# Expansivity for optical Hamiltonian systems with two degrees of freedom

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# EXPANSIVITY FOR OPTICAL HAMILTONIAN SYSTEMS WITH TWO DEGREES OF FREEDOM

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ABSTRACT. Let $H$ be an optical Hamiltonian on $T^*M$ where $M$ is a compact oriented surface. Let $h$ be the maximum of $H$ over the zero section of $T^*M$. We show that if the Hamiltonian flow of $H$ is expansive on the compact regular energy level $H^{-1}(\sigma)$, then there are no conjugate points provided $\sigma > h$. We also show that if $H$ is symmetric and $\sigma < h$ then the Hamiltonian flow of $H$ on $H^{-1}(\sigma)$ cannot be expansive.

## 1. *Introduction*

A flow $\phi_t : X \to X$ on a compact metric space $(X, d)$ is said to be *expansive* if given $\epsilon > 0$ there exists $\delta > 0$ such that if there is an homeomorphism $\tau : \mathbf{R} \to \mathbf{R}$, $\tau(0) = 0$, such that

$$d(\phi_{\tau(t)}(y), \phi_t(x)) < \delta$$

then $y = \phi_{\bar{t}}(x)$ where $|\ \bar{t}\ | < \epsilon$. Anosov flows and suspension of Pseudo-Anosov maps are expansive.

In [10] the second author showed that if $M$ is a closed riemannian surface whose geodesic flow is expansive, then $M$ has no conjugate points and hence all expansive geodesic flows are topologically equivalent. In this note we announce a generalization of these results to the wider open class of optical Hamiltonians and we give outlines of the proofs.

Let $M$ be a closed oriented surface and let $H : T^*M \to \mathbf{R}$ be a smooth Hamiltonian with associated Hamiltonian flow $\phi_t$. We will say that $H$ is *optical or convex* (cf.[1]) if for each $q \in M$ the function $H(q, .)$ regarded as a function on the linear space $T_q^*M$ has positive definite Hessian; equivalently the second fibre derivative $D_F^2 H$ is positive definite. Let $\sigma$ be a regular value of $H$ and let $\Sigma_\sigma = H^{-1}(\sigma)$. It is well known that $\phi_t$ leaves $\Sigma_\sigma$ invariant. Let $h$ be the maximum of $H$ over the zero section of $T^*M$.

Let $\pi : T^*M \to M$ be the canonical projection. For $s \in (-\epsilon, \epsilon)$ consider a curve $s \to p_s \in T_q^*M$ with $p_0 \in \Sigma_\sigma$. Let $Y(t)$ be the variational field

$$Y(t) = \frac{d}{ds}\ |_{s=0}\ \pi \circ \phi_t(p_s).$$

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We will say that $\Sigma_\sigma$ has *no conjugate points* if for all fields $Y$ as above we have that $Y(t) \neq 0$ for all $t \neq 0$.

**Theorem 1.1.** *Let $M$ be a closed oriented surface and let $H$ be an optical Hamiltonian on $T^*M$. Suppose $\phi_t$ is expansive on the compact energy level $\Sigma_\sigma$. Then $\Sigma_\sigma$ has no conjugate points provided $\sigma > h$.*

The Hamiltonian $H$ is said to be *symmetric* if it is invariant under the involution $(q,p) \to (q,-p)$. The following theorem describes the situation when $\sigma < h$ under the additional assumption of symmetry (note that $\sigma = h$ is not a regular value of $H$).

**Theorem 1.2.** *Let $M$ be a closed oriented surface and let $H$ be an optical symmetric Hamiltonian on $T^*M$. Then if $\sigma < h$ the Hamiltonian flow $\phi_t$ on $\Sigma_\sigma$ cannot be expansive.*

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## 2. *Sketches of proofs*

We start by describing a few important facts about expansive flows. Let $X$ be a closed oriented 3-manifold endowed with a riemannian metric. Let $\phi_t : X \to X$ be an expansive smooth flow with associated vector field $W$. Suppose $W(x) \neq 0$ for all $x \in X$. Define

$$H_\epsilon(x) = \{exp_x v : \| v \| < \epsilon \ \ and \ < W(x), v >= 0\}.$$

It is easy to see that $\phi_t$ is expansive if there exists $0 < \alpha < \epsilon$ so that if there is a smooth increasing surjective function $\tau_{x,y} : \mathbf{R} \to \mathbf{R}$, $\tau_{x,y}(0) = 0$ for which

$$\phi_t(y) \in H_\alpha(\phi_{\tau_{x,y}(t)}(x))$$

for all $t \in \mathbf{R}$ then $x = y$.

For $\delta < \alpha$ define the stable set $S_\delta(x)$ as

$$S_\delta(x) = \{y \in H_\delta(x) : \ \phi_t(y) \in H_\delta(\phi_{\tau_{x,y}(t)}(x)) \ for\ t \geq 0$$
$$for\ some\ increasing\ surjective\ function\ \tau_{x,y} : \mathbf{R}^+ \to \mathbf{R}^+\}.$$

Analogously for $t \leq 0$ we define the unstable set $U_\delta(x)$.

**Definition 2.1.** We say that $x \in X$ has a *local product structure* if there exists a homeomorphism of $\mathbf{R}^2$ onto an open neighborhood of $x$ in $H_\alpha(x)$ that maps horizontal (vertical) lines onto open subsets of local stable (unstable) sets.

The main consequence of expansivity is the following proposition which is proved in [11] (see [8] for the discrete version).

**Proposition 2.2.** *Except for a finite number of periodic orbits, whose points we call singular, every point of $X$ has a local product structure. If $x$ is a singular point, $S_\delta(x)$ is the union of $r$ arcs, $r \geq 3$, that only meet at $x$.*

In what follows, lemmas 2.3, 2.6 and 2.7 are due to Marco Brunella and are borrowed from the paper [2] in which he classifies all expansive flows on Seifert manifolds and torus bundles.

**Lemma 2.3.** *There is no periodic orbit of $\phi_t$ belonging to a homotopy class which is a central element of $\pi_1(X)$.*

*Proof*: Is an addaptation of Lemma 2.4 in [3] to the expansive case; this addaptation is possible on account of results in [7, 11]. □

**Remark 2.4.** *A periodic orbit is not null-homotopic.*

Suppose from now on that $X$ is a circle bundle over an oriented surface $M$.

**Corollary 2.5.** *A periodic orbit of $\phi_t$ is not homotopic to a circle of the fibration.*

*Proof*: The fundamental group of a fibre is central in $\pi_1(X)$ (cf.[6]). □

Now let $\omega \subset M$ be a simple closed curve not null-homotopic and disjoint from the projection of the singular orbits. Denote by $T^2_\omega$ the incompressible torus $\pi^{-1}(\omega) \subset X$, where $\pi : X \to M$ is the fibration map. The following lemma is straightforward:

**Lemma 2.6.** *If $\gamma \subset X$ is a simple closed curve such that for all $\omega \subset M$ as above $\gamma$ is homotopic to a curve which does not intersect $T^2_\omega$, then $\pi \circ \gamma$ is null-homotopic.*

Now let $S$ denote the set of singular orbits of $\phi_t$.

**Lemma 2.7.** *The set $S$ is empty.*

*Proof*: Assume that $S$ is not empty. Let $\gamma \subset S$ be a singular orbit of $\phi_t$ and let $\omega \subset M$ be any closed curve as before. Let us disjoin $\gamma$ and $T^2_\omega$ and thus obtain a contradiction to Corollary 2.5 and Lemma 2.6. First we put (as in [11, 7] by techniques of [13, 5]) $T^2_\omega$ in general position with respect to the stable foliation. By this procedure, the stable foliation induces a 1-dimensional foliation on $T^2_\omega$ whose singularities are centers and saddles due to the tangencies, and a finite number of prongs due to the transverse intersection of $T^2_\omega$ and $S$. This can be done in such a way that the only connection between these elements are saddle self-connections ([7, 11]). A combinatorial argument as in [12] shows that the number of centers is less than or equal than the number of self-connected saddles and this, in turn implies -via the Poincaré-Hopf formula- that there are no prongs and consequently that $T^2_\omega$ does not intersect $S$: a contradiction. □

We are now ready to sketch the proof of the theorems.

*Proof of Theorem 1.1:* Since $\sigma > h$ then $\Sigma_\sigma$ admits a fibration by circles where the circles are just $T^*_qM \cap \Sigma_\sigma$. By Proposition 2.2 and Lemma 2.7 the stable sets give rise to a continuous foliation on $\Sigma_\sigma$ without singularities (in [10] the symmetry of the riemannian metric was heavily used to show the absence of singular orbits; as Lemma 2.7 shows, the symmetry is in fact not needed). Hence we can attach to each continuous closed curve $\alpha : S^1 \to \Sigma_\sigma$ an index, *ind* $\alpha$, just as it was done in [10] for the geodesic flow case (this definition of the index is inspired by Mañé's paper [9] and is a continuous version of the Maslov-Arnold index). This index is roughly the winding number of the stable foliation around $\alpha$ and its basic properties are (the last two properties depend strongly on the fact that $H$ is optical):

- It is invariant under homotopies
- If $\alpha$ is a closed orbit of $\phi_t$ then $ind\ \alpha \geq 0$.
- If $\alpha$ is a closed orbit of $\phi_t$ then $ind\ \alpha > 0$ if and only $\pi \circ \alpha$ has conjugate points within one period.

Since $H$ is convex on each fibre and $\sigma > h$ we can assume without loss of generality that $H$ is the Hamiltonian corresponding to a Finsler metric. Suppose $\Sigma_\sigma$ has conjugate points. Since the closed orbits of $\phi_t$ are dense we can find a closed orbit $\alpha$ such that $\pi \circ \alpha$ possesses conjugate points and thus positive index. Let us consider the homotopy class $[\pi \circ \alpha]$ and observe that it is not trivial (cf. Remark 2.4 and Corollary 2.5). Hence on account of Morse theory we can find an orbit $\beta$ of $\phi_t$ so that $\pi \circ \beta$ is homotopic to $\pi \circ \alpha$ and it minimizes the length in its homotopy class. By the Morse index theorem $\pi \circ \beta$ has no conjugate points within one period and thus $ind\ \beta = 0$. But this is a contradiction because $ind\ \beta = ind\ \alpha > 0$ since $\alpha$ and $\beta$ are homotopic. □

*Proof of Theorem 1.2:* Since $H$ is convex and symmetric if $\sigma < h$ there exists [4, Theorem A'] a break orbit in $\Sigma_\sigma$; that is, an orbit of $\phi_t$ whose projection connects boundary points of $\pi\Sigma_\sigma$ but otherwise runs through the interior. But such an orbit is contractible within $\Sigma_\sigma$ which contradicts Remark 2.4. □

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