Regularity on metric spaces: Hölder continuity of extremal functions in axiomatic and Poincaré-Sobolev spaces

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Abstract

Using arguments developed by De Giorgi in the 1950's, it is possible to prove the regularity of the solutions to a vast class of variational problems in the Euclidean space. The main goal of the present paper is to extend these results to the more abstract context of metric spaces with a measure. In particular, working in the axiomatic framework of Gol'dshtein-Troyanov, we establish the interior regularity of quasi-minimizers of the p-Dirichlet energy. Our proof works for quite general domains, assuming some natural hypotheses on the (axiomatic) D-structure. Furthermore, we prove analogous results for extremal functions lying in the class of Poincaré-Sobolev functions, i.e. functions characterized by the single condition that a Poincaré inequality be satisfied.

Introduction

The problem of the regularity of solutions to partial differential equations with prescribed boundary values and of regular variational problems constitutes one of the most interesting chapters in analysis, which has its origins mostly starting from the year 1900, when D. Hilbert formulated his famous 23 problems in an address delivered before the International Congress of Mathematicians at Paris. The essential parts of the twentieth problem on existence of solutions and its related nineteenth problem about the regularity itself read as follows:

19th problem: "Are the solutions of regular problems in the calculus of variations always necessarily analytic?"

20th problem: "Has not every regular variational problem a solution, provided certain assumptions regarding the given boundary conditions are satisfied, and provided also if need be that the notion of a solution shall be suitably extended?"

It is known that in the Euclidean space the problem of minimizing a variational integral in a set of functions with prescribed boundary values is closely related to solving the corresponding Dirichlet problem for its Euler-Lagrange equation. In particular, for the Dirichlet p-energy integral

$$\int_{\Omega\subset\mathbb{R}^n} |\nabla u(x)|^p dx$$

the corresponding Euler-Lagrange equation, for $1 < p < \infty$, is

$$\mathrm{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = 0.$$

Starting with the remarkable result of S. Bernstein in 1904 that any C^3 solution of an elliptic nonlinear analytic equation in two variables is necessarily analytic, and through the works of many authors, in particular in the works of Leray and Schauder in 1934, it was proved that every sufficiently smooth, say $C^{0,\alpha}$ (Hölder continuous), stationary point of a regular variational problem with analytic integrand is analytic. On the other hand, by direct methods of the calculus of variations one can prove in general the existence of solutions which have derivatives only in a generalized sense and satisfy the equation only in a correspondingly weak form.

Thus arose the problem of proving that such "generalized solutions" are "regular", namely possess enough smoothness so as to satisfy the differential equation in a classical sense. In this respect, Hilbert's twentieth problem of existence of classical solutions becomes precisely the problem of regularity of generalized solutions.

This *problem of regularity*, by which we now mean the problem to show that solutions, or extremals, which belong to a Sobolev space, are in fact Hölder continuous, resisted many attempts, but finally in 1957, E. De Giorgi [2] and J. Nash [16], independently of each other, provided a proof of it. Later, in 1960, J. Moser [15], by entirely different methods, gave another proof of their result. The Moser's argument was later extended by J. Serrin,

N. S. Trudinger and by others. While this approach (known as Moser's iteration technique), which is based on differential equation, has proved to be very useful for investigating various problems in the Euclidian spaces, it is not readily generalized to the case when one wants to deal with regularity questions on a general metric space (see, however, [1]), since the concept of a partial derivative is (generally) meaningless on a metric space, and thus there is no differential (Euler-Lagrange) equation. However, since it is possible to define a substitute for the modulus of the usual gradient to the case of general metric spaces, the approach of De Giorgi, which is essentially a variational one, can be used. This approach was developed and generalized to certain cases of non-linear equations by O. Ladyzhenskaya, N. Ural'tseva, G. Stampacchia and by others. Later, in the 80s, M. Giaquinta [4] (see also [5]), and then, in the 90s, J. Malý, W. P. Ziemer [14] tried to give to the method of De Giorgi a more transparent form.

The question of the regularity on a general metric space appeared for the first time in the paper [12] of J. Kinnunen and N. Shanmugalingam. As it is known, there exist several approaches to generalize the notion of Sobolev spaces to a metric space. Among the most important ones are the Sobolev spaces of Hajlasz [8], the Sobolev spaces via the upper gradients [11], the axiomatic Sobolev spaces of Gol'dshtein-Troyanov [6],[7] and the Sobolev spaces based on a Poincaré inequality first considered in [13] and extensively studied in [9]. Let us mention that the last two are more general ones including the two first. In the paper [12] the authors applying the De Giorgi's method studied the Hölder continuity of the quasi-minimizers of the p-Dirichlet integral on general metric spaces using the notion of upper gradients. Note however that this approach to Sobolev spaces is restricted to length spaces or quasi-convex metric spaces, the spaces which have sufficiently many rectifiable curves.

One of the objects of this note is to show that the De Giorgi's method might be applied to a very general situation, when there is not any (analog of) Sobolev space to work with. In particular, in the first part of the present paper we try to further formalize the method reducing it to the form when, for checking the Hölder continuity of a function u on a metric measure space, it is sufficient only to verify some natural hypotheses for this function and the eventual Sobolev space of functions we are going to deal with later in concrete situations. These hypotheses (see Hypotheses (H1)-(H3) in Section 1.1) are expressed in terms which do not assume that u belongs to a class of

Sobolev functions.

Then, the obtained "machinery" (checking Hypotheses (H1)-(H3)) is used to establish the main results of the present work. Namely, in the second part of the paper, we prove that if the D-structure in the sense of Gol'dshtein-Troyanov on a metric space X equipped with a Borel regular doubling measure μ is strongly local and supports a weak $(1, q)$-Poincaré inequality for some q, $q < p$, then the capacitary function minimizing the Dirichlet p-energy satisfies our hypotheses in the pair with its minimal pseudo-gradient and thus is Hölder continuous.

Finally, in the last part of the paper, we show that on a metric space X with a doubling measure μ, a function u from the Poincaré-Sobolev space on X, i.e. a function satisfying a Poincaré inequality in the pair with some function g, such that the pair (u, g) enjoys the De Giorgi condition, is Hölder continuous, provided the truncation property is valid for this Poincaré-Sobolev space.

The paper is organized as follows. In the first section we formulate the hypotheses (H1)-(H3) we are going to work with. Then we show that a function u satisfying the hypotheses (H1) and (H2) in the pair with some function g, is locally bounded. At the end of the first section we prove that if, in addition, the hypothesis (H3) is satisfied for the functions u and g, then the function u is locally Hölder continuous. Section 2 focuses on preliminaries on the axiomatic Sobolev spaces. Here we repeat some of the main definitions and constructions from [6] and [7]. In Subsection 2.3 we introduce a new notion of locality in axiomatic Sobolev spaces (the strong locality) which we need in order to establish our main results and we prove some auxiliary results (Propositions 2.11 and 2.12) which we use in the sequel. The third section is devoted to the regularity of a quasi-minimizer of the energy functional in the axiomatic setting. We prove that all three hypotheses (H1)-(H3) are satisfied by the quasi-minimizer and its minimal pseudo-gradient and, therefore, that the quasi-minimizer is Hölder continuous. In the forth section we recall the approach to Sobolev spaces on a metric space via Poincaré inequalities from [9] and show that functions from the Poincaré-Sobolev space, which have an additional property (De Giorgi condition), satisfy the hypotheses (H1)-(H3) and, thus, are Hölder continuous.

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1 De Giorgi Argument in an Abstract Setting

Throughout the paper (X, d) will be a metric space equipped with a Borel regular outer measure μ such that $0 < \mu(B) < \infty$ for any ball $B = B(R) = B(z, R) = \{x \in X : d(x, z) < R\}$ in X of positive radius. If $\sigma > 0$ and $B = B(z, R)$ is a ball, we denote by σB the ball $B(z, \sigma R)$.

For convenience we will suppose that the space X is locally compact and separable. For $1 \leq p < \infty$, $L^p_{loc}(X) = L^p_{loc}(X, d, \mu)$ is the space of measurable functions on X which are p-integrable on every relatively compact subset of X.

We will also assume that the measure μ is *doubling*, i.e. that there exists a constant $C_d \geq 1$ such that for all balls $B \subset X$ we have

$$\mu(2B) \leq C_d \mu(B)\,.$$

C_d is called the *doubling constant*.

At the beginning of this section we want to underline that in the sequel the notation $g_{(u)}$ for a function from $L^p(X)$ means no a priori dependence of this function on the given function $u \in L^p_{loc}(X)$, whereas g_u stands for the minimal pseudo-gradient of the function u (see Section 2.2).

Let Ω be an open subset of X and u be a function in $L^p(\Omega)$. In this section we prove that if the functions u and $-u$ satisfy Hypotheses (H1) and (H2) stated below in the pairs with some functions $g_{(u)}, g_{(-u)} \in L^p(\Omega)$ respectively, and if, in addition, the pair $(u, g_{(u)})$ satisfies Hypothesis (H3), then u (and, of course, $-u$) is Hölder continuous inside the set Ω.

Unless otherwise stated, C denotes a positive constant whose exact value is unimportant, can change even within a line and depends only on fixed parameters, such as X, d, μ, p and others.

1.1 List of hypotheses

The hypotheses for two functions $u, g_{(u)} \in L^p(\Omega)$, which we shall need are the following:

Hypothesis (H1) (De Giorgi condition) There exist constants $C > 0$ and $k^* \in \mathbb{R}$, such that for all $k \geq k^*,\ z \in \Omega$, and $0 < \rho < R \leq \operatorname{diam}(X)/3$ so that $B(z,R) \subset \Omega$, the following Caccioppoli type inequality on the "upper-level" sets of the function u holds

$$\int_{A(k,\rho)} g_{(u)}^p d\mu \leq \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u-k)^p d\mu, \tag{1}$$

where $A(k,r) = A_z(k,r) = \{x \in B(z,r) = B(r) \colon u(x) > k\}$ with $z \in \Omega$ being fixed.

Let η be a $\frac{C}{(R-\rho)}$-Lipschitz (cutoff) function for some $C > 0$, such that $0 \leq \eta \leq 1$, the support of η is contained in $B(\frac{R+\rho}{2})$ and $\eta = 1$ on $B(\rho)$.

Hypothesis (H2) There exists a constant $C > 0$ such that for functions $v = \eta\,(u-k)_+$ and $g_{(v)} = g_{(u)}\,\chi_{A(k,\frac{R+\rho}{2})} + \frac{C}{R-\rho}(u-k)_+$ and for some t and q, $t > p > q$, we have

$$\left(\fint_{B(\frac{R+\rho}{2})} v^t d\mu \right)^{\frac{1}{t}} \leq CR \left(\fint_{B(\frac{R+\rho}{2})} g_{(v)}^q d\mu \right)^{\frac{1}{q}}, \tag{2}$$

where k, ρ and R are as in Hypothesis (H1). Here, as usual, $(u-k)_+ = \max\{u-k, 0\}$, $\chi_{A(k,\frac{R+\rho}{2})}$ is the characteristic function of the set $A(k, \frac{R+\rho}{2})$.

Hypothesis (H3) There exist constants $C > 0$ and $\sigma \geq 1$, such that for all $h, k \in \mathbb{R}$, $h > k \geq k^*$, for the functions

$$w = u_k^h := \min\{u,h\} - \min\{u,k\} = \begin{cases} h-k & \text{if} \quad u \geq h \\ u-k & \text{if} \quad k < u < h \\ 0 & \text{if} \quad u \leq k \end{cases}$$

and $g_{(w)} = g_{(u)}\,\chi_{\{k<u\leq h\}}$ we have

$$\left(\int_{B(R)} w^q d\mu \right)^{\frac{1}{q}} \leq CR \left(\int_{B(\sigma R)} g_{(w)}^q d\mu \right)^{\frac{1}{q}}, \tag{3}$$

where q is as in Hypothesis (H2).

Note that Hypotheses (H2) and (H3) are the characteristics of the Sobolev space of functions we will work with in the next sections, whereas Hypothesis (H1) is the property of some particular functions, the functions whose regularity we want to establish. Hypotheses (H2) and (H3) are Sobolev-type inequalities that are typically true for pairs $(u, g_{(u)})$ in a sufficiently nice metric measure space: they essentially assert that the associated Poincaré inequality remains stable under cutoffs and truncations.

1.2 Boundedness

In this subsection we prove that a function $u \in L^p_{loc}(\Omega)$ satisfying Hypotheses (H1) and (H2) with some function $g_{(u)} \in L^p(\Omega)$ is locally bounded in Ω.

Theorem 1.1 *Suppose that a pair of functions $(u, g_{(u)})$ satisfies Hypotheses* (H1),(H2). *If $k' \geq k^*$, then there exist constants $C > 0$ and $\theta > 1$ such that*

$$\operatorname*{ess\,sup}_{B(\frac{R}{2})} u \leq k' + C \left(\fint_{B(R)} (u - k')_+^p d\mu \right)^{\frac{1}{p}} \left(\frac{\mu(A(k', R))}{\mu(B(\frac{R}{2}))} \right)^{\frac{\theta}{p}},$$

for all $z \in \Omega$ and $0 < R \leq diam(X)/3$

Proof Suppose that the functions $u, g_{(u)} \in L^p(\Omega)$ satisfy the conditions of the theorem, $k \in \mathbb{R}$; $\rho, R \in \mathbb{R}$ are such that $0 < \rho < R \leq \operatorname{diam}(X)/3$ and $B(R) \subset \Omega$. Replacing ρ by $\frac{R+\rho}{2}$ and C by $C/2^p$ we may rewrite the inequality (1) in the form

$$\int_{A(k,\frac{R+\rho}{2})} g_{(u)}^p d\mu \leq \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u-k)^p d\mu,$$

which is equivalent to

$$\int_{B(\frac{R+\rho}{2})} g_{(u)}^p \, \chi_{A(k,\frac{R+\rho}{2})} d\mu \leq \frac{C}{(R-\rho)^p} \int_{B(R)} (u-k)_+^p d\mu. \tag{4}$$

Let $\eta, v, g_{(v)}$ be as in Hypothesis (H2), i.e. η is Lipschitz, $v = \eta\,(u-k)_+$ and $g_{(v)} = g_{(u)}\, \chi_{A(k,\frac{R+\rho}{2})} + \frac{C}{(R-\rho)}(u-k)_+$. The Minkowski inequality and the inequality (4) imply that

$$\begin{aligned}
&\left(\int_{B(\frac{R+\rho}{2})} g_{(v)}^p d\mu\right)^{\frac{1}{p}} \\
&\quad \le \left(\int_{B(\frac{R+\rho}{2})} g_{(u)}^p \,\chi_{A(k,\frac{R+\rho}{2})} d\mu\right)^{\frac{1}{p}} + \frac{C}{(R-\rho)} \left(\int_{B(\frac{R+\rho}{2})} (u-k)_+^p d\mu\right)^{\frac{1}{p}} \\
&\quad \le \left(\frac{C}{(R-\rho)^p} \int_{B(R)} (u-k)_+^p d\mu\right)^{\frac{1}{p}} + \frac{C}{(R-\rho)} \left(\int_{B(\frac{R+\rho}{2})} (u-k)_+^p d\mu\right)^{\frac{1}{p}} \\
&\quad \le \frac{C}{(R-\rho)} \left(\int_{B(R)} (u-k)_+^p d\mu\right)^{\frac{1}{p}}.
\end{aligned}$$

From this last inequality, the inequality (2) and the Hölder inequality we obtain (recall that $q < p < t$)

$$\begin{aligned}
\left(\int_{B(\rho)} (u-k)_+^t d\mu\right)^{\frac{1}{t}} &\le \left(\int_{B(\frac{R+\rho}{2})} v^t d\mu\right)^{\frac{1}{t}} \\
&\le C\,R\,\frac{\mu(B(\frac{R+\rho}{2}))^{\frac{1}{t}}}{\mu(B(\frac{R+\rho}{2}))^{\frac{1}{q}}} \left(\int_{B(\frac{R+\rho}{2})} g_{(v)}^q d\mu\right)^{\frac{1}{q}} \\
&\le C\,R\,\left(\mu(B(\tfrac{R+\rho}{2}))\right)^{\frac{1}{t}-\frac{1}{p}} \left(\int_{B(\frac{R+\rho}{2})} g_{(v)}^p d\mu\right)^{\frac{1}{p}} \\
&\le C\,R\,\left(\mu(B(\tfrac{R+\rho}{2}))\right)^{\frac{1}{t}-\frac{1}{p}} \frac{C}{(R-\rho)} \left(\int_{B(R)} (u-k)_+^p d\mu\right)^{\frac{1}{p}} \\
&\le C\,\frac{R}{(R-\rho)}\,\left(\mu(B(\tfrac{R+\rho}{2}))\right)^{\frac{1}{t}-\frac{1}{p}} \left(\int_{B(R)} (u-k)_+^p d\mu\right)^{\frac{1}{p}}. \qquad (5)
\end{aligned}$$

The Hölder inequality implies that

$$\left(\int_{B(\rho)} (u-k)_+^p d\mu\right)^{\frac{1}{p}} \le \mu(A(k,\rho))^{\frac{1}{p}-\frac{1}{t}} \left(\int_{B(\rho)} (u-k)_+^t d\mu\right)^{\frac{1}{t}}.$$

Therefore, the inequality (5) gives us

$$\left(\int_{B(\rho)}(u-k)_+^p d\mu\right)^{\frac{1}{p}} \leq C\,\frac{R}{(R-\rho)}\left(\frac{\mu(A(k,\rho))}{\mu(B(\frac{R+\rho}{2}))}\right)^{\frac{1}{p}-\frac{1}{t}}\left(\int_{B(R)}(u-k)_+^p d\mu\right)^{\frac{1}{p}}. \qquad (6)$$

If $h>k\geq k^*$, then $h-k\leq u-k$ on $A(h,\rho)$. Therefore, as $A(h,\rho)\subset A(k,\rho)$, we conclude that

$$(h-k)^p\mu(A(h,\rho)) = \int_{A(h,\rho)}(h-k)^p d\mu \leq \int_{A(h,\rho)}(u-k)^p d\mu \leq \int_{A(k,\rho)}(u-k)^p d\mu. \qquad (7)$$

Let

$$a(h,\rho)=\mu(A(h,\rho)) \quad \text{and} \quad u(h,\rho)=\int_{A(h,\rho)}(u-h)^p d\mu.$$

Note that if $h\leq k$ and $\rho\leq r$, then $a(k,\rho)\leq a(h,r)$ and $u(k,\rho)\leq u(h,r)$.

Let $h>k\geq k^*$ and $R>\rho>0$. Then, by inequality (7) we have

$$a(h,\rho)\leq \frac{1}{(h-k)^p}u(k,\rho)\leq \frac{1}{(h-k)^p}u(k,R),$$

and by inequality (6) we obtain

$$u(h,\rho)\leq u(k,\rho)\leq C\left(\frac{R}{R-\rho}\right)^p\left(\frac{\mu(A(k,\rho))}{\mu(B(\frac{R}{2}))}\right)^{1-\frac{p}{t}}u(k,R)\,.$$

Let α be the positive solution of the equation $(t-p)\alpha^2-t(\alpha+1)=0$. From the last two inequalities we have

$$u(h,\rho)^\alpha a(h,\rho)\leq C\left(\frac{R}{R-\rho}\right)^{p\alpha}\left(\frac{\mu(A(k,\rho))}{\mu(B(\frac{R}{2}))}\right)^{\alpha(1-\frac{p}{t})}\frac{1}{(h-k)^p}\,u(k,R)^{\alpha+1}\,.$$

Let

$$\phi(h,\rho):=u(h,\rho)^\alpha\,a(h,\rho).$$

Then, by the above, we conclude that

$$\phi(h,\rho) \le C\left(\frac{R}{R-\rho}\right)^{p\alpha} \mu(B(\tfrac{R}{2}))^{\theta} \frac{1}{(h-k)^p}\,\phi(k,R)^{\theta}\,,$$

with $\theta = \alpha\left(1-\frac{p}{t}\right) = 1+\frac{1}{\alpha} > 1$.

Now, for some $k' \ge k^*$ and $j \in \mathbb{N}$, set

$$\rho_j := \frac{R}{2}\left(1+\frac{1}{2^j}\right)\,,$$

$$k_j := k' + d - \frac{d}{2^j} \ge k',$$

where

$$d^p = C2^{p(1+\alpha)^2+p\alpha}\mu(B(\tfrac{R}{2}))^{\theta}\phi(k',R)^{\theta-1}.$$

Since $p,\alpha > 0$,

$$\begin{aligned}\phi(k_j,\rho_j) &\le C\left(\frac{1+\frac{1}{2^{j-1}}}{\frac{1}{2^j}}\right)^{p\alpha} \mu(B(\tfrac{R}{2}))^{\theta}\frac{1}{\left(\frac{d}{2^j}\right)^p}\,\phi(k_{j-1},\rho_{j-1})^{\theta}\\ &\le \frac{C\,2^{p(j\alpha+\alpha+j)}}{d^p}\mu(B(\tfrac{R}{2}))^{\theta}\phi(k_{j-1},\rho_{j-1})^{\theta}\\ &= 2^{\frac{\beta}{\alpha}(j-1-\alpha)}\,\phi(k',R)^{1-\theta}\,\phi(k_{j-1},\rho_{j-1})^{\theta},\end{aligned}$$

with $\beta = p\alpha(\alpha+1)$.

By induction, conclude that

$$\phi(k_j,\rho_j) \le \frac{\phi(k',R)}{2^{\beta j}}.$$

Letting $j \to \infty$ we obtain

$$a(k'+d,R/2)\,u(k'+d,R/2)^{\alpha} = \phi(k'+d,R/2) = 0.$$

It follows that either $u(k'+d,R/2) = 0$ or $a(k'+d,R/2) = 0$. Thus,

$$\operatorname*{ess\,sup}_{B(\frac{R}{2})} u \le k'+d = k' + C\left(\int_{B(R)} (u-k')_+^p d\mu\right)^{\frac{1}{p}} \frac{\mu(A(k',R))^{\frac{\theta-1}{p}}}{\mu(B(\frac{R}{2}))^{\frac{\theta}{p}}}.$$

Q.E.D.

Corollary 1.2 *If the functions u and $-u$ satisfy Hypotheses* H1 *and* H2 *with some functions $g_{(u)}$ and $g_{(-u)}$ respectively, $u, g_{(u)}, g_{(-u)} \in L^p(\Omega)$, then*

$$\operatorname*{ess\,sup}_{B(\frac{R}{2})} |u| \le k + C \left(\fint_{B(R)} |u|^p d\mu \right)^{\frac{1}{p}}, \tag{8}$$

for all $z \in \Omega$, $k \ge k^*$, $0 < R \le \operatorname{diam}(X)/3$ and some $C > 0$.

1.3 Hölder continuity

The goal of this subsection is to prove the Hölder continuity of a function satisfying all of Hypotheses (H1)-(H3). We have the following

Theorem 1.3 *Assume that u and $-u \in L^p(\Omega)$ satisfy in the pairs with some $g_{(u)}, g_{(-u)} \in L^p(\Omega)$ Hypotheses* (H1) *and* (H2)*. If Hypothesis* (H3) *is also satisfied for the pair $(u, g_{(u)})$, then u is locally Hölder continuous.*

Proof Using the inequality (3) for our auxiliary functions w and $g_{(w)}$ with some h and k, $h > k \ge k^*$, we obtain

$$\begin{aligned}
(h-k)\mu(A(h,R)) &= \int_{A(h,R)} w d\mu \ \le \int_{B(R)} w d\mu \\
&\le \left(\int_{B(R)} w^q d\mu \right)^{\frac{1}{q}} \mu(B(R))^{1-\frac{1}{q}} \\
&\le C\, R \left(\int_{B(\sigma R)} {g_{(w)}}^q d\mu \right)^{\frac{1}{q}} \mu(B(R))^{1-\frac{1}{q}} \\
&= C\, R \left(\int_{B(\sigma R)} {g_{(u)}}^q \chi_{\{k<u\le h\}} d\mu \right)^{\frac{1}{q}} \mu(B(R))^{1-\frac{1}{q}} \\
&= C\, R \left(\int_{A(k,\sigma R)\setminus A(h,\sigma R)} {g_{(u)}}^q d\mu \right)^{\frac{1}{q}} \mu(B(R))^{1-\frac{1}{q}}.
\end{aligned}$$

Hence, by Hölder inequality we have

$$\begin{aligned}
(h-k)\mu(A(h,R)) \le C\, R &\left(\int_{A(k,\sigma R)} {g_{(u)}}^p d\mu \right)^{\frac{1}{p}} \\
&\times \left(\mu(A(k,\sigma R)) - \mu(A(h,\sigma R)) \right)^{\frac{1}{q}-\frac{1}{p}} \mu(B(R))^{1-\frac{1}{q}}.
\end{aligned}$$

As the functions u and $g_{(u)}$ satisfy the inequality (1), we conclude that

$$(h-k)\mu(A(h,R)) \leq C\left(\int_{A(k,2\sigma R)} (u-k)^p d\mu\right)^{\frac{1}{p}} \times \left(\mu(A(k,\sigma R)) - \mu(A(h,\sigma R))\right)^{\frac{1}{q}-\frac{1}{p}} \mu(B(R))^{1-\frac{1}{q}}. \tag{9}$$

Let

$$m(R) = \operatorname*{ess\,inf}_{B(R)} u \qquad \text{and} \qquad M(R) = \operatorname*{ess\,sup}_{B(R)} u.$$

Denote

$$M = M(2\sigma R), \quad m = m(2\sigma R) \ \text{ and } \ k_0 = \frac{(M+m)}{2}.$$

By Corollary 1.2, m and M are finite for small enough R (recall that we suppose that the function $-u$ satisfies Hypotheses (H1) and (H2) in the pair with some function $g_{(-u)}$). Using Theorem 1.1 with k' replaced by $k_\nu = M - 2^{-\nu-1}(M-m)$, $\nu = 0, 1, 2, ...$, we get

$$M(R/2) \leq k_\nu + C(M-k_\nu)\left(\frac{\mu(A(k_\nu,R))}{\mu(B(\frac{R}{2}))}\right)^{\frac{\theta}{p}}.$$

By Proposition 1.4 stated after the proof, it is possible to choose an integer ν, independent of z, R and u, large enough so that

$$C\left(\frac{\mu(A(k_\nu,R))}{\mu(B(\frac{R}{2}))}\right)^{\frac{\theta}{p}} < \frac{1}{2}.$$

Hence

$$M(R/2) < k_\nu + \frac{1}{2}(M-k_\nu) = M - \frac{M-m}{2^{\nu+2}},$$

and therefore

$$M(R/2) - m(R/2) \leq M(R/2) - m < (M-m)\left(1 - 2^{-(\nu+2)}\right).$$

Let

$$\operatorname{osc}(r) = M(r) - m(r)$$

denote the oscillation of u on $B(z,r)$. Then by the above inequality

$$\operatorname{osc}(R/2) < \lambda \operatorname{osc}(2\sigma R), \tag{10}$$

where $\lambda = 1 - 2^{-(\nu+2)} < 1$.

To complete the proof we iterate the inequality (10). Choose an integer $j \geq 1$ so that $(4\sigma)^{j-1} \leq \frac{R}{r} < (4\sigma)^j$. Inequality (10) implies that

$$\operatorname{osc}(r) \leq \lambda^{j-1}\operatorname{osc}((4\sigma)^{j-1}r) \leq \lambda^{j-1}\operatorname{osc}(R).$$

By the choice of j we conclude that

$$\lambda^{j-1} = (4\sigma)^{(j-1)(\log\lambda)/\log(4\sigma)} \leq (4\sigma)^{\alpha}\left(\frac{R}{r}\right)^{-\alpha},$$

where $\alpha = -(\log\lambda)/\log(4\sigma)$. Note that $0 < \alpha \leq 1$.

Finally, we have

$$\operatorname{osc}(r) \leq (4\sigma)^{\alpha}\left(\frac{r}{R}\right)^{\alpha}\operatorname{osc}(R) \leq H\, r^{\alpha},$$

with $H = (4\sigma)^{\alpha} \sup\limits_{\varrho\in(\frac{R}{4\sigma},R)} \dfrac{\operatorname{osc}(\varrho)}{\varrho^{\alpha}}$.

Therefore, after a redefinition on a set of measure zero, u is locally Hölder continuous on Ω.

Q.E.D.

Proposition 1.4 *Under the conditions of Theorem* 1.3 *there exists a sequence* $\{\alpha_\nu\} \subset \mathbb{R}$, *such that* $\alpha_\nu \to 0$ *when* $\nu \to \infty$, *and*

$$\frac{\mu(A(k_\nu, R))}{\mu(B(\frac{R}{2}))} \leq \alpha_\nu\,.$$

Proof Let

$$k_i = M - 2^{-(i+1)}(M-m),\ i = 0,1,2,\ldots$$

Then $k_i \nearrow M$ as $i \to \infty$ and $k_0 = \frac{(M+m)}{2}$. Note that

$$M - k_{i-1} = 2^{-i}(M-m) \quad \text{and} \quad k_i - k_{i-1} = 2^{-(i-1)}(M-m)\,.$$

By the inequality (9) we have

$$(k_i - k_{i-1})\mu(A(k_i, R)) \le C \left(\int_{A(k_{i-1}, 2\sigma R)} (u - k_{i-1})^p d\mu \right)^{\frac{1}{p}}$$
$$\times \left(\mu(A(k_{i-1}, \sigma R)) - \mu(A(k_i, \sigma R))\right)^{\frac{1}{q} - \frac{1}{p}} \mu(B(R))^{1 - \frac{1}{q}}.$$

Therefore, as $u - k_{i-1} \le M - k_{i-1}$ on $A(k_{i-1}, 2\sigma R)$, we conclude that

$$2^{-(i+1)} (M - m)\, \mu(A(k_i, R))$$
$$\le C\, \mu(B(2\sigma R))^{1 - \frac{1}{q} + \frac{1}{p}}\, 2^{-i}(M - m)(\mu(A(k_{i-1}, \sigma R)) - \mu(A(k_i, \sigma R)))^{\frac{1}{q} - \frac{1}{p}}.$$

Note that if $\nu \ge i$, then $\mu(A(k_\nu, R)) \le \mu(A(k_i, R))$. Hence

$$\mu(A(k_\nu, R)) \le 2\, C\, \mu(B(2\sigma R))^{1 - \frac{1}{q} + \frac{1}{p}} \left(\mu(A(k_{i-1}, \sigma R)) - \mu(A(k_i, \sigma R))\right)^{\frac{1}{q} - \frac{1}{p}}.$$

Now raising the last inequality to the power $\frac{pq}{p-q}$ and then summing the result over $i = 1, 2, ..., \nu$, we get

$$\nu \mu(A(k_\nu, R))^{\frac{pq}{p-q}} \le C\, \mu(B(2\sigma R))^{\frac{pq}{p-q} - 1} \left(\mu(A(k_0, \sigma R)) - \mu(A(k_\nu, \sigma R))\right)$$
$$\le C\, \mu(B(2\sigma R))^{\frac{pq}{p-q}}.$$

Dividing both parts of the last inequality by $\mu(B(\frac{R}{2}))^{\frac{pq}{p-q}}$ and using the doubling property of μ, we obtain the result.

Q.E.D.

2 Preliminaries on Axiomatic Sobolev Spaces

In this section we recall basic definitions and give a brief summary of the axiomatic theory of Sobolev spaces developed by V.M. Gol'dshtein and M. Troyanov in [6], which will constitute the general setup of our study in Section 3. We refer the reader to this paper and to the paper [7] for more details on the axiomatic theory of Sobolev spaces.

2.1 D-structure on a metric measure space

Definition 2.1 (D-structure) *A D-structure on (X, d, μ) is an operation which associates to each function $u \in L^p_{loc}(X)$ a collection $D[u]$ of measurable functions $g : X \to \mathbb{R}_+ \cup \{\infty\}$ (called the pseudo-gradients of u). The correspondence $u \to D[u]$ is supposed to satisfy the following axioms A1-A5:*

Axiom A1 (Non triviality) If $u : X \to \mathbb{R}$ is non-negative and k-Lipschitz, then the function

$$g := k\chi_{\text{supp}(u)} = \begin{cases} k & \text{on} \quad \text{supp}(u) \\ 0 & \text{on} \quad X \setminus \text{supp}(u) \end{cases}$$

belongs to $D[u]$.

Axiom A2 (Upper linearity) If $g_1 \in D[u_1]$, $g_2 \in D[u_2]$ and $g \geq |\alpha|g_1 + |\beta|g_2$ almost everywhere, then $g \in D[\alpha u_1 + \beta u_2]$.

Axiom A3 (Strong Leibnitz rule) Let $u \in L^p_{loc}(X)$. If $g \in D[u]$, then for any bounded Lipschitz function $\varphi : X \to \mathbb{R}$ the function

$$h(x) = (|\varphi|g(x) + \text{Lip}(\varphi)|u(x)|)$$

belongs to $D[\varphi u]$.

Axiom A4 (Lattice property) Let $u := \max\{u_1, u_2\}$ and $v := \min\{u_1, u_2\}$ where $u_1, u_2 \in L^p_{loc}(X)$. If $g_1 \in D[u_1]$, $g_2 \in D[u_2]$, then

$$g := \max\{g_1, g_2\} \in D[u] \cap D[v] \,.$$

Axiom A5 (Completeness) Let $\{u_i\}$ and $\{g_i\}$ be two sequences of functions such that $g_i \in D[u_i]$ for all i. Assume that $u_i \to u$ in $L^p_{loc}(X)$ topology and $(g_i - g) \to 0$ in L^p topology, then $g \in D[u]$.

Remark Originally, in [6] in the place of Axiom A3 stated here one postulates the following

Axiom A3*(Leibnitz rule) Let $u \in L^p_{loc}(X)$. If $g \in D[u]$, then for any bounded Lipschitz function $\varphi : X \to \mathbb{R}$ the function

$$h(x) = (\sup|\varphi|g(x) + \text{Lip}(\varphi)|u(x)|)$$

belongs to $D[\varphi u]$ (The absolute value of φ is replaced by $\sup|\varphi|$).

This "weaker" version of the Leibnitz rule allows the authors to include in the class of axiomatic Sobolev spaces such spaces as graphs (combinatorial Sobolev spaces) and Sobolev spaces of Hajłasz. Note, however, that these "global" spaces do not satisfy certain localization properties without which it is not clear how it would be possible to achieve the regularity results of the present paper.

2.2 Some properties of D-structure. Axiomatic Sobolev space.

Definition 2.2 (Poincaré inequality) *One says that a D-structure on a metric measure space X supports a weak (s,q)-Poincaré inequality, $s,q \geq 1$, if there exist two constants $\sigma \geq 1$ and $C_P > 0$ such that*

$$\left(\fint_B |u - u_B|^s d\mu \right)^{1/s} \leq C_P \, r \left(\fint_{\sigma B} g^q d\mu \right)^{1/q} \tag{11}$$

for any ball $B \subset X$, any $u \in L^p_{loc}(X)$ and any $g \in D[u]$. Here r is the radius of B. Recall that

$$u_B = \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u d\mu.$$

By the Hölder inequality, a weak (s,q)-Poincaré inequality implies weak (s',q')-Poincaré inequalities with the same σ for all $s' \leq s$ and $q' \geq q$. On the other hand, by Theorem 5.1 in [9], a weak $(1,q)$-Poincaré inequality implies a weak (s,q)-Poincaré inequality for some $s > q$ and possibly a new σ.

We define a notion of energy and the associated Sobolev space as follows:

Definition 2.3 (Energy and Sobolev space) *The p-Dirichlet energy of a function $u \in L^p_{loc}(X)$ is defined to be*

$$\mathcal{E}_p(u) = \inf \left\{ \int_X g^p d\mu : g \in D[u] \right\},$$

and the p-Dirichlet space is the space $\mathcal{L}^{1,p}(X)$ of functions from $L^p_{loc}(X)$ with finite p-energy. The Sobolev space is then the space

$$W^{1,p}(X) := \mathcal{L}^{1,p}(X) \cap L^p(X).$$

Theorem 2.4 *$W^{1,p}(X)$ is a Banach space with norm*

$$\|u\|_{W^{1,p}(X)} = \left(\int_X |u|^p d\mu + \mathcal{E}_p(u) \right)^{1/p}.$$

Proof See [6].

Proposition 2.5 *Assume that* $1 < p < \infty$*. Then for any function* $u \in \mathcal{L}^{1,p}(X)$*, there exists a unique function* $g_u \in D[u]$ *such that* $\int_X g_u^p d\mu = \mathcal{E}_p(u)$*.*

Proof See [6].

The function g_u is called the *minimal pseudo-gradient* of u.

2.3 Locality in axiomatic Sobolev space

Definition 2.6 (Locality) *We say that a* D*-structure is* local *if, in addition to Axioms A1-A5, the following property holds:* If u is constant a.e. on a relatively compact subset A$\subset X$, then $\mathcal{E}_p(u|A) = 0$, where

$$\mathcal{E}_p(u|A) := \inf \left\{ \int_A g^p d\mu \,\middle|\, g \in D[u] \right\}$$

is the local p-Dirichlet energy of u.

Definition 2.7 (Strict locality) *We say that a* D*-structure is* strictly local *if, in addition to Axioms A1-A5, we have* $(g\chi_{\{v>0\}}) \in D[v_+]$ *for any* $v \in \mathcal{L}^{1,p}(X)$ *and* $g \in D[v]$*.*

Lemma 2.8 *If the* D*-structure is strictly local, then it is local.*

Proof See [6].

Lemma 2.9 *If the* D*-structure is strictly local and a pair of functions* $u, v \in L^p_{loc}(X)$ *is such that* $u = v$ *on a relatively compact set* $A \subset X$*, then*

$$\mathcal{E}_p(v|A) = \mathcal{E}_p(u|A)\,.$$

Proof See [6].

For the proof of the theorem 3.2 we will need a still stronger notion of locality which we introduce in the following

Definition 2.10 (Strong locality) *We say that a* D*-structure is* strongly local *if, in addition to Axioms A1-A5, the following property holds:*

Let $u_1, u_2 \in \mathcal{L}^{1,p}(X)$. If $g_1 \in D[u_1], g_2 \in D[u_2]$ and

$$g(x) = \begin{cases} g_1(x) & \text{if} \quad u_1(x) < u_2(x) \\ g_2(x) & \text{if} \quad u_1(x) > u_2(x) \\ \min\{g_1(x), g_2(x)\} & \text{if} \quad u_1(x) = u_2(x)\,, \end{cases}$$

then $g \in D[\min\{u_1, u_2\}]$.

This property enables one to "paste" two Sobolev functions along the set where they coincide. Note that if we take one of the functions u_1 or u_2 to be identically zero in the last definition, we obtain the strict locality of the D-structure.

Proposition 2.11 *Let $u \in \mathcal{L}^{1,p}(X)$ and $A \subset X$ be a relatively compact set. If the D-structure on the space X is strongly local, then*

$$\mathcal{E}_p(u|A) = \int_A g_u^p d\mu\ ,$$

in particular, if $u_1, u_2 \in \mathcal{L}^{1,p}(X)$ are such that $u_1 = u_2$ a.e. on A, then

$$\int_A g_{u_1}^p d\mu = \int_A g_{u_2}^p d\mu$$

Proof The result will easily follow if we would show that for all $g \in D[u]$

$$g_u \leq g \quad \text{a.e. on } X.$$

Suppose that the last assertion is not true, i.e. there exist a subset $A \subset X$, $\mu(A) > 0$, and $g \in D[u]$ such that $g < g_u$ on A. Let B, $A \subset B \subset X$, be the subset of X such that $g_u \leq g$ on $X \setminus B$ and $g < g_u$ on B. From the strong locality of the D-structure it will follow then that the function $h = \min\{g_u, g\}$ belongs to $D[u = \min\{u, u\}]$ and we will have

$$\int_X h^p\, d\mu = \int_B g^p\, d\mu + \int_{X\setminus B} g_u^p\, d\mu < \int_B g_u^p\, d\mu + \int_{X\setminus B} g_u^p\, d\mu = \int_X g_u^p\, d\mu\,,$$

which contradicts the minimality of g_u.

Q.E.D.

The locality of the D-structure together with a Poincaré inequality imply certain connectedness of the space X. Namely, in the sequel we will need the following

Proposition 2.12 *If the space X admits a D-structure which is strictly local and supports a weak $(1,q)$-Poincaré inequality for some $0<q<p$, then for every $z\in X$ and $0<r<R<diam(X)/3$, we have*

$$\mu\left(B(z,R)\setminus B(z,r)\right)>0\,.$$

In other words, the measure of sufficiently small annuli in X is positive. If, in addition, the measure μ is doubling, then there exists γ, $0<\gamma<1$, independent of R such that

$$\frac{\mu(B(z,\frac{R}{2}))}{\mu(B(z,R))}\le\gamma\,.$$

Proof With $z\in X$ fixed, for some δ, $0<\delta<\frac{R-r}{2}$, let us denote

$$S:=\left\{x\in X\mid d(x,z)=\tfrac{R+r}{2}\right\}$$

the sphere of radius $\frac{R+r}{2}$ centered at z, and

$$S_\delta:=\{y\in X\mid d(y,x)\le\delta\ \text{ for some }\ x\in S\}$$

its δ-neighborhood. Denote also

$$E=B\left(\tfrac{R+r}{2}\right)\setminus S_\delta\quad\text{ and }\quad F=X\setminus(S_\delta\cup E)\,.$$

Suppose now that the set S_δ is empty. Then the function

$$u:=\begin{cases}1 & \text{on}\quad E\\ 0 & \text{on}\quad F\end{cases}$$

is a k-Lipschitz with $k=\frac{1}{2\delta}$. Hence, by Axiom A1, the function

$$g:=k\chi_{\mathrm{supp}(u)}=\tfrac{1}{2\delta}\chi_E$$

belongs to $D[u]$. Therefore, u has a finite p-energy, i.e. $u\in\mathcal{L}^{1,p}(X)$.

As the strict locality of the D-structure implies its locality, it will follow then that for any $\varepsilon>0$, there exists a function $g_1\in D[u]$ such that

$$\int_E g_1^p d\mu<\varepsilon.$$

From the strict locality itself, it will follow further that the function

$$g_2 = g_1 \chi_{\{u>0\}} \in D[u^+] = D[u],$$

since $u^+ = u$. Note that

$$g_2 = \begin{cases} g_1 & \text{on} \quad E \\ 0 & \text{on} \quad F\,. \end{cases}$$

As $R < \operatorname{diam}(X)/3$, there are points in X lying in the complement of $B(R)$. Let $R_1 > R$ be large enough so that some of these points lie inside the ball $B(R_1)$.

The right-hand side of the weak $(1,q)$-Poincaré inequality applied to the functions u and $g_2 \in D[u]$ on the ball $B(R_1)$ with $\sigma > 1$ can be estimated as follows

$$\begin{aligned} C_P R_1 \left(\fint_{B(\sigma R_1)} g_2^q d\mu \right)^{1/q} &\leq C_P R_1 \left(\fint_{B(\sigma R_1)} g_2^p d\mu \right)^{1/p} \\ &= C_P R_1 \left(\frac{1}{\mu(B(\sigma R_1))} \int_E g_1^p d\mu \right)^{1/p} \\ &< C_P R_1 \left(\frac{\varepsilon}{\mu\left(B(r)\right)} \right)^{1/p}, \end{aligned}$$

and, thus, can be made arbitrarily small by varying ε. The Poincaré inequality will imply then that the function u is a.e. constant on the ball $B(R_1)$.

This contradiction shows that the set S_δ is non-empty and, hence, there exists a point $x_0 \in S_\delta$ and we have

$$\mu\left(B(z,R) \setminus B(z,r)\right) \geq \mu(B(x_0,\rho)) > 0\,,$$

for some $\rho < \left(\frac{R-r}{2} - \delta\right)$.

Suppose now that the measure μ is doubling. Taking $r = \frac{R}{2}$ and $\delta = \frac{R}{8}$ we see that there exists a point x_0 in $S_{\frac{R}{8}}$. As $B(z, \frac{R}{2}) \subset B(z,R) \setminus B(x_0, \frac{R}{8})$ and $B(z,R) \subset 15\, B(x_0, \frac{R}{8})$, the doubling property of μ implies

$$\frac{\mu(B(z,\frac{R}{2}))}{\mu(B(z,R))} \leq 1 - \frac{\mu(B(x_0,\frac{R}{8}))}{\mu(B(z,R))} \leq \gamma\,,$$

where $0 < \gamma < 1$.

Q.E.D.

2.4 The variational capacity

For the open set $\Omega \subset X$ we denote by $C_0(\Omega)$ the set of continuous functions $u : \Omega \to \mathbb{R}$ such that $\mathrm{supp}(u) \Subset \Omega$, i.e. $\mathrm{supp}(u)$ is a compact subset of Ω. $\mathcal{L}_0^{1,p}(\Omega)$ is then the closure of $C_0(\Omega) \cap \mathcal{L}^{1,p}(X)$ in $\mathcal{L}^{1,p}(X)$ for the norm

$$\|u\|_{\mathcal{L}^{1,p}(\Omega,Q)} = \left(\int_Q |u|^p d\mu + \mathcal{E}_p(u) \right)^{1/p},$$

where $Q \Subset \Omega$ is a fixed relatively compact subset of positive measure.

Definition 2.13 (Capacity) *The variational p-capacity of a pair $F \subset \Omega \subset X$ (where F is arbitrary) is defined as*

$$\mathrm{Cap}_p(F,\Omega) := \inf\{\mathcal{E}_p(u) |\ u \in \mathcal{A}_p(F,\Omega)\},$$

where the set of admissible functions is defined by

$$\mathcal{A}_p(F,\Omega) := \left\{ u \in \mathcal{L}_0^{1,p}(\Omega) \,\middle|\, u \geq 1 \text{ on a neighbourhood of } F \text{ and } u \geq 0 \text{ a.e.} \right\}.$$

If $\mathcal{A}_p(F,\Omega) = \emptyset$, then we set $\mathrm{Cap}_p(F,\Omega) = \infty$. If $\Omega = X$, we simply write $\mathrm{Cap}_p(F,\Omega) = \mathrm{Cap}_p(F)$.

We now state a result about the existence and uniqueness of extremal functions for p-capacity. We first need two definitions:

Definition 2.14 (a) *A set $S \subset X$ is p-polar (or p-null) if for any pair of open relatively compact sets $\Omega_1 \subset \Omega_2 \neq X$ such that $\mathrm{dist}(\Omega_1, X \setminus \Omega_2) > 0$, we have $\mathrm{Cap}_p(S \cap \Omega_1, \Omega_2) = 0$.*

(b) *A property is said to hold p-quasi-everywhere if it holds everywhere except on a p-polar set.*

Definition 2.15 *A Borel measure τ is said to be absolutely continuous with respect to p-capacity if $\tau(S) = 0$ for all p-polar subsets $S \subset X$*

For any Borel subset $F \subset X$ we denote by $\mathcal{M}_p(F)$ the set of all probability measures τ on X which are absolutely continuous with respect to p-capacity and whose support is contained in F.

Definition 2.16 *A subset F is said to be p-fat if it is a Borel subset and $\mathcal{M}_p(F) \neq \emptyset$.*

Theorem 2.17 *Let $F \subset X$ be a p-fat subset $(1 < p < \infty)$ of the space X, such that $\mathrm{Cap}_p(F) < \infty$. Then there exists a unique function $u^* \in \mathcal{L}_0^{1,p}(X)$ such that $u^* = 1$ p-quasi-everywhere on F and $\mathcal{E}_p(u^*) = \mathrm{Cap}_p(F)$. Furthermore $0 \le u^* \le 1$ for all $x \in X$.*

The function u^* is called the *capacitary function* of the condenser F.

Proof See [7].

Definition 2.18 (Quasi-minimizer) *A function $u \in L^p_{loc}(\Omega)$ is called a* quasi-minimizer *of the energy $\mathcal{E}_p$ on the set $\Omega \subset X$ if there exists a constant $K > 0$ such that for all functions $\varphi \in \mathcal{L}^{1,p}(X)$ with $supp(\varphi) \Subset \Omega$ the inequality*

$$\int_{supp(\varphi)} g_u^p d\mu \le K \int_{supp(\varphi)} g_{u+\varphi}^p d\mu$$

holds (where, as usual, $g_{u+\varphi}$ is the minimal pseudo-gradient of $u+\varphi$). When $K = 1$, the corresponding quasi-minimizer is called the minimizer *of the energy functional $\mathcal{E}_p$.*

Proposition 2.19 *Assume that the D-structure on X is strongly local. Then the capacitary function u^* of the condenser F is a minimizer of $\mathcal{E}_p$ on the set $X \setminus F$.*

Proof Let $\varphi \in \mathcal{L}^{1,p}(X)$ with $\mathrm{supp}(\varphi) \Subset X \setminus F$ and $v = u^* + \varphi$. Then

$$v = u^* \text{ on } X \setminus \mathrm{supp}(\varphi)\,,$$

and the strong locality implies that

$$\int_{X \setminus \mathrm{supp}(\varphi)} g_v^p d\mu = \int_{X \setminus \mathrm{supp}(\varphi)} g_{u^*}^p d\mu\,.$$

As the function $v^+ \in \mathcal{A}_p(F, X)$, by the energy minimizing property of u^* we have

$$\begin{aligned}\int_{\mathrm{supp}(\varphi)} g_{u^*}^p d\mu + \int_{X \setminus \mathrm{supp}(\varphi)} g_{u^*}^p d\mu = \int_X g_{u^*}^p d\mu \le \int_X g_{v^+}^p d\mu \\ \le \int_X g_v^p d\mu = \int_{\mathrm{supp}(\varphi)} g_v^p d\mu + \int_{X \setminus \mathrm{supp}(\varphi)} g_v^p d\mu\,.\end{aligned}$$

Thus,

$$\int_{\mathrm{supp}(\varphi)} g_{u^*}^p d\mu \le \int_{\mathrm{supp}(\varphi)} g_{u^*+\varphi}^p d\mu$$

and u^* is a minimizer.

Q.E.D.

3 Regularity of Quasi-minimizers in Axiomatic Sobolev Spaces

In this section we assume that the metric measure space (X, d, μ) is equipped with a D-structure and we derive the Hölder continuity inside the domain $\Omega \subset X$ of a quasi-minimizer of the p-Dirichlet energy of the axiomatic setting of the previous section.

For the proof of Propositions 3.3 below we will need the following

Lemma 3.1 *Let $f(r)$ be a nonnegative function defined on the interval $[R_1, R_2]$, where $R_1 \ge 1$. Suppose that for all $R_1 \le r_1 < r_2 \le R_2$,*

$$f(r_1) \le \theta f(r_2) + \frac{A}{(r_2 - r_1)^\alpha} + B\,,$$

where $A, B \ge 0$, $\alpha > 0$ and $0 \le \theta < 1$. Then there exists $C > 0$ depending only on α and θ such that for all $R_1 \le r_1 < r_2 \le R_2$,

$$f(r_1) \le C\left(\frac{A}{(r_2 - r_1)^\alpha} + B\right)\,.$$

Proof See, e.g., Lemma 5.1 in [5].

The main result of this section is the following

Theorem 3.2 *Assume that the D-structure on X is strongly local. If it also supports a weak $(1, q)$-Poincaré inequality for some q, $q < p$, then a quasi-minimizer u^* of the energy functional $\mathcal{E}_p$ on the set Ω is locally Hölder continuous inside the set $\Omega \subset X$.*

The result of this theorem follows from Theorem 1.3 and the following

Proposition 3.3 *Under the conditions of Theorem* 3.2*, both the quasi-minimizer u^* with its minimal pseudo-gradient g_{u^*} and the pair $(-u^*, g_{u^*})$ satisfy Hypotheses* (H1) *and* (H2)*. In addition, Hypothesis* (H3) *is satisfied either by the pair (u^*, g_{u^*}) or by $(-u^*, g_{u^*})$.*

Proof Hypothesis (H1): Let $B(z, R) \subset \Omega$, $0 < \rho < R$ and η be a $\frac{1}{(R-\rho)}$-Lipschitz cutoff function so that $0 \le \eta \le 1$, $\eta = 1$ on $B(z, \rho)$ and the support of η is contained in $B(z, R)$. Set

$$v = u^* - \eta \max\{u^* - k, 0\},$$

where $k \ge k^*$, k^* will be chosen in the proof of Hypothesis (H3). Observe that

$$v = \begin{cases} (1-\eta)(u^* - k) + k & \text{on} \quad A(k, R) \\ u^* & \text{on} \quad \Omega \setminus A(k, R)\,, \end{cases}$$

where $A(k, R) = \{x \in B(R) : \ u^*(x) > k\}$.

Note that obviously $v = u^* + (v - u^*)$ and that $(v - u^*) = 0$ on $\Omega \setminus A(k, R)$. As u^* is a quasi-minimizer we have

$$\int_{A(k,R)} g_{u^*}^p d\mu \le K \int_{A(k,R)} g_v^p d\mu\,,$$

where K is the constant in the definition the quasi-minimizer u^*.

From the strong locality of the D-structure it follows (see Proposition 2.11) that

$$\int_{A(k,R)} g_v^p d\mu = \int_{A(k,R)} g_{(1-\eta)(u^*-k)+k}^p d\mu\,,$$

Note that $\frac{1}{(R-\rho)} \in D[1-\eta]$. Axioms A1, A2 and A3 imply

$$(u^* - k)\tfrac{1}{(R-\rho)} + (1-\eta) g_{u^*} \in D[(1-\eta)(u^* - k) + k].$$

From this and the last two inequalities we obtain

$$\begin{aligned} \int_{A(k,\rho)} g_{u^*}^p d\mu &\le K \int_{A(k,R)} \left((u^* - k)\frac{1}{(R-\rho)} + (1-\eta) g_{u^*} \right)^p d\mu \\ &\le \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u^* - k)^p d\mu + C \int_{A(k,R)\setminus A(k,\rho)} g_{u^*}^p d\mu, \end{aligned}$$

where $C = K\,2^{p-1}$. Here we used the fact that $1-\eta = 0$ on $A(k,\rho)$. Adding the term $C\int_{A(k,\rho)} g_{u^*}^p d\mu$ to the left and right hand sides of the inequality above, we see that

$$(1+C)\int_{A(k,\rho)} g_{u^*}^p d\mu \le C\int_{A(k,R)} g_{u^*}^p d\mu + \frac{C}{(R-\rho)^p}\int_{A(k,R)} (u^*-k)^p d\mu,$$

or

$$\int_{A(k,\rho)} g_{u^*}^p d\mu \le \frac{C}{1+C}\int_{A(k,R)} g_{u^*}^p d\mu + \frac{C}{1+C}\frac{1}{(R-\rho)^p}\int_{A(k,R)} (u^*-k)^p d\mu.$$

Hence, if $\rho < r \le R$, then

$$\int_{A(k,\rho)} g_{u^*}^p d\mu \le \frac{C}{1+C}\int_{A(k,r)} g_{u^*}^p d\mu + \frac{C}{1+C}\frac{1}{(r-\rho)^p}\int_{A(k,R)} (u^*-k)^p d\mu.$$

From the last inequality and Lemma 3.1 we conclude that there is a constant C depending on p and K only so that

$$\int_{A(k,\rho)} g_{u^*}^p d\mu \le \frac{C}{(R-\rho)^p}\int_{A(k,R)} (u^*-k)^p d\mu$$

and hence the pair u^* and g_{u^*} satisfies Hypothesis (H1).

<u>Hypothesis (H2)</u> Let η be the Lipschitz function from Hypothesis (H2) and $v = \eta\,(u^*-k)_+$. Axioms A1, A2, A3 and the strict locality of D-structure imply that function $\eta\, g_{u^*}\,\chi_{\{u^*>k\}} + \frac{C}{(R-\rho)}(u^*-k)_+ \in D[v]$. Obviously, $v = v\chi_{\{v>0\}}$. Hence from the strict locality it follows that

$$\Big(\eta\, g_{u^*}\,\chi_{\{u^*>k\}} + \tfrac{C}{(R-\rho)}(u^*-k)_+\Big)\,\chi_{\{v>0\}} \in D[v]\,.$$

Therefore, since the D-structure of the space X supports a weak $(1,q)$-Poincaré inequality and thus a weak (t,q)-Poincaré inequality for some t, $t > p > q$, and $\tau > 1$ (see Section 2.2), we have by the Minkowski inequality and the facts that $g_{u^*} \geq 0$, $0 \leq \eta \leq 1$,

$$
\begin{aligned}
&\left(\fint_{B(R+\rho)} v^t d\mu\right)^{\frac{1}{t}} \\
&\quad \leq \left(\fint_{B(R+\rho)} |v - v_{B(R+\rho)}|^t d\mu\right)^{\frac{1}{t}} + |v_{B(R+\rho)}| \\
&\quad \leq C(R+\rho)\left(\fint_{B(\tau(R+\rho))} \left(\eta\, g_{u^*}\, \chi_{\{u^*>k\}} + \frac{C}{(R-\rho)}(u^*-k)_+\right)^q \chi_{\{v>0\}}\, d\mu\right)^{\frac{1}{q}} \\
&\qquad\qquad + |v_{B(R+\rho)}| \\
&\quad \leq C(R+\rho)\left(\fint_{B(\tau(R+\rho))} \left(g_{u^*}\, \chi_{A(k,\frac{R+\rho}{2})} + \frac{C}{(R-\rho)}(u^*-k)_+\right)^q \chi_{\{v>0\}}\, d\mu\right)^{\frac{1}{q}} \\
&\qquad\qquad + |v_{B(R+\rho)}|
\end{aligned}
$$

$$
\leq CR\left(\fint_{B(\frac{R+\rho}{2})} g_{(v)}^q d\mu\right)^{\frac{1}{q}} + |v_{B(R+\rho)}|. \tag{12}
$$

In the last inequality we denoted $g_{(v)} = g_{u^*}\, \chi_{A(k,\frac{R+\rho}{2})} + \frac{C}{(R-\rho)}(u^*-k)_+$ and used the doubling property of μ and the fact that $\{v>0\} \subset B(\frac{R+\rho}{2})$.

Since $v = \eta\,(u-k)_+$ is non-negative, by the Hölder inequality we obtain

$$
\begin{aligned}
|v_{B(R+\rho)}| &= \frac{1}{\mu(B(R+\rho))}\int_{B(R+\rho)} v d\mu = \frac{1}{\mu(B(R+\rho)}\int_{B(R+\rho)} v\, \chi_{\{v>0\}} d\mu \\
&\leq \left(\fint_{B(R+\rho)} v^t d\mu\right)^{\frac{1}{t}} \left(\frac{\mu\left(\{x \in B\left(R+\rho\right) : v(x) > 0\}\right)}{\mu\left(B\left(R+\rho\right)\right)}\right)^{1-\frac{1}{t}}.
\end{aligned}
$$

As $\operatorname{supp} v \subset B\left(\frac{R+\rho}{2}\right)$, the property of the measure μ proved under assumptions of the theorem in Proposition 2.12 implies that

$$
\frac{\mu\left(\{v>0\}\right)}{\mu\left(B\left(R+\rho\right)\right)} \leq \frac{\mu\left(B\left(\frac{R+\rho}{2}\right)\right)}{\mu\left(B\left(R+\rho\right)\right)} \leq \gamma
$$

for some γ, $0 < \gamma < 1$.

Hence from the previous inequality and the inequality (12) we obtain

$$(1-\gamma^{1-\frac{1}{t}})\left(\fint_{B(R+\rho)} v^t d\mu\right)^{\frac{1}{t}} \leq CR\left(\fint_{B(\frac{R+\rho}{2})} g_{(v)}^q d\mu\right)^{\frac{1}{q}}.$$

From the doubling property of μ finally we will have

$$\left(\fint_{B(\frac{R+\rho}{2})} v^t d\mu\right)^{\frac{1}{t}} \leq CR\left(\fint_{B(\frac{R+\rho}{2})} g_{(v)}^q d\mu\right)^{\frac{1}{q}},$$

for some constant $C > 0$.

Note that since by Axiom A2, $D[-u] = D[u]$ for any function $u \in L^p_{loc}(X)$, the function $-u^*$ is also a quasi-minimizer of the energy functional $\mathcal{E}_p$. Thus Hypotheses (H1) and (H2) are also true for the function $-u^*$ and the pseudo-gradient g_{u^*}.

Hypothesis (H3): By Corollary 1.2 the function u^* is now locally bounded. Let

$$k^* := \frac{\operatorname{ess\,sup}_{B(\sigma R)} u^* + \operatorname{ess\,inf}_{B(\sigma R)} u^*}{2}.$$

For $h > k \geq k^*$, the function $g_{(w)} := g_{u^*}\, \chi_{\{k<u^*\leq h\}}$ belongs to the $D[w]$ for $w = u_k^h = \min\{u^*, h\} - \min\{u^*, k\} = h - k - (h - k - (u^* - k)_+)_+$. Indeed, from the strict locality of D-structure, we have $g_{u^*}\, \chi_{\{u^*>k\}} \in D[(u^* - k)_+]$, and thus $g_{u^*}\, \chi_{\{u^*>k\}}\, \chi_{\{h-k-(u^*-k)_+>0\}} \in D[(h - k - (u^* - k)_+)_+]$. Axioms A1 and A2 imply finally that $g_{u^*}\, \chi_{\{k<u^*\leq h\}} \in D[w]$, as $\chi_{\{u^*>k\}}\, \chi_{\{h-k-(u^*-k)_+>0\}} = \chi_{\{k<u^*<h\}}$ and $g_{u^*}\, \chi_{\{k<u^*\leq h\}} \geq g_{u^*}\, \chi_{\{k<u^*<h\}}$.

If $\mu(\{x \in B(R) : u^*(x) > k^*\}) > \frac{1}{2}\mu(B(R))$, then

$$\mu(\{x \in B(R) : -u^*(x) \leq -k^*\}) > \tfrac{1}{2}\mu(B(R)).$$

Consequently we have

$$\mu(\{x \in B(R) : -u^*(x) > -k^*\}) \leq \tfrac{1}{2}\mu(B(R)),$$

and hence we could consider $-u^*$ rather then u^* in our discussion. Then if we prove that $-u^*$ is Hölder continuous, obviously the function u^* will be Hölder continuous too. Therefore, without loss of generality, we may assume that $\mu(\{x \in B(R) : u^*(x) > k^*\}) \le \frac{1}{2}\mu(B(R))$, and thus, for $k \ge k^*$, that

$$\begin{aligned}\mu(\{x \in B(R) : w > 0\}) &\le \mu(\{x \in B(R) : u^*(x) > k\}) \\ &\le \mu(\{x \in B(R) : u^*(x) > k^*\}) \le \tfrac{1}{2}\mu(B(R)). \quad (13)\end{aligned}$$

Since $g_{(w)} = g_{u^*}\chi_{\{k<u^*\le h\}} \in D[w]$ and because a weak (t,q)-Poincaré inequality, $t > q$, implies a weak (q,q)-Poincaré inequality, we have

$$\begin{aligned}\left(\fint_{B(R)} w^q d\mu\right)^{\frac{1}{q}} &\le \left(\fint_{B(R)} |w - w_{B(R)}|^q d\mu\right)^{\frac{1}{q}} + |w_{B(R)}| \\ &\le C_P R \left(\fint_{B(\sigma R)} g_{(w)}^q d\mu\right)^{\frac{1}{q}} + |w_{B(R)}|.\end{aligned}$$

By the Hölder inequality we obtain

$$\begin{aligned}|w_{B(R)}| &= \frac{1}{\mu(B(R))}\int_{B(R)} w d\mu = \frac{1}{\mu(B(R))}\int_{B(R)} w\,\chi_{\{w>0\}} d\mu \\ &\le \left(\fint_{B(R)} w^q d\mu\right)^{\frac{1}{q}} \left(\frac{\mu\left(\{x \in B(R) : w(x) > 0\}\right)}{\mu(B(R))}\right)^{1-\frac{1}{q}}.\end{aligned}$$

The inequality (13) will then imply

$$\left(\fint_{B(R)} w^q d\mu\right)^{\frac{1}{q}} \le C_w R \left(\fint_{B(\sigma R)} g_{(w)}^q d\mu\right)^{\frac{1}{q}},$$

where the constant $C_w = C_P/(1 - \left(\frac{1}{2}\right)^{1-\frac{1}{q}})$. Thus, finally we obtain

$$\left(\int_{B(R)} w^q d\mu\right)^{\frac{1}{q}} \le C_w R \left(\int_{B(\sigma R)} g_{(w)}^q d\mu\right)^{\frac{1}{q}},$$

and hence Hypothesis (H3) is satisfied.

Q.E.D.

4 Regularity in the Class of Poincaré-Sobolev Functions

In this section we shortly recall the approach to Sobolev spaces on a metric space using Poincaré inequalities (see [9] for the definitions given below) and prove the Hölder continuity of certain extremal functions in these spaces.

4.1 Poincaré-Sobolev functions

Definition 4.1 (Poincaré inequality) *Let $u \in L^1_{loc}(X)$ and $g : X \to [0, \infty]$ be Borel measurable functions. We say that the pair (u, g) satisfies a (s, q)-Poincaré inequality in $\Omega \subset X$, $s, q \geq 1$, if there exist two constants $\sigma \geq 1$ and $C_P > 0$ such that the inequality*

$$\left(\fint_B |u - u_B|^s d\mu \right)^{1/s} \leq C_P \, r \left(\fint_{\sigma B} g^q d\mu \right)^{1/q} \tag{14}$$

holds on every ball B with $\sigma B \subset \Omega$, where r is the radius of B.

Definition 4.2 (Poincaré-Sobolev functions) *A function $u \in L^1_{loc}(X)$ for which there exists $0 \leq g \in L^q(X)$ such that the pair (u, g) satisfies a $(1, q)$-Poincaré inequality in X is called a Poincaré-Sobolev function. We denote by $PW^{1,q}(X)$ the set of all Poincaré-Sobolev functions.*

The Poincaré inequality (14) is the only relationship between the functions u and g. Working in this setting P. Hajlasz and P. Koskela developed in [9] quite a rich theory of these Sobolev-type functions on metric spaces.

Given a function v and $\infty > t_2 > t_1 > 0$, we set

$$v_{t_1}^{t_2} = \min\{\max\{0, v - t_1\}, t_2 - t_1\}.$$

In the sequel we will need also the following definitions.

Definition 4.3 (Truncation property) *Let the pair (u, g) satisfies a $(1, q)$-Poincaré inequality in Ω. Assume that for every $b \in \mathbb{R}$, $\infty > t_2 > t_1 > 0$, and $\varepsilon \in \{-1, 1\}$, the pair $(v_{t_1}^{t_2}, g\chi_{\{t_1 < v \leq t_2\}})$, where $v = \varepsilon(u - b)$, satisfies the $(1, q)$-Poincaré inequality in Ω (with fixed constants C_P, σ). Then we say that the pair (u, g) has the truncation property.*

The truncation property for Poincaré-Sobolev functions is the notion similar to the one of the strict locality in axiomatic Sobolev spaces, which also reflects some localization properties of the Sobolev space under consideration. Note that in the Euclidean space $\mathbb{R}^n$ both conditions mean that the gradient of a function, which is constant on some set, equals zero a.e. on that set.

As we have seen in Section 3, in the case of axiomatic Sobolev spaces the quasi-minimizers of the p-Dirichlet energy satisfy the De Giorgi condition

(Hypothesis (H1)). Note that this is also the case for the quasi-minimizers in the approach to Sobolev spaces on a metric space via upper gradients (see [12]). For the class of Poincaré-Sobolev functions the possible notion of energy is not consistent, in particular it is not clear how it would be possible to prove the existence of corresponding minimizers, since in this case the corresponding Sobolev space is not a Banach space (it is, in fact, only a quasi-Banach space). But the De Giorgi condition is still legitimate for the Poincaré-Sobolev functions. Thus, as it seems that there exists an intimate connection between extremal functions and the functions satisfying the De Giorgi condition, in the case of Poincaré-Sobolev functions, the functions whose regularity we are going to establish will be those who satisfy the following property:

Definition 4.4 (p-De Giorgi condition) *We say that a Poincaré-Sobolev function u (satisfying a $(1,q)$-Poincaré inequality with some function g) enjoys the p-De Giorgi condition on the set Ω if for all $k \in \mathbb{R}$, $z \in X$ and $0 < \rho < R \le diam(X)/3$, the following inequality*

$$\int_{A(k,\rho)} g^p d\mu \le \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u-k)^p d\mu, \tag{15}$$

holds, provided $\mu(\Omega \setminus A(k,R)) = 0$, where $A(k,r) = B(z,r) \cap \{x \colon u(x) > k\}$, $p \in \mathbb{R}$, $p > q$.

4.2 Regularity of extremal functions

In this subsection we impose the following condition on the measure μ. For every $z \in X$ and $0 < R \le \operatorname{diam}(X)/3$ we assume that there exists γ, $0 < \gamma < 1$, such that

$$\frac{\mu(B(z, \frac{R}{2}))}{B(z,R)} \le \gamma\,.$$

Note that in the axiomatic setting this condition is proved in Proposition 2.12.

We will also assume that any pair (u,g), $u \in L^1_{loc}(X)$, $g \in L^q(X)$, satisfying a $(1,q)$-Poincaré inequality in X has the truncation property.

We have the following

Theorem 4.5 *Let $u \in PW^{1,q}(X)$ (satisfying a $(1,q)$-Poincaré inequality with some function $g \in L^q(X)$). Suppose that the pairs (u,g) and $(-u,g)$*

enjoy the p-De Giorgi condition on the set Ω*. Then both* (u, g) *and* $(-u, g)$ *satisfy Hypotheses* (H1) *and* (H2)*. In addition, one of these pairs satisfies Hypothesis* (H3) *and, thus, the function* u *is locally Hölder continuous in* Ω*.*

Proof Hypothesis (H1) is the definition of the p-De Giorgi condition.

Hypothesis (H2): Let η be the Lipschitz function as in Hypothesis (H2) and $v = \eta(u-k)_+$ (k, ρ and R are fixed). Since the pair (u, g) satisfies a $(1, q)$-Poincaré inequality, by the truncation property, for every $h \in \mathbb{R}, k < h < \infty$, the functions

$$u_k^h = \min\{\max\{0, u-k\}, h-k\} = \begin{cases} h-k & \text{if } \ u \geq h, \\ u-k & \text{if } \ k < u < h, \\ 0 & \text{if } \ u \leq k, \end{cases}$$

and $g\,\chi_{\{k<v\leq h\}}$ satisfy this $(1, q)$-Poincaré inequality as well. Hence they satisfy a (t, q)-Poincaré inequality for some t, $t > q$, and thus a (q, q)-Poincaré inequality (see [9] and Section 2.2).

Let $\{h_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers such that $h_i > k, i \in \mathbb{N}$, and $h_i \to \infty$ as $i \to \infty$. Denote $u_i := u_k^{h_i}$. Then, the sequence of functions $\{u_i\}_{i\in\mathbb{N}}$ converges in L^q_{loc} topology to the function $(u-k)_+$. Indeed, for any $i \in \mathbb{N}$,

$$0 \leq u_i \leq (u-k)_+\,,$$

and the fact follows from the dominated convergence theorem.

Similarly, the functions $g_i := g\,\chi_{\{k<v\leq h_i\}}$ converge in L^q_{loc} topology to the function $g\,\chi_{\{u>k\}}$.

As for every $i \in \mathbb{N}$ the pair (u_i, g_i) satisfies a (q, q)-Poincaré inequality, it follows that the pair $((u-k)_+, g\,\chi_{\{u>k\}})$ also satisfies it.
Denote $\varphi := (u-k)_+$. For all $x, y \in \Omega$ and some ball $B \subset \Omega$ we have

$$\begin{aligned} |\eta(x)\varphi(x) - (\eta\varphi)_B| &\leq |\eta(x)\varphi(x) - \eta(x)\varphi_B| + |\eta(x)\varphi_B - (\eta\varphi)_B| \\ &\leq \sup|\eta||\varphi(x) - \varphi_B| + |\eta(x)\varphi_B - (\eta\varphi)_B| \\ &\leq |\varphi(x) - \varphi_B| + |\eta(x)\varphi_B - (\eta\varphi)_B| =: \Psi(x). \end{aligned}$$

Integrating the last expression $\Psi(x)$ to the power q and using classical inequalities and the definition of Lipschitz functions we get

$$\begin{aligned}
&\fint_B \Psi(x)^q d\mu(x)\\
&\quad= \fint_B \{|\varphi(x)-\varphi_B| + |\eta(x)\varphi_B - (\eta\varphi)_B|\}^q d\mu(x)\\
&\quad= \fint_B \left\{|\varphi(x)-\varphi_B| + \left|\eta(x)\fint_B \varphi(y)d\mu(y) - \fint_B \eta(y)\varphi(y)d\mu(y)\right|\right\}^q d\mu(x)\\
&\quad= \fint_B \left\{|\varphi(x)-\varphi_B| + \left|\fint_B (\eta(x)\varphi(y)) - \eta(y)\varphi(y))d\mu(y)\right|\right\}^q d\mu(x)\\
&\quad\le \fint_B \left\{|\varphi(x)-\varphi_B| + \fint_B |\varphi(y)||\eta(x)-\eta(y)|d\mu(y)\right\}^q d\mu(x)\\
&\quad\le \fint_B \left\{|\varphi(x)-\varphi_B| + \mathrm{Lip}(\eta)\mathrm{diam}(B)\fint_B |\varphi(y)|d\mu(y)\right\}^q d\mu(x)\\
&\quad\le 2^{q-1}\fint_B \left\{|\varphi(x)-\varphi_B|^q + (\mathrm{Lip}(\eta)\mathrm{diam}(B))^q \left(\fint_B |\varphi(y)|d\mu(y)\right)^q\right\} d\mu(x)\\
&\quad= 2^{q-1}\fint_B |\varphi(x)-\varphi_B|^q d\mu(x) + 2^{q-1}(\mathrm{Lip}(\eta)\mathrm{diam}(B))^q \left(\fint_B |\varphi(y)|d\mu(y)\right)^q.
\end{aligned}$$

Now, the pair $(\varphi = (u-k)_+, g\,\chi_{\{u>k\}})$ satisfies the following (q,q)-Poincaré inequality

$$\fint_B |\varphi - \varphi_B|^q d\mu \le \left(C_P \frac{\mathrm{diam}(B)}{2}\right)^q \fint_{\tau B} \left(g\,\chi_{\{u>k\}}\right)^q d\mu,$$

where $\tau \ge 1$.

Hence

$$\fint_B \Psi(x)^q d\mu(x) \le C(\mathrm{diam}(B))^q \fint_{\tau B} \left\{\left(g\,\chi_{\{u>k\}}\right)^q + (\mathrm{Lip}(\eta)|\varphi|)^q\right\} d\mu,$$

where the constant C depends only on τ, q, C_P and on the doubling constant C_d. We thus have proved that

$$\begin{aligned}
\left(\fint_B |\eta(x)(u(x)-k)_+ - (\eta(u-k)_+)_B|^q d\mu(x)\right)^{1/q} &\le \left(\fint_B \Psi(x)^q d\mu(x)\right)^{1/q}\\
&\le C\mathrm{diam}(B)\left(\fint_{\tau B} \left(g\,\chi_{\{u>k\}} + \mathrm{Lip}(\eta)(u-k)_+\right)^q d\mu\right)^{1/q}.
\end{aligned}$$

In particular, recalling that $v = \eta(u-k)_+$, for the ball $B\,(R+\rho)$ we have

$$\left(\fint_{B(R+\rho)} |v - v_{B(R+\rho)}|^q d\mu\right)^{1/q} \leq C(R+\rho)\left(\fint_{B(\tau(R+\rho))} (g\,\chi_{\{u>k\}} + \frac{C}{R-\rho}(u-k)_+)^q d\mu\right)^{1/q},$$

for some $C > 0$.

Obviously, $v = v\chi_{\{v>0\}}$. Repeating the argument in the very beginning of the proof of Hypothesis (H2), it is easy to show that the truncation property implies that a (q,q)-Poincaré inequality holds for the pair of v and $(g\,\chi_{\{u>k\}} + \frac{C}{R-\rho}(u-k)_+)\chi_{\{v>0\}}$. A (t,q)-Poincaré inequality also holds for these functions. Therefore, we have

$$\begin{aligned}
&\left(\fint_{B(R+\rho)} v^t d\mu\right)^{\frac{1}{t}} \\
&\quad \leq \left(\fint_{B(R+\rho)} |v - v_{B(R+\rho)}|^t d\mu\right)^{\frac{1}{t}} + |v_{B(R+\rho)}| \\
&\quad \leq C(R+\rho)\left(\fint_{B(\lambda(R+\rho))} \left(g\,\chi_{\{u>k\}} + \frac{C}{(R-\rho)}(u-k)_+\right)^q \chi_{\{v>0\}}\, d\mu\right)^{\frac{1}{q}} \\
&\qquad\qquad + |v_{B(R+\rho)}| \\
&\quad \leq C(R+\rho)\left(\fint_{B(\lambda(R+\rho))} \left(g\,\chi_{A(k,\frac{R+\rho}{2})} + \frac{C}{(R-\rho)}(u-k)_+\right)^q \chi_{\{v>0\}}\, d\mu\right)^{\frac{1}{q}} \\
&\qquad\qquad + |v_{B(R+\rho)}| \\
&\quad \leq CR\left(\fint_{B(\frac{R+\rho}{2})} g_{(v)}^q d\mu\right)^{\frac{1}{q}} + |v_{B(R+\rho)}|,
\end{aligned} \tag{16}$$

for some $\lambda > 0$. In the last inequality we denoted $g_{(v)} = g\,\chi_{A(k,\frac{R+\rho}{2})} + \frac{C}{(R-\rho)}(u- \ k)_+$ and used the doubling property of μ and the fact that $\{v > 0\} \subset B(\frac{R+\rho}{2})$.

By the Hölder inequality we obtain

$$
\begin{aligned}
|v_{B(R+\rho)}| &= \frac{1}{\mu(B(R+\rho))}\int_{B(R+\rho)} v d\mu = \frac{1}{\mu(B(R+\rho)}\int_{B(R+\rho)} v\,\chi_{\{v>0\}} d\mu \\
&\leq \left(\fint_{B(R+\rho)} v^t d\mu\right)^{\frac{1}{t}} \left(\frac{\mu\left(\{x\in B\left(R+\rho\right) : v(x)>0\}\right)}{\mu\left(B\left(R+\rho\right)\right)}\right)^{1-\frac{1}{t}} .
\end{aligned}
$$

Then, the condition for the measure μ stated at the beginning of this section implies that

$$
\frac{\mu\left(\{v>0\}\right)}{\mu\left(B\left(R+\rho\right)\right)} \leq \frac{\mu\left(B\left(\frac{R+\rho}{2}\right)\right)}{\mu\left(B\left(R+\rho\right)\right)} \leq \gamma\,,
$$

for some γ, $0<\gamma<1$.

Hence from the previous inequality and the inequality (16) we obtain

$$
(1-\gamma^{1-\frac{1}{t}})\left(\fint_{B(R+\rho)} v^t d\mu\right)^{\frac{1}{t}} \leq CR\left(\fint_{B(\frac{R+\rho}{2})} g_{(v)}^q d\mu\right)^{\frac{1}{q}} .
$$

From the doubling property of μ finally we have

$$
\left(\fint_{B(\frac{R+\rho}{2})} v^t d\mu\right)^{\frac{1}{t}} \leq CR\left(\fint_{B(\frac{R+\rho}{2})} g_{(v)}^q d\mu\right)^{\frac{1}{q}} ,
$$

for some $C>0$.

Hypothesis (H2) is thus verified.

Hypothesis (H3) follows from the truncation property, the doubling condition and the fact that a $(1,q)$-Poincaré inequality on the doubling metric measure space implies a (t,q)-Poincaré inequality with some $t>q$ and, thus, a (q,q)-Poincaré inequality. Indeed, take in the definition 4.3 of the truncation property $\varepsilon=1$, $b=0$, $t_1=k$, $t_2=h$ and note that, in this case,

$$
v_{t_1}^{t_2} = u_k^h = \min\{\max\{0,u-k\},h-k\} = \min\{u,h\}-\min\{u,k\} = w.
$$

Then we repeat the proof of Hypothesis (H3) in Proposition 3.3 with the functions u^* replaced by u and g_{u^*} replaced by g.

Q.E.D.

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