ON FOLIATIONS OF SEMI-SIMPLICIAL MANIFOLDS AND THEIR HOLONOMY

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Introduction

Semi-simplicial foliations have appeared in [2]-[3], where they form a linking between a group action on a manifold and characteristic classes of foliations and produce that way characteristic invariants of the group actions. In fact, the present paper arose while the author was trying to prepare a unified treatment of characteristic classes of foliations - on a manifold and, at the same time - on classifying spaces, which would cover the characteristic invariants studied in [1] (in future, we hope to continue that direction of an investigation). One of the questions was then: what is a geometrical setting in which the semi-simplicial manifold $N\Gamma_q$ (nerve of Γ_q) could play the role of a classifying space? The solution found by the author is presented in Chapter IV of the paper (cf IV; proposition 1.10 and corollary 2.3.1) and given by a special category of semi-simplicial manifolds. Another question was: how to extend the notion of holonomy to foliations of arbitrary ss-manifolds? A solution to that problem, given in Chapter III, is also an attempt to "close the category" , since it is reasonable (cf eg [4]) to look for the "manifold" of leaves of any foliation F among the nerves $N\Gamma_{F,T}$ of the holonomy groupoids. The results of Chapter IV allow one to identify with each other all the ss-manifolds $N\Gamma_{F,T}$, T being any complete transversal for F.

The paper consists of four chapters. Chapter I is of an introductory character and contains a brief account of foliations and Γ-structures (for a more complete treatment, see [11],[12],[13], and [14] as well as [7] and [8]).

Chapter II deals with semisimplicial objects and extends some ideas of [5].

In Chepter III we develop the notion of a holonomy pseudogroup $G_{F,T}$ (and groupoid $\Gamma_{F,T}$) of an ss-foliation with respect to a complete transversal T. Our main result here is theorem 2.2, in which the existence and minimality of a canonical semi-simplicial $\Gamma_{F,T}$-structure are established. In examples 3.1 - 3.3, the theorem is then applied to classical situations (a standard foliation, a pointwise foliation of $N\Gamma$, and a flag) and the corresponding holonomy objects are computed.

Chapter IV contains a presentation of a special category of ss-morphisms and their relations to ss-foliations. In particular, theorem 3.2 of the chapter assures a consistency of the notions of ss-foliations and ss-morphisms.

The author aknowledges the hospitality of the Max-Planck Institute für Mathematik, where the paper has been written, as well as the SFB 40 Universität Bonn for a financial support.

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<u>Symbols and notations</u>

f^*F	- I; 1.2
Γ_q	- 1.3
E_F	- 1.3&1.9
α, β	- 1.4
f^*E	- 1.6&1.8
$[\gamma,x]$	- 1.8
$f^*\sigma$	- 1.12
$\tilde{f}$	- 1.12
$h_{c,T_1}^{T_0}$	- 2.1
h_c	- 2.2
Γ_F	- 2.3
$\Gamma_{F,T}$	- 2.5
$h_{c,T}$	- 2.5
$G_{F,T}$	- 2.5
$\Phi: G \to G'$	- 2.6
$G \approx G'$	- 2.7
$\omega_{F,T}$	- 2.8
$H_{c,T}$	- 2.8
$E_{F,T}$	- 2.8
ε_i, η_i	-II; 1.1
$N\mathcal{U}$, $N_n\mathcal{U}$	- 1.3
NM	- 1.3
$N\Gamma$, $N_n\Gamma$	- 1.3
$X_{\mathcal{U}}$	- 1.4
$(a_0,\dots,a_n;x)$	- 1.4
$X_n^{(K)}$	- 2.5
$X^{(K)}$	- 2.5.1
E_F	- 3.1
$\Sigma: \Gamma \longrightarrow \Gamma'$	- 3.4
ω_Σ	- 3.7
$\Sigma' \circ \Sigma$	- 3.8
$\Gamma \approx \Gamma'$	- 3.8
$\Sigma_*\omega$	- 3.10
Σ_*E	- 3.10
ω_Γ	- 3.11.1
$\bar{N}\Gamma$, $\bar{N}_n\Gamma$	- 3.11.1
$\lvert\Phi\rvert$	- 3.12
$\Phi_*\omega$	- 3.12.1
T^0, T^1	-III; 1.1
$G_{F,T^0\cup T^1}$	- 1.1
id_1^0	- 1.1
$G_{F,T}$	- 1.1
Φ_T	- 1.1.1
id_0^1	- 1.1.1
$\omega_{F,T}$	- 2.2&2.3
Ψ_T^ω	- 2.2&2.6
$\omega'_{F,T}$	- 2.3
$E'_{F,T}$	- 2.3
Φ^T	- 2.3
$\{\gamma_{ab}\}$	- 2.5
E^γ	- 2.5.2
$[a;x,g]$	- 2.5.2
$\Sigma_{T',T}$	- 3.3
F'_T	- 3.3
$F_{\mathcal{U}}$	- 4.1
$\omega_{\mathcal{U}}$	- 4.1
$\rho_\#$	-IV; 1.2
$f \sim g$	- 1.3
$f \propto g$	- 1.3
$f: X \longrightarrow Y$	- 1.3
$f^{-1}V$	- 1.6
f_V	- 1.6
$g \circ f$	- 1.6.1
1_X	- 1.6.1
$\lambda: X_{\mathcal{U}} \to X$	- 1.6.1
$[f]$	- 1.7
$f^*\omega$	- 1.9
$f^*\omega$	- 1.9.2
f^*E	- 1.9.3
$X \approx Y$	- 1.12
f^*F	- 2.2
f_{TS}	- 3.1

I. Foliations, pseudogroups, groupoids

In the present paper all manifolds, maps etc are of class C^∞ (unless otherwise specified), although we do not require the manifolds to be neither paracompact nor even Hausdorff. We propose the following:

Definition 1.1. A q-codimensional (smooth) foliation F of an n-dimensional manifold M $(n \geq q \geq 0)$ is a topology in M such that every point of M admits a chart $\varphi: U \to \mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q$ trivializing F, ie inducing a homeomorphism of $(U, F|U)$ into $\mathbb{R}^{n-q} \times (\mathbb{R}^q)_\delta$, where δ means the discrete topology.

On leaves of F, ie the connected components of (M,F), M induces a structure of (n-q)-dimensional submanifolds.

1.2 Any map $f: M' \to M$ transverse to F (ie to the leaves) induces a q-codimensional foliation f^*F of M' which is the topology generated by the pull-back of F and by the manifold topology of M'. For any leaf L' of f^*F there is a unique leaf L of F such that $f(L') \subset L$; the restriction $f|L': L' \to L$ is a smooth map.

1.3 The above modification of the definition has little influence on standard results and constructions associated with foliations, as there is still a one-to-one correspondence between q-codimensional foliations of M and some Γ_q-structures on M, Γ_q being the groupoid of germs of local diffeomorphisms of $\mathbb{R}^q$. If one interprets a Γ_q-structure as a collection of Γ_q-cocycles which are transition maps for a common principal Γ_q-bundle, then the Γ_q-bundle E_F that corresponds to a foliation F consists of all germs of submersions $M \supset U \to \mathbb{R}^q$ locally constant on the leaves ([7]). E_F is an example of highly non-Hausdorff smooth manifolds which come into consideration while studying foliations

We recall the definitions which will be exploited throughout the paper.

Definition 1.4. A (differentiable) groupoid Γ is a small category with only invertible elements (morphisms), such that both Γ and its set of objects U are equipped with differentiable structures, and

(i) the source $\alpha: \Gamma \to U$ and the target $\beta: \Gamma \to U$ are submersions;

(ii) the composition $\Gamma \times_{(\alpha,\beta)} \Gamma \ni (g_1, g_2) \to g_1 g_2 \in \Gamma$ and the inverse map $\Gamma \ni g \to g^{-1} \in \Gamma$ are both smooth (by (i) the Whitney product

$$\Gamma \times_{(\alpha,\beta)} \Gamma = \{(g_1,g_2) \in \Gamma \times \Gamma;\ \alpha(g_1) = \beta(g_2)\}$$

is a manifold).

If one identifies objects with units, then U becomes a submanifold of Γ.

Definition 1.5. A principal Γ-bundle E over a manifold M is a manifold endowed with two maps, the projection $\pi: E \to M$ and the source $\alpha: E \to U$ (units of Γ) and with a right Γ-action $E \times_{(\alpha,\beta)} \Gamma \to E$ in the fibers of π (ie, $z.g = z$ if $g \in U$, and $(z.g_1)g_2 = z(g_1g_2)$). One requires a local triviality condition: on a neighbourhood of each point $x \in M$ there is a section $\sigma: M \supset V \to E$ of π such that the map

$$V \times_{(\alpha\sigma,\beta)} \Gamma \ni (y,g) \to \sigma(y)g \in \pi^{-1}(V) \subset E$$

is a diffeomorphism.

1.6 Many properties of principal G-bundles (G - a Lie group) carry over to the above, more general setting. In particular, any map $f: M' \to M$ pulls back Γ-bundles over M to Γ-bundles over M',

$$\begin{array}{ccc} E & & f^*E := M' \times_{(f,\pi)} E \\ \downarrow & \rightsquigarrow & \downarrow \\ M & & M' \end{array}$$

and the classical result that a homomorphism of a G-bundle E' into a G-bundle E, ie a commuting square

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ \downarrow & & \downarrow \\ M' & \xrightarrow{f} & M \end{array}$$

(where $\bar{f}$ preserves the G-actions) yields an isomorphism $E' \cong f^*E$, remains valid if G is replaced with Γ. An important exception here is that a Γ-bundle over $M \times \mathbb{R}$ cannot be, in general, induced from a single bundle over M.

1.7 Given a family of local sections of E with domains covering M, the corresponding transition functions with values in Γ form a so-called Γ-cocycle over the covering of M. All the Γ-cocycles obtained in that way from a fixed principal Γ-bundle E are in some natural sense equivalent ([7]) and constitute a Γ-structure on M. The last notion means essentially the same as "an isomorphy class of principal Γ-bundles over M" and the distinction comes from the tradition only.

1.8 Apart from Lie groups, a large and in some sense opposite class of groupoids constitute groupoids of germs, namely of all germs of elements of a pseudogroup of diffeomorphisms. We accept the notation $[\gamma,x]$ for the germ of a diffeomorphism γ at a point x, so that the composition $[\gamma,x][\gamma',y]$ is defined and equal to $[\gamma\gamma',y]$ iff $x = \gamma'(y)$.

While dealing with a groupoid of germs Γ it is convenient (for the notational reason) to consider principal Γ-bundles equipped with a left action of the groupoid. Such a bundle $\pi: E \to M$ is then endowed with a target map $\beta: E \to N$ (N - units of Γ) and the action $(g,z) \to gz$ is defined on the Whitney product $\Gamma \times_{(\alpha,\beta)} E$. If $f: M' \to M$ is any map, then for the same reason we define the pull-back left Γ-bundle to be $f^*E = E \times_{(\pi,f)} M'$.

1.9 We have mentioned (cf 1.3.) that a q-codimensional foliation F of M gives rise to a principal Γ_q-bundle E_F over M. E_F is the set of all germs $[\varphi,x]$ of submersions locally defining F at $x \in M$, equipped with the sheaf topology and the differentiable structure induced from M, so that the projection $E_F \ni [\varphi,x] \overset{\pi}{\to} x \in M$ be locally a diffeomorphism. Γ_q acts on E_F (from the left!) by the formula $[\gamma,y][\varphi,x] = [\gamma\varphi,x]$ iff $y = \varphi(x)$.

1.10 Let Γ be any groupoid of germs and $\pi: E \to M$ a principal Γ-bundle such that the target map β of E into N, the manifold of units of Γ, is a submersion. Then there exists a unique foliation F of M such that at the level of E the foliation π^*F coincides with the one induced by β from the pointwise foliation of N. If this is the case, we shall call F the foliation defined by E or, by the Γ-structure corresponding to E. Clearly, codim F = dim N. This definition agrees with the classical one which assumes the Γ-cocycles representing the Γ-structure to be generated by submersions of open subsets of M in N ([7]).

1.11 To any Γ-bundle $\pi: E \to M$ defining F one associates its canonical form $\tilde{E} \to M$ composed of all germs $[\varphi,x]$ of the submersions $\varphi: M \supset U \overset{\sigma}{\to} E \overset{\beta}{\to} N$ (the distinguished submersions for E), where σ ranges over local sections of π (note that π is locally invertible for Γ a groupoid of germs) and $x \in U$ (cf [7]). Γ acts on $\tilde{E}$ from the left by the composition of maps. It can be easily seen that the map

$$(1.11.1) \qquad E \ni g\sigma(x) \to g[\beta\sigma,x] \in \tilde{E}$$

is a well-defined canonical isomorphism. Clearly, the canonical form is the same for all Γ-bundles isomorphic to E and is canonically distinguished by the corresponding Γ-structure.

1.12 If $f: M' \to M$ is a map transverse to F and a Γ-bundle $\pi: E \to M$ defines F on M, then the pull-back bundle $f^*E = E \times_{(\pi,f)} M'$ defines on M' the induced foliation f^*F. Clearly, f pulls any section $\sigma: U \to E$ back to a section $f^*\sigma = (\sigma \circ f, id): f^{-1}U \to f^*E$; thus any distinguished submersion φ for E gives rise to a distinguished submersion φf for f^*E. In terms of the canonical forms of the bundles the projection $f^*E \to E$ takes the form

$$[\varphi f, x'] \to [\varphi, f(x')].$$

In particular, since E_{f^*F} is necessarily the canonical form of the Γ_q--bundle f^*E_F, $q = \operatorname{codim} F$, f lifts functorially to a Γ_q-equivariant map $\tilde{f}: E_{f^*F} \ni [\varphi f, x'] \to [\varphi, f(x')] \in E_F$.

For any foliation F there is a family of pseudogroups of diffeomorphisms (thus also: groupoids of germs) which are related to F more closely than any other pseudogroup. These are holonomy pseudogroups (resp.: holonomy groupoids) and describe the transverse structure of the foliation.

2.1 Given a continuous path c in a leaf of F, any local transversal $T = T_0$ at $c(0)$ can be continuously transported along c in such a way that each point of T remains all the time in the same leaf. As a result one gets a locally defined diffeomorphism of the transversal T_0 at $c(0)$ to a transversal T_1 at $c(1)$ which is called the holonomy translation along c (cf [12]; notation $h_{c,T_1}^{T_0}$).

Any map $f: M' \to M$ transverse to the foliation F carries transversals for the induced foliation f^*F to transversals for F. If c is a path in a leaf of f^*F, then for any two local transversals T and T' at the ends of c,

$$h_{c,T'}^{T} = h_{fc,T'}^{T} .$$

2.2 An invariant description of the holonomy refers to the Γ_q-bundle E_F (cf 1.9): when restricted to a single leaf L of F, E_F becomes a covering of L, hence any path c in L lifts uniquely to a path in E_F starting from an arbitrary but fixed point of the fiber through $c(0)$. As a result, c yields a Γ_q-equivariant bijection

$$h_c: \pi^{-1}(c(0)) \to \pi^{-1}(c(1))$$

which is just an alternative description of the holonomy ([7]; note that the target projection $\beta: E_F \to \mathbb{R}^q$ is locally constant on $\pi^{-1}(L) \subset E_F$

and thus $\beta\circ h_c = \beta$ on $\pi^{-1}(c(0))$). For a connection between the two notions of holonomy translation, $h_{c,T'}^{T}$ and h_c, see (2.4.1) below.

2.3 The fact that h_c depends on the homotopy class of c only suggests that the set Γ_F of all h_c's can be organized in a manifold. This has been done eg in [15]. In fact, Γ_F with the obviously defined composition rule ($h_c . h_{c'} = h_{c'*c}$) is even a differentiable groupoid with M as the manifold of units; it is the graph of F. Topologically, h_c is close to h_{c_0} in Γ_F if there is a path $\bar{c}$ close to c_0 such that $h_{\bar{c}} = h_c$; evidently, $\alpha(h_c) = c(o)$ and $\beta(h_c) = c(1)$.

2.4 The action of Γ_F can be canonically extended to the general case of any principal Γ-bundle $\pi: E \to M$ (Γ - a groupoid of germs) defining the foliation F. A synthetic definition of holonomy translations in E is similar to the one given in 2.2: for any leaf L of F the deck transformations in the covering space $E\supset\pi^{-1}(L) \to L$ depend on the holonomy of paths in L only. Consequently, any path c in L yields a Γ-equivariant bijection $\pi^{-1}(c(0)) \to \pi^{-1}(c(1))$ understood as a (left) action of the holonomy $h_c \in \Gamma_F$.

Lemma. For an arbitrary Γ-structure ω defining the foliation F and any path c in a leaf of F, the holonomy h_c acts on fibers of the canonical Γ-bundle $E \to M$ distinguished by ω (cf 1.11) according to the rule

(2.4.1) $$(h_c[\varphi,c(0)])|T' = [(\varphi|T)\circ(h_{c,T'}^{T})^{-1},c(1)] \text{ for } [\varphi,c(0)]\in\pi^{-1}(c(0))$$

where T and T' are any local transversals at $c(0)$ and $c(1)$ resp., and φ is an arbitrary distinguished submersion for ω over a neighbourhood of $c(0)$ (see figure 1).

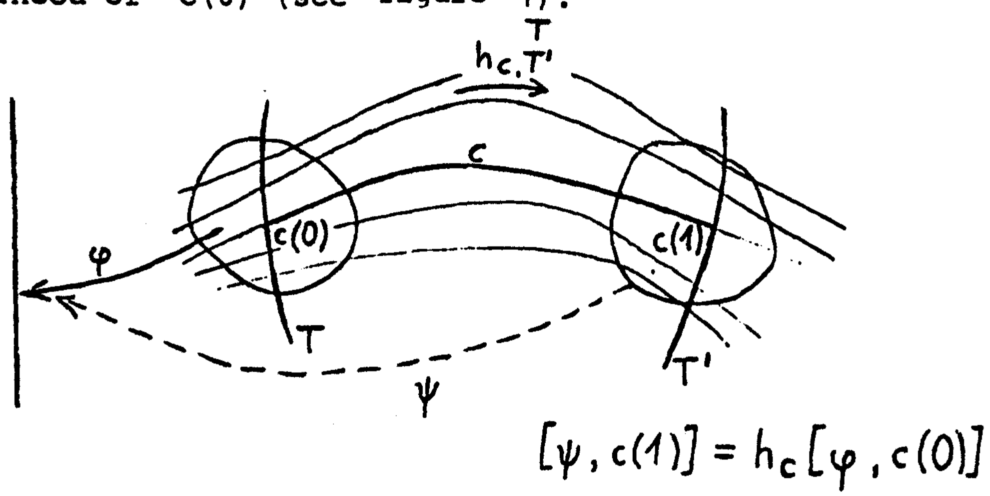


Figure 1.

Proof. When starting from a collection of sections $\sigma_a: M\supset U_a \to E$, $a \in A$, such that the U_a's cover M we get a Γ-cocycle $\{\gamma_{ab}\}$ representing ω. In particular, the maps $\gamma_{aa} = \beta \circ \sigma_a : U_a \to N$ (N - the units of Γ) are distinguished submersions such that for any $x \in U_a \cap U_b$, $a,b \in A$, one has

$$\gamma_{aa} = \gamma_{ab}^{(x)} \gamma_{bb} \quad \text{over a neighbourhood of } x,$$

if $\gamma_{ab}^{(x)}$ is a diffeomorphism representing the germ $\gamma_{ab}(x) \in \Gamma$. We fix a sequence of indices $a_0,\ldots,a_r$ such that $c([\tau_i,\tau_{i+1}]) \in U_{a_i}$, $i = 0,\ldots,r$, for some partition $0 = \tau_0 \le \tau_1 \le \ldots \le \tau_{r+1} = 1$ of the unit interval. At every point $c(\tau_i)$, $i = 0,1,\ldots,r+1$, we choose a transversal S_i ($S_0 = T$ and $S_{r+1} = T'$). Let for shortness γ_{ij} denote $\gamma_{a_i a_j}$. Since γ_{ii} is a submersion locally constant on leaves of F, the composition $(\gamma_{ii}|S_{i+1})^{-1}(\gamma_{ii}|S_i)$, well-defined over a neighbourhood of $c(\tau_i)$ in S_i, is the holonomy translation along $c|[\tau_i,\tau_{i+1}]$,

$$(\gamma_{ii}|S_{i+1})^{-1}(\gamma_{ii}|S_i) = h_{c|[\tau_i,\tau_{i+1}],S_{i+1}}^{S_i} \quad \text{for } i = 0,\ldots,r.$$

Consequently, at the level of germs one has

$$\begin{aligned}
&\left[h_{c,T'}^{T}, c(0)\right] = \\
&= \left[h_{c|[\tau_r,1],T'}^{S_r}, c(\tau_r)\right] \cdots \left[h_{c|[\tau_1,\tau_2],S_2}^{S_1}, c(\tau_1)\right]\left[h_{c|[0,\tau_1],S_1}^{T}, c(0)\right] \\
&= [\gamma_{rr}|T', c(1)]^{-1} \cdots [\gamma_{11}|S_1, c(\tau_1)][\gamma_{00}|S_1, c(\tau_1)]^{-1}[\gamma_{00}|T, c(0)] \\
&= [\gamma_{rr}|T', c(1)]^{-1}\, \gamma_{r,r-1}(c(\tau_r)) \ \cdots\ \gamma_{10}(c(\tau_1))[\gamma_{00}|T, c(0)] \\
&= \Big(\gamma_{01}(c(\tau_1)) \ \cdots\ \gamma_{r-1,r}(c(\tau_r))[\gamma_{rr}|T', c(1)]\Big)^{-1}[\gamma_{00}|T, c(0)].
\end{aligned}$$

On the other hand, we are able to write down explicitely a lift $\tilde{c}$ of c to E starting at $\tilde{c}(0) = [\gamma_{00}, c(0)]$. Namely, for $\tau \in [\tau_i,\tau_{i+1}]$ we define $\tilde{c}(\tau)$ to be the germ

$$\gamma_{01}(c(\tau_1)) \ \cdots\ \gamma_{i-1,i}(c(\tau_i))[\gamma_{ii}, c(\tau)] \in E$$

(observe that $\gamma_{ii}(c(\tau)) = \gamma_{ii}(c(\tau_i)) = \gamma_{ii}(c(\tau_{i+1}))$). Clearly $\tilde{c}$ is well-defined and thus continuous. By definition of the action of h_c one has

$$\begin{aligned}
h_c[\gamma_{00}, c(0)] &= c(1) \\
&= \gamma_{01}(c(\tau_1)) \ \cdots\ \gamma_{r-1,r}(c(\tau_r))[\gamma_{rr}, c(1)]
\end{aligned}$$

and a comparison of the above two formulas gives

$$h_c[\gamma_{00}, c(0)]|T' = [(\gamma_{00}|T)\circ(h_{c,T'}^{T})^{-1}, c(1)].$$

The general formula follows now from the Γ-equivariance of h_c.

2.5 By fixing a complete transversal $i: T \hookrightarrow M$ for F (complete means that $i(T)$ cuts every leaf) one reduces Γ_F to a (transverse) holonomy groupoid $\Gamma_{F,T}$ which is a groupoid of germs on T. The underlying holonomy pseudogroup $G_{F,T}$ of diffeomorphisms of T is generated by all the holonomy translations $h_{c,T}^T$ of open subsets of T; we accept the notation $h_{c,T} := [h_{c,T}^T, c(0)]$ for the element of $\Gamma_{F,T}$ determined by an arbitrary path c. The holonomy pseudogroups for F with respect to different complete transversals are all canonically equivalent ([9]).

2.6 We recall here that a morphism $\Phi: G \rightharpoonup G'$ of two pseudogroups of diffeomorphisms (on manifolds, resp., N and N') is a maximal collection of diffeomorphisms $\varphi: N \supset U_\varphi \to N'$ of open subsets of N on open subsets of N' subject to the conditions

(i) the collection of all U_φ, $\varphi \in \Phi$, covers N;
(ii) $\varphi\gamma^{-1}\psi \in G'$ if $\varphi, \psi \in \Phi$, $\gamma \in G$;
(iii) $\gamma'\varphi\gamma \in \Phi$ if $\varphi \in \Phi$, $\gamma \in G$ and $\gamma' \in G'$;

(we emphasize the fact that Φ is not a map by using the half-arrows).

One defines the composition $G \xrightharpoonup{\Phi} G' \xrightharpoonup{\Phi'} G''$ to be the unique morphism containing the diffeomorphisms $\varphi'\varphi$ ($\varphi \in \Phi$, $\varphi' \in \Phi'$). Clearly, any collection of diffeomorphisms (of open subsets of N on open subsets of N') which satisfies the conditions (i)-(ii) is contained in (hence: generates) a unique morphism $G \rightharpoonup G'$.

2.7 Given any two complete transversals $T, T' \hookrightarrow M$ for a foliation F, the collection of all the holonomy translations from T to T' is a canonical invertible morphism $G_{F,T} \rightharpoonup G_{F,T'}$. In general, invertible morphisms will be called equivalences (notation " $\approx$ ").

2.8 For any complete transversal $T \hookrightarrow M$, there is a canonical $\Gamma_{F,T}$-structure $\omega_{F,T}$ defining F on M, namely the one generated by those submersions of subsets of M on T which are holonomy projections $H_{c,T}$ along paths c such that $c(1) \in T \hookrightarrow M$ (see figure 2).

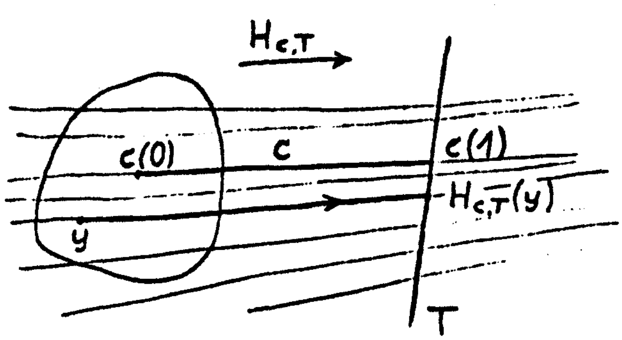


Figure 2.

The canonical $\Gamma_{F,T}$-bundle $E_{F,T}$ over M distinguished by $\omega_{F,T}$

consists of all germs $[H_{c,T},x]$ of the submersions. Note that $\Gamma_{F,T}$ acts (from the left) on $E_{F,T}$ by the formula

$$h_{c,T}[H_{c',T},c'(0)] = [H_{c'*c,T},c'(0)] \quad \text{iff} \quad c(0) = c'(1) \text{ in } T.$$

2.9 The following minimality property of $G_{F,T}$ and $\omega_{F,T}$ belongs to the "folklore" although is rarely formulated in this way.

Under the above notations let Γ be a groupoid of germs such that some Γ-structure ω defines on M the foliation F. Then the distinguished submersions for ω restricted to $T \hookrightarrow M$ form a morphism $\Psi: G_{F,T} \longrightarrow G$ such that ω coincides with the induced Γ-structure $\Psi^*\omega_{F,T}$ (cf II.3.12.1 below; here G stands for pseudogroup underlying Γ).

It is worth noticing here that if the transversal T varies, then all the above morphisms $G_{F,T} \longrightarrow G$ come from the unique common morphism of groupoids described in 2.4, $\Gamma_F \longrightarrow \Gamma$ (cf also II.3.4 below).

II. Semi-simplicial structures

Any Γ-structure on a paracompact manifold M can be obtained as a pull-back of the so-called universal Γ-structure ω_o on $B\Gamma$ - the classifying space of Γ, which unfortunately is no longer a manifold. One constructs $B\Gamma$ as a realization of some semi-simplicial manifold $N\Gamma$ (nerve of Γ). We recall some of the definitions ([5]).

1.1 An ss- (semi-simplicial-) manifold $X = (X_n)_{n=0,1,...}$ is a sequence of manifolds endowed with structure operators $\varepsilon_i = \varepsilon_i^{(n)}: X_n \to X_{n-1}$ (i-th face operators; we require them to be submersions) and $\eta_i = \eta_i^{(n)}: X_n \to X_{n+1}$ (i-th degeneracy operators), $i = 0,1,...n$, which are subject to the conditions

$$\text{(1.1.1)} \quad \begin{aligned} &\text{(i)} \quad \varepsilon_i\varepsilon_j = \varepsilon_{j-1}\varepsilon_i \quad \text{if } i < j \\ &\text{(ii)} \quad \varepsilon_i\eta_j = \begin{cases} \eta_{j-1}\varepsilon_i & \text{if } i < j \\ \mathrm{id} & \text{if } i = j,j+1 \\ \eta_j\varepsilon_{i-1} & \text{if } i > j+1 \end{cases} \\ &\text{(iii)} \quad \eta_i\eta_j = \eta_{j+1}\eta_i \quad \text{if } i \leq j \end{aligned}$$

(which originate from purely combinatorical relations among face and degeneracy assignments for standard n-simplices).

Obviously, an ss-map (or homomorphism) $f: X \to Y$ of $X = (X_n)$ into another ss-manifold $Y = (Y_n)$ is a sequence $f = (f_n)$ of maps $f_n: X_n \to Y_n$ commuting with the structure operators.

1.2 After introducing the i-th face imbeddings $\varepsilon^i: \Delta^{n-1} \to \Delta^n$ and i-th face projections $\eta^i: \Delta^{n+1} \to \Delta^n$ between geometrical simplices (cf [5; p. 6]), one defines the (fat) geometrical realization $[\![X]\!]$ of X as the topological quotient of the disjoint union

$$\bigsqcup X_n \times \Delta^n$$

by the relations $(\varepsilon_i x, t) \sim (x, \varepsilon^i t)$ for $x \in X_{n+1}$, $t \in \Delta^n$.

Although the topological space $[\![X]\!]$ is less abstract than the ss-manifold X itself, it is lacking in the "smoothness" of X. In the present paper we shall treat ss-manifolds themselves as a sufficiently good completion of the category of manifolds (cf eg IV. proposition 1.10).

1.3 Keeping the above plan in mind, we begin with some examples ([5]).

Example 1.3.1. Any open covering $\mathcal{U} = \{U_a\}$ of a manifold M gives rise to an ss-manifold $N\mathcal{U} = (N_n\mathcal{U})$, the nerve of $\mathcal{U}$, such that

$$N_n\mathcal{U} = \bigsqcup_{(a_0,\dots,a_n)} U_{a_0} \cap \dots \cap U_{a_n}$$

and the structure operators are the suitable inclusions

$$U_{a_0} \cap \dots \cap U_{a_n} \ni x \begin{cases} \xrightarrow{\varepsilon_i} x \in U_{a_0} \cap \dots \cap U_{a_{i-1}} \cap U_{a_{i+1}} \cap \dots \cap U_{a_n} \\ \xrightarrow{\eta_i} x \in U_{a_0} \cap \dots \cap U_{a_i} \cap U_{a_i} \cap \dots \cap U_{a_n} \end{cases}$$

Roughly speaking, $N\mathcal{U}$ has the same differentiable structure as M whereas some part of the topological complexity of M is expressed in a combinatorial language.

The nerve NM of the trivial covering $\{M\}$ can be identified with M itself.

Example 1.3.2. The nerve $N\Gamma = (N_n\Gamma)$ of an arbitrary groupoid Γ. Here $N_0\Gamma = U$ (units of Γ),

$$N_n\Gamma = \{(g_1,\dots,g_n) \in \Gamma \times \dots \times \Gamma;\ \alpha(g_1) = \beta(g_2), \dots, \alpha(g_{n-1}) = \beta(g_n)\}$$

and the structure operators are defined as follows:

on $N_1\Gamma = \Gamma$, $\varepsilon_0 = \alpha$ and $\varepsilon_1 = \beta$,

$$\varepsilon_i(g_1,\dots,g_n) = \begin{cases} (g_2,\dots,g_n) & \text{if } i = 0 \\ (\dots,g_i g_{i+1},\dots) & \text{if } i = 1,2,\dots,n-1 \\ (g_1,\dots,g_{n-1}) & \text{if } i = n \end{cases}$$

$\eta_0: U \hookrightarrow \Gamma$,

$$\eta_i(g_1,\dots,g_n) = \begin{cases} (\beta(g_1), g_1,\dots,g_n) & \text{if } i = 0 \\ (\dots,g_i,\alpha(g_i),g_{i+1},\dots) & \text{if } i > 0 \end{cases}$$

Clearly, all the $N_n\Gamma$'s are manifolds.

<u>1.4</u> <u>Localization</u>. For an arbitrary ss-manifold $X = (X_n)$ let $U = \{U_a\}$ be any open covering of X_0. By the <u>localization of</u> X <u>to</u> U we shall mean an ss-manifold X_U such that

$$(X_U)_n = \coprod_{(a_0,\ldots,a_n)} \bigcap_{i=n}^{n} (\varepsilon_1^{n-i}\varepsilon_0^{i})^{-1} U_{a_i}$$

and the structure operators are the maps

$$(a_0,\ldots,a_n;x) \xrightarrow{\bar{\varepsilon}_i} (\ldots,a_{i-1},a_{i+1},\ldots;\ \varepsilon_i x)$$
$$(a_0,\ldots,a_n;x) \xrightarrow{\bar{\eta}_i} (\ldots,a_i,a_i,\ldots;\ \eta_i x)$$

where, for shortness, the coordinetes before the semicolon point at the appropriate disjoint summand. A direct application of the axioms (1.1.1) proves that the maps $\bar{\varepsilon}_i$ and $\bar{\eta}_i$ are well-defined and again satisfy (1.1.1). Note that, given U as in example 1.3.1, the ss-manifolds NU and $(NM)_U$ are evidently isomorphic. In general, the localization replaces a part of the topological structure of X_0 with a combinatorial construction. This statement will be made more precise in IV.

We return to foliations now. An obvious observation is that, given any covering U of a manifold M, each foliation F of M induces a foliation, say F_n, on each N_nU. Note that a suitably chosen covering can even trivialize all the foliations. We pose a problem which will be our starting point.

<u>Problem</u> 2.1. Reconstruct the holonomy pseudogroup of F in terms of the foliations $(F_n)_{n\geq 0}$ and the ss-structure of NU.

In fact, we shall solve a more general problem. Namely,

1° we shall give a construction of a holonomy pseudogroup for foliations of arbitrary ss-manifolds (cf III, theorem 2.2), and then

2° we prove its invariance under localizations (cf III, proposition 4.2).

<u>Definition</u> 2.2. A q-<u>codimensional foliation</u> F <u>of</u> an ss-<u>manifold</u> $X = (X_n)$ is a sequence (F_n) of foliations such that

(i) for each n, F_n is a q-codimensional foliation of X_n;

(ii) $\varepsilon_i{}^*F_{n-1} = F_n$ for $i \leq n$ (recall that all the ε_i's are submersions).

<u>2.3</u> The condition 2.2 (ii) together with (1.1.1;ii) implies transversality of all the degeneracy operators $\eta_i\colon X_{n-1} \to X_n$ to F_n as well as the equalities $\eta_i{}^*F_n = F_{n-1}$, for $n = 1,2,\ldots$. Moreover, F is completely determined by F_0 which is subject to the only condition $\varepsilon_0{}^*F_0 = \varepsilon_1{}^*F_0$ on X_1; then $F_n = (\varepsilon_0{}^n)^*F_0$ for every n. This allows

one to expect that the transverse structure of F (whatever this means) is not more complicated than that of ordinary foliations and can be described in terms of some differentiable groupoids.

2.4 We list a few naturally arising examples of ss-foliations.

Example 2.4.1. The foliation of NU induced from an ordinary foliation of M - already mentioned.

Example 2.4.2. If the structure operators of $X = (X_n)$ are locally invertible, then X carries a pointwise foliation which consists of the pointwise foliations on every stage X_n. In particular, this is the case of the nerve $N\Gamma$ of an arbitrary groupoid of germs Γ.

Example 2.4.3. Consider a flag F, F' of foliations of a manifold M (ie, codim $F \geq$ codim F', and the leaves of F' are foliated by leaves of F). Every complete transversal T for F is transverse to F', and the induced foliation $F'|T$ of T lifts to a foliation of $N\Gamma_{F,T}$. We shall see later that differently chosen transversals give rise to nerves $N\Gamma_{F,T}$ equivalent as ss-manifolds, and that the corresponding foliations are in some sense identical (cf IV; corollary 3.3 & proposition 2.4).

Example 2.4.4. Any ss-manifold carries a unique 0-codimensional foliation.

2.5 Given an arbitrary foliation $F = (F_n)$ of an ss-manifold $X = (X_n)$, let for $n = 0,1,\ldots$, L_n denote the set of all leaves of F_n. We consider an equivalence relation " $\sim$ " in L_0 defined as the smallest one containing the pairs (L,L'), $L,L' \in L_0$, such that there is a leaf $\tilde{L} \in L_1$ with the property $L \supset \varepsilon_0\tilde{L}$ and $L' \supset \varepsilon_1\tilde{L}$. For an arbitrary equivalence class $K \subset L_0$ and each $n \geq 0$ we define a manifold $X_n^{(K)}$ to be the disjoint union of the leaves $L \in L_n$ contained in the subset

$$\bigcup_{i=0}^{n} (\varepsilon_1^{\,n-i}\varepsilon_0^{\,i})^{-1}(\cup K) \subset X_n.$$

Lemma 2.5.1. The manifolds $X_n^{(K)}$ and $X_n^{(K')}$ are disjoint if K and K' are different equivalence classes. Furthermore, for any class K, the structure operators of X induce on $X^{(K)} := (X_n^{(K)})$ a semi-simplicial structure.

Proof. Suppose $L \in L_n$ is a leaf such that

$$\varepsilon_1^{\,n-i}\varepsilon_0^{\,i}L \subset L' \quad \text{and} \quad \varepsilon_1^{\,n-j}\varepsilon_0^{\,j}L \subset L'' \qquad i < j$$

where $L' \in K \subset L_0$ and $L'' \in K' \subset L_0$. For $h = 1,\ldots,j-i$, let $L_h \in L_1$ be the leaf in X_1 such that $\varepsilon_2^{\,n-i-h}\varepsilon_0^{\,i+h-1}L \subset L_h$. Then the relations

$$\varepsilon_0(\varepsilon_2^{\,n-i-h}\varepsilon_0^{\,i+h-1}) = \varepsilon_1^{\,n-i-h}\varepsilon_0^{\,i+h}$$

and

$$\varepsilon_1(\varepsilon_2^{n-i-h}\varepsilon_0^{i+h-1}) = \varepsilon_1^{n-i-h+1}\varepsilon_0^{i+h-1}$$

prove that $\varepsilon_1 L_1 \subset L'$ and $\varepsilon_0 L_{j-i} \subset L''$, whereas for $h = 1,\dots,j-i-1$, $\varepsilon_0 L_h$ and $\varepsilon_1 L_{h+1}$ are contained in the same leaf. This immediately implies $L' \sim L''$, and thus $K = K'$.

Now we fix K. We have already shown that if a leaf L is in $X_n^{(K)}$ then for every i the set $\varepsilon_1^{n-i}\varepsilon_0^{i}L$ is contained in a leaf $\tilde{L} \in K$. Consequently, if $\varepsilon_h L \subset L' \in \mathcal{L}_{n-1}$ then $L' \subset X_{n-1}^{(K)}$, and if $\eta_h L \subset L'' \in \mathcal{L}_{n+1}$ then $L'' \subset X_{n+1}^{(K)}$, for $h = 0,1,\dots$. It follows that the structure operators induce maps

$$\tilde{\varepsilon}_h\colon X_n^{(K)} \to X_{n-1}^{(K)}, \quad \tilde{\eta}_h\colon X_n^{(K)} \to X_{n+1}^{(K)}$$

subject to the commutativity relations (1.1.1). By I.1.2, the maps are smooth.

<u>2.5.2</u> We shall call the ss-manifolds $X^{(K)}$ <u>leaves</u> of the ss-foliation $\mathcal{F}$. In view of lemma 2.5.1, any foliation divides the foliated ss-manifold into "disjoint" leaves which are again ss-manifolds.

Like in the classical situation (cf I.1.3 - 1.9) a q-codimensional foliation $\mathcal{F}$ of an ss-manifold X yields a Γ_q-structure on X. The last notion could be naively understood as a sequence of Γ_q-structures, say ω_n on X_n, such that $\varepsilon_i{}^*\omega_{n-1} = \omega_n$. In 3.3 below, we shall see that this condition would be too weak for our purposes.

<u>3.1</u> Let us consider the distinguished Γ_q-bundles $\pi_n\colon E_{\mathcal{F}_n} \to X_n$ associated with the foliations $\mathcal{F}_n$, forming an arbitrary ss-foliation $\mathcal{F}$ of $X = (X_n)$. In view of I.1.12, the sequence $E_{\mathcal{F}} = (E_{\mathcal{F}_n})$ carries a canonical structure of an ss-manifold given by the lifts

$$\tilde{\varepsilon}_i\colon E_{\mathcal{F}_n} \ni [\varphi\circ\varepsilon_i, x] \to [\varphi, \varepsilon_i x] \in E_{\mathcal{F}_{n-1}}$$

and

$$\tilde{\eta}_i\colon E_{\mathcal{F}_n} \ni [\psi\circ\eta_i, x] \to [\psi, \eta_i x] \in E_{\mathcal{F}_{n+1}}.$$

It follows directly from the above definitions that the structure operators commute with the actions of Γ_q, and that $\pi = (\pi_n)$ is an ss-map of $E_{\mathcal{F}}$ in X.

<u>Definition</u> 3.2 (cf [5] for Γ a Lie group). Given any groupoid Γ, a (<u>principal</u>) ss-Γ-<u>bundle</u> E over an ss-manifold $X = (X_n)$ is an ss-manifold $E = (E_n)$ together with an ss-map $\pi = (\pi_n)\colon E \to X$ such that

(i) for every n, $\pi_n\colon E_n \to X_n$ is a principal Γ-bundle;

(ii) the structure operators of E are Γ-equivariant.

In other words, the structure operators yield homomorphisms of Γ-bundles

$$\begin{array}{ccccc} E_{n-1} & \xleftarrow{\varepsilon_i} & E_n & \xrightarrow{\eta_j} & E_{n+1} \\ \downarrow & & \downarrow & & \downarrow \\ X_{n-1} & \xleftarrow{\varepsilon_i} & X_n & \xrightarrow{\eta_j} & X_{n+1}. \end{array}$$

We shall call isomorphy classes of principal ss-Γ-bundles over X Γ-<u>structures</u> on X (isomorphisms being meant as invertible Γ-equvariant ss-maps inducing the identity ss-map on the base X).

<u>3.3</u> Consider an arbitrary principal ss-Γ-bundle $\pi: E \to X$. According to I;1.6, the maps

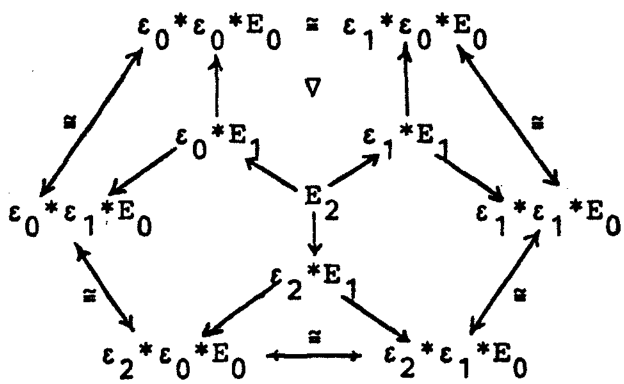
$$E_1 \ni e \begin{array}{l} \xrightarrow{(\pi_1,\varepsilon_0)} (\pi_1 e, \varepsilon_0 e) \in \varepsilon_0{}^*E_0 \\ \xrightarrow[(\pi_1,\varepsilon_1)]{} (\pi_1 e, \varepsilon_1 e) \in \varepsilon_1{}^*E_0 \end{array}$$

as well as

$$E_2 \ni e \longmapsto (\pi_2 e, \varepsilon_i e) \in \varepsilon_i{}^*E_1, \quad i = 0,1,2$$

are isomorphisms of Γ-bundles. Together with the axioms (1.1.1), this gives rise to the following commuting diagram of isomorphisms of principal Γ-bundles:

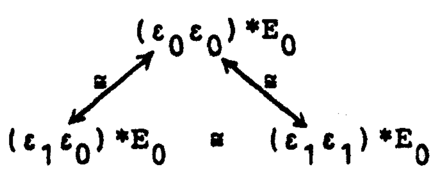
(3.3.1)



where, for example in the pentagon ∇, the arrows map an arbitrary $e \in E_2$ to $(\pi_2 e, (\pi_1\varepsilon_0 e, \varepsilon_0\varepsilon_0 e)) \in \varepsilon_0{}^*\varepsilon_0{}^*E_0$ or to $(\pi_2 e, (\pi_1\varepsilon_1 e, \varepsilon_0\varepsilon_1 e)) \in \varepsilon_1{}^*\varepsilon_0{}^*E_0$, and these two elements correspond to each other under the isomorphism

$$\varepsilon_0{}^*\varepsilon_0{}^*E_0 \cong (\varepsilon_0\varepsilon_0)^*E_0 = (\varepsilon_0\varepsilon_1)^*E_0 \cong \varepsilon_1{}^*\varepsilon_0{}^*E_0.$$

The resulted commutativity of the triangle



is precisely what differs the above definition of an ss-Γ-structure from the "naive" one (cf III;2.5 below).

3.4 We have already got an example of a Γ-structure on an ss-manifold, namely - the one associated with an ss-foliation F and represented by E_F. The next example is provided by morphisms of groupoids (ie generalized homomorphisms in the sense of [10]). We recall the definition.

Definition. A morphism $\Sigma: \Gamma \rightharpoonup \Gamma'$ of a groupoid Γ in a groupoid Γ' is the isomorphy class represented by a principal Γ'-bundle Σ over units of Γ, equipped with a Γ-action (a left one if Γ' acts from the right) $\Gamma \times_{(\alpha,\pi)} \Sigma \to \Sigma$ such that

(i) the two actions commute with each other;

(ii) $g.z = z$ for $z \in \Sigma$ if g is a unit;

(iii) $g_1(g_2z) = (g_1g_2)z$ if the elements are composable

(we use the half-arrows to stress the fact that morphisms are not maps and to distinguish them from homomorphisms ie smooth functors).

Example 3.4.1. Let Γ and Γ' be arbitrary groupoids. For any principal Γ'-bundle E over the nerve $N\Gamma$ let μ_E denote the product

$$\Gamma \times_{(\alpha,\pi_0)} E_0 = \varepsilon_0{}^*E_0 \xleftarrow{\cong} E_1 \xrightarrow{\varepsilon_1} E_0.$$

We claim that the assignment $E \rightsquigarrow (E_0,\mu_E)$ establishes a bijection between Γ'-structures on $N\Gamma$ and morphisms $\Gamma \rightharpoonup \Gamma'$. This follows (cf 3.7 below) from the following useful lemma.

Lemma 3.5. Let $X = (X_n)$ be an arbitrary ss-manifold, Γ any groupoid and $\pi: E_0 \to X_0$ a principal Γ-bundle over X_0. Assume that a map $\overline{\varepsilon}_1: \varepsilon_0{}^*E_0 \to E_0$ satisfies the conditions

(i) $\overline{\varepsilon}_1$ is Γ-equivariant and induces the map $\varepsilon_1: X_1 \to X_0$ on the bases;

(ii) $\overline{\varepsilon}_1(\eta_0\pi e,e) = e$ for $e \in E_0$;

(iii) $\overline{\varepsilon}_1(\varepsilon_2x,\overline{\varepsilon}_1(\varepsilon_0x,e)) = \overline{\varepsilon}_1(\varepsilon_1x,e)$ for every $(x,e) \in (\varepsilon_0\varepsilon_0)^*E_0$.

Then the projection $\overline{\varepsilon}_0: \varepsilon_0{}^*E_0 \to E_0$ together with $\overline{\varepsilon}_1$ and the maps

$$(\varepsilon_0{}^n)^*E_0 \ni (x,e) \xrightarrow{\overline{\varepsilon}_i} \begin{cases} (\varepsilon_ix,e) \in (\varepsilon_0{}^{n-1})^*E_0 & \text{if } i < n \\ (\varepsilon_nx,\overline{\varepsilon}_1(\varepsilon_0{}^{n-1}x,e)) \in (\varepsilon_0{}^{n-1})^*E_0 & \text{if } i = n \end{cases}$$

and

$$E_0 \ni e \xrightarrow{\eta_0} (\eta_0\pi e,e) \in \varepsilon_0{}^*E_0$$

$$(\varepsilon_0{}^n)^*E_0 \ni (x,e) \xrightarrow{\eta_i} (\eta_ix,e) \in (\varepsilon_0{}^{n+1})^*E_0 \quad \text{if } i \geq 1$$

make the sequence $E = ((\varepsilon_0{}^n)^*E_0)$ an ss-manifold which, when equipped with the projections $(\varepsilon_0{}^n)^*E_0 \to X_n$, is a principal ss-$\Gamma$-bundle over X.

Conversely, if $E = (E_n)$ is any principal ss-Γ-bundle over X, then the map $\overline{\varepsilon}_1 \colon \varepsilon_0{}^*E_0 \xleftarrow{\cong} E_1 \xrightarrow{\varepsilon_1} E_0$ verifies the conditions (i) - (iii), and the maps

$$E_n \xrightarrow{(\pi,\varepsilon_0{}^n)} X_n \times_{(\varepsilon_0{}^n,\pi)} E_0 = (\varepsilon_0{}^n)^*E_0$$

form an isomorphism of E and the ss-Γ-bundle reconstructed from E_0 and $\overline{\varepsilon}_1$.

<u>Proof</u>. The maps $\overline{\varepsilon}_i$ and $\overline{\eta}_i$ are well-defined since

$$\begin{aligned}\varepsilon_0{}^{n-1}\varepsilon_i x &= \begin{cases}\varepsilon_0{}^n x = \pi e & \text{if } i < n\\ \varepsilon_1\varepsilon_0{}^{n-1}x & \text{if } i = n\end{cases}\\ &= \varepsilon_1\pi_1(\varepsilon_0{}^{n-1}x,e) = \pi_1\overline{\varepsilon}_1(\varepsilon_0{}^{n-1}x,e)\end{aligned}$$

for $(x,e) \in (\varepsilon_0{}^n)^*E_0$, and

$$\varepsilon_0{}^{n+1}\eta_i x = \varepsilon_0{}^n x \quad \text{for all } i\text{'s}.$$

In a similar way we check that the maps are subject to the commutativity axioms (1.1.1). The only nontrivial relations are those involving $\overline{\varepsilon}_n \colon (\varepsilon_0{}^n)^*E_0 \to (\varepsilon_0{}^{n-1})^*E_0$. One has $\overline{\varepsilon}_{n-1}\overline{\varepsilon}_i = \overline{\varepsilon}_i\overline{\varepsilon}_n$ on $(\varepsilon_0{}^n)^*E_0$ $(i<n-1)$, as both the maps applied to $(x,e) \in (\varepsilon_0{}^n)^*E_0$ give

$$(\varepsilon_{n-1}\varepsilon_i x,\overline{\varepsilon}_1(\varepsilon_0{}^{n-2}\varepsilon_i x,e)) = (\varepsilon_i\varepsilon_n x,\overline{\varepsilon}_1(\varepsilon_0{}^{n-1}x,e)).$$

Similarly, for $(x,e) \in (\varepsilon_0{}^n)^*E_0$,

$$\begin{aligned}\overline{\varepsilon}_{n-1}\overline{\varepsilon}_n(x,e) &= (\varepsilon_{n-1}\varepsilon_n x,\overline{\varepsilon}_1(\varepsilon_0{}^{n-2}\varepsilon_n x,\overline{\varepsilon}_1(\varepsilon_0{}^{n-1}x,e)))\\ &= (\varepsilon_{n-1}\varepsilon_n x,\overline{\varepsilon}_1(\varepsilon_2\varepsilon_0{}^{n-2}x,\overline{\varepsilon}_1(\varepsilon_0\varepsilon_0{}^{n-2}x,e)))\\ &= (\varepsilon_{n-1}\varepsilon_n x,\overline{\varepsilon}_1(\varepsilon_1\varepsilon_0{}^{n-2}x,e))\\ &= (\varepsilon_{n-1}\varepsilon_{n-1}x,\overline{\varepsilon}_1(\varepsilon_0{}^{n-2}\varepsilon_{n-1}x,e))\\ &= \overline{\varepsilon}_{n-1}\overline{\varepsilon}_{n-1}(x,e).\end{aligned}$$

The remaining relations involve η_i's. Clearly, for $(x,e) \in (\varepsilon_0{}^n)^*E_0$,

$$\begin{aligned}\overline{\varepsilon}_{n+1}\overline{\eta}_i(x,e) &= (\varepsilon_{n+1}\eta_i x,\overline{\varepsilon}_1(\varepsilon_0{}^n\eta_i x,e))\\ &= (\eta_i\varepsilon_n x,\overline{\varepsilon}_1(\varepsilon_0{}^{n-1}x,e)) = \overline{\eta}_i\overline{\varepsilon}_n(x,e) \quad \text{if } i < n,\end{aligned}$$

whereas

$$\overline{\varepsilon}_{n+1}\overline{\eta}_n(x,e) = (x,\overline{\varepsilon}_1(\eta_0\varepsilon_0{}^n x,e)) = (x,e)$$

as $\varepsilon_0{}^n x = \pi e$. This implies that $E' = ((\varepsilon_0{}^n)^*E_0)$ is an ss-manifold. The fact that $E' \to X$ is an ss-Γ-bundle follows now immediately from (i).

In order to prove the second part of the lemma it sufficies to show that the isomorphisms $E_n \to (\varepsilon_0{}^n)^*E_0$ form an ss-map, ie commute with the structure operators. This is again a direct computation of symbols. For $e \in E_n$, one has

$$\overline{\varepsilon}_i(\pi e,\varepsilon_0{}^n e) = (\varepsilon_i\pi e,\varepsilon_0{}^n e) = (\pi,\varepsilon_0{}^{n-1})\varepsilon_i e \quad \text{if} \quad i < n$$

and

$$\overline{\varepsilon}_n(\pi e,\varepsilon_0{}^n e) = (\varepsilon_n\pi e,\overline{\varepsilon}_1(\pi\varepsilon_0{}^{n-1}e,\varepsilon_0{}^n e)) = (\varepsilon_n\pi e,\varepsilon_1\varepsilon_0{}^{n-1}e) = (\pi,\varepsilon_0{}^{n-1})\varepsilon_n e.$$

Similarly, if $e \in E_n$, then

$$(\pi,\varepsilon_0{}^{n+1})\eta_i e = (\pi\eta_i e,\varepsilon_0{}^n e) = \overline{\eta}_i(\pi,\varepsilon_0{}^n)e$$

as was to be shown. □

<u>Corollary</u> 3.6. Let $f\colon X \to Y$ be an ss-map and $E \to X$ and $E' \to Y$ any principal Γ-bundles over X and Y. Assume that $I_0\colon E_0 \to E_0'$ is a Γ-equivariant map inducing $f_0\colon X_0 \to X_0'$ on the bases. Then the maps

$$I_n\colon E_n \xrightarrow{(\pi,\varepsilon_0{}^n)} X_n \times_{(\varepsilon_0{}^n,\pi)} E_0 \xrightarrow{f_n\times I_0} Y_n \times_{(\varepsilon_0{}^n,\pi)} E_0' \xrightarrow{(\pi,\varepsilon_0{}^n)^{-1}} E_n'$$

constitute an ss-map $I = (I_n)\colon E \to E'$ iff the condition

$$\overline{\varepsilon}_1(f_n\times I_0) = I_0\overline{\varepsilon}_1 \quad \text{over} \quad \varepsilon_0{}^*E_0$$

is fulfilled; here $\overline{\varepsilon}_1$ stands for the maps $\varepsilon_1(\pi,\varepsilon_0)^{-1}$ induced from ε_1.

<u>3.7</u> <u>Proof</u> of 3.4.1. According to lemma 3.2, we know how to reconstruct the ss-Γ'-bundle E from E_0 and the product $\overline{\varepsilon}_1 = \mu_E\colon \Gamma \times_{(\alpha,\pi)} E_0 \to E_0$. It remains to observe that the conditions (i) - (iii) of the lemma correspond precisely to the conditions (i) - (iii) of definition 3.4. □

We reserve the symbol ω_Σ for the Γ'-structure on $N\Gamma^-$ that corresponds to a morphism $\Sigma\colon \Gamma \longrightarrow \Gamma'$.

<u>3.8</u> Morphisms of groupoids form a category, in which the <u>identity</u> $\Gamma \longrightarrow \Gamma$ is represented by Γ itself, and the <u>composition</u> $\Gamma \xrightarrow{\Sigma} \Gamma' \xrightarrow{\Sigma'} \Gamma''$ is given by the Γ''-bundle (cf [10])

$$\Sigma'\circ\Sigma := \Sigma \times_{\Gamma'} \Sigma' = \Sigma \times_{(\alpha,\pi)} \Sigma' / (\text{the diagonal action of } \Gamma')$$

Any two groupoids Γ and Γ' are said to be <u>equivalent</u> (notation

$\Gamma \approx \Gamma'$) if there exists an invertible morphism (an equivalence) of Γ in Γ'.

3.9 Remarks. 1. A simple criterion [10] of invertibility of a morphism $\Sigma\colon \Gamma \longrightarrow \Gamma'$ which says that the morphism is invertible iff the Γ-action makes Σ a principal Γ-bundle over units of Γ' (then the inverse morphism is represented by Σ with the transposed actions of the groupoids) allows one to treat the actions of the groupoids in a more symmetric manner. In particular, it is somethimes more convenient (especially, when the groupoids are groupoids of germs) to represent a morphism $\Gamma \longrightarrow \Gamma'$ by a left principal Γ'-bundle equipped with a (right or left) Γ-action.

2. As shown in [10], for any foliation F of a manifold M and an arbitrary complete transversal T, the canonical $\Gamma_{F,T}$-bundle $E_{F,T}$ over M endowed with the Γ_F-action (see I.2.4) establishes an equivalence of the graph Γ_F and $\Gamma_{F,T}$.

3.10 Like in the classical situation (cf [10]), every morphism $\Sigma\colon \Gamma \longrightarrow \Gamma'$ transforms any Γ-structure ω on $X = (X_n)$ to a Γ'-structure $\Sigma_*\omega$ on X; by definition, if ω is represented by an ss-Γ-bundle $E = (E_n)$, then $\Sigma_*\omega$ is the ss-Γ'-structure represented by

$$\Sigma_*E := (E_n \times_\Gamma \Sigma)_{n\geq 0}$$

where the structure operators are induced from those of E.

3.11 Turning back to the bijective correspondence

$$\text{„ } \Sigma\colon \Gamma \longrightarrow \Gamma' \text{ "} \quad \rightsquigarrow \quad \text{„ } \omega_\Sigma \text{ - a } \Gamma'\text{-structure on } N\Gamma \text{ "}$$

described in example 3.4.1 and 3.7, we can actually prove what follows.

Proposition 3.11.1. (i) For any groupoid Γ, the Γ-structure ω_Γ on $N\Gamma$ corresponding to the identity morphism $\Gamma\colon \Gamma \longrightarrow \Gamma$ is the one represented by the universal ss-Γ-bundle $\bar{N}\Gamma \to N\Gamma$, where $\bar{N}\Gamma = (\bar{N}_n\Gamma)$,

$$\bar{N}_n\Gamma = \{(g_0,\ldots,g_n) \in \Gamma\times \ldots \times\Gamma;\ \alpha(g_0) = \ldots = \alpha(g_n)\},$$

the projection is given by the maps

$$\bar{N}_n\Gamma \ni (g_0,\ldots,g_n) \to (g_0g_1^{-1},\ldots,g_{n-1}g_n^{-1}) \in N_n\Gamma$$

and both the structure operators and the Γ-action are the classical ones ([5]).

(ii) For any two morphisms of groupoids $\Sigma\colon \Gamma \longrightarrow \Gamma'$ and $\Sigma'\colon \Gamma' \longrightarrow \Gamma''$ there is

$$\Sigma'_*\omega_\Sigma = \omega_{\Sigma\circ\Sigma'}.$$

<u>Proof</u>. One has $\bar{N}_0\Gamma = \Gamma$ with the standard right action of Γ. A computation of the corresponding left Γ-action,

$$\Gamma \times_{(\varepsilon_0,\pi)} \bar{N}_0\Gamma \xrightarrow{(\pi,\varepsilon_0)^{-1}} \bar{N}_1\Gamma \xrightarrow{\varepsilon_1} \bar{N}_0\Gamma$$

gives

$$(g,g_0) \to (gg_0,g_0) \to gg_0$$

as was to be shown. As for (ii), we recall that ω_Σ is represented by the ss-Γ'-bundle $((\varepsilon_0{}^n)*\Sigma)$ with the structure operators given by lemma 3.5. Consequently, $\Sigma^!_*\omega_\Sigma$ is represented by the ss-Γ''-bundle $((\varepsilon_0{}^n)*\Sigma\times_\Gamma\Sigma')$ One can now easily see that the isomorphisms $(\varepsilon_0{}^n)*\Sigma \times_\Gamma\Sigma' \cong (\varepsilon_0{}^n)*(\Sigma \times_\Gamma\Sigma')$ preserve the structure operators (actually the crucial ε_1).

<u>3.12</u> Important examples of morphisms of groupoids are supplied by morphisms of pseudogroups of diffeomorphisms (cf I.2.6). Namely, such a morphism $\Phi\colon G \longrightarrow G'$ yields functorially a morphism

$$|\Phi| : \Gamma \longrightarrow \Gamma'$$

of the corresponding groupoids of germs, where

$$|\Phi| := \{[\varphi,x];\ \varphi \in \Phi,\ x \in U_\varphi\}$$

is a manifold of germs (with the sheaf topology and the suitable differentiable structure), and the groupoids act by composition of germs.

We note that $|\Phi|$ is a <u>left</u> principal Γ'-bundle endowed with a <u>right</u> action of Γ (cf remark 3.9.1).

<u>3.12.1</u> For any Γ-structure ω, Φ transfers ω to the Γ'-structure $\Phi_*\omega := |\Phi|_*\omega$. Let us note here that ω defines a foliation iff the induced Γ'-structure $\Phi_*\omega$ defines a foliation. If this is the case, then the distinguished submersions for $\Phi_*\omega$ are locally of the form $\varphi\circ\gamma$, where $\varphi\in\Phi$ and γ ranges over the distinguished submersions for ω.

<u>3.13</u> In IV below we extend the notion of a morphism to ss-manifolds in such a way that

1° a morphism $N\Gamma \rightharpoonup N\Gamma'$ (again a half-arrow) means exactly a morphism $\Gamma \longrightarrow \Gamma'$;

2° a Γ-structure on any ss-manifold X means a morphism $X \rightharpoonup N\Gamma$;

3° the transformation of ss-Γstructures to ss-Γ'-structures along an arbitrary morphism of groupoids $\Gamma \longrightarrow \Gamma'$ corresponds to composition of morphisms of ss-manifolds.

The new category happens to be especially useful while studying semisimplicial foliations.

III. Holonomy in the semi-simplicial context

In this paragraph we modify the notion of the holonomy groupoid in order that the minimality property I.2.9 be valid for ss-foliations.

1.1 Let us fix a q-codimensional foliation F of an ss-manifold M and consider an arbitrary complete transversal $i: T \hookrightarrow X_1$ for F_1 (cf remark 1.4.2 below). Clearly, the disjoint union $T^0 \sqcup T^1$ of two copies of T immersed in X_0 by the map $\varepsilon_0 i \sqcup \varepsilon_1 i$ is a complete transversal for F_0 (in fact, so are both T^0 and T^1). Indeed, for any leaf L of F_0 there is a leaf $\bar{L}$ of F_1 such that $\eta_0 L \subset \bar{L}$; since both $\varepsilon_0 \bar{L}$ and $\varepsilon_1 \bar{L}$ contain L and are contained in a leaf of F_0, the equalities $\varepsilon_0 \bar{L} = L = \varepsilon_1 \bar{L}$ hold. Consequently, if $\bar{L} \cap iT \neq 0$ then also $L \cap \varepsilon_i iT \neq 0$, for $i = 0,1$.

We denote by $G_{F,T^0 \sqcup T^1}$ the pseudogroup (of diffeomorphisms of $T^0 \sqcup T^1$) generated by the holonomy pseudogroup $G_{F_0,T^0 \sqcup T^1}$ and the identification map $id_1^0: T^0 \to T^1$. Finally, let $G_{F,T}$ stand for the pseudogroup

$$G_{F,T} = \{\gamma \in G_{F,T^0 \sqcup T^1};\ \text{domain } \gamma,\ \text{image } \gamma \subset T^0\}.$$

Lemma 1.1.1. (i) The inclusion $T^0 \hookrightarrow T^0 \sqcup T^1$ generates an equivalence of pseudogroups

$$\Phi_T: G_{F,T} \longrightarrow G_{F,T^0 \sqcup T^1}.$$

Furthermore, if $\bar{T} \hookrightarrow X_1$ is another complete transversal for F_1, then: (ii) the holonomy translations $h^{T}_{c,\bar{T}}$ of T in $\bar{T}$ along the paths in leaves of F_1 generate an equivalence $\Phi: G_{F,T} \longrightarrow G_{F,\bar{T}}$;

(iii) the holonomy translations $h^{T^0 \sqcup T^1}_{c,\bar{T}^0 \sqcup \bar{T}^1}$ of $T^0 \sqcup T^1$ in $\bar{T}^0 \sqcup \bar{T}^1$ along the paths in leaves of F_0 generate an equivalence $\Phi': G_{F,T^0 \sqcup T^1} \longrightarrow G_{F,\bar{T}^0 \sqcup \bar{T}^1}$; and

(iv) the diagram

$$\begin{array}{ccc} G_{F,T} & \xrightarrow{\Phi_T} & G_{F,T^0 \sqcup T^1} \\ \Phi \downarrow & & \downarrow \Phi' \\ G_{F,\bar{T}} & \xrightarrow[\Phi_{\bar{T}}]{} & G_{F,\bar{T}^0 \sqcup \bar{T}^1} \end{array}$$

commutes.

Proof. (i) The fact that Φ_T is invertible is a direct consequence of another one, that $G_{F,T^0 \sqcup T^1}$ contains the identification map $id_0^1: T^1 \to T^0$; clearly, the two diffeomorphisms

$$T^0 \sqcup T^1 \begin{cases} T^0 \xrightarrow{id} T^0 \\ T^1 \xrightarrow{id_0^1} T^0 \end{cases}$$

generate a morphism $\Phi^T\colon G_{F,T^0\sqcup T^1} \longrightarrow G_{F,T}$ inverse to Φ_T.

We left the part (ii) for a moment and first prove (iii). Since the holonomy translations $h_{c,\bar{T}}^{T} := h_{c,\bar{T}^0\sqcup\bar{T}^1}^{T^0\sqcup T^1}$ establish a morphism $G_{F_0,T^0\sqcup T^1} \to G_{F_0,\bar{T}^0\sqcup\bar{T}^1}$, it sufficies to check (cf I;2.6(ii)) that they transfer id_1^0 to an element of $G_{F,\bar{T}^0\sqcup\bar{T}^1}$ (invertibility of the resulted morphism follows then from the symmetry arguments). So let c and c' be any two paths in X_0 such that each one is contained in a leaf of F_0, and that $c(0) = \varepsilon_0 t$ and $c'(0) = \varepsilon_1 t$ for a $t \in T \hookrightarrow X_1$. Moreover, we require both the paths to have their ends in $\bar{T}^0 \sqcup \bar{T}^1 \hookrightarrow X_0$, and examine the diffeomorphism $h_{c',\bar{T}}^{T}\, id_1^0 (h_{c,\bar{T}}^{T})^{-1}$ defined over a neighbourhood of $c(1)$ in $\bar{T}^0 \sqcup \bar{T}^1$. If one choses a path C connecting t to a point of $\bar{T}$ in a leaf of F_1, then one immediately gets

$$h_{c',\bar{T}}^{T}\, id_1^0 (h_{c,\bar{T}}^{T})^{-1} = h_{(\varepsilon_1 C)^{-1} * c',\bar{T}}^{\bar{T}}\; h_{\varepsilon_1 C,\bar{T}}^{T}\, id_1^0 (h_{\varepsilon_0 C,\bar{T}}^{T})^{-1}\; h_{c^{-1} * (\varepsilon_0 C),\bar{T}}^{\bar{T}}$$

In view of I.2.1, the holonomy translations along $\varepsilon_0 C$ or $\varepsilon_1 C$ are the same as those along C in X_1 (up to the identification maps); this means the composition

$$h_{\varepsilon_1 C,\bar{T}}^{T}\, id_1^0 (h_{\varepsilon_0 C,\bar{T}}^{T})^{-1} \;:\; \bar{T}^0 \to \bar{T}^1$$

is just the map id_1^0. Consequently, being a composition of three maps from $G_{F,\bar{T}^0\sqcup\bar{T}^1}$ the examined diffeomorphism is an element of the same pseudogroup.

Turning back now to the assertion (ii) we note that the domains of the holonomy translations from T to $\bar{T}$ (in X_1!) cover the whole T. Furthermore, since one has $h_{c\bar{T}}^{T} = h_{\varepsilon_0 c,\bar{T}^0}^{T^0}$ for c in X_1, it follows from the above proof of (iii) that the collection of all the $h_{c,\bar{T}}^{T}$'s transfers the subset $G_{F,T} \subset G_{F,T^0\sqcup T^1}$ to the subset $G_{F,\bar{T}} \subset G_{F,\bar{T}^0\sqcup\bar{T}^1}$, and thus generates a morphism $\Phi\colon G_{F,T} \longrightarrow G_{F,\bar{T}}$ such that the square of (iv) commutes. Evidently, Φ must be an equivalence.

<u>Definition</u> 1.2. We shall call $G_{F,T}$ and $G_{F,T^0\sqcup T^1}$ respectively the (reduced) <u>holonomy pseudogroup</u> and the <u>non-reduced holonomy pseudogroup</u> of F with respect to the complete transversal $T \hookrightarrow X_1$. Similarly, the corresponding groupoids of germs $\Gamma_{F,T}$ and $\Gamma_{F,T^0\sqcup T^1}$ will be called the <u>holonomy groupoid</u> and the <u>non-reduced holonomy groupoid</u> of F with respect to T.

<u>1.3</u> The lemma proves not only that the holonomy pseudogroups with respect to different transversals are all mutually equivalent, but also points, each time, at a particular <u>canonical equivalence</u>. Since no one invariant form of the holonomy pseudogroups or groupoids is known (at

least to the author; for standard foliations such a role plays the graph) let us put emphasis on the <u>consistency</u> of the canonical equivalences. Namely, for any three complete transversals T, $\overline{T}$ and $\widetilde{T} \hookrightarrow X_1$ the corresponding triangle

$$\begin{array}{ccc} & G_{F,T} & \\ \swarrow\approx & & \approx\searrow \\ G_{F,\overline{T}} & \xrightarrow{\ \approx\ } & G_{F,\widetilde{T}} \end{array}$$

commutes; clearly, the same holds for any such triangle composed of either reduced or non-reduced holonomy pseudogroups and their canonical equivalences.

In what follows, we generally formulate our results in terms of the reduced holonomy pseudogroups or groupoids, whereas the non-reduced notions appear to be more convenient for the proofs.

<u>1.4</u> <u>Remarks</u>. 1. It can be shown that if one choses a complete transversal T for F_n, n arbitrary ≥ 1, then maps it along all the boundary maps into X_0 and, finally, extends the corresponding holonomy pseudogroup of F_0 with the help of all the identification maps, then the resulted pseudogroup is equivalent to the holonomy pseudogroups of F.

2. Like in the standard (ie non-semi-simplicial) case the orbits $\beta(\alpha^{-1}(t))$, $t \in T$, of the holonomy groupoid $\Gamma_{F,T}$ are in a one-to-one correspondence with the leaves of F; the correspondence being given by the assignment

$$\beta(\alpha^{-1}(t)) \rightsquigarrow X^{(K_t)}$$

where K_t denotes the equivalence class containing the leaves of F_0 that pass through $\varepsilon_i \iota t$, $i = 0,1$ (cf II.2.5). Consequently, we may call T a <u>complete transversal for</u> F.

In order to justify the name: holonomy pseudogroup or holonomy groupoid, we shall now consider arbitrary groupoids of germs related to the foliation F as in I;1.10.

<u>Definition</u> 2.1. Let Γ be an arbitrary groupoid of germs such that $\dim \Gamma = \operatorname{codim} F$. An ss-$\Gamma$-bundle $E = (E_n)$ over X <u>defines</u> the <u>foliation</u> F of X (and so does the ss-Γ-structure represented by E) if the Γ-bundle $E_0 \to X_0$ defines F_0.

By I.1.10 the condition means that the target map β_0 of E_0 to the units of Γ is a submersion, and the foliation $\pi^* F_0$ of E_0 coincides with the foliation induced by β_0 from the pointwise one. Clearly, this implies that for any n the Γ-bundle $E_n \to X_n$ defines F_n.

Our main result is the following one.

<u>Theorem</u> 2.2. Let $F = (F_n)$ be an arbitrary foliation of an ss-manifold $X = (X_n)$. For any complete transversal $T \hookrightarrow X_1$ there exists a canonical principal $\Gamma_{F,T}$-structure $\omega_{F,T}$ defining F on X and such that

(i) if Γ is any groupoid of germs and an ss-Γ-structure ω on X defines F, the there is a canonically induced morphism $\Psi = \Psi_T^\omega$ of $G_{F,T}$ in G, the pseudogroup underlying Γ, such that

$$\omega = \Psi_* \omega_{F,T}$$

(ii) if $\bar{T} \hookrightarrow X_1$ is another complete transversal for F, then the canonically induced morphism

$$\Psi_T^{\omega_{F,\bar{T}}} \colon G_{F,T} \longrightarrow G_{F,\bar{T}}$$

is the canonical equivalence of holonomy pseudogroups (cf 1.3);

(iii) if ω and $\bar{T}$ are as in (i) - (ii) and, moreover, $\Lambda\colon G \longrightarrow G'$ is an arbitrary morphism of pseudogroups, then the square

$$\begin{array}{ccc} G_{F,T} & \xrightarrow{\ \Psi_T^{\omega}\ } & G \\ \Phi \big\downarrow & & \big\downarrow \Lambda \\ G_{F,\bar{T}} & \xrightarrow[\ \Psi_{\bar{T}}^{\Lambda_*\omega}\ ]{} & G' \end{array}$$

commutes, Φ being the canonical equivalence.

<u>Definition</u> 2.2.1. We shall call the morphism $\Psi_T^\omega\colon G_{F,T} \longrightarrow G$ constructed in the proof of theorem 2.2 (i) (cf 2.4 - 2.6 below), as well as the composition of Ψ_T^ω and the canonical equivalence $G_{F,T^0\sqcup T^1} \longrightarrow G_{F,T}$, the <u>holonomy morphism for</u> F <u>with respect to</u> T.

Our proof of theorem 2.2 will consist of several parts (cf 2.3 - 2.7 below).

<u>2.3</u> We begin the proof with a construction of a $\Gamma_{F,T^0\sqcup T^1}$-structure $\omega^!_{F,T}$ canonically associated with F. Namely, let us consider a (left!) $\Gamma_{F,T^0\sqcup T^1}$-bundle over X_0,

(2.3.1) $$E_{F,0} := \Gamma_{F,T^0\sqcup T^1} \times_{\Gamma_{F_0,T^0\sqcup T^1}} E_{F_0,T^0\sqcup T^1}$$

which is the result of an extension of the structure groupoid of $E_{F_0,T^0\sqcup T^1}$ (the canonical principal bundle associated with F_0 and the transversal $T^0 \sqcup T^1 \hookrightarrow X_0$, cf I;2.8).

In view of lemma II;3.5, we may let $E_{F,n} := (\varepsilon_0{}^n)^* E_{F,0}$ and the question is how to construct the operator $\bar{\varepsilon}_1 : E_{F,1} \to E_{F,0}$. To do that, we write an arbitrary element of

$$E_{F,1} = E_{F,0} \times_{(\pi,\varepsilon_0)} X_1$$

in a special way. Namely, for any $x \in X_1$ we choose a path c in a lea of F_1, connecting x to a point of $T \hookrightarrow X_1$. Then the path $\varepsilon_0 c$ connects $\varepsilon_0 x$ to a point of $T^0 \hookrightarrow X_0$, and the germ $[H_{\varepsilon_0 c, T^0}, \varepsilon_0 x]$ (cf I.2.8) is an element of $E_{F_0, T^0 \sqcup T^1}$. Consequently, any element of $\pi^{-1}(\varepsilon_0 x) \subset E_{F,0}$ is of the form $g[H_{\varepsilon_0 c, T^0}, \varepsilon_0 x]$ for a $g \in \Gamma_{F, T^0 \sqcup T^1}$ such that $\alpha(g) = c(1) \in T^0$. We set

$$\text{(2.3.2)} \qquad E_{F,1} \ni (g[H_{\varepsilon_0 c, T^0}, \varepsilon_0 x], x) \xmapsto{\ \bar{\varepsilon}_1\ } g\, \mathrm{id}_0^1 [H_{\varepsilon_1, T^1}, \varepsilon_1 x] \in E_{F,0}$$

where id_0^1 stands for the suitable germ of the identification map. The resulted map $\bar{\varepsilon}_1$ is well-defined. Indeed, if c' is another path in X_1 starting at x, then (cf I.2.8)

$$[H_{\varepsilon_i c', T^i}, \varepsilon_i x] = h_{\varepsilon_i (c^{-1} * c'), T^i} [H_{\varepsilon_i c, T^i}, \varepsilon_i x] \quad \text{for} \quad i=0,1,$$

where, by I.2.1,

$$\mathrm{id}_0^1 h_{\varepsilon_1 (c^{-1} * c'), T^1} = h_{\varepsilon_0 (c^{-1} * c'), T^0}\, \mathrm{id}_0^1$$

as both the holonomy translations are equal to $h_{c^{-1} * c', T}$. Now, an equality

$$g[H_{\varepsilon_0 c, T^0}, \varepsilon_0 x] = g'[H_{\varepsilon_0 c', T^0}, \varepsilon_0 x]$$

implies $g = g' h_{\varepsilon_0 (c^{-1} * c'), T^0}$, and thus the equalities

$$\begin{aligned} g\, \mathrm{id}_0^1 [H_{\varepsilon_1 c, T^1}, \varepsilon_1 x] &= g' \mathrm{id}_0^1 h_{\varepsilon_1 (c^{-1} * c'), T^1} [H_{\varepsilon_1 c, T^1}, \varepsilon_1 x] \\ &= g' \mathrm{id}_0^1 [H_{\varepsilon_1 c', T^1}, \varepsilon_1 x] \end{aligned}$$

prove correctness of the definition (2.3.2). Clearly, $\bar{\varepsilon}_1$ is $\Gamma_{F, T^0 \sqcup T^1}$-equivariant and the map it induces on the bases is $\varepsilon_1 : X_1 \to X_0$.

Let us show now smoothness of $\bar{\varepsilon}_1$. Namely, if x is an arbitrary point of X_1 and c is a fixed path in a leaf of F_1 connecting x to a point of T, then it sufficies to prove that $\bar{\varepsilon}_1$ maps the section of $E_{F,1}$,

$$y \xrightarrow{\ \sigma\ } ([H_{\varepsilon_0 c, T^0}, \varepsilon_0 y], y) \text{ for } y \text{ in a neighbourhood of } x \in X_1,$$

to the smooth map $y \longrightarrow \mathrm{id}_0^1 [H_{\varepsilon_1 c, T^1}, \varepsilon_1 y]$.

There is a neighbourhood U of x such that any point $y \in U$ can be connected to a point of T by a path c^y continuously depending on y and contained in the leaf of F_1 through y (with the condition $c^x = c$). Now the paths $\varepsilon_0 c^y$ as well as $\varepsilon_1 c^y$ continuously depend on y, and it follows from the definition I.2.8 that for any y in a (possibly smaller) neighbourhood $V \subset U$ of x, one has

$$[H_{\varepsilon_i c^y, T^i}, \varepsilon_i y] = [H_{\varepsilon_i c, T^i}, \varepsilon_i y] \quad \text{for} \quad i = 0,1.$$

Consequently, if $y \in V$, then

$$\bar{\varepsilon}_1 \sigma(y) = \bar{\varepsilon}_1([H_{\varepsilon_0 c^y, T^0}, \varepsilon_0 y], y)$$

$$= id_0^1 [H_{\varepsilon_1 c^y, T^1}, \varepsilon_1 y] = id_0^1 [H_{\varepsilon_1 c, T^1}, \varepsilon_1 y]$$

as was to be shown.

We verify the conditions (ii) - (iii) of lemma II.3.5 (according to I.1.8 we use the notation for left principal bundles).

"(ii)" For an arbitrary $x \in X_0$ let c be a path connecting $\eta_0 x \in X_1$ to a point of T in a leaf of F_1. Then

$$\bar{\varepsilon}_1([H_{\varepsilon_0 c, T^0}, x], \varepsilon_0 x) = id_0^1 [H_{\varepsilon_1 c, T^1}, x]$$

$$= id_0^1 h_{(\varepsilon_0 c)^{-1} * (\varepsilon_1 c), T^0 \sqcup T^1} [H_{\varepsilon_0 c, T^0}, x].$$

By choosing a local transversal $R \hookrightarrow X_0$ for F_0 at x, we get

$$h_{(\varepsilon_0 c)^{-1} * (\varepsilon_1 c), T^0 \sqcup T^1} = [h^R_{\varepsilon_1 c, T^1}, x][h^R_{\varepsilon_0 c, T^0}, x]^{-1}$$

where (cf I.2.1) both the holonomy translations in X_0 are equal to $h^{\eta_0 R}_{c,T}$, a holonomy translation in X_1. Consequently, the above composition of germs is a germ of the identification map $T^0 \to T^1$. In view of the $\Gamma_{F, T^0 \sqcup T^1}$-equivariance of $\bar{\varepsilon}_1$, the resulted equality

$$\bar{\varepsilon}_1([H_{\varepsilon_0 c, T^0}, x], \varepsilon_0 x) = [H_{\varepsilon_0 c, T^0}, x]$$

immediately implies (ii) of lemma II;3.5.

"(iii)" We fix an arbitrary element x of X_2 and choose for each $i = 0,1,2$, a path c_i connecting the point $\varepsilon_i x$ to some point of T in a leaf of F_1 (see figure 3). Let $e = [H_{\varepsilon_0 c_0, T^0}, \varepsilon_0 \varepsilon_0 x]$, then one has $(e, \varepsilon_0 x), (e, \varepsilon_1 x) \in E_{F,1}$, and

$$\bar{\varepsilon}_1(e, \varepsilon_0 x) = id_0^1 [H_{\varepsilon_1 c_0, T^1}, \varepsilon_1 \varepsilon_0 x]$$

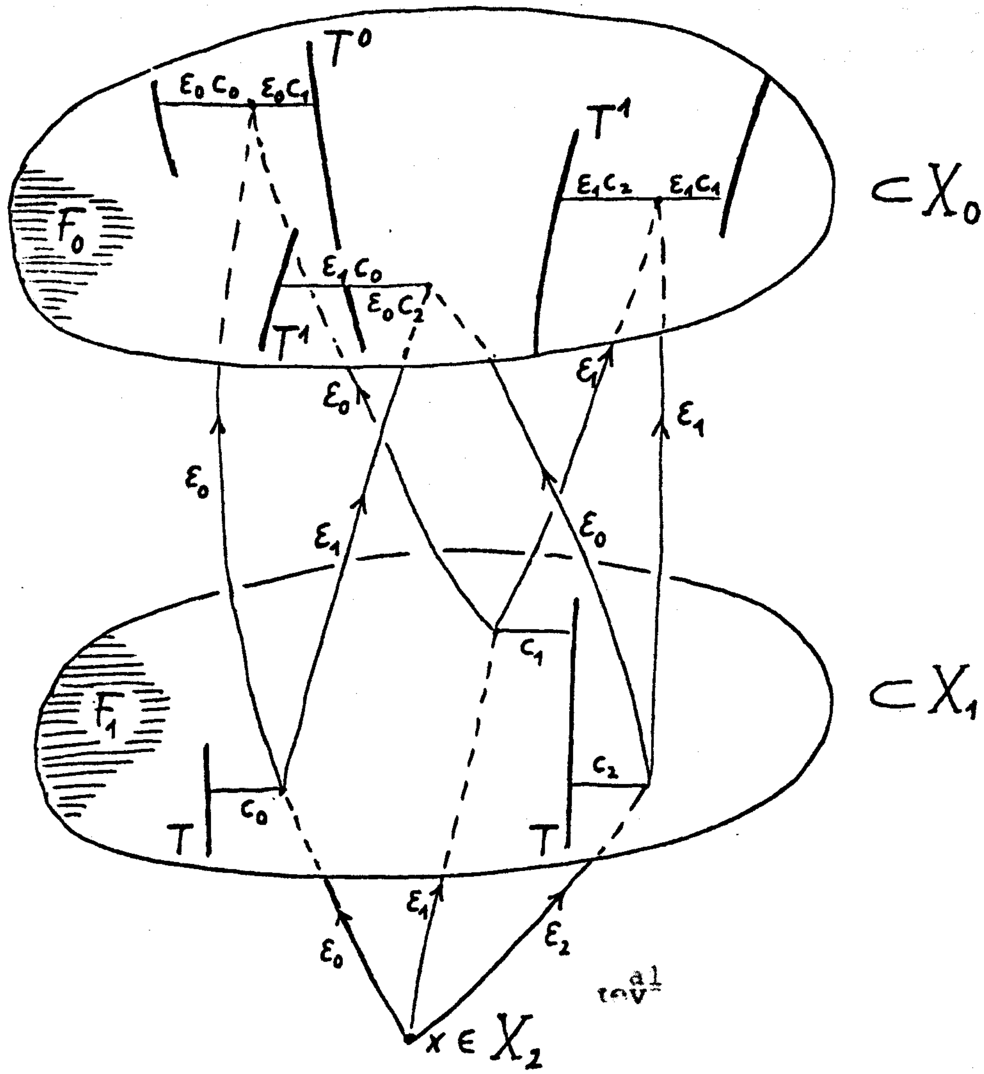


Figure 3.

whereas

$$\bar{\varepsilon}_1(e,\varepsilon_1 x) = \bar{\varepsilon}_1(h_{(\varepsilon_0 c_1)^{-1}*(\varepsilon_0 c_0),T^0}[H_{\varepsilon_0 c_1,T^0},\varepsilon_0\varepsilon_1 x],\varepsilon_1 x)$$

$$= h_{(\varepsilon_0 c_1)^{-1}*(\varepsilon_0 c_0)}\,id_0^1[H_{\varepsilon_1 c_1,T^1},\varepsilon_1\varepsilon_1 x].$$

According to (2.3.2), we get also

$$\bar{\varepsilon}_1(\bar{\varepsilon}_1(e,\varepsilon_0 x),\varepsilon_2 x) = \bar{\varepsilon}_1(id_0^1 h_{(\varepsilon_0 c_2)^{-1}*(\varepsilon_1 c_0),T^0\sqcup T^1}[H_{\varepsilon_0 c_2,T^0},\varepsilon_0\varepsilon_2 x],\varepsilon_2 x)$$

$$= id_0^1 h_{(\varepsilon_0 c_2)^{-1}*(\varepsilon_1 c_0),T^0\sqcup T^1}\,id_0^1[H_{\varepsilon_1 c_2,T^1},\varepsilon_1\varepsilon_2 x]$$

and the condition (iii) of lemma II.3.5 reduces to the equality

(2.3.3) $\quad h_{(\varepsilon_0 c_1)^{-1}*(\varepsilon_0 c_0),T^0}\,id_0^1 h_{(\varepsilon_1 c_2)^{-1}*(\varepsilon_1 c_1),T^1} =$

$$= \mathrm{id}_0^1 h_{(\varepsilon_0 c_2)^{-1} * (\varepsilon_1 c_0), T^0 \sqcup T^1} \mathrm{id}_0^1$$

which remains to be shown. Keeping this goal in mind, we fix a local transversal $Q \hookrightarrow X_2$ for the foliation F_2 at x. This allows us to consider all the holonomy translations $h_{c_i}: \varepsilon_i Q \to T$, $i = 0,1,2$, as well as $h_{\varepsilon_0 c_0}, h_{\varepsilon_0 c_1}: \varepsilon_0 \varepsilon_0 Q \to T^0$, and $h_{\varepsilon_1 c_0}, h_{\varepsilon_0 c_2}: \varepsilon_1 \varepsilon_0 Q \to T$ etc (for shortness we use abbreviated notations) which are, by I.2.1, subject to the relations (at the level of germs)

$$h_{\varepsilon_0 c_0} = \mathrm{id}_0 \circ h_{c_0} \circ (\varepsilon_0 | \varepsilon_0 Q)^{-1}$$
$$h_{\varepsilon_0 c_1} = \mathrm{id}_0 \circ h_{c_0} \circ (\varepsilon_0 | \varepsilon_1 Q)^{-1} \qquad \text{etc,}$$

id_i standing for the identification map $T \to T^i$ as $i = 0,1$. Consequently, the left-hand side of (2.3.3) is a germ of the composition

$$(\mathrm{id}_0 \circ h_{c_0} \circ (\varepsilon_0 | \varepsilon_0 Q)^{-1})(\mathrm{id}_0 \circ h_{c_1} \circ (\varepsilon_0 | \varepsilon_1 Q)^{-1})^{-1} \circ \mathrm{id}_0^1 \circ$$
$$\circ (\mathrm{id}_1 \circ h_{c_1} \circ (\varepsilon_1 | \varepsilon_1 Q)^{-1})(\mathrm{id}_1 \circ h_{c_2} \circ (\varepsilon_1 | \varepsilon_2 Q)^{-1})^{-1}$$
$$= \mathrm{id}_0 h_{c_0} (\varepsilon_0 | \varepsilon_0 Q)^{-1} (\varepsilon_0 | \varepsilon_1 Q)(\varepsilon_1 | \varepsilon_1 Q)^{-1} (\varepsilon_1 | \varepsilon_2 Q) h_{c_2}^{-1} (\mathrm{id}_1)^{-1}$$

whereas the right-hand side is a germ of

$$\mathrm{id}_0^1 (\mathrm{id}_1 h_{c_0} (\varepsilon_1 | \varepsilon_0 Q)^{-1} (\mathrm{id}_0 h_{c_2} (\varepsilon_0 | \varepsilon_2 Q)^{-1})^{-1} \ \mathrm{id}_0^1$$
$$= \mathrm{id}_0 h_{c_0} (\varepsilon_1 | \varepsilon_0 Q)^{-1} (\varepsilon_0 | \varepsilon_2 Q) h_{c_2}^{-1} (\mathrm{id}_1)^{-1}.$$

Since now both the intrinsic compositions of the ε_i's are identical and equal to the map $\varepsilon_2 Q \ni \varepsilon_2 z \mapsto \varepsilon_0 z \in \varepsilon_0 Q$, the equality (2.3.3) holds. This finishes the above construction of an ss-$\Gamma_{F, T^0 \sqcup T^1}$-bundle $E'_{F,T} = (E_{F,n})$ associated with the foliation $F = (F_n)$ and the transversal $T \hookrightarrow X_1$. The fact that $E_{F,T}$ defines F on X is obvious. Let $\omega'_{F,T}$ denote the corresponding ss-$\Gamma_{F,T^0 \sqcup T^1}$-structure; we end the construction by setting

$$\omega_{F,T} := \Phi^T{}_* \omega'_{F,T}$$

where $\Phi^T: G_{F,T^0 \sqcup T^1} \longrightarrow G_{F,T}$ is the canonical equivalence.

<u>2.4</u> We consider now an arbitrary groupoid of germs Γ such that some ss-Γ-structure given by a (left) principal ss-Γ-bundle $E \to X$, $E = (E_n)$, defines on X the foliation F. The submersions of subsets of X_0 to the manifold of units of Γ which are distinguished by E_0 can be restricted to $T^0 \sqcup T^1 \hookrightarrow X_0$; we claim that the resulted collection of diffeomorphisms of open subsets of $T^0 \sqcup T^1$ into the units of Γ

generates a morphism

$$\Psi' : G_{F,T^0 \sqcup T^1} \longrightarrow G$$

such that $\omega = \Psi'_* \omega'_{F,T}$. Before proving this, we first give a cocycle description of an arbitrary ss-Γ-bundle.

Definition 2.5. Let Γ be an arbitrary groupoid, $X = (X_n)$ an ss-manifold, and $\mathcal{U} = \{U_a, a \in A\}$ any open covering of X_0. A Γ-cocycle on X with respect to $\mathcal{U}$ is any collection $\{\gamma_{ab}, (a,b) \in A \times A\}$ of maps

$$\gamma_{ab} : X_1 \supset \varepsilon_1^{-1} U_a \cap \varepsilon_0^{-1} U_b \to \Gamma$$

such that

$$(2.5.1) \quad \gamma_{ab}(\varepsilon_2 x)\gamma_{bc}(\varepsilon_0 x) = \gamma_{ac}(\varepsilon_1 x)$$

for $x \in (\varepsilon_1\varepsilon_1)^{-1}U_a \cap (\varepsilon_1\varepsilon_0)^{-1}U_b \cap (\varepsilon_0\varepsilon_0)^{-1}U_c$ and for all triples of indices.

2.5.2 Any Γ-cocycle $\gamma = \{\gamma_{ab}\}$, as above, determines a (right) principa ss-Γ-bundle over X. Indeed, when applied to $\eta_0\eta_0 x \in X_2$ the condition (2.5.1) implies that $\{\gamma_{ab} \circ \eta_0\}$ is a Γ-cocycle on X_0. Consequently, the corresponding Γ-bundle

$$E_0^\gamma := \bigsqcup_{a \in A} U_a \times_{(\gamma_{aa}\eta_0, \beta)} \Gamma / \sim$$

where

$$(a;x,g) \sim (b;x',g') \quad \text{iff} \quad x = x' \in U_a \cap U_b \quad \text{and} \quad g' = \gamma_{ba}(\eta_0 x)g,$$

together with the map

$$E_1^\gamma := X_1 \times_{(\varepsilon_0,\pi)} E_0 \ni (x,[a;\varepsilon_0 x,g]) \xrightarrow{\bar{\varepsilon}_1} [b;\varepsilon_1 x, \gamma_{ba}(x)g] \in E_0^\gamma$$

where the braces "[ ]" stand for the equivalence classes and $b \in A$ is an index such that $U_b \ni \varepsilon_1 x$, fulfill the assumptions of lemma II.3.5:

"(i)" If $\varepsilon_1 x \in U_b \cap U_{b'}$, then one has

$$\eta_0 x \in (\varepsilon_1^2)^{-1}U_{b'} \cap (\varepsilon_1\varepsilon_0)^{-1}U_b \cap (\varepsilon_0^2)^{-1}U_a$$

and the equality $\gamma_{b'b}(\varepsilon_2\eta_0 x)\gamma_{ba}(\varepsilon_0\eta_0 x) = \gamma_{b'a}(\varepsilon_1\eta_0 x)$ implies

$$[b;\varepsilon_1 x, \gamma_{ba}(x)g] = [b';\varepsilon_1 x, \gamma_{b'b}(\eta_0\varepsilon_1 x)\gamma_{ba}(x)g]$$
$$= [b';\varepsilon_1 x, \gamma_{b'a}(x)g].$$

Hence $\bar{\varepsilon}_1$ is well-defined (its smoothness and Γ-equivariance are obvious).

(ii) If $x \in U_a$ and $[a;x,g] \in E_0^\gamma$, then

$$\bar{\varepsilon}_1(\eta_0 x,[a;x,g]) = [a;\varepsilon_1\eta_0 x,\gamma_{aa}(\eta_0 x)g] = [a;x,g].$$

(iii) If $x \in (\varepsilon_1{}^2)^{-1}U_a \cap (\varepsilon_1\varepsilon_0)^{-1}U_b \cap (\varepsilon_0{}^2)^{-1}U_c$ and $[c;\varepsilon_0\varepsilon_0 x,g] \in E_0^\gamma$, then

$$\bar{\varepsilon}_1(\varepsilon_1 x,[c;\varepsilon_0\varepsilon_0 x,g]) = [a;\varepsilon_1\varepsilon_1 x,\gamma_{ac}(\varepsilon_1 x)g]$$

whereas

$$\bar{\varepsilon}_1(\varepsilon_2 x,\bar{\varepsilon}_1(\varepsilon_0 x,[c;\varepsilon_0\varepsilon_0 x,g])) = \bar{\varepsilon}_1(\varepsilon_2 x,[b;\varepsilon_1\varepsilon_0 x,\gamma_{bc}(\varepsilon_0 x)g]) = [a;\varepsilon_1\varepsilon_2 x,\gamma_{ab}(\varepsilon_2 x)\gamma_{bc}(\varepsilon_0 x)g].$$

In view of the lemma, E_0^γ and $\bar{\varepsilon}_1$ give rise to an ss-Γ-bundle $E^\gamma \to X$.

Lemma 2.5.3. Every principal ss-Γ-bundle $E \to X$ is isomorphic to a bundle E^γ, where γ is a Γ-cocycle on X with respect to a suitably chosen covering $\mathcal{U}$ of X_0.

Proof. Consider any collection of sections $\sigma_a\colon X_0 \supset U_a \to E_0$ such that $\mathcal{U} = \{U_a\}$ covers X_0. If $x \in \varepsilon_1{}^{-1}U_a \cap \varepsilon_0{}^{-1}U_b$ then the elements $\sigma_a(\varepsilon_1 x)$, $\sigma_b(\varepsilon_0 x) \in E_0$ are defined. Since there are canonical isomorphisms

$$\varepsilon_1^* E_0 \xleftarrow{(\pi,\varepsilon_1)} E_1 \xrightarrow{(\pi,\varepsilon_0)} \varepsilon_0^* E_0$$

one can compare the inverse images $(\pi,\varepsilon_1)^{-1}(x,\sigma_a(\varepsilon_1 x))$ and $(\pi,\varepsilon_0)^{-1}(x,\sigma_b(\varepsilon_0 x))$ which are in the same fiber of E_1. Clearly, one has

$$(2.5.4)\qquad (\pi,\varepsilon_0)^{-1}(x,\sigma_b(\varepsilon_0 x)) = (\pi,\varepsilon_1)^{-1}(x,\sigma_a(\varepsilon_1 x))\cdot\gamma_{ab}(x)$$

for a $\gamma_{ab}(x) \in \Gamma$. We claim that the maps $\gamma_{ab}\colon \varepsilon_1{}^{-1}U_a \cap \varepsilon_0{}^{-1}U_b \to \Gamma$ form the desired cocycle and that the cocycle condition (2.5.1) is just an alternative description of the commutativity observed in II.3.3. Indeed, any element x of $(\varepsilon_1{}^2)^{-1}U_a \cap (\varepsilon_1\varepsilon_0)^{-1}U_b \cap (\varepsilon_0{}^2)^{-1}U_c$ determines three elements of E_0: $\sigma_a(\varepsilon_1\varepsilon_1 x)$, $\sigma_b(\varepsilon_1\varepsilon_0 x)$, and $\sigma_c(\varepsilon_0\varepsilon_0 x)$, six elements of E_1:

$(\pi,\varepsilon_0)^{-1}(\varepsilon_0 x,\sigma_c(\varepsilon_0\varepsilon_0 x))$, $(\pi,\varepsilon_1)^{-1}(\varepsilon_0 x,\sigma_b(\varepsilon_1\varepsilon_0 x))$, $(\pi,\varepsilon_0)^{-1}(\varepsilon_1 x,\sigma_c(\varepsilon_0\varepsilon_1 x))$

etc, and three of E_2: $(\pi,\varepsilon_1\varepsilon_1)^{-1}(x,\sigma_a(\varepsilon_1\varepsilon_1 x))$, etc. Starting from the equality (in E_1)

$$(\pi,\varepsilon_1)^{-1}(\varepsilon_1 x,\sigma_a(\varepsilon_1\varepsilon_1 x))\gamma_{ac}(\varepsilon_1 x) = (\pi,\varepsilon_0)^{-1}(\varepsilon_1 x,\sigma_c(\varepsilon_0\varepsilon_1 x))$$

we get in E_2 (cf II, diagram (3.3.1))

$$(\pi,\varepsilon_1\varepsilon_1)^{-1}(x,\sigma_a(\varepsilon_1\varepsilon_1 x))\gamma_{ac}(\varepsilon_1 x) =$$
$$= (\pi,\varepsilon_1)^{-1}(x,(\pi,\varepsilon_1)^{-1}(\varepsilon_1 x,\sigma_a(\varepsilon_1\varepsilon_1 x))\gamma_{ac}(\varepsilon_1 x))$$
$$= (\pi,\varepsilon_1)^{-1}(x,(\pi,\varepsilon_0)^{-1}(\varepsilon_1 x,\sigma_c(\varepsilon_0\varepsilon_1 x)))$$
$$= (\pi,\varepsilon_0\varepsilon_1)^{-1}(x,\sigma_c(\varepsilon_0\varepsilon_1 x)).$$

In a similar way, we check the equalities

$$(\pi,\varepsilon_1\varepsilon_2)^{-1}(x,\sigma_a(\varepsilon_1\varepsilon_2 x))\gamma_{ab}(\varepsilon_2 x) = (\pi,\varepsilon_0\varepsilon_2)^{-1}(x,\sigma_b(\varepsilon_0\varepsilon_0 x))$$

and

$$(\pi,\varepsilon_1\varepsilon_0)^{-1}(x,\sigma_b(\varepsilon_1\varepsilon_0 x))\gamma_{bc}(\varepsilon_0 x) = (\pi,\varepsilon_0\varepsilon_0)^{-1}(x,\sigma_c(\varepsilon_0\varepsilon_0 x)).$$

Since $\varepsilon_0\varepsilon_0 = \varepsilon_0\varepsilon_1$, $\varepsilon_1\varepsilon_0 = \varepsilon_0\varepsilon_2$ and $\varepsilon_1\varepsilon_1 = \varepsilon_1\varepsilon_2$, this implies

$$\gamma_{ab}(\varepsilon_2 x)\gamma_{bc}(\varepsilon_0 x) = \gamma_{ac}(\varepsilon_1 x).$$

It remains to indicate an isomorphism between E and the reconstructed ss-Γ-bundle E^γ. Clearly, when applied to $\eta_0 x$, $x \in U_a \cap U_b$, the equality (2.5.4) reduces to

$$\sigma_b(x) = \sigma_a(x)\gamma_{ab}(\eta_0 x)$$

what means that $\{\gamma_{ab}\circ\eta_0\}$ is a cocycle describing E_0. In other words, the map

$$E_0^\gamma \ni [a;x,g] \xrightarrow{I_0} \sigma_a(x)g \in E_0$$

is a well-defined isomorphism. The only extension of I_0 to an ss-map $I\colon E^\gamma \to E$ must be of the form $I = (I_n)$, where I_n is the isomorphism

$$E_n^\gamma = X_n \times_{(\varepsilon_0^n,\pi)} E_0 \xrightarrow{id\times I_0} X_n \times_{(\varepsilon_0^n,\pi)} E_0 \xrightarrow{(\pi,\varepsilon_0^n)^{-1}} E_n,\ n=1,2,\ldots$$

and by corollary II.3.6, it sufficies to check the commutativity relatio[n] $I_0\bar{\varepsilon}_1 = \varepsilon_1 I_1$. For $x \in \varepsilon_1^{-1}U_a \cap \varepsilon_0^{-1}U_b$ and $(x,[b;\varepsilon_0 x,g]) \in E_1$, one has

$$\varepsilon_1 I_1(x,[b;\varepsilon_0 x,g]) = \varepsilon_1(\pi,\varepsilon_0)^{-1}(x,\sigma_b(\varepsilon_0 x)g)$$
$$= \varepsilon_1(\pi,\varepsilon_1)^{-1}(x,\sigma_a(\varepsilon_1 x))\gamma_{ab}(x)g$$
$$= \sigma_a(\varepsilon_1 x)\gamma_{ab}(x)g$$
$$= I_0[a;\varepsilon_1 x,\gamma_{ab}(x)g]$$
$$= I_0\bar{\varepsilon}_1(x,[b;\varepsilon_0 x,g])$$

as was to be shown.

2.6 Now we turn back to our assertion that the restrictions to $T^0 \sqcup T^1 \hookrightarrow X_0$ of of the submersions distinguished by E_0 generate a morphism $G_{F,T^0\sqcup T^1} \longrightarrow G$. The fact that the restrictions form a morphism $G_{F_0,T^0\sqcup T^1} \longrightarrow G$ is a part of the minimality property in the non-semisimplicial case (I;2.9) and follows from the relation I(2.4.1). In order to prove that the morphism can be extended to $G_{F,T^0\sqcup T^1}$ we consider any Γ-cocycle $\{\gamma_{ab}\}$, $\gamma_{ab}: X_1 \supset \varepsilon_1^{-1}U_a \cap \varepsilon_0^{-1}U_b \to \Gamma$ associated with the examined ss-Γ-structure. If an arbitrary but fixed point t of $T \hookrightarrow X_1$ is in $\varepsilon_1^{-1}U_a \cap \varepsilon_0^{-1}U_b$, then clearly

$$\eta_0 t \in (\varepsilon_1{}^2)^{-1}U_a \cap (\varepsilon_1\varepsilon_0)^{-1}U_a \cap (\varepsilon_0{}^2)^{-1}U_b$$

and

$$\eta_1 t \in (\varepsilon_1{}^2)^{-1}U_a \cap (\varepsilon_1\varepsilon_0)^{-1}U_b \cap (\varepsilon_0{}^2)^{-1}U_b.$$

Consequently, the cocycle condition (2.5.1) gives

$$\gamma_{aa}(\varepsilon_2\eta_0 t)\gamma_{ab}(t) = \gamma_{ab}(t)$$

and

$$\gamma_{ab}(t)\gamma_{bb}(\varepsilon_0\eta_1 t) = \gamma_{ab}(t),$$

and it follows that

(2.6.1) $\gamma_{ab}(t) \in \Gamma$ is a germ of the locally defined diffeomorphism $(\gamma_{aa}\eta_0\varepsilon_1|T)\circ(\gamma_{bb}\eta_0\varepsilon_0|T)^{-1}$.

In other words, the diffeomorphism, which is precisely the map $(\gamma_{aa}\eta_0|T^1)\circ id_1^0\circ(\gamma_{bb}\eta_0|T^0)^{-1}$ (see figure 4)

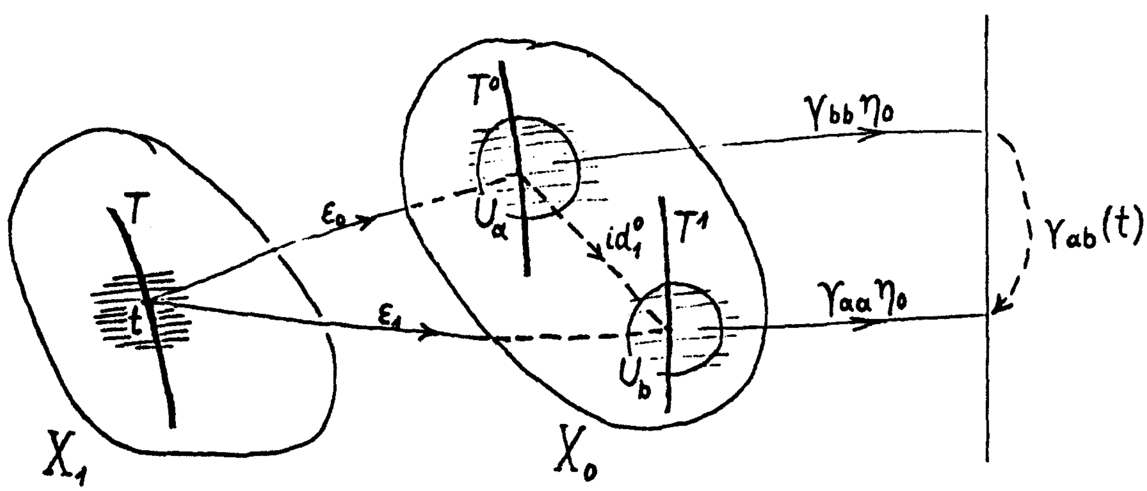


Figure 4.

belongs to G, the pseudogroup underlying Γ. This concludes our assertion; let $\Psi' = \Psi'_T: G_{F,T^0\sqcup T^1} \longrightarrow G$ denote the resulted morphism of pseudogroups.

2.6.2 We have to find an isomorphism between $E = (E_n)$ and the ss-Γ-bundle

$$\Psi^!_*E'_{F,T} = (\Psi^!_*E_{F,0} \times_{(\pi,\varepsilon_0 n)} X_n)$$

(cf 2.3; we use the convention of I;1.8 for left principal Γ-bundles).

At the zero level, one has

$$\Psi^!_*E_{F,0} = |\Psi'| \times_{\Gamma_{F,T^0\sqcup T^1}} (\Gamma_{F,T^0\sqcup T^1} \times_{\Gamma_{F_0,T^0\sqcup T^1}} E_{F_0,T^0\sqcup T^1})$$

$$\cong |\Psi'| \times_{\Gamma_{F_0,T^0\sqcup T^1}} E_{F_0,T^0\sqcup T^1}$$

(with the multiplicative notation). We identify E_0 with the corresponding Γ-bundle of germs of the distinguished submersions (the canonical form of E_0, cf I;1.11) and define $J_0: E_0 \to \Psi^!_*E_{F,0}$ to be the map

(2.6.3) $\quad [\phi,x] \to [\psi|T^0 \sqcup T^1, c(1)]\cdot[H_c,x]$

where $[\phi,x]$ is the germ at $x \in X_0$ of a submersion ϕ, c is any path connecting x to a point of $T^0 \sqcup T^1$ in a leaf of F_0, and $[\psi|T^0 \sqcup T^1,c(1)] = h_c([\phi,x])|T^0 \sqcup T^1$ is the image of $[\phi,x] \in E_0$ under the holonomy translation h_c (hence a germ $[\psi,c(1)]$ of a distinguished submersion) restricted to the transversal. If $\bar{c}$ is another path from x to $T^0 \sqcup T^1$, then

$$[H_{\bar{c}},x] = h_{c^{-1}*\bar{c},T^0\sqcup T^1}[H_c,x]$$

and

$$h_{\bar{c}}([\phi,x])|T^0 \sqcup T^1 = h_{c^{-1}*\bar{c}}(h_c[\phi,x])|T^0 \sqcup T^1$$

$$= (h_c[\phi,x]|T^0 \sqcup T^1)\cdot h^{-1}_{c^{-1}*\bar{c},T^0\sqcup T^1}$$

by I(2.4.1). This proves correctness of the definition of J_0. Smoothness of that map follows from the same argument as the one applied in 2.3. Now J_0 is a Γ-equivariant map (because h_c acts Γ-equivariantly) inducing the identity map on X_0, hence an isomorphism. In view of corollary II;3.6, J_0 can be extended to an isomorphism $J: E \to \Psi^!_*E'_{F,T}$ if the commutativity relation $\bar{\varepsilon}_1 J_1 = J_0\varepsilon_1: E_1 \to \Psi^!_*E_{F,0}$ holds, J_1 standing for the map

$$E_1 \cong E_0 \times_{(\pi,\varepsilon_0)} X_1 \xrightarrow{J_0\times id} \Psi^!_*E_{F,0} \times_{(\pi,\varepsilon_0)} X_1 \cong \Psi^!_*E_{F,1}.$$

We fix an arbitrary element e of E_1 and a path c connecting $x = \pi e$ to a point of $T \hookrightarrow X_1$ in a leaf of F_1; then $\varepsilon_i c$ connects $\varepsilon_i x$ to $T^i \hookrightarrow X_0$, for $i = 0,1$. By definition

$$J_0\varepsilon_1 e = (h_{\varepsilon_1 c}(\varepsilon_1 e)|T^1)\cdot[H_{\varepsilon_1 c},\varepsilon_1 x]$$

whereas (cf 2.3.2)

$$\bar{\varepsilon}_1 J_1 e = (h_{\varepsilon_0 c}(\varepsilon_0 e)|T^0)\ \mathrm{id}_0^1[H_{\varepsilon_1 c},\varepsilon_1 x];$$

it remains to show the equality

$$(2.6.4)\qquad (h_{\varepsilon_0 c}(\varepsilon_0 e)|T^0)\ \mathrm{id}_0^1 = h_{\varepsilon_1 c}(\varepsilon_1 e)|T^1.$$

To do that, we once more take advantage of lemma 2.5.3 (modified suitably for left Γ-bundles). So we fix a covering $\{U_a;\ a\in A\}$ of X_0 and a Γ-cocycle $\{\gamma_{ab}\}$, $\gamma_{ab}\colon X_1 \supset \varepsilon_1^{-1}U_a \cap \varepsilon_0^{-1}U_b \to \Gamma$ associated with the examined ss-Γ-structure. Since the submersions $\gamma_{aa}\eta_0$ are distinguished for E_0 and since $E_1 \cong \varepsilon_i{}^*E_0$ for $i = 0,1$, we see that among the distinguished submersions for E_1 there are $\gamma_{aa}\eta_0\varepsilon_0$ as well as $\gamma_{aa}\eta_0\varepsilon_1$, $a \in A$. If $x = \pi e \in \varepsilon_1^{-1}U_a \cap \varepsilon_0^{-1}U_b$ then, after the identification of E_1 with its canonical form (cf I;1.11), one may write

$$e = g[\gamma_{bb}\eta_0\varepsilon_0,x] = g'[\gamma_{aa}\eta_0\varepsilon_1,x]$$

for some $g, g' \in \Gamma$, and the double description of e implies

$$\varepsilon_0 e = g[\gamma_{bb}\eta_0,\varepsilon_0 x]$$

and

$$\varepsilon_1 e = g'[\gamma_{aa}\eta_0,\varepsilon_1 x]$$

(cf I;1.12). In order to find a relationship between g and g' we apply the canonical isomorphism I(1.11.1) to the equality

$$(\varepsilon_0,\pi)^{-1}(\sigma_b(\varepsilon_0 x),x) = \gamma_{ab}(x)^{-1}(\varepsilon_1,\pi)^{-1}(\sigma_a(\varepsilon_1 x),x)$$

obtained from (2.5.4) for left principal bundles. The resulted equality

$$[\gamma_{bb}\eta_0\varepsilon_0,x] = \gamma_{ab}(x)^{-1}[\gamma_{aa}\eta_0\varepsilon_1,x]$$

implies $g = g'\gamma_{ab}(x)$. Consequently, (2.6.4) reduces to

$$(2.6.5)\qquad \gamma_{ab}(x)\cdot(h_{\varepsilon_0 c}[\gamma_{bb}\eta_0,\varepsilon_0 x]|T^0)\,\mathrm{id}_0^1 = h_{\varepsilon_1 c}[\gamma_{aa}\eta_0,\varepsilon_1 x]|T^1.$$

Now we take any local transversal $S \hookrightarrow X_1$ for F_1 at x and apply the relation I(2.4.1). It follows that the left-hand side of (2.6.5) is a germ of the map

$$\gamma_{ab}^{(x)}\circ(\gamma_{bb}\eta_0|\varepsilon_0 S)\circ(h_{\varepsilon_0 c,T^0}^{\varepsilon_0 S})^{-1}\circ\mathrm{id}_0^1 = \gamma_{ab}^{(x)}\gamma_{bb}\eta_0(\varepsilon_0|S)(h_{c,T}^{S})^{-1}\circ\mathrm{id}^1$$

whereas the right-hand side -- of the map

$$(\gamma_{aa}\eta_0|\varepsilon_1 S)\circ(h_{\varepsilon_1 c,T^1}^{\varepsilon_1 S})^{-1} = \gamma_{aa}\eta_0(\varepsilon_1|S)(h_{c,T}^{S})^{-1}\circ id^1$$

where $\gamma_{ab}^{(x)} \in G$ is a diffeomorphism representing $\gamma_{ab}(x)$, and id^1 is the identification map $T^1 \to T$. According to (2.6.1) the two maps yield the same germ.

To conclude our proof of theorem 2.2(i) and complete the constructior we define the holonomy morphism $\Psi = \Psi_T^{\omega}: G_{F,T} \longrightarrow G$ to be the composition of $\Psi': G_{F,T^0\sqcup T^1} \longrightarrow G$ and the canonical equivalence $\Phi_T: G_{F,T} \longrightarrow G_{F,T^0\sqcup T^1}$; by the equality

$$\Psi_*\omega_{F,T} = (\Psi'_*\Phi_{T*})\Phi^T{}_*\omega'_{F,T} = \Psi'_*\omega'_{F,T} = \omega$$

we are done.

<u>2.7</u> We come to the proof of theorem 2.2(ii) now. By the above construction, the morphism

$$\Phi' = \Psi_T^{\omega'_{F,\overline{T}}}: G_{F,T} \longrightarrow G_{F,\overline{T}^0\sqcup\overline{T}^1}$$

is the composition of the canonical equivalence $G_{F,T} \longrightarrow G_{F,T^0\sqcup T^1}$ and the morphism $\Psi': G_{F,T^0\sqcup T^1} \longrightarrow G_{F,\overline{T}^0\sqcup\overline{T}^1}$ generated by restrictions to $T^0 \sqcup T^1$ of the distinguished submersions $H_{c,\overline{T}^0\sqcup\overline{T}^1}$ for $\omega'_{F,\overline{T}}$. Since the restriction of $H_{c,\overline{T}^0\sqcup\overline{T}^1}$ to the transversal $T^0 \sqcup T^1$ is a holonomy translation along c, we see that Ψ' as well as Φ' are both the canonical equivalences. To and the proof of (ii) it sufficies to show that

$$\Psi_T^{\omega_{F,\overline{T}}}: G_{F,T} \longrightarrow G_{F,\overline{T}}$$

is the composition of Φ' and the equivalence $\Phi^{\overline{T}}: G_{F,\overline{T}^0\sqcup\overline{T}^1} \longrightarrow G_{F,\overline{T}}$. This will clearly follow from theorem 2.2(iii) (cf 2.8 below) as $\omega_{F,\overline{T}}$ is, by the construction, equal to $\Phi^{\overline{T}}{}_*\omega'_{F,\overline{T}}$.

<u>2.8</u> The last step is a proof of theorem 2.2(iii). In fact (cf lemma 1.1.1(iv)), it sufficies to prove commutativity of the modified diagram

$$\begin{array}{ccc} G_{F,T^0\sqcup T^1} & \xrightarrow{\Psi_T^{\omega}\circ\Phi_T^{-1}} & G \\ \approx\big\downarrow\Phi & & \big\downarrow\Lambda \\ G_{F,\overline{T}^0\sqcup\overline{T}^1} & \xrightarrow{\Psi_{\overline{T}}^{\Lambda_*\omega}\circ\Phi_{\overline{T}}^{-1}} & G' \end{array}$$

Φ standing for the canonical equivalence. By the construction 2.4 (and 2.6) $\Lambda\circ(\Psi_T^{\omega}\circ\Phi_T^{-1})$ is the morphism generated by the compositions $\lambda\circ(\varphi|T^0\sqcup T^1)$ where $\lambda \in \Lambda$, and φ ranges over the distinguished submersions for E_0. One has to compare the above collection of diffeomorphisms with the one generating the morphism $\Psi\frac{\Lambda}{T}*^{\omega} \circ \Phi$ and composed of the superpositions $(\lambda\circ\varphi|\overline{T}^0\sqcup\overline{T}^1)\circ h$, where λ and φ are as above (cf II;3.12-3.12.1) and h ranges over all the holonomy translations $h_{c,\overline{T}^0\sqcup\overline{T}^1}^{T^0\sqcup T^1}$. We shall show that the second collection is contained by the first one.

Let c be an arbitrary path connecting $T^0 \sqcup T^1$ to $\overline{T}^0 \sqcup \overline{T}^1$ in a leaf of F_0, and φ any distinguished submersion over a neighbourhood of $c(1)$. By lemma I.2.4, one has (in the canonical form $\widetilde{E}$ of E_0)

$$[(\varphi|\overline{T}^0 \sqcup \overline{T}^1)\circ h_{c,\overline{T}^0\sqcup\overline{T}^1}^{T^0\sqcup T^1}, c(0)] = (h_{c^{-1}}[\varphi,c(1)])|T^0 \sqcup T^1$$
$$= [\psi|T^0 \sqcup T^1, c(0)]$$

where $h_{c^{-1}}$ is the holonomy translation and ψ is a distinguished submersion representing the germ $h_{c^{-1}}[\varphi,c(1)] \in \widetilde{E}$. Consequently, for any $\lambda \in \Lambda$, one has

$$(\lambda\circ\varphi|\overline{T}^0\sqcup\overline{T}^1)\circ h_{c,\overline{T}^0\sqcup\overline{T}^1}^{T^0\sqcup T^1} = \lambda\circ(\psi|T^0\sqcup T^1) \quad \text{over a nbhd of } c(0),$$

and thus the morphism $G_{F,T^0\sqcup T^1} \longrightarrow G$ generated by the first collection is the only one containing the second.

This concludes our proof of theorem 2.2.

According to theorem 2.2, foliations of ss-manifolds give rise to some pseudogroups of diffeomorphisms, canonical ss-Γ-structures, morphisms, etc. We now "compute" those objects in several particular cases.

<u>Example</u> 3.1. $X = NM$ is the trivial ss-manifold (cf II; example 1.3.1) endowed with a foliation F_0 of M; the foliation extends trivially to a foliation F of X. If $T \hookrightarrow X_1 = M$ is any complete transversal for F, then the pseudogroup $G_{F_0,T^0\sqcup T^1}$ (cf 1.1) contains the identification map $T^0 \to T^1$ (as there is $\varepsilon_1 = \varepsilon_0$ on X_1); hence the holonomy pseudogroups of F and of F_0 are the same

$$G_{F,T^0\sqcup T^1} = G_{F_0,T^0\sqcup T^1}, \text{ and } G_{F,T} = G_{F_0,T}.$$

Furthermore (and more precisely), it follows from the construction that also the canonical equivalences are the same. Similarly, a principal ss-Γ-bundle over X is nothing more than a Γ-bundle over M, and it can be easily verified that the canonical $\Gamma_{F,T}$-structure on X as well as the holonomy morphisms $G_{F,T} \longrightarrow G$ (cf theorem 2.2(i)) are exactly the same as those corresponding to M and F_0.

<u>Example</u> 3.2. $X = N\Gamma$ is the nerve of an arbitrary groupoid of germs Γ; F is the pointwise foliation of X. Let N denote the manifold of

units of Γ and G the underlying pseudogroup of diffeomorphisms of N. We take $T = \Gamma = X_1$ as a complete transversal for F. Since leaves of F_0 are just points of $X_0 = N$, the holonomy translations of $T^0 \sqcup T^1$ are all of the form $\alpha^{-1}\alpha$, $\alpha^{-1}\beta$, $\beta^{-1}\alpha$, or $\beta^{-1}\beta$ and correspond to those parts of Γ where the source and target maps overlap. Precisely, the pseudogroup $G_{F_0,T^0\sqcup T^1}$ consists of the diffeomorphisms

$$[\gamma_0,x]^0 \to [\gamma_1,x]^0 \ , \quad x \in \text{domain } \gamma_0 \cap \text{domain } \gamma_1,$$

$$[\gamma_0,x]^0 \to [\gamma_1,\gamma_1^{-1}(x)]^1 \ , \quad x \in \text{domain } \gamma_0 \cap \text{image } \gamma_1,$$

etc, where γ_0 and γ_1 range over elements of G and the upper indices indicate the appropriate copy of $T = \Gamma$. The non-reduced holonomy pseudogroup $G_{F,T^0\sqcup T^1}$ is generated by $G_{F_0,T^0\sqcup T^1}$ and the identification

$$[\gamma,x]^0 \to [\gamma,x]^1 \ .$$

We apply theorem 2.2(i) to the Γ-structure ω_Γ over X (cf II.3.11) which clearly defines the pointwise foliation; the resulted holonomy morphism Φ: $G_{F,T^0\sqcup T^1} \longrightarrow G$ is generated by the identity $N \to N$ restricted to the transversal, ie by those restrictions of the projections $\alpha \sqcup \beta$: $\Gamma \sqcup \Gamma \to N$ which are diffeomorphisms.

<u>Proposition</u> 3.2.1. (i) Φ: $G_{F,T^0\sqcup T^1} \longrightarrow G$ is an equivalence.
(ii) The canonical $\Gamma_{F,T^0\sqcup T^1}$-structure $\omega^!_{F,T}$ (cf 2.3) on $X = N\Gamma$ is equal to $\omega_{|\Phi^{-1}|}$, the one corresponding to the morphism of groupoids $|\Phi^{-1}|$: $\Gamma \longrightarrow \Gamma_{F,T^0\sqcup T^1}$ (cf II;3.7).
(iii) If ω is any ss-Γ'-structure defining F (Γ' - a groupoid of germs) then the holonomy morphism Ψ': $G_{F,T^0\sqcup T^1} \longrightarrow G'$ is the composition $\Psi\circ\Phi$, where Ψ: $G \longrightarrow G'$ is the only morphism of the pseudogroups underlying Γ and Γ' such that $\omega = \omega_{|\Psi|}$.

<u>Proof</u>. (i) By [10], it sufficies to show that the inverses

$$N \supset \text{domain } \varphi \ni x \xrightarrow{\varphi^0} [\varphi,x]^0 \in T^0 \ , \quad \varphi \in G$$

and

$$N \supset \text{image } \varphi \ni x \xrightarrow{\varphi^1} [\varphi,\varphi^{-1}(x)]^1 \in T^1 \ , \ \varphi \in G$$

of the projections generating Φ again generate a morphism $G \longrightarrow G_{F,T^0\sqcup T^1}$. We verify the condition (ii) of I;2.6; one has

$$\varphi^1\circ \gamma \circ(\psi^0)^{-1}: [\psi,x]^0 \to x \to \gamma(x) \to [\varphi,\varphi^{-1}\gamma(x)]^1 \quad \text{for } \gamma \in G$$

which is exactly the composition

$$[\psi,x]^0 \to [\gamma,x]^0 \to [\gamma,x]^1 \to [\varphi,\varphi^{-1}\gamma(x)]^1$$

of elements of $G_{F,T^0\sqcup T^1}$; clearly, the same holds for the maps $\varphi^i \circ \gamma \circ(\psi^i)^{-1}$, $i = 0,1$.

(ii) By theorem 2.2(i), $\omega_\Gamma = \Phi_*\omega^!_{F,T}$; on applying II;3.11, one immediately gets

$$\omega^!_{F,T} = (\Phi^{-1})_*\Phi_*\omega^!_{F,T} = (\Phi^{-1})_*\omega_\Gamma = \omega_{|\Phi^{-1}|\circ\Gamma} = \omega_{|\Phi^{-1}|}.$$

(iii) It follows from II.3.7 that there is $\omega = \omega_\Sigma$ for a morphism $\Sigma: \Gamma \longrightarrow \Gamma'$. ω defines the foliation F iff the target map β of the (left) principal Γ'-bundle Σ to the units of Γ' is locally a diffeomorpphism; hence the only morphism of pseudogroups $\Psi: G \longrightarrow G'$ which underlies Σ (if any; cf II;3.12) must be generated by sections of $\pi: \Sigma \to N$ followed by β. According to theorem 2.2(i) and to part (ii) already proved,

$$\omega = \Psi'_*\omega^!_{F,T} = \omega_{|\Psi'\circ\Phi^{-1}|}$$

and we are done.

Remark. We have actually shown (cf the proof of (i) above) that the holonomy pseudogroup $G_{F,T}$ consists of all the diffeomorphisms of the form $[\varphi,x] \to [\psi,\gamma(x)]$, for φ, ψ and $\gamma \in G$.

Example 3.3 Flags. Let (F,F') be an arbitrary flag of foliations of a manifold M (cf II; example 2.4.3). We recall that, given any transversals $T, T' \hookrightarrow M$ for F and F', respectively, there is a canonical morphism $\Sigma = \Sigma_{T',T}: \Gamma_{F,T} \longrightarrow \Gamma_{F',T'}$; when considered as a $\Gamma_{F',T'}$-bundle over T, Σ consists of all the holonomy translations h'_c (prime " ' " means: with respect to F') along paths in leaves of F', such that $c(0) \in T$ and $c(1) \in T'$. Note that in terms of the graphs (cf II; remark 3.9.2) the corresponding morphism $\Gamma_F \longrightarrow \Gamma_{F'}$ comes from a smooth map (functor) $h_c \to h'_c$ and thus is represented by $\Gamma_{F'}$ with the left Γ_F-action $(h_{c_1},h'_{c_2}) \to h'_{c_1}h'_{c_2}$.

We fix a complete transversal $\iota: T \hookrightarrow M$ for F. As mentioned in II;2.4.3, the foliation $F'|T = \iota^*F'$ of T extends to a foliation F'_T of the ss-manifold $N\Gamma_{F,T}$. Now we choose a complete transversal $\overline{T}$ for F'_T in a special way described below. Let $\{V_a;\ a \in A\}$ be a fixed open covering of $\Gamma_{F,T}$ such that for every a the restrictions $\alpha|V_a$ and $\beta|V_a$ be diffeomorphisms. If φ_a denotes the composition $\beta\circ(\alpha|V_a)^{-1}: \alpha V_a \to \beta V_a$, then there is $\varphi_a \in G_{F,T}$ and $V_a = \{[\varphi_a,t];\ t\in\alpha V_a\}$ for $a \in A$. In every set αV_a we fix a complete transversal

$$\iota_a: T_a \hookrightarrow \alpha V_a \subset T;$$

then $\overline{T} := \bigsqcup T_a$ immersed in $\Gamma_{F,T}$ with the help of the map

$$\overline{\iota} = \bigsqcup(\alpha|V_a)^{-1}\iota_a$$

is a complete transversal for F'_T. Obviously, being complete transversals for $F'|T$, both $\bar{T}^0$ and $\bar{T}^1$ are also complete transversals for F'

Proposition 3.3.1. (i) Given any complete transversal $T' \hookrightarrow M$ for F', the $\Gamma_{F',T'}$-structure ω_Σ on $N\Gamma_{F,T}$ that corresponds to the canonical morphism $\Sigma = \Sigma_{T',T}: \Gamma_{F,T} \rightharpoonup \Gamma_{F',T'}$ defines on the nerve $N\Gamma_{F,T}$ the foliation F'_T. Furthermore, the corresponding holonomy morphism with respect to any complete transversal $S \hookrightarrow \Gamma_{F,T}$ for F'_T,

$$\Psi: G_{F'_T,S} \rightharpoonup G_{F',T'}$$

is an equivalence of pseudogroups; if $\tilde{S}$ and $\tilde{T}'$ are any other complete transversals for F'_T and F', then the square

$$\begin{array}{ccc} G_{F'_T,S} & \longrightarrow & G_{F',T'} \\ \downarrow & & \downarrow \\ G_{F'_T,\tilde{S}} & \longrightarrow & G_{F',\tilde{T}'} \end{array}$$

where the vertical half-arrows are the canonical equivalences, commutes.

(ii) With respect to the specially chosen complete transversals, the holonomy morphism

$$G_{F'_T,\bar{T}} \longrightarrow G_{F',\bar{T}^0}$$

is the identity, and the canonical $\Gamma_{F'_T,\bar{T}}$-structure on $N\Gamma_{F,T}$ is the one corresponding (cf II;3.4.1) to the morphism $\Sigma_{\bar{T}^0,T}: \Gamma_{F,T} \rightharpoonup \Gamma_{F',\bar{T}^0}$.

Proof. In order to verify the first assertion of (i), we consider an arbitrary continuous section of Σ. Since such a section can be obtained by taking the locally defined projection in T onto a local transversal for the foliation $F'|T$ (along leaves of $F'|T$) followed by a holonomy translation to T' (see figure 5), we conclude that Σ defines the foliation $F'|T$.

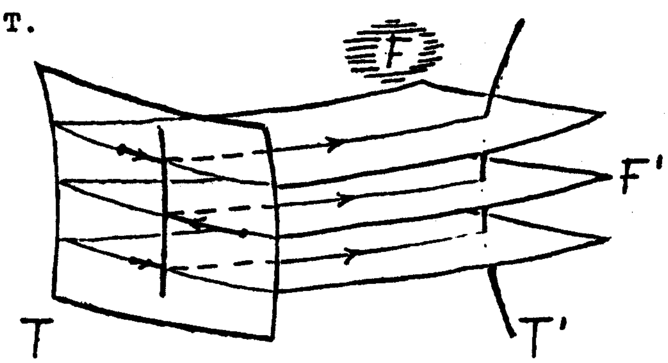


Figure 5

By theorem 2.2(iii), invertibility of the holonomy morphism Ψ (in general) follows from the particular case (ii), whereas commutativity of the square is guaranteed by theorem 2.2(iii) and naturality of the morphism $\Sigma\colon \Gamma_{F,T} \longrightarrow \Gamma_{F',T'}$ with respect to T'. Furthermore, in view of theorem 2.2(i) the second assertion of (ii) is a straightforward consequence of the first one. Thus it remains to examine the holonomy morphism $\Phi\colon G_{F'_T,\overline{T}} \longrightarrow G_{F',\overline{T}^0}$. Clearly,

$$G_{F',\overline{T}^0} = \{\gamma \in G_{F',\overline{T}^0\sqcup\overline{T}^1};\ \text{domain } \gamma,\ \text{image } \gamma \subset \overline{T}^0\}$$

and we want to reduce the assertion to the non-reduced case of the holonomy morphism

$$\Phi'\colon G_{F'_T,\overline{T}^0\sqcup\overline{T}^1} \longrightarrow G_{F',\overline{T}^0\sqcup\overline{T}^1}.$$

First of all, we shall show that Φ' is the identity morphism. By construction, $\overline{T}^0$ and $\overline{T}^1$ coincide, respectively, with the transversals

$$\bigsqcup i_a\colon \bigsqcup T_a \hookrightarrow T$$

and

$$\bigsqcup \varphi_a i_a\colon \bigsqcup T_a \hookrightarrow T.$$

Consequently, the subpseudogroup

$$G_{F'|T,\overline{T}^0\sqcup\overline{T}^1} \subset G_{F'_T,\overline{T}^0\sqcup\overline{T}^1}$$

consists of all the holonomy translations $T_a \to T_b$, $T_a \leftrightarrows \varphi_b T_b$, and $\varphi_a T_a \to \varphi_b T_b$ along paths in leaves of $F'|T$, whereas the identification $\overline{T}^0 \to \overline{T}^1$ adds all the holonomy translations along paths in leaves of F. In particular, one has

$$G_{F'_T,\overline{T}^0\sqcup\overline{T}^1} \subset G_{F',\overline{T}^0\sqcup\overline{T}^1}$$

and we claim that the holonomy morphism Φ' is generated by (ie contains) the identity map on $\overline{T}^0 \sqcup \overline{T}^1$. Indeed, this is so, since among the distinguished submersions for Σ there are locally defined projections of T on $\overline{T}^0 \sqcup \overline{T}^1$ along leaves of $F'|T$; when restricted to $\overline{T}^0 \sqcup \overline{T}^1$, they reduce to the identity maps. Now we prove that actually the equality of the pseudogroups takes place. Namely, we fix an arbitrary path $c\colon [0,1] \to M$ connecting two points of $\overline{T}^0\sqcup\overline{T}^1 \hookrightarrow T \hookrightarrow M$ in a leaf L of F' and denote by Ω the set of all $\tau \in [0,1]$ for which there are paths c_1 and c_2 in a leaf of, resp., $F'|T$ and F such that (see figure 6)

$$h'_{c,\overline{T}^0\sqcup\overline{T}^1} = h'_{c_1 * c_2 * c|[\tau,1],\overline{T}^0\sqcup\overline{T}^1} \cdot g \quad \text{for a } g \in \Gamma_{F'_T,\overline{T}^0\sqcup\overline{T}^1}.$$

Clearly, $0 \in \Omega$. Let us assume for the moment that $1 \in \Omega$. Then the

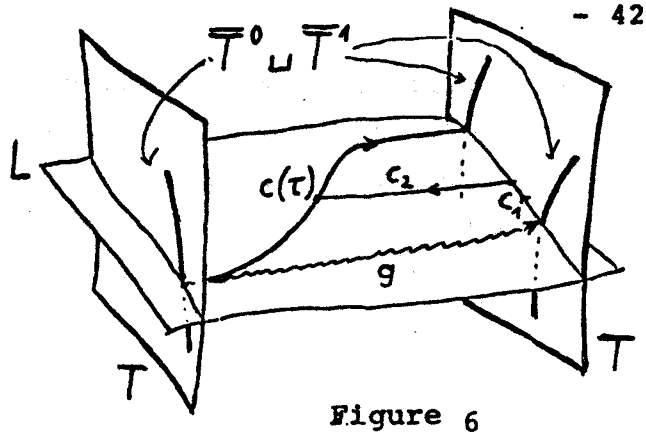


Figure 6

equality

$$h'_{c,\overline{T}^0\sqcup\overline{T}^1} = h'_{c_1 * c_2,\overline{T}^0\sqcup\overline{T}^1} g$$

holds for some c_1 and c_2 as above; clearly, $c_1(0) = \beta(g) \in \overline{T}^0 \sqcup \overline{T}^1$ and $c_2(1) = c(1) \in \overline{T}^0 \sqcup \overline{T}^1$. In order to show that

$$h'_{c_1 * c_2,\overline{T}^0\sqcup\overline{T}^1} \in \Gamma_{F'_T,\overline{T}^0\sqcup\overline{T}^1}$$

we consider the holonomy translation $h_{c_2,T} \in \Gamma_{F,\underline{T}}$ (observe that both ends of c_2 are in T). By the construction of $\overline{T} = \bigsqcup T_a$, there is an $a \in A$ such that $h_{c_2,T} \in V_a$, and a path $\overline{c}$ in $\alpha V_a \subset T$ connecting $c_2(0) = \alpha h_{c_2,T}$ to a point of T_a (in a leaf of $F'|T$). Consequently, $\varphi_a\overline{c}$ is a path in $\beta V_a \subset T$ connecting $c_2(1) = \beta h_{c_2,T}$ to a point of $\varphi_a T_a$, and we may write (see figure 7)

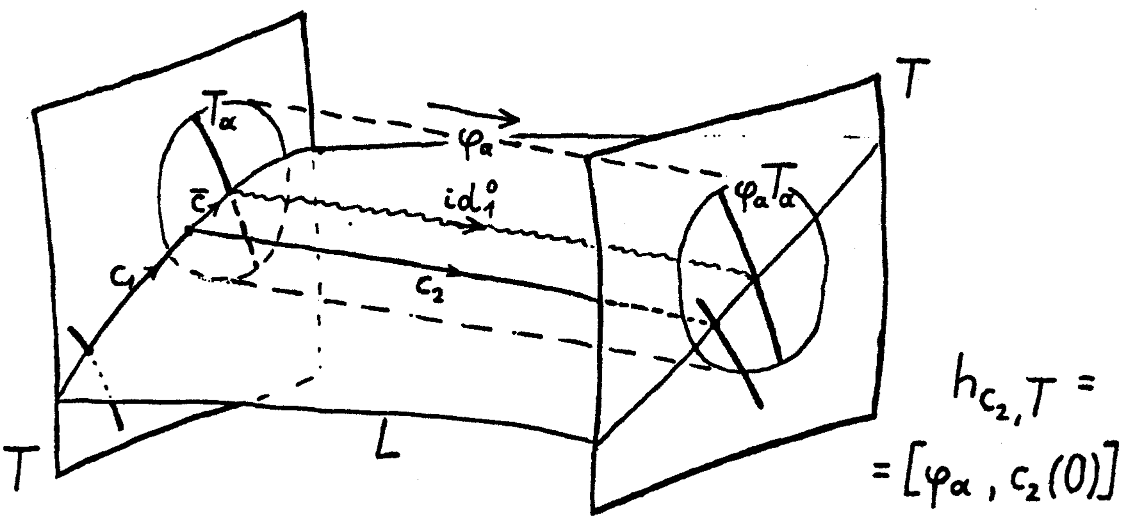


Figure 7

$$h'^{\,\overline{T}^0\sqcup\overline{T}^1}_{c_1 * c_2,\overline{T}^0\sqcup\overline{T}^1} = h'^{\,T_a}_{\overline{c}^{-1} * c_2,\overline{T}^0\sqcup\overline{T}^1} h'^{\,\overline{T}^0\sqcup\overline{T}^1}_{c_1 * \overline{c},T_a}$$

$$= h'^{\,\varphi_a T_a}_{\varphi_a\overline{c}^{-1},\overline{T}^0\sqcup\overline{T}^1} id_1^0 h'^{\,\overline{T}^0\sqcup\overline{T}^1}_{c_1 * \overline{c},T_a} \in G_{F'_T,\overline{T}^0\sqcup\overline{T}^1}$$

as was to be shown.

Continuing our proof of the proposition, we shall show now that the upper bound $\tau_0 = \sup \Omega$ is an element of Ω. Let c_0 be any path in a leaf of F connecting the point $c(\tau_0)$ to a point of T. In a neighbourhood U of $c(\tau_0)$ on which the submersion $H_{c_0,T}$ is defined, every point x can be connected to the point $H_{c_0,T}(x) \in T$ by a path close to c_0, contained in a leaf of F, and continuously depending on x. In particular, if a $\tau \in \Omega$ satisfies $c([\tau,\tau_0]) \subset U$, then there exists such a path $\tilde{c}_0$ connecting $c(\tau)$ to a point of T. Let a path c' in T be the projection of $c|[\tau,\tau_0]$ with respect to $H_{c_0,T}$, and

$$h'_{c,\overline{T}^0\sqcup\overline{T}^1} = h'_{c_1*c_2*c|[\tau,1],\overline{T}^0\sqcup\overline{T}^1}g$$

be a decomposition guaranteed by the property $\tau \in \Omega$. We consider the holonomy translation $h_{c_2*\tilde{c}_0,T} \in \Gamma_{F,T}$. Again, one has

$$h_{c_2*\tilde{c}_0,T} = [\varphi_a, c_2(0)] \in V_a$$

for an $a \in A$, and there is a path $\bar{c}$ in a leaf of $F'|\alpha V_a$ connecting $c_2(0)$ to a point of T_a (see figure 8)

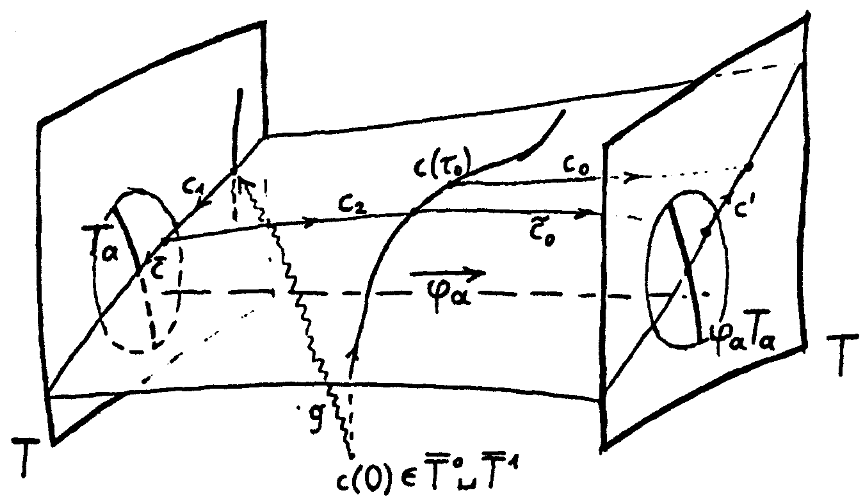


Figure 8

Since the path $c|[\tau,\tau_0]*c_0$ is homotopic to $\tilde{c}_0*c'$, we may write

$$h'^{\overline{T}^0\sqcup\overline{T}^1}_{c_1*c_2*c|[\tau,1],\overline{T}^0\sqcup\overline{T}^1} = h'^{\overline{T}^0\sqcup\overline{T}^1}_{c_1*c_2*\tilde{c}_0*c'*c_0^{-1}*c|[\tau_0,1],\overline{T}^0\sqcup\overline{T}^1}$$

$$= h'^{\varphi_aT_a}_{(\varphi_a\bar{c})^{-1}*c'*c_0^{-1}*c|[\tau_0,1],\overline{T}^0\sqcup\overline{T}^1}\; h'^{T_a}_{\bar{c}^{-1}*c_2*\tilde{c}_0*(\varphi_a\bar{c}),\varphi_aT_a}\; h'^{\overline{T}^0\sqcup\overline{T}^1}_{c_1*\bar{c},T_a}$$

$$= h'^{\varphi_aT_a}_{\bar{c}_1*\bar{c}_2*c|[\tau_0,1],\overline{T}^0\sqcup\overline{T}^1}\,\mathrm{id}^0_1\, h'^{\overline{T}^0\sqcup\overline{T}^1}_{c_1*\bar{c},T_a}$$

where $\bar{c}_1 = (\varphi_a\bar{c})^{-1}*c'$ and $\bar{c}_2 = c_0^{-1}$. Since $c_1*\bar{c}$ is in a leaf of $F'|T$, we are done. It remains to prove that τ_0 cannot be less than 1.

Indeed, if it was, then a decomposition

$$h'_{c,\overline{T}^0\sqcup\overline{T}^1} = h'_{c_1*c_2*c|[\tau_0,1],\overline{T}^0\sqcup\overline{T}^1}g$$

would immediately imply

$$h'_{c,\overline{T}^0\sqcup\overline{T}^1} = h'_{(c_1*\bar{c})*c_2'*c|[\tau,1],\overline{T}^0\sqcup\overline{T}^1}g$$

for each τ close enough to τ_0 (and $> \tau_0$), $\bar{c} := H_{c_2^{-1},T}(c|[\tau_0,\tau])$, and c_2' being a path close to c_2 and connecting $H_{c_2^{-1},T}(c(\tau))$ to $c(\tau)$ in a leaf of F. The resulted contradiction to the maximality of τ_0 proves that $\tau_0 = 1$.

We have come to the point that the holonomy morphism Φ': $G_{F_T',\overline{T}^0\sqcup\overline{T}^1} \longrightarrow G_{F',\overline{T}^0\sqcup\overline{T}^1}$ is the identity. In particular, the equality

$$G_{F_T',\overline{T}} = G_{F',\overline{T}^0}$$

holds, and it follows from the construction of the holonomy morphisms that the reduced holonomy morphism

$$\Phi'': G_{F_T',\overline{T}} \longrightarrow G_{F',\overline{T}^0\sqcup\overline{T}^1}$$

is generated by the inclusion $\overline{T}^0 \hookrightarrow \overline{T}^0 \sqcup \overline{T}^1$. By theorem 2.2(iii)

$$\Phi: G_{F_T',\overline{T}} \longrightarrow G_{F',\overline{T}^0}$$

is the composition of Φ'' and the canonical equivalence $G_{F',\overline{T}^0\sqcup\overline{T}^1} \longrightarrow G_{F',\overline{T}^0}$ which contains the diffeomorphism $\overline{T}^0 \sqcup \overline{T}^1 \supset \overline{T}^0 \xrightarrow{id} \overline{T}^0$. Hence Φ contains the superposition id: $\overline{T}^0 \to \overline{T}^0$, as was to be shown.

<u>Remark</u> 3.4. Roughly speaking, the above examples can be summarized as follows.

1° For standard manifolds and foliations the semi-simplicial constructions coincide with the classical ones.

2° If Γ is any groupoid of germs, then then the holonomy groupoid of the pointwise foliation of the nerve $N\Gamma$ is Γ itself; the canonical ss-Γ-structure associated with the foliation is the one corresponding to the identity morphism $\Gamma \longrightarrow \Gamma$.

3° If (F,F') is a flag of foliations, then for any pair T, T' of complete transversals $\Gamma_{F',T'}$ is a holonomy groupoid of the ss-foliation induced on $N\Gamma_{F,T}$ from F', and the canonical $\Gamma_{F',T'}$-structure on $N\Gamma_{F,T}$ comes from the classical morphism $\Sigma_{T',T}$: $\Gamma_{F,T} \longrightarrow \Gamma_{F',T'}$.

In II;2.1 we have posed a problem how to compute the holonomy pseudogroup for a foliation F of a manifold M, starting from the foliation induced from F on the nerve $N\mathcal{U}$ of an arbitrary open covering $\mathcal{U}$ of M. Clearly, theorem 2.2 gives us a recipe for such a computation, and the fact that what we get is precisely the holonomy groupoid for F is a straightforward consequence of the general result formulated below.

4.1 Let us observe that in view of II;1.4 any foliation F of an ss-manifold X yields a foliation $F_{\mathcal{U}}$ of the localization $X_{\mathcal{U}}$, for every covering $\mathcal{U}$ of X_0 (each $(X_{\mathcal{U}})_n$ is a disjoint union of open subsets of X_n). Similarly, if $\pi\colon E \to X$ is any ss-Γ-bundle over X, and $\mathcal{U} = \{U_a\}$ is a covering of X_0, then the localization $E_{\pi^{-1}\mathcal{U}}$ of E with respect to the covering

$$\pi^{-1}\mathcal{U} = \{\pi_0^{-1}U_a\}$$

carries an induced from E structure of an ss-Γ-bundle over $X_{\mathcal{U}}$. In this way any Γ-structure ω on X induces a well-defined Γ-structure $\omega_{\mathcal{U}}$ on $X_{\mathcal{U}}$.

Proposition 4.2. Let F be an arbitrary foliation of an ss-manifold $X = (X_n)$ and $\mathcal{U} = \{U_a;\ a \in A\}$ an open covering of X_0. If for each pair $(a,b) \in A \times A$, $i_{ab}\colon T_{ab} \hookrightarrow \varepsilon_1^{-1}U_a \cap \varepsilon_0^{-1}U_b$ is a complete transversal for the restriction of F_1, then the disjoint union $T = \bigsqcup T_{ab}$ is a complete transversal for F as well as for the foliation $F_{\mathcal{U}}$ induced by F on the localization $X_{\mathcal{U}}$. Moreover,

(i) the corresponding holonomy pseudogroups are equal,

$$G_{F,T} = G_{F_{\mathcal{U}},T}$$

(ii) the canonical $\Gamma_{F_{\mathcal{U}},T}$-structure on $X_{\mathcal{U}}$ associated with $F_{\mathcal{U}}$ and T is equal to the one induced from the $\Gamma_{F,T}$-structure on X associated with F and T;

(iii) let G be a pseudogroup and Γ the groupoid of germ; a Γ-structure ω on X defines the foliation F iff the induced Γ-structure $\omega_{\mathcal{U}}$ on $X_{\mathcal{U}}$ defines the foliation $F_{\mathcal{U}}$. If this is the case, then both the ss-Γ-structures give rise to the same holonomy morphism $G_{F,T} \longrightarrow G$.

Furthermore, if $\bar{T} = \bigsqcup \bar{T}_{ab}$ is another complete transversal for $F_{\mathcal{U}}$, then the canonical equivalences of holonomy pseudogroups

$$G_{F,T} \approx G_{F,\bar{T}} \quad \text{and} \quad G_{F_{\mathcal{U}},T} \approx G_{F_{\mathcal{U}},\bar{T}}$$

are given by the same invertible morphism.

Proof. Completeness of T as a transversal is obvious.

(i) We shall prove that the non-reduced holonomy pseudogroups are equal. Clearly,

$$G_{F_U, T^0 \sqcup T^1} \subset G_{F, T^0 \sqcup T^1}$$

and it remains to show the converse inclusion. Let $c: [0,1] \to X_0$ be an arbitrary path in a leaf of F_0 such that

$$c(0) = \varepsilon_0 i_{kl} t \in U_l \quad \text{and} \quad c(1) = \varepsilon_1 i_{mn} \bar{t} \in U_m$$

for some $t \in T_{kl}$, $\bar{t} \in T_{mn}$, and some indices $k,l,m,n \in A$ (observe that, since both T^0 and T^1 are complete transversals, it sufficies to consider the holonomy translations from T^0 to T^1 only). There is a sequence of indices $a_0, a_1, \ldots, a_s \in A$, $a_0 = l$ and $a_s = m$, and a partition $0 = \tau_0 < \tau_1 < \ldots < \tau_{s+1} = 1$ of the unit interval such that

$$c([\tau_i, \tau_{i+1}]) \subset U_{a_i} \quad \text{for} \quad i = 0,1,\ldots,s.$$

For each i, we choose a path c_i in a leaf of the foliation $(F_U)_1$ of $(X_U)_1$ connecting the point

$$\eta_0 c(\tau_i) \in \varepsilon_0^{-1} U_{a_{i-1}} \cap \varepsilon_1^{-1} U_{a_i}$$

to a point

$$i_{a_i a_{i-1}} t_i \in (X_U)_1 \ , \quad t_i \in T_{a_i a_{i-1}} .$$

Now,

$c'_0 = (c|[0,\tau_1]) * (\varepsilon_0 c_1)$ connects $c(0)$ to $\varepsilon_0 i_{a_1 a_0} t_1$ in a leaf of $F_0|U_l$,

$c'_i = (\varepsilon_1 c_i)^{-1} * (c|[\tau_i, \tau_{i+1}]) * (\varepsilon_0 c_{i+1})$ connects $\varepsilon_1 i_{a_i a_{i-1}} t_i$ to $\varepsilon_0 i_{a_{i+1} a_i} t_{i+1}$ in a leaf of $F_0|U_{a_i}$, for $i = 1,2,\ldots,s-1$, and

$c'_s = (\varepsilon_1 c_s)^{-1} * (c|[\tau_s, 1])$ connects $\varepsilon_1 i_{a_s a_{s-1}} t_s$ to $c(1)$ in a leaf of $F_0|U_m$.

We claim that the holonomy translation $h_{c,T^1}^{T^0}$ (for F_0) is identical with the composition

$$h_{c'_s, T^1}^{T^1} \, id_1^0 \, h_{c'_{s-1}, T^0}^{T^1} \cdots id_1^0 \, h_{c'_1, T^0}^{T^1} \, id_1^0 \, h_{c'_0, T^0}^{T^0} \in G_{F_U, T^0 \sqcup T^1} .$$

Indeed, by choosing local transversals at the points $c(\tau_i)$ and lifting them to $\eta_0 c(\tau_i)$, $i = 1,\ldots,s$, one immediately reduces our assertion to I;2.1.

(ii) Again, it sufficies to compare the canonical $\Gamma_{F_U, T^0 \sqcup T^1}$-structure on X_U with the $\Gamma_{F, T^0 \sqcup T^1}$-structure on X. We recall the notation of the proof of theorem 2.2. In view of (i), there is a well-defined map at the zero level,

$$\Gamma_{F,T^0\sqcup T^1} \times_{\Gamma_{(F_{\mathcal{U}})_0,T^0\sqcup T^1}} E_{(F_{\mathcal{U}})_0,T^0\sqcup T^1} \xrightarrow{I_0} \Gamma_{F,T^0\sqcup T^1} \times_{\Gamma_{F_0,T^0\sqcup T^1}} E_{F_0,T^0\sqcup T^1}$$

$$E_{F_{\mathcal{U}},0} \ni g\cdot[H_{c,T^0\sqcup T^1},x] \longrightarrow g\cdot[H_{\lambda_0 c,T^0\sqcup T^1},\lambda_0 x] \in E_{F,0}$$

where λ_0 stands for the projection $(X_{\mathcal{U}})_0 \to X_0$. Let us consider the following map

$$E_{F_{\mathcal{U}},0} \ni e \xrightarrow{J_0} (a_e;I_0 e) \in \bigsqcup \pi^{-1}U_a = \big((E_{F,T})_{\pi^{-1}\mathcal{U}}\big)_0$$

where a_e is the unique index such that $\pi(e) \in \{a_e\} \times U_a \subset (X_{\mathcal{U}})_0$. Clearly, J_0 is $\Gamma_{F,T^0\sqcup T^1}$-equivariant and projects to the identity map on $(X_{\mathcal{U}})_0$. Hence J_0 is an isomorphism of $\Gamma_{F,T^0\sqcup T^1}$-bundles. According to II; corollary 3.6, J_0 can be extended along the projection $\lambda = (\lambda_n)$: $X_{\mathcal{U}} \to X$ to an ss-map

$$J\colon E_{F_{\mathcal{U}},T} \longrightarrow (E_{F,T})_{\pi^{-1}\mathcal{U}}$$

(necessarily an isomorphism) if the commutativity relation

$$\bar{\varepsilon}_1(I_0 \times \lambda_1) = I_0\,\bar{\varepsilon}_1$$

holds. So we consider an arbitrary element e of $E_{F_{\mathcal{U}},1}$ written in the form

$$e = (g[H_{\varepsilon_0 c,T^0}(b;\varepsilon_0 x)],(a,b;x))$$

where $(a,b;x) \in (X_{\mathcal{U}})_1$, c is a path in $\varepsilon_0^{-1}U_b \cap \varepsilon_1^{-1}U_a \hookrightarrow (X_{\mathcal{U}})_1$ connecting x to a point of T, and $g \in \Gamma_{F,T^0\sqcup T^1}$. One has

$$\begin{aligned} I_0\,\bar{\varepsilon}_1\, e &= I_0(g\ \mathrm{id}_0^1[H_{\varepsilon_1 c,T^1}(a;\varepsilon_1 x)]) \\ &= g\ \mathrm{id}_0^1[H_{\lambda_0\varepsilon_1 c,T^1},\varepsilon_1 x] \end{aligned}$$

and

$$\begin{aligned} \bar{\varepsilon}_1(I_0 \times \lambda_1)e &= \bar{\varepsilon}_1(g[H_{\lambda_0\varepsilon_0 c,T^0}\varepsilon_0 x],x) \\ &= g\ \mathrm{id}_0^1[H_{\varepsilon_1\lambda_1 c,T^1},\varepsilon_1 x] = I_0\,\bar{\varepsilon}_1\, e \end{aligned}$$

as λ commutes with the structure operators.

(iii) We shall show that both the ss-Γ-structures yield the same holonomy morphism $G_{F,T\sqcup T} \longrightarrow G$. Let an ss-$\Gamma$-bundle $E \to X$ represent ω. As $\omega_{\mathcal{U}}$ is represented by $E_{\pi^{-1}\mathcal{U}}$ and

$$(E_{\pi^{-1}\mathcal{U}})_0 \cong \bigsqcup E_0|U_a$$

the first statement of (iii) is obvious. By definition, the distinguished submersions for $\bigsqcup E_0|U_a$ are local sections of E_0 over subsets of the U_a's followed by the target projection on units of Γ. Hence they are distinguished also for E_0, and their restrictions to the transversal generate the same morphism $G_{F,T^0\sqcup T^1} \longrightarrow G$ as the (possibly larger) collection obtained for E_0 does.

A similar argument proves the last assertion of the proposition. Indeed, if $\overline{T}$ is another complete transversal for $F_{\mathcal{U}}$, then the equivalence

$$G_{F_{\mathcal{U}},T} \longrightarrow G_{F_{\mathcal{U}},\overline{T}}$$

is generated by the holonomy translations from T to $\overline{T}$ along paths contained in some $\varepsilon_1^{-1}U_a \cap \varepsilon_0^{-1}U_b$, $a,b \in A$ (cf III, lemma 1.1.1). Sinc the collection of such translations generates a morphism Ψ: $G_{F,T} \longrightarrow G_{F,\overline{T}}$, Ψ must be the morphism generated by the collection (larger, in general) of all the holonomy translations from T to $\overline{T}$. This ends the proof of proposition 4.2.

IV Morphisms of ss-manifolds

Proposition III;4.2 above suggests that an ss-manifold X carries the same information (whatever that means) as any of its localizations $X_{\mathcal{U}}$. A hint in the same direction is also lemma III;2.5.3:

1.1 A direct verification shows that Γ-cocycles $\{\gamma_{ab};\ (a,b) \in A \times A\}$ on an ss-manifold X with respect to a covering $\mathcal{U} = \{U_a;\ a \in A\}$ of X_0 are in a bijective correspondence with ss-maps $X_{\mathcal{U}} \to N\Gamma$, the ss-ma corresponding to a cocycle $\{\gamma_{ab}\}$ being given by the formulas

(1.1.1)
$$(X_{\mathcal{U}})_0 \ni (a;x) \to \gamma_{aa}(x) \in N_0\Gamma, \text{ and}$$

$$(X_{\mathcal{U}})_n \ni (a_0,\dots,a_n;x) \longrightarrow$$

$$(\gamma_{a_0a_1}(\varepsilon_2^{\,n-1}x), \gamma_{a_1a_2}(\varepsilon_2^{\,n-2}\varepsilon_0 x), \dots, \gamma_{a_{n-1}a_n}(\varepsilon_0^{\,n-1}x)) \in N_n\Gamma$$

for $n = 1,2,\dots$.

We follow that suggestion. This is also an attempt to deal with the non-uniqueness problem that has arisen in II; example 2.4.3. In that example we have examined foliated ss-manifolds of the form $N\Gamma_{F,T}$ where F is fixed and T is arbitrarily chosen. The problem is: to what extent and in which sense are the ss-manifolds $N\Gamma_{F,T}$ as well as the ss-foliations induced on them from F' really equivalent.

<u>1.2</u> Given any two open coverings $\mathcal{U} = \{U_a;\ a \in A\}$ and $\mathcal{V} = \{V_i;\ i \in I\}$ of a manifold X_0, we say that $\mathcal{V}$ is <u>subordinate</u> to $\mathcal{U}$ if there is a <u>comparing map</u> $\rho\colon I \to A$ such that

$$V_i \subset U_{\rho(i)} \quad \text{for} \quad i \in I.$$

If $X = (X_n)$ is an ss-manifold, then any comparing map $\rho\colon I \to A$ induces an ss-map $\rho_{\#}\colon X_{\mathcal{V}} \to X_{\mathcal{U}}$,

$$(X_{\mathcal{V}})_n \ni (i_0,\dots,i_n;x) \to (\rho(i_0),\dots,\rho(i_n);x) \in (X_{\mathcal{U}})_n .$$

<u>Definition</u> 1.3. Let $X = (X_n)$ and $Y = (Y_n)$ be any ss-manifolds. In the collection Θ of all the ss-maps $X_{\mathcal{U}} \to Y$ ($\mathcal{U}$ ranges over all open coverings of X_0) we consider the smallest equivalence relation " $\sim$ " generated by the relation " $\simeq$ ", where $f \simeq g$ for some ss-maps $f\colon X_{\mathcal{U}} \to Y$ and $g\colon X_{\mathcal{V}} \to Y$ if there are:

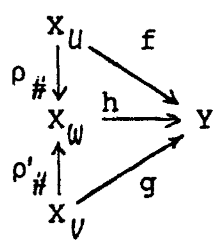
(i) a covering $\mathcal{W}$, to which both $\mathcal{U}$ and $\mathcal{V}$ are subordinate,

(ii) an ss-map $h\colon X_{\mathcal{W}} \to Y$, and

(iii) comparing maps ρ and ρ',

such that the following diagram commutes.

(1.3.1)



An ss-<u>morphism</u> f of X in Y (notation $f\colon X \longrightarrow Y$) is any equivalence class of the relation " $\sim$ ".

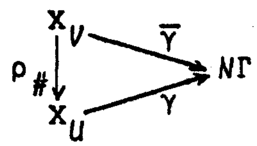
<u>Remark</u> 1.4. The above notion of an ss-morphism $X \longrightarrow Y$ can be defined in terms of equivalent ss-maps $X_{\mathcal{U}} \to Y$, where $\mathcal{U}$ ranges over <u>non-indexed</u> coverings only (such a covering is indexed by its elements). Unfortunately, this would complicate the equivalence relation too much. Nevertheless, by considering only such elements of the ss-morphisms, one can assume ss-morphisms to be sets; then the collection of all the ss-morphisms of X in Y is clearly a set too.

<u>Proposition</u> 1.5. The description II;2.5.2 - 2.5.3 of an arbitrary ss-Γ-structure through Γ-cocycles yields a bijective correspondence between Γ-structures on an arbitrary ss-manifold X and ss-morphisms $X \longrightarrow N\Gamma$.

<u>Proof</u>. Let $E \to X$ be an arbitrary but fixed principal ss-Γ-bundle over X, and $\{\gamma_{ij}\}$ and $\{\bar{\gamma}_{ij}\}$ any two Γ-cocycles on X such that

$E \cong E^{\gamma} \cong E^{\bar{\gamma}}$. We shall show that the corresponding ss-maps $\gamma: X \to N\Gamma$ and $\bar{\gamma}: X \to N\Gamma$ (cf 1.1) yield the same ss-morphism $X \longrightarrow N\Gamma$. Indeed, by II;2.5.2 -2.5.3 the cocycles correspond to some collections of sections $\sigma_a: U_a \to E$, $a \in A$, and $\bar{\sigma}_i: V_i \to E$, $i \in I$. Clearly, both $\mathcal{U}$ and $\mathcal{V}$ are subordinate to the disjoint union $\mathcal{W} := \mathcal{U} \sqcup \mathcal{V}$, and all the sections together give rise to a Γ-cocycle such that the corresponding ss-map $\gamma: X_{\mathcal{W}} \to N\Gamma$ closes the diagram (1.3.1) (the comparing maps ρ and ρ' being the inclusions).

Conversely, it sufficies to show that any commuting triangle of the form



yields an isomorphism of the ss-Γ-bundles E^{γ} and $E^{\bar{\gamma}}$ (cf II;2.5.3). Since for $\gamma = \{\gamma_{ab}\}$ and $\bar{\gamma} = \{\bar{\gamma}_{ij}\}$, the commutativity means

$$\bar{\gamma}_{ij} = \gamma_{\rho(i)\rho(j)} | (\varepsilon_1{}^{-1} V_i \cap \varepsilon_0{}^{-1} V_j),$$

the map

$$E_0^{\bar{\gamma}} \ni [i;x,g] \to [\rho(i);x,g] \in E_0^{\gamma}$$

is a well-defined isomorphism at the zero level. By II; corollary 3.6, the isomorphism extends to an ss-map $E^{\bar{\gamma}} \to E^{\gamma}$, which is again an isomorphism.

Corollary 1.5.1. The bijection established in proposition 1.5 restricts to a canonical bijection between morphisms of groupoids and ss-morphisms of their nerves.

Proof. Cf II;3.4.1.

1.6. Suppose now that X, Y and Z are arbitrary ss-manifolds and consider any ss-maps $f: X_{\mathcal{U}} \to Y$ and $g: Y_{\mathcal{V}} \to Z$, where $\mathcal{U} = \{U_a; a \in A\}$ and $\mathcal{V} = \{V_i; i \in I\}$. Let $f^{-1}\mathcal{V}$ denote the covering

$$\{f_{0a}^{-1} V_i;\ (a,i) \in A \times I\}$$

where

$$f_0 = \bigsqcup f_{0a}: (X_{\mathcal{U}})_0 = \bigsqcup U_a \to Y .$$

We define an ss-map $f_{\mathcal{V}}: X_{f^{-1}\mathcal{V}} \to Y_{\mathcal{V}}$ as follows

$$(X_{f^{-1}\mathcal{V}})_n \ni ((a_0,i_0),\ldots,(a_n,i_n);x) \to (i_0,\ldots,i_n; f_n(a_0,\ldots,a_n;x)) \in (Y_{\mathcal{V}})_n$$

for $n = 0,1,\ldots$.

Definition 1.6.1. (i) By the composition $g \circ f: X \longrightarrow Z$ of ss-morphisr

$f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ represented, respectively, by ss-maps $f: X_{\mathcal{U}} \to Y$ and $g: Y_{\mathcal{V}} \to Z$ we shall mean the equivalence class of the ss-map $g \circ f_{\mathcal{V}}: X_{f^{-1}\mathcal{V}} \to Z$.

(ii) An <u>identity</u> ss-<u>morphism</u> $1_X: X \longrightarrow X$ is the one represented by the projections $\lambda : X_{\mathcal{U}} \to X$,

$$(X_{\mathcal{U}})_n \ni (a_0, \dots, a_n; x) \longrightarrow x \in X_n .$$

<u>Proposition</u> 1.7. Semi-simplicial manifolds and their ss-morphisms (cf remark 1.4) form a category with products.

<u>Proof</u> is straightforward but rather tedious and we left it to the reader.

Clearly, there is a canonical functor carrying any ss-map $f: X \to Y$ to the ss-morphism

$$[f]: X \longrightarrow Y$$

represented by the composition $X_{\{X\}} \equiv X \xrightarrow{f} Y$.

<u>Proposition</u> 1.8. If ω is an ss-Γ-structure on X and $\Sigma: \Gamma \longrightarrow \Gamma'$ is an arbitrary morphism of groupoids, then the induced ss-Γ'-structure $\Sigma_*\omega$ (cf II.3.10) corresponds, under the bijection established in proposition 1.5, to the composition

$$X \xrightarrow{\omega} N\Gamma \xrightarrow{\Sigma} N\Gamma' .$$

<u>Proof</u>. Let ω be represented by an ss-Γ-bundle $E \to X$. We fix a collection of local sections

$$X_0 \supset U_a \xrightarrow{s_a} E_0 , \quad a \in A$$

such that $\mathcal{U} = \{U_a\}$ covers X_0, and a similar collection of sections

$$N \supset V_i \xrightarrow{\sigma_i} \Sigma$$

of the Γ'-bundle $\Sigma \to N$ (N = units of Γ). Each pair (s_a, σ_i) gives rise to a section

$$X_0 \supset (\alpha s_a)^{-1} V_i \ni x \xrightarrow{s'_{ai}} s_a(x)\sigma_i(\alpha s_a(x)) \in E_0 \times_\Gamma \Sigma = \Sigma_* E_0 .$$

Now, if $\{\gamma_{ab}\}$ (resp., $\{\Sigma_{ij}\}$) is the Γ-cocycle over X (resp., the Γ'-cocycle over N) defined by the equalities

in E_1: $(\pi, \varepsilon_0)^{-1}(x, s_b(\varepsilon_0 x)) = (\pi, \varepsilon_1)^{-1}(x, s_a(\varepsilon_1 x))\gamma_{ab}(x)$,

(resp., in Σ: $g \cdot \sigma_j(\alpha g) = \sigma_i(\beta g) \cdot \Sigma_{ij}(g)$), then, in the induced Γ'-bundle $\Sigma_* E_1 = E_1 \times_\Gamma \Sigma$, one has

$$(\pi,\varepsilon_0)^{-1}(x, s'_{bj}(\varepsilon_0 x)) = (\pi,\varepsilon_0)^{-1}(x, s_b(\varepsilon_0 x))\sigma_j(\alpha s_b(\varepsilon_0 x))$$
$$= (\pi,\varepsilon_1)^{-1}(x, s_a(\varepsilon_1 x))\gamma_{ab}(x)\sigma_j(\alpha s_b(\varepsilon_0 x))$$
$$= (\pi,\varepsilon_1)^{-1}(x, s_a(\varepsilon_1 x))\sigma_i(\alpha s_a(\varepsilon_1 x))\Sigma_{ij}(\gamma_{ab}(x))$$
$$= (\pi,\varepsilon_1)^{-1}(x, s'_{ai}(\varepsilon_1 x))(\Sigma_{ij}\circ\gamma_{ab})(x)$$

for $x \in \varepsilon_1^{-1}(\alpha s_a)^{-1}V_i \cap \varepsilon_0^{-1}(\alpha s_b)^{-1}V_j$. Consequently, the maps $\Sigma_{ij}\circ\gamma_{ab}$ form a Γ'-cocycle over X generating the Γ'-bundle $\Sigma_* E$.

Turning back to ss-morphisms, we recall that the cocycles $\gamma = \{\gamma_{ab}\}$ and $\sigma = \{\Sigma_{ij}\}$ are shortened descriptions (cf 1.1) of ss-maps $\gamma\colon X_{\mathcal{U}} \to N\Gamma$ and $\sigma : (N\Gamma)_{\mathcal{V}} \to N\Gamma'$ which represent the ss-morphisms $\omega\colon X \longrightarrow N\Gamma$ and, respectively, $\Sigma\colon N\Gamma \longrightarrow N\Gamma'$ (precisely, the ss-morphisms corresponding to ω and to Σ). Now, according to 1.6 $\gamma^{-1}\mathcal{V}$ is precisely the covering $\{(\alpha s_a)^{-1}V_i;\ (a,i) \in A \times I\}$, and the composition

$$X_{\gamma^{-1}\mathcal{V}} \xrightarrow{\gamma_{\mathcal{V}}} (N\Gamma)_{\mathcal{V}} \xrightarrow{\sigma} N\Gamma'$$

corresponds to the Γ'-cocycle $\{\Sigma_{ij}\circ\gamma_{ab}\}$. In view of the previous considerations we are done.

<u>Corollary</u> 1.8.1. The bijective correspondence of 1.5 describes groupoids and their morphisms as a complete subcategory of the category of ss-manifolds and ss-morphisms.

<u>Proof</u>. Cf corollary 1.5.1 and II; proposition 3.11.1.

<u>Definition</u> 1.9. For any ss-Γ-structure ω on X and any ss-morphism $f\colon Y \longrightarrow X$, a <u>pull-back</u> ss-$\Gamma$-structure $f^*\omega$ on Y is the one corresponding to the composition

$$Y \xrightarrow{f} X \xrightarrow{\omega} N\Gamma.$$

<u>Corollary</u> 1.9.1. Let ω be an arbitrary Γ-structure on an ss-manifold X. (i) If $f\colon Y \longrightarrow X$ and $g\colon Z \longrightarrow Y$ are any ss-morphisms, then

$$(f\circ g)^*\omega = g^* f^*\omega .$$

(ii) If $f\colon Y \longrightarrow X$ is an ss-morphism and $\Sigma\colon \Gamma \longrightarrow \Gamma'$ an arbitrary morphism of groupoids, then

$$f^*(\Sigma_*\omega) = \Sigma_*(f^*\omega).$$

<u>1.9.2</u> For any ss-map $f\colon Y \to X$ the induced ss-morphism $[f]\colon Y \longrightarrow X$

pulls ω back to an ss-Γ-structure on Y which we shall denote simply by $f^*\omega$.

Corollary 1.9.3. (i) If $f: Y \to X$ is any ss-map and if an ss-Γ-structure ω on X is represented by a principal ss-Γ-bundle $E = (E_n)$, then the ss-Γ-structure $f^*\omega$ on Y is represented by the ss-Γ-bundle $f^*E := (f_n^*E_n)$; the structure operators being induced from those of E and Y.

(ii) For any ss-map $f: Y_{\mathcal{U}} \to X$ representing an arbitrary ss-morphism $f: Y \longrightarrow X$ the pull-back ss-Γ-structures $f^*\omega$ on $Y_{\mathcal{U}}$ and $f^*\omega$ on Y correspond to each other under the localization of Y (cf III;4.1)

Proof. (i) We fix a collection of local sections $s_a: X \supset U_a \to E_0$, $a \in A$, such that $\mathcal{U} = \{U_a\}$ covers X. Let $\{\gamma_{ab}\}$ be the corresponding Γ-cocycle on X ; according to the formula (1.1.1), the cocycle extends to an ss-map $\gamma: X_{\mathcal{U}} \to N\Gamma$ representing the ss-morphism $\omega: X \longrightarrow N\Gamma$. Now the pull-back ss-Γ-structure $f^*\omega$ is, as an ss-morphism, represented by the composition

$$Y_{f^{-1}\mathcal{U}} \xrightarrow{f_{\mathcal{U}}} X_{\mathcal{U}} \xrightarrow{\gamma} N\Gamma$$

which clearly corresponds to the Γ-cocycle $\{\gamma_{ab} \circ f_1\}$ on Y with respect to the covering $f^{-1}\mathcal{U} = \{f_0^{-1}U_a;\ a \in A\}$. It sufficies to observe that the same cocycle comes from a collection $\{f_0 s_a\}$ of sections

$$f^{-1}U_a \ni y \to (y, s_a(f_0(y))) \in f_0^*E_0$$

of the pull-back Γ-bundle.

(ii) In view of proposition 1.11.1(ii) below and corollary 1.9.1(i),

$$f^*\omega = \lambda^*(f^*\omega)$$

where $\lambda: Y_{\mathcal{U}} \to Y$ is the projection. Consequently, it sufficies to prove that for any ss-Γ-bundle E on Y the pull-back bundle $\lambda^*E = (\lambda_n^*E_n)$ is isomorphic to the localization $E_{\pi^{-1}\mathcal{U}}$. We wright down an isomorphism $I: E_{\pi^{-1}\mathcal{U}} \to \lambda^*E$ explicitely, as follows

$$(E_{\pi^{-1}\mathcal{U}})_n \ni (a_0,\dots,a_n;e) \xrightarrow{I_n} ((a_0,\dots,a_n;\pi e),e) \in (Y_{\mathcal{U}})_n \times_{(\lambda_n,\pi)} E_n .$$

Clearly, $I = (I_n)$ is an invertible ss-map commuting with the two right actions of Γ induced from E.

Proposition 1.10. For any groupoid Γ and arbitrary ss-manifold X, the bijective correspondence (cf proposition 1.5) between ss-Γ-structures on X and ss-morphisms $X \longrightarrow N\Gamma$ is given by the assignment

$$\text{„ } f\colon X \longrightarrow N\Gamma \text{ "} \rightsquigarrow \text{ „ } f^*\omega_\Gamma \text{ "}$$

where ω_Γ is the universal Γ-structure on $N\Gamma$ represented by the ss-Γ-bundle $\overline{N}\Gamma$ (cf II;3.11.1).

Proof. Let $E \to X$ be an arbitrary principal ss-Γ-bundle over X. We fix a collection of local sections $\sigma_a\colon X_0 \supset U_a \to E_0$, $a \in A$, such that $\mathcal{U} = \{U_a\}$ covers X_0; the corresponding Γ-cocycle $\{\gamma_{ab}\}$ extends to an ss-map $\gamma\colon X_{\mathcal{U}} \to N\Gamma$. In view of corollary 1.9.3(ii), it sufficies to prove that the localization $E_{\pi^{-1}\mathcal{U}}$ and the pull-back $\gamma^*\overline{N}\Gamma$ are isomorphic ss-Γ-bundles over $X_{\mathcal{U}}$. Since, for every n, $\gamma_n^*\overline{N}_n$ consists of the pairs

$$((a_0,\ldots,a_n;x),(g_0,\ldots,g_n)) \in (X_{\mathcal{U}})_n \times \overline{N}_n\Gamma$$

such that $g_0g_1^{-1} = \gamma_{a_0a_1}$, $g_1g_2^{-1} = \gamma_{a_1a_2}$ etc, one can easily see that the maps

$$\gamma_n^*\overline{N}_n\Gamma \ni ((a_0,\ldots,a_n;x),(g_0,\ldots,g_n)) \to (a_0,\ldots,a_n;\sigma_{a_0}(x)g_0) \in (E_{\pi^{-1}\mathcal{U}})_n$$

are well-defined and give rise to a desired isomorphism of the ss-Γ-bundles.

1.11 We end the above categorical considerations by proving a proposition that makes precise an intuitive equivalence between an arbitrary ss-manifold and any of its localizations.

Proposition 1.11.1. Let X be an arbitrary ss-manifold and $\mathcal{U} \doteq \{U_a;\ a \in A\}$ an open covering of X_0.

(i) The ss-morphism $[\lambda]\colon X_{\mathcal{U}} \longrightarrow X$ defined (cf 1.7) by the projection $\lambda\colon X_{\mathcal{U}} \to X$ is invertible.

(ii) If $f\colon X \longrightarrow Y$ is an arbitrary ss-morphism and $f\colon X_{\mathcal{U}} \to Y$ any ss-map representing f ,then the triangle

$$\begin{array}{ccc} X_{\mathcal{U}} & \xrightarrow{[f]} & Y \\ {\scriptstyle [\lambda]}\downarrow & \nearrow{\scriptstyle f} & \\ X & & \end{array}$$

commutes.

Proof. (i) We shall show that the ss-morphism $\mathfrak{h}\colon X \longrightarrow X_{\mathcal{U}}$ represented by $\mathrm{id}\colon X_{\mathcal{U}} \to X_{\mathcal{U}}$ is inverse to $[\lambda]$. Clearly, the composition $[\lambda]\circ\mathfrak{h}\colon X \longrightarrow X$ is represented by λ and thus equal to the identity 1_X.

On the other hand, the composition $\mathfrak{h}\circ[\lambda]\colon X_{\mathcal{U}} \longrightarrow X_{\mathcal{U}}$ is represented by the ss-map

$$(X_{\mathcal{U}})_{\pi^{-1}\mathcal{U}} \xrightarrow{\lambda_{\mathcal{U}}} X_{\mathcal{U}} \xrightarrow{\mathrm{id}} X_{\mathcal{U}}$$

One has

$$\left((X_{\mathcal{U}})_{\lambda^{-1}\mathcal{U}}\right)_n = \coprod_{(a_0,\ldots,a_n)} (\varepsilon_1{}^n)^{-1}\lambda_0^{-1}U_{a_0} \cap \ldots \cap (\varepsilon_0{}^n)^{-1}\lambda_0^{-1}U_{a_n}$$

where every summand is a subset of

$$(X_{\mathcal{U}})_n = \coprod_{(b_0,\ldots,b_n)} (\varepsilon_1{}^n)^{-1}U_{b_0} \cap \ldots \cap (\varepsilon_0{}^n)^{-1}U_{b_n} .$$

Thus an arbitrary element of $\left((X_{\mathcal{U}})_{\lambda^{-1}\mathcal{U}}\right)_n$ is of the form $(a_0,\ldots,a_n;(b_0,\ldots,b_n;x))$, where

$x \in (\varepsilon_1{}^n)^{-1}(U_{a_0} \cap U_{b_0}) \cap \ldots \cap (\varepsilon_0{}^n)^{-1}(U_{a_n} \cap U_{b_n})$,

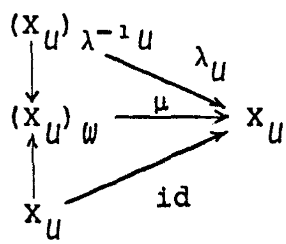
and the ss-map $\lambda_{\mathcal{U}}$ acts by the formulas

$$(a_0,\ldots,a_n;(b_0,\ldots,b_n;x)) \longrightarrow (a_0,\ldots,a_n;x)$$

for $n = 0,1,\ldots$. We consider the disjoint union $\mathcal{W} = \lambda^{-1}\mathcal{U} \sqcup \{(X_{\mathcal{U}})_0\}$, the last set being indexed by a star " * ". For $n \geq 0$, let μ_n denote the map $((X_{\mathcal{U}})_{\mathcal{W}})_n \to (X_{\mathcal{U}})_n$,

$$(\ldots,a_{i-1},*,a_{i+1},\ldots;(b_0,\ldots,b_n;x)) \to (\ldots,a_{i-1},b_i,a_{i+1},\ldots;x)$$

where we replace all stars with the corresponding b_i's. Now the sequence (μ_n) is an ss-map $(X_{\mathcal{U}})_{\mathcal{W}} \to X_{\mathcal{U}}$ such that the diagram



commutes (the details are left to the reader). In conclusion, we get $h\circ[\lambda] = 1_X$.

(ii) Since $[\lambda]$ is invertible, it sufficies to observe the equality

$$f = [f][\lambda]^{-1} .$$

As follows from the proof of (i), the right-hand ss-morphism is represented by $f\circ\mathrm{id} = f$.

<u>Definition</u> 1.12. Two ss-manifolds X and Y are <u>equivalent</u> (notation $X \approx Y$) if there is an invertible ss-morphism (an <u>equivalence</u>) $X \longrightarrow Y$.

<u>Corollary</u> 1.12.1. Two groupoids are equivalent iff their nerves are.

We now turn back to foliations and shall show that some ss-morphisms act on ss-foliations, generalizing the action of transverse maps in the standard (ie non-ss-) case. We begin with an important

Lemma 2.1. Let F be a foliation of an ss-manifold X. For an arbitrary ss-morphism $f: Y \rightharpoonup X$ the following four conditions are equivalent: (i) there is an ss-map $f: Y_{\mathcal{U}} \to X$ representing f, such that the map $f_0: (Y_{\mathcal{U}})_0 \to X_0$ is transverse to the foliation F_0;

(ii) for every ss-map $f: Y_{\mathcal{U}} \to X$ representing f, f_0 is transverse to F_0;

(iii) there exist a foliation F' of Y and a Γ-structure ω on X defining F, such that the pull-back Γ-structure $f^*\omega$ defines F' on Y;

(iv) there exists a foliation F' of Y such that for every Γ-structure ω defining F on X $f^*\omega$ defines F' on Y.

Proof. (i) $\Rightarrow$ (iv): We shall prove that the foliation $f_0^*F_0$ of $(Y_{\mathcal{U}})_0 = \bigsqcup U_a$ comes from a foliation F_0' of Y_0. Let us observe first that the foliations $\varepsilon_0^*(f_0^*F_0)$ and $\varepsilon_1^*(f_0^*F_0)$ of $(Y_{\mathcal{U}})_1$ are equal, as they both coincide with $f_1^*F_1$. Next, for any pair a,b of indices, let us consider the maps η_0' and η_0'',

$$\begin{array}{l} (Y_{\mathcal{U}})_0 \supset \{a\}\times U_a\cap U_b \ni (a;x) \xrightarrow{\eta_0'} \\ \qquad\qquad\qquad\qquad\qquad\qquad (a,b;\eta_0 x) \in (Y_{\mathcal{U}})_1 . \\ (Y_{\mathcal{U}})_0 \supset \{b\}\times U_a\cap U_b \ni (b;x) \xrightarrow{\eta_0''} \end{array}$$

Since

$$\varepsilon_1\eta_0' = id_{\{a\}\times U_a\cap U_b} \quad \text{and} \quad \varepsilon_0\eta_0'' = id_{\{b\}\times U_a\cap U_b}$$

η_0' is transverse to any foliation pulled-back by ε_1, and η_0'' - to any one pulled-back by ε_0. In particular,

$$f_0^*F_0|\{a\}\times U_a\cap U_b = (\eta_0'')^*\varepsilon_0^*(f_0^*F_0) = (\eta_0'')^*f_1^*F_1$$

and

$$f_0^*F_0|\{b\}\times U_a\cap U_b = (\eta_0')^*\varepsilon_1^*(f_0^*F_0) = (\eta_0')^*f_1^*F_1 ,$$

and finally

$$f_0^*F_0|\{b\}\times U_a\cap U_b = (id_a^b)^*(f_0^*F_0|\{a\}\times U_a\cap U_b)$$

where $id_a^b: \{b\}\times U_a\cap U_b \to \{a\}\times U_a\cap U_b$ is the identification map. Being the same over the overlaps, the foliations $f_0^*F_0|\{a\}\times U_a$ are restrictions of a uniquely defined foliation F_0' of Y_0.

Our next step is the equality $\varepsilon_0^*F_0' = \varepsilon_1^*F_0'$ on Y_1. Let, again, a and b be arbitrary indices. By the construction of F_0', the restrict ion

$$\varepsilon_0^* F_0' | \varepsilon_1^{-1} U_a \cap \varepsilon_0^{-1} U_b = (\varepsilon_0 | \varepsilon_1^{-1} U_a \cap \varepsilon_0^{-1} U_b)^* (F_0' | U_b)$$

is the same as the foliation

$$\varepsilon_0^* f_0^* F_0 | \{(a,b)\} \times \varepsilon_1^{-1} U_a \cap \varepsilon_0^{-1} U_b$$

of a summand of $(Y_U)_1$; similarly, $\varepsilon_1^* F_0' | \varepsilon_1^{-1} U_a \cap \varepsilon_0^{-1} U_b$ is the same as a suitable restriction of $\varepsilon_1^* f_0^* F_0$. Since the two last foliations are equal, we get

$$\varepsilon_0^* F_0' = \varepsilon_1^* F_0' \quad \text{over} \quad \varepsilon_1^{-1} U_a \cap \varepsilon_0^{-1} U_b$$

and it remains to observe that the sets $\varepsilon_1^{-1} U_a \cap \varepsilon_0^{-1} U_b$ cover Y_1.

Let us consider now an arbitrary ss-Γ-bundle $E \to X$ defining F. By corollary 1.9.3, the bundles $(f_0|U_a)^* E_0 \to U_a$ come from an ss-Γ-bundle $E' \to Y$ representing the pull-back ss-Γ-structure. If $\sigma: V \to E_0$ is any local section, then the target map β' of E_0' into the units of Γ evaluated on the pull-back section $(f_0|U_a)^*\sigma$ yields $\beta \circ \sigma \circ (f_0|U_a) = \varphi \circ (f_0|U_a)$, where φ is a distinguished submersion for E_0. Now, the pull-back sections cover E_0' and thus the bundle defines a foliation of Y_0. Furthermore, the submersions $\varphi \circ (f_0|U_a)$ are distinguished for E_0' and define exactly the foliation F_0'.

(iii) ⇒ (ii): Let an ss-Γ-bundle $E \to X$ define F and $f: Y_U \to X$ be an arbitrary ss-map representing f. Reasoning as above we see that if φ is any distinguished submersion for E_0 then $\varphi \circ f_0$ is again a <u>submersion</u>. Hence f_0 is transverse to the foliation defined by E_0.

The parts (iv) ⇒ (iii), and (ii) ⇒ (i) are both trivial.

<u>Definition</u> 2.2. An ss-morphism $f: Y \to X$ is <u>transverse</u> to a foliation F of X, if one of the conditions (i) - (iv) of lemma 2.1 is fulfilled; the corresponding foliation $f^*F = F'$ of Y is the <u>pull-back foliation induced by</u> F and f.

<u>Corollary</u> 2.3.1. A Γ-structure ω on an ss-manifold X defines a foliation iff the corresponding ss-morphism $X \xrightarrow{\omega} N\Gamma$ is transverse to the pointwise foliation of $N\Gamma$. If this is the case, then the foliation ω defines is exactly the pull-back foliation.

<u>Proof</u>: follows immediately from lemma 2.1(iii)-(iv), proposition 1.10, and the fact that the ss-Γ-bundle $\overline{N\Gamma} \to N\Gamma$ defines the pointwise foliation.

<u>Corollary</u> 2.3.2. Let F be a foliation of an ss-manifold X, and $f: Y \to X$ and $g: Z \to Y$ be arbitrary ss-morphisms. Then, the compo-

sition $f\circ g$ is transverse to F iff f is transverse to F and g - to the induced foliation f^*F. If this is the case, then the equality

$$(f\circ g)^*F = g^*f^*F$$

holds.

Proof. Clearly, if $f\colon Y_U \to X$ and $g\colon Z_V \to Y$ represent f and g, respectively, then g_0 is transverse to the foliation $(f^*F)_0$ iff $(g_U)_0$ is transverse to $f_0^*F_0$, the foliation obtained from $(f^*F)_0$ by a localization. Consequently, the first assertion of the corollary follows from its standard, non-semi-simplicial version. The second assertion follows easily from lemma 2.1(iii)-(iv).

We are now ready to solve the problem posed at the beginning of this chapter.

Proposition 2.4. Let (F,F') be any flag of foliations of a manifold M. For each pair $T, \overline{T} \hookrightarrow M$ of complete transversals for F, let $N\Phi\colon N\Gamma_{F,\overline{T}} \longrightarrow N\Gamma_{F,T}$ be the ss-morphism induced from the canonical equivalence $\Phi\colon G_{F,\overline{T}} \longrightarrow G_{F,T}$. Then one has

$$N\Phi^*F'_T = F'_{\overline{T}}$$

F'_T and $F'_{\overline{T}}$ being the foliations induced by F' on the ss-manifolds $N\Gamma_{F,T}$ and $N\Gamma_{F,\overline{T}}$, respectively.

Proof. Since Φ is the composition

$$G_{F,\overline{T}} \xrightarrow{\Psi_{\overline{T}}} G_{F,T\sqcup\overline{T}} \xrightarrow{\Psi_T^{-1}} G_{F,T}$$

where both $\Psi_{\overline{T}}$ and Ψ_T are equivalences induced by an inclusion, we may restrict ourselves to the case $\overline{T} \subset T$. Consequently, Φ is generated by the inclusion $\overline{T} \hookrightarrow T$, and the ss-morphism $N\Phi$ is represented by an ss-map $N\Phi\colon N\Gamma_{F,\overline{T}} \to N\Gamma_{F,T}$ which is just a sequence of inclusions. Evidently the map $(N\Phi)_0\colon \overline{T} \hookrightarrow T$ is transverse to $F'|T$, and one has

$$(N\Phi)_0^*(F'|T) = F'|\overline{T}$$

which implies $N\Phi^*F'_T = F'_{\overline{T}}$. Turning back to the general case, we thus get

$$N\Phi^*F'_T = N\Phi^*N\Psi_T^*F'_{T\sqcup\overline{T}} = N\Psi_{\overline{T}}^*F'_{T\sqcup\overline{T}} = F'_{\overline{T}}$$

where the transversality of $N\Phi$ follows from that of $N\Psi_{\overline{T}} = N\Psi_T\circ N\Phi$ (cf corollary 2.3.2).

3.1 If an ss-morphism $f: Y \longrightarrow X$ is transverse to a foliation F of X, and $S \hookrightarrow Y_1$ and $T \hookrightarrow X_1$ are arbitrary complete transversals for $F' = f^*F$ and F, respectively, then by lemma 2.1, the pull-back $\Gamma_{F,T}$-structure on Y induced by f from the canonical one on X defines the foliation F'. In view of III; theorem 2.2(i), this gives rise to a holonomy morphism

$$f_{TS} = \Psi_S^{f^*\omega_{F,T}}: G_{F',S} \longrightarrow G_{F,T}$$

induced by the foliation from f with respect to the pair S, T of transversals.

Theorem 3.2. (i) Let an ss-morphism $f: Y \longrightarrow X$ be transverse to a foliation F of X, and $S,\overline{S} \hookrightarrow Y_1$ and $T,\overline{T} \hookrightarrow X_1$ be arbitrary complete transversals for, resp., $F' = f^*F$ and F. Then the square

$$\begin{array}{ccc} G_{F',S} & \xrightarrow{f_{TS}} & G_{F,T} \\ {\scriptstyle \Phi'}\downarrow & & \downarrow{\scriptstyle \Phi} \\ G_{F',\overline{S}} & \xrightarrow{f_{\overline{T}\overline{S}}} & G_{F,\overline{T}} \end{array}$$

commutes, Φ and Φ' being the canonical equivalences.

(ii) If $T,\overline{T}$ are any two complete transversals for F, then the holonomy morphism

$$(1_X)_{\overline{T}T}: G_{F,T} \longrightarrow G_{F,\overline{T}}$$

is the canonical equivalence of holonomy pseudogroups.

(iii) If $f: Y \longrightarrow X$ and $g: Z \longrightarrow Y$ are ss-morphisms such that the composition fg is transverse to a foliation F of X (cf corollary 2.3.2), then for any triple of complete transversals R,S,T for, respectively, $F'' = (fg)^*F$, $F' = f^*F$, and F, the corresponding holonomy morphisms form a commuting triangle

$$\begin{array}{ccc} G_{F'',R} & \xrightarrow{(fg)_{TR}} & G_{F,T} \\ & {\scriptstyle g_{SR}}\searrow \quad \nearrow{\scriptstyle f_{TS}} & \\ & G_{F',S} & \end{array}$$

(iv) If f and F are as in (i), then for any pair T,S of complete transversals for F and f^*F, respectively, and any ss-Γ-structure ω defining F, the following diagram composed of holonomy morphisms commutes.

$$\begin{array}{ccc} G_{F',S} & \xrightarrow{\Psi_S^{f^*\omega}} & G \\ {\scriptstyle f_{TS}}\downarrow & \nearrow{\scriptstyle \Psi_T^{\omega}} & \\ G_{F,T} & & \end{array}$$

Here G stands for the pseudogroup that underlies Γ.

<u>Proof</u>. (i) The equality $\omega_{F,\overline{T}} = \Phi_*\omega_{F,T}$ of theorem III;2.2(ii) impli (cf corollary 1.9.1)

$$f^*\omega_{F,\overline{T}} = \Phi_*(f^*\omega_{F,T}).$$

Consequently, (i) follows from theorem III;2.2(iii).

(ii) is a part of theorem III;2.2(ii).

(iii) We first prove the commutativity for appropriately chosen transversals. Namely, let $f\colon Y_{\mathcal{U}} \to X$ and $g\colon Z_{\mathcal{V}} \to Y$ be any ss-maps representing f and g, $\mathcal{U} = \{U_a\}$ and $\mathcal{V} = \{V_i\}$; by definition, $f\circ g$ is represented by the composition

$$Z_{g^{-1}\mathcal{U}} \xrightarrow{g_{\mathcal{U}}} Y_{\mathcal{U}} \xrightarrow{f} X.$$

If $\overline{R} \hookrightarrow (Z_{g^{-1}\mathcal{U}})_1$ is a complete transversal for $(F''_{g^{-1}\mathcal{U}})_1$, then $g_{\mathcal{U}}$ maps $\overline{R}$ to a transversal for $(F'_{\mathcal{U}})_1$. The new transversal need not be complete, so we extend it (by adding a suitable disjoint summand) to a complete one, say $\overline{S} \hookrightarrow (Y_{\mathcal{U}})_1$. In a similar vein, we extend $\overline{S}$ trans- ferred by f to a complete transversal $\overline{T} \hookrightarrow X_1$ for F_1. Clearly,

$$G_{F''_{g^{-1}\mathcal{U}},\overline{R}} \subset G_{F'_{\mathcal{U}},\overline{S}} \subset G_{F,\overline{T}}$$

and the holonomy morphisms induced from the ss-maps f, $g_{\mathcal{U}}$, and $f\circ g_{\mathcal{U}}$ are generated by the inclusions $\overline{S} \subset \overline{T}$, $\overline{R} \subset \overline{S}$, and $\overline{R} \subset \overline{T}$, respectively; at the moment the commutativity holds.

$$\begin{array}{ccc} G_{F''_{g^{-1}\mathcal{U}},\overline{R}} & \xrightarrow{[g_{\mathcal{U}}f]_{\overline{T}\overline{R}} \;=\; \Psi_{\overline{R}}^{(g_{\mathcal{U}}f)^*\omega_{F,\overline{T}}}} & G_{F,\overline{T}} \\ \Psi_{\overline{R}}^{g_{\mathcal{U}}^*\omega_{F'_{\mathcal{U}},\overline{S}}} = [g_{\mathcal{U}}]_{\overline{S}\overline{R}} \searrow & & \nearrow [f]_{\overline{T}\overline{S}} = \Psi_{\overline{S}}^{f^*\omega_{F,\overline{T}}} \\ & G_{F'_{\mathcal{U}},\overline{S}} & \end{array}$$

By III; proposition 4.2(i)-(ii), the morphism

$$[g_{\mathcal{U}}]_{\overline{S}\overline{R}}\colon G_{F''_{g^{-1}\mathcal{U}},\overline{R}} \longrightarrow G_{F'_{\mathcal{U}},\overline{S}} = G_{F',\overline{S}}$$

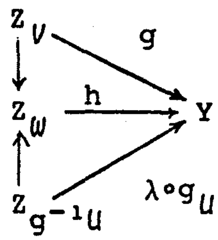
is the holonomy morphism for the ss-$\Gamma_{F',\overline{S}}$-structure

$$g_{\mathcal{U}}^*\lambda^*\omega_{F',\overline{S}} = (\lambda\circ g_{\mathcal{U}})^*\omega_{F',\overline{S}}$$

where $\lambda: Y_U \to Y$ is the projection. We claim that the ss-map $\lambda \circ g_U: Z_{g^{-1}U} \to Y$ represents the ss-morphism $fg: Z \longrightarrow Y$. Indeed, one can easily check that for $W = g^{-1}U \sqcup V$ (the disjoint union) the maps

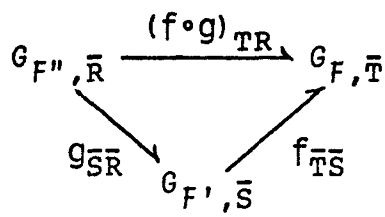
$$(Z_W)_n \ni (\ldots,i_r,\ldots,(i_s,a_s),\ldots;z) \to g(\ldots,i_r,\ldots,i_s,\ldots;z) \in Y_n$$

constitute an ss-map $h: Z_W \to Y$ such that the diagram



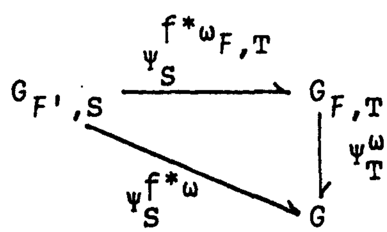
commutes; being equivalent to g, $\lambda \circ g_U$ represents the same ss-morphism $Z \longrightarrow Y$. In conclusion, the $\Gamma_{F',\bar{S}}$-structure $(\lambda \circ g_U)^*\omega_{F',\bar{S}}$ on $Z_{g^{-1}U}$ is the one induced from $g^*\omega_{F',\bar{S}}$ (cf corollary 1.9.3(ii)).

We are now able to apply theorem III;4.2(iii) to $g^*\omega_{F',\bar{S}}$ and to the ss-$\Gamma_{F,\bar{T}}$-structures $f^*\omega_{F,\bar{S}}$ and $(f \circ g)^*\omega_{F,\bar{S}}$. As a result, we get the commuting triangle



By the part (i) already proved, the specially chosen transversals can be replaced by arbitrary R, S and T.

(iv) We can apply theorem III;2.2(iii) to the morphism Ψ_T and to $f^*\omega_{F,T}$ instead of ω. The resulted commuting triangle is



as

$$(\Psi_T^\omega)_* f^*\omega_{F,T} = f^*(\Psi_T^\omega)_*\omega_{F,T} = f^*\omega$$

by corollary 1.9.1(ii). According to 3.1, the upper morphism is exactly f_{TS}.

<u>Corollary</u> 3.3. If X and Y are equivalent ss-manifolds, then any equivalence $Y \longrightarrow X$ is transverse to every foliation of X. Furthermore, let $f: Y \longrightarrow X$ be such an equivalence, F a fixed foliation of

X , and $T \hookrightarrow X_1$ and $S \hookrightarrow Y_1$ arbitrary complete transversals for F and $F' = f^*F$, respectively. Then the holonomy morphisms

$$f_{TS}: G_{F',S} \longrightarrow G_{F,T}$$

and

$$(f^{-1})_{ST}: G_{F,T} \longrightarrow G_{F',S}$$

are inverse to each other, ie $(f_{TS})^{-1} = (f^{-1})_{ST}$.

Proof. Let $f: Y \longrightarrow X$ be any equivalence and F an arbitrary foliation of X. By corollary 2.3.2, f is transverse to F as the composition $1_X = f \circ f^{-1}$ is. The rest follows directly from theorem 3.2(iii)

3.4 Remarks. 1. It can be shown that any ss-morphism $f: Y \longrightarrow X$ transverse to a foliation F of X induces well-defined ss-morphisms between appropriate leaves of f^*F and F. In particular, the assertion of corollary 3.3 completed by adding the following one:

If f is an equivalence, then leaves of f^*F are equivalent to the corresponding leaves of F (and there is a one-to-one correspondence between the leaves).

2. Our definition of ss-morphisms is a solution to the equivalence problem. The construction can be easily modified so as to cover the notion of simplicially homotopic ss-maps ([6]) which identifies any ss-set (in particular: any ss-manifold) $X = (X_n)$ with $X' = (X'_n)$ such that

$$X'_n = \coprod_{\sigma} X_n$$

where σ ranges over non-decreasing $(n+1)$-sequences with the only elements 0 and 1, and the structure operators are

$$X'_n \ni (i_0,\dots,i_n;x) \begin{cases} \xrightarrow{\varepsilon_h} (\dots,\hat{i}_h,\dots;\varepsilon_h x) \in X'_{n-1} \\ \xrightarrow{\eta_h} (\dots,i_h,i_h,\dots;\eta_h x) \in X'_{n+1} \end{cases} .$$

Let us sketch the modification. We consider ordered coverings, ie indexed coverings $U = \{U_a;\ a \in A\}$ endowed with a preorder $<$ in the set A of indices, and generalize II;1.4 by requiring $(X_U)_n$ to be the disjoint union over the ordered $(n+1)$-tuples $a_0 < \dots < a_n$ (an unordered covering can be trivially ordered by taking the total preorder relation). The above X' is now equal to the localization of X to the ordered covering $\{U_0, U_1\}$ of X_0 such that $U_0 = U_1 = X_0$.

Another modification requires the equivalence relation 1.3. Namely, we say that an ordered covering $U = \{U_a;\ a \in A\}$ has a supremum (infimum) if the subset

$$A^{+} := \{a \in A;\ \text{for every}\ \ b \in A,\ \ b < a\}$$

(resp. $\quad A^{-} := \{a \in A;\ \text{for every}\ \ b \in A,\ \ a < b\}$)

is non-empty and $\mathcal{U}^{+} = \{U_a;\ a \in A^{+}\}$ ($\mathcal{U}^{-} = \{U_a;\ a \in A^{-}\}$) covers X_0.

The modified ss-morphism of X in Y is again an equivalence class of ss-maps $X_{\mathcal{U}} \to Y$ where $\mathcal{U}$ are non-ordered, but in the diagram (1.3.1) $\mathcal{W} = \{W_s;\ s \in S\}$ can be an arbitrary ordered covering having both supremum and infimum, whereas ρ and ρ' should map the indices into, resp., S^{+} and S^{-}.

It can be shown that the modified ss-morphisms form a category with products, in which a localization $X_{\mathcal{U}}$ is equivalent to X if $\mathcal{U}$ has at least the supremum or infimum. Furthermore, the results: 1.5, 1.5.1, 1.8, 1.8.1, 1.9.1, 1.9.3, 1.10, and 1.12.1 concerning ss-Γ-structures and morphisms of groupoids, as well as: 2.1, 2.3.1, 2.3.2, 2.4, 3.2, and 3.3 concerning semi-simplicial foliations remain valid.

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