Sz] Szpiro, L., *Propriété numériques de faisceau dualisant relatif*, in Séminaire sur les pinceaux de courbes de genre au moins deux, ed. L. Szpiro, Astérisque **86** (1981), 44–78.

[Ta] Tan, S.-L., *On the invariants of base changes of pencils of curves*, I, Manuscripta Math. (to appear).

[Vo] Vojta, P., *Diophantine inequalities and Arakelov theory*, in Lang, S., Introduction to Arakelov Theory (1988), Springer-Verlag, 155–178.

[X1] Xiao, G., *Fibered algebraic surfaces with low slope*, Math. Ann. **276** (1987), 449–466.

[X2] Xiao, G., *Problem list*, In: Birational geometry of algebraic varieties: open problems. The 23rd International Symposium of the Taniguchi Foundation, (1988), pp.36–41.

[X3] Xiao, G., *On the stable reduction of pencils of curves*, Math. Z. **203** (1990), 379–389.

[X4] Xiao, G., *The fibrations of algebraic surfaces*, Shanghai Scientific & Technical Publishers, 1992. (Chinese)

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