A finite morphism which does not preserve rational equivalence

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Zwischenoriginal!

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Let $f : X \to Y$ be any proper morphism of noetherian schemes and let $f_* : ZX \to ZY$ be the corresponding push-forth homomorphism of groups of cycles. We say that f *preserves rational equivalence*, if f_* sends cycles rationally equivalent to zero on X to cycles rationally equivalent to zero on Y. This means that f_* induces a homomorphism $f_* : AX \to AY$ of the corresponding groups modulo rational equivalence ('Chow groups'). It is not hard to show that this is so, if Y is *universally catenary*. When Y is not universally catenary, the situation is not so clear (see for example [**FULTON**], p 396 and [**KLEIMAN**], pp 327 - 329). With the following example we propose to show that being universally catenary is just about the best condition one can impose on Y in order that all proper morphisms from a noetherian scheme to Y should preserve rational equivalence. We shall construct a finite and birational morphism $f : X \to Y$ of two-dimensional noetherian, integral and catenary schemes such that

i) X is regular, ii) Y is local,

iii) f does not preserve rational equivalence.

We get our f by modifying Example (5.6.11) of [**GD**] pp 101 - 102, in the following manner:

Choose a field extension $K_0 \subset K_1$ which has a countable transcendence base $x, y, x_0, x_1, x_2, \ldots$. Set

$$K = K_0(x_0, x_1, \ldots), \quad \Lambda = K[x]_{xK[x]} \quad \text{and} \quad A = \Lambda[y].$$

Then Λ is a discrete valuation ring with parameter x and A is a polynomial algebra over Λ. In particular $dim A = 2$. Define K-epimorphisms $\epsilon : A \to K$

and $\epsilon' : A \to K(x)$ by

$$\epsilon(x) = \epsilon(y) = 0, \quad \epsilon'(x) = x, \quad \epsilon'(y) = 1/x. \tag{1}$$

The corresponding kernels $\mathbf{m} := Ker(\epsilon)$ and $\mathbf{m}' := Ker(\epsilon')$ are maximal ideals of A. Obviously

$$\mathbf{m} = xA + yA, \quad \mathbf{m}' = (xy-1)A, \tag{2}$$

$$\mathbf{m}' \cap \mathbf{m} = \mathbf{m}'\mathbf{m}, \quad ht(\mathbf{m}') = 1, \quad ht(\mathbf{m}) = 2.$$

The conditions

$$\epsilon''(x) = x_0^2, \quad \epsilon''(x_i) = x_{i+1} \quad , \quad (i \geq 0), \quad L = Im(\epsilon'')$$

define a K_0-homomorphism $\epsilon'' : K(x) \to K$ and a subfield L of K such that $K = L \oplus x_0 L$. Put $\phi = \epsilon'' \circ \epsilon' : A \to L$. Then

$$Im(\phi) = L, \quad Ker(\phi) = \mathbf{m}'. \tag{3}$$

Set $\mathbf{n} = \mathbf{m}'\mathbf{m}$, $B = Ker(\phi - \epsilon)$, and $\psi = \epsilon|B = \phi|B$. Clearly B is a subring of A such that

$$\mathbf{m} \cap B = \mathbf{n} = \mathbf{m}' \cap B, \quad \mathbf{n} = Ker(\psi). \tag{4}$$

Lemma 1. *i)* $Im(\psi) = L$, *ii)* $\mathbf{n}$ *is a maximal ideal of* B.

Proof: Suppose $z \in K(x)$ and $t = \epsilon''(z) \in L$. Since $K(x)$ is the quotient field of Λ and x is a parameter of Λ, we can find $n >> 0$ such that $x^n z \in \Lambda$. Set $a = t + (z-t)x^n y^n \in K_1$. Then $a \in A$, $\epsilon(a) = t$, and $\phi(a) = \epsilon''(t + (z-t)) = t$. Thus $a \in B$ and $\psi(a) = \phi(a) = t$. This proves *i*). The assertion *ii*) now follows immediately from (4). □

Lemma 2. $A = B + (xy-1)B + x_0(xy-1)B$.

Proof : Let $a \in A$. Since $\psi = \phi|B$ and $Im(\psi) = L = Im(\phi)$, we can choose $b \in B$ such that $\phi(a) = \psi(b) = \phi(b)$. Then $a - b \in Ker(\phi) = (xy-1)A$ and $a - b = (xy-1)a'$ where $a' \in A$. Because $Im(\psi) = L$ and $K = L \oplus x_0 L$, $\epsilon(a') = \psi(c) + x_0\psi(d) = \epsilon(c + x_0 d)$ for some $c, d \in B$. Set $e = a' - c - x_0 d$. Then $e \in Ker(\epsilon) = \mathbf{m}$ and $(xy-1)e \in \mathbf{m}'\mathbf{m} = \mathbf{n} \subset B$. Because $b, c, d \in B, (xy-1)e \in B$ and $a = (b + (xy-1)e) + (xy-1)c + x_0(xy-1)d$, the assertion follows. ▫

Now Eakin's theorem and lemma 2 imply immediately:

Lemma 3. *The ring B is noetherian and A is a finite extension of B.*

Set

$$M = B\backslash\mathbf{n}, \quad S = M^{-1}B, \quad J = \mathbf{n}S, \quad Y = SpecS, \tag{5}$$

$$R = M^{-1}A, \quad I = \mathbf{m}R, \quad I' = \mathbf{m}'R, \quad X = SpecR. \tag{6}$$

Because R and S are noetherian subrings of K_1, X and Y are integral and noetherian. The inclusion $S \subset R$ defines a finite and dominant morphism $f: X \to Y$. Since $\mathbf{n}$ is maximal in B, Y is local and X is semilocal. It follows from (2) - (5) and (6) that I and I' are the only closed points of X and that

$$ht(I) = 2, \quad ht(I') = 1. \tag{7}$$

Because Λ is a discrete valuation ring, it follows from (6), (7) that X is regular and that

$$dimY = dimX = 2. \tag{8}$$

Being integral and local Y is then catenary.

Lemma 4. *The morphism f is birational.*

Proof : The field $\mathcal{R}(X)$ of rational functions on X is the common quotient field $K(x, y)$ of the subrings $K[x, y]$, A and R of K_1. For any $r = a/b \in \mathcal{R}(X)$ ($a, b \in K[x, y]$) we have $x(xy-1)a \in \mathbf{n} \subset B$ and $x(xy-1)b \in B$. Consequently

$r = (x(xy-1)a/(x(xy-1)b) \in \mathcal{R}(Y)$ and $\mathcal{R}(X) = \mathcal{R}(Y)$. □

Denote

$$r = xy - 1 \in \mathcal{R}(X)^*$$

and

$$\alpha = f_*[div(r)] \in ZY.$$

To show that f does not preserve rational equivalence it is enough to show that $\alpha \not\sim 0$ in ZY. The degrees of the residue-field extensions

$$\mathbf{k}(I) \leftarrow \mathbf{k}(J) \rightarrow \mathbf{k}(I') \tag{9}$$

are calculated using the natural identifications $\mathbf{k}(I) = A/\mathbf{m}$, $\mathbf{k}(I') = A/\mathbf{m}'$ and $\mathbf{k}(J) = B/\mathbf{n}$. From (2), (3), (4) and lemma 1 we then see that the sequence (9) can be identified with $K \leftarrow L \rightarrow L$. Because $K = L \oplus x_0 L$, it follows that

$$[\mathbf{k}(I) : \mathbf{k}(J)] = 2, \quad [\mathbf{k}(I') : \mathbf{k}(J)] = 1. \tag{10}$$

Since $r = xy - 1$ is a parameter of the discrete valuation ring $\mathcal{O}_{X,I'} = A_{\mathbf{m}'}$, it follows that $ord_{I'}(r) = 1$. By (7) I' is the only point of codimension 1 in X lying over J. It then follows from (10) that the coefficient of $[J]$ in α equals $1 \cdot 1 = 1$. Let $\mathbf{q}$ be any point of dimension 1 in Y. Then $W := \{\mathbf{q}, J\}$ is the closure of $\{\mathbf{q}\}$ in Y. For each $s \in \mathcal{R}(W)^*$ the coefficient of $[J]$ in $[div(s)]$ is equal to $ord_{J/\mathbf{q}}(s)$. It is enough to show that this is always *even*. Since X is regular and J is clearly the conductor of R in S, it follows that f restricts to an isomorphism $X \backslash \{I, I'\} \rightarrow Y \backslash \{J\}$. In particular there is exactly one prime $\mathbf{p}$ in X over $\mathbf{q}$, and $R_{\mathbf{p}} = S_{\mathbf{q}}$ in $K(x,y)$. Since $\mathbf{p} \cap S = \mathbf{q} \neq J$, we see that $(0) \neq \mathbf{p} \notin \{I, I'\}$. Because $\mathbf{q} \neq J$ and $J = I' \cap S$ and $ht(I') = 1$, it follows that $\mathbf{p} \not\subset I' \not\subset \mathbf{p}$ and $xy - 1 \notin \mathbf{p}$. Since I and I' are the only maximal ideals of R, it follows that $\mathbf{p}$ is a proper subideal of I. The ring $R/\mathbf{p}$ is then local and one-dimensional. Set $U = V(\mathbf{p}) = Spec(R/\mathbf{p}) \subset X$. The local ring $R_{\mathbf{p}} = S_{\mathbf{q}}$ has the residue field $L := \mathcal{R}(W) = \mathcal{R}(U)$ so that the restriction $g : U \to W$ is birational and bijective. Let E be the common integral closure of $S/\mathbf{q}$ and $R/\mathbf{p}$ in L. Set $V = SpecE$ and look at the morphisms $V \to U \to W$ induced by the imbeddings $S/\mathbf{q} \to R/\mathbf{p} \to E$. Since the local ring $R/\mathbf{p}$ is algebraic, the morphism $V \to U$ is finite and E is semilocal. Let $\mathbf{m}_1, \ldots, \mathbf{m}_e$ be the maximal ideals of E and $\mathbf{k}(\mathbf{m}_i)$ $(i = 1, \ldots, e)$ the corresponding residue fields. Because $R/\mathbf{p}$ is local, it follows that the $\mathbf{m}_i$ lie over its maximal ideal $I/\mathbf{p}$. Of the residue field extensions

$$\mathbf{k}(J/\mathbf{q}) = \mathbf{k}(J) \to \mathbf{k}(I) = \mathbf{k}(I/\mathbf{p}) \to \mathbf{k}(\mathbf{m}_i)$$

the first is of degree two by (10). Hence the degrees $[\mathbf{k}(\mathbf{m}_i) : \mathbf{k}(J/\mathbf{q})]$ are all even. Because the morphisms $V \to U \to W$ are finite, we know that

$$ord_{J/\mathbf{q}}(s) = \sum_{i=1}^{e} ord_{\mathbf{m}_i}(s)[\mathbf{k}(\mathbf{m}_i) : \mathbf{k}(J/\mathbf{q})]$$

(see [**FULTON**], A.3.1 p 412, for example). The assertion follows immediately.

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