Generalized Exponents via Hall-Littlewood Symmetric Functions

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The generalized exponents of finite-dimensional irreducible representations of a compact Lie group are important invariants first constructed and studied by Kostant in the early 1960's. Their actual computation has remained quite enigmatic. What was known ([K] and [H, Th. 1]) suggested to us that their computation lies at the heart of a rich combinatorially flavored theory.

This note announces several results all tied together by Theorem 2.3 below which selects the natural generalizations of Hall-Littlewood symmetric functions, rather than irreducible characters, as the best basis of the character ring. Full details will appear elsewhere.

1. Statement of Problem. Let $\underline{g}$ be a complex semi-simple Lie algebra with adjoint group G. Via the adjoint action, the symmetric algebra $S(\underline{g})$ becomes a graded representation of G. Kostant studied this representation in his fundamental paper [K] ; his results are well-known. $S(\underline{g}) = I \otimes H$ is a free module over the G-invariants I generated by the harmonics H. Moreover, I is a polynomial ring on homogeneous generators of known degrees, and $H = \bigoplus_{p \geq 0} H^p$ is a graded, locally-finite G-representation.

Hence, to study the isotypic decomposition of $S(\underline{g})$, one forms for each irreducible G-representation V the polynomial in an indeterminate q:

(1.1) $$F(V) := \sum_{p\geq 0} \langle V , H^p \rangle \, q^p .$$

Here $\langle \, , \rangle$ is the usual form dim $\mathrm{Hom}_{\underline{g}}(\, , \,)$ on the representation ring of $\underline{g}$. Kostant's problem asks us to determine F(V); he called the integers $e_1, \ldots, e_s$ with $F(V) = \sum_{i=1}^{s} q^{e_i}$ the <u>generalized exponents of V</u>.

The polynomial F(V) turns out to be a rather deep invariant of the representation V. For instance, the F(V) are certain Kazhdan-Lusztig polynomials for the affine Weyl group (combine [H, Th. 1] and [Ka, Th. 1.8]), and they describe certain group cohomology ([FP, Th. 6.1]).

2. <u>A Bilinear Form</u>. Our idea is to interpret F as a bilinear form on the character ring Λ of $\underline{g}$. Precisely, <u>define</u> a $\mathbb{Z}[q]$-valued symmetric bilinear form $\langle\langle \, , \rangle\rangle$ on $\Lambda[q]$ by setting

(2.1) $$\langle\langle \mathrm{ch}(V_1) \, , \, \mathrm{ch}(V_2) \rangle\rangle := F(V_1 \otimes V_2^*),$$

for any two $\underline{g}$-representations V_1 and V_2, and extending q-linearly. (Here ch(V) and V^* mean the character and dual of V.) Our (2.1) makes sense as (1.1) actually defines F on any representation of $\underline{g}$.

We will present a basis in which our new form $\langle\langle\, ,\rangle\rangle$ diagonalizes. First fix a Cartan subalgebra $\underline{h}$ of $\underline{g}$ and some familiar associated objects. Let Φ be the root system with Φ^+ a choice of positive roots. Form the lattice $\underline{P}$ of integral weights and its subset $\underline{P}^{++}$ of dominant ones. Let W be the Weyl group with length function l. Set $t_\pi(q) := \sum_{\substack{w \in W \\ w\cdot\pi = \pi}} q^{l(w)}$, for $\pi \in \underline{P}$. Use exponential notation for characters.

Define, for $\pi \in \underline{P}^{++}$, the Hall-Littlewood characters

$$P_\pi := t_\pi(q)^{-1} \sum_{w \in W} w\left(e^{\pi} \prod_{\varphi \in \Phi^+} \frac{1 - qe^{-\varphi}}{1 - e^{-\varphi}}\right). \tag{2.2}$$

These characters are classical objects when $\underline{g} = \underline{sl}_n$; they appear in this more general form in work of Kato ([Ka]).

Theorem 2.3. The P_π, $\pi \in \underline{P}^{++}$, form an orthogonal $\mathbb{Z}[q]$-basis of $\Lambda[q]$ with respect to the form $\langle\langle\, ,\rangle\rangle$, and

$$\langle\langle P_\pi\, ,\, P_\pi\rangle\rangle = t_0(q)/t_\pi(q).$$

To prove this, we compare $\langle\langle , \rangle\rangle$ to the usual form $\langle , \rangle$ via the expansion $\sum_{p\geq 0} ch(H^p)q^p = t_0(q) \prod_{\varphi\in\Phi}(1-qe^{\varphi})^{-1}$. Then we extend $\langle , \rangle$ to a form on $\Lambda[[q]]$, where we know ([G, Th. 2.5]) the basis dual to $\{P_\pi\}_{\pi\in \underline{P}^{++}}$

3. Stability for PGL_n. Let us concentrate on $\underline{g}=\underline{sl}_n$ to illustrate (§5) the effective use of 2.3 in evaluating F on irreducibles.

We have formulated a stability theory (1981) for the generalized exponents based on a "mixed tensor" parameterization $V^n_{\alpha,\beta}$ of the irreducible PGL_n-representations, for certain pairs α,β of partitions. (See §4, but for example, $\mathbb{C}=V^n_{(0),(0)}$ and $\underline{g}=V^n_{(1),(1)}$.) Write H^p_n for the degree p harmonics.

Theorem 3.1. Fix $p\geq 0$. Then the number of irreducible PGL_n-components of H^p_n is constant for $n\geq 2p$. Moreover, the decomposition stabilizes: for some finite set J^p of partition pairs and integers $c^p_{\alpha,\beta}$,

$$H^p_n \simeq \bigoplus_{(\alpha,\beta)\in J^p} c^p_{\alpha,\beta}\; V^n_{\alpha,\beta}, \qquad \text{for } n\geq 2p.$$

Our original proof worked by a combinatorial analysis of the pieces in S(End $\mathbb{C}^n$) using the Cauchy and Littlewood-Richardson rules. We, R. Stanley, and P. Hanlon then studied the stable series $\lim_{n\to\infty} F(V^n_{\alpha,\beta})$.

The main question raised by 3.1, however, is the determination of the $F(V^n_{\alpha,\beta})$ as functions of two variables q and n (with the proviso $n\geq l(\alpha)+l(\beta)$ always implicit).

4. Combinatorics of SL_n-Representations. As $\underline{g}=\underline{sl}_n$, the character ring Λ now identifies with the ring of symmetric functions in variables $x_1,\ldots,x_n$ modulo the relation $x_1 \cdots x_n = 1$. The set $\underline{P}^{++}$ identifies with the set Q_n of partitions of at most n-1 rows. The Schur function $s_\pi(x_1,\ldots,x_n)$ is the character of the irreducible highest weight representation V^n_π, $\pi \in Q_n$. Also, $P_\pi = P_\pi(x_1,\ldots,x_n;q)$ is the classical Hall-Littlewood symmetric function.

Write partitions γ as non-decreasing sequences $\gamma=(\gamma_1,\gamma_2,\ldots)$, ignoring trailing zeros, with magnitude $|\gamma|=\gamma_1+\gamma_2+\ldots$ and length $l(\gamma)$.

Given partitions α and β with $l(\alpha)+l(\beta)\leq n$, we defined $V^n_{\alpha,\beta}$ as the Cartan piece (the "highest" irreducible component) in $V^n_\alpha \otimes V^{n*}_\beta$. So $V^n_{\alpha,\beta} = V^n_\gamma$ when γ is the component-wise sum (put $s=l(\alpha)$, $t=l(\beta)$):

$$\gamma = \mathrm{prt}_n(\alpha,\beta) := (\alpha_1,\ldots,\alpha_s,\underbrace{0,\ldots,0}_{n-s-t},-\beta_t,\ldots,-\beta_1) + (\underbrace{\beta_1,\ldots,\beta_1}_{n}).$$

Lemma 4.1. Fix $n\geq 1$. Then the $V^n_{\alpha,\beta}$, where α and β satisfy $l(\alpha)+l(\beta)\leq n$ and $|\alpha|=|\beta|$, form an exhaustive, repetition free, list of the irreducible, finite-dimensional representations of PGL_n.

For each value of n, $F(V^n_{\alpha,\beta})\in \mathbb{Z}[q]$ is controlled by the partitions $\lambda=prt_n(\alpha,\beta)$ and $\mu=(\beta_1^n)$ of magnitude $\beta_1 n$. In fact, we observed that $F(V^n_{\alpha,\beta})$ equals the combinatorial Kostka-Foulkes polynomial $K_{\lambda,\mu}(q)$ attached to Young tableaux of shape λ and weight μ (see [M, III, 6]).

However, in §5 we prove that $F(V^n_{\alpha,\beta})$ as a function of q and n is really "controlled" just by α and β (symmetrically, as $F(V^n_{\alpha,\beta})=F(V^n_{\alpha,\beta})$). Let $h_1(\alpha),\ldots,h_{|\alpha|}(\alpha)$ be the hook numbers and $\tilde{\alpha}$ be the conjugate of α (see [M, I, 1]). Set $e(\alpha):=\sum_{i\geq 1} i\alpha_i$. Previously (1982), we knew only

Proposition 4.2. Assume $|\alpha|=r$.

(i) If $\beta=(1^r)$, then $F(V^n_{\alpha,\beta}) = q^{e(\tilde{\alpha})}\prod_{i=1}^{r}(1-q^{n-r-\tilde{\alpha}_i+i})/(1-q^{h_i(\alpha)})$.

(ii) If $\beta=(r)$, then $F(V^n_{\alpha,\beta}) = s_\alpha(q,\ldots,q^{n-1})$.

5. A Formula for $F(V^n_{\alpha,\beta})$. Let us extend our notation $K_{\lambda,\mu}(q)$ to skew-partitions (i.e., skew-diagrams) $\lambda=\alpha/\theta$. Cf. [M, I, 5]. Set $b_\pi(q) := \prod_{i\geq 1}(1-q)\cdots(1-q^{m_i(\pi)})$, for $m_i(\pi)$ the multiplicity of i in π.

Theorem 5.1. Fix α and β with $|\alpha|=|\beta|=r$. Then

$$F(V^n_{\alpha,\beta}) = \sum (-1)^{|\theta|} K_{\alpha/\theta,\pi}(q)\, K_{\beta/\tilde{\theta},\pi}(q)\, \frac{(1-q^n)\cdots(1-q^{n-l(\pi)+1})}{b_\pi(q)}$$

summed over all partition pairs θ,π with $|\theta|+|\pi|=r$.

To prove this, we first compute $F(V^n_\gamma\otimes V^{n*}_\delta)$ using Th. 2.3 by writing s_γ and s_δ in terms of the P_π. Then we express $V^n_{\alpha,\beta}$ in terms of the $V^n_\gamma\otimes V^{n*}_\delta$ using essentially a formula of Littlewood.

Th. 5.1 leads to new, unified proofs of several old results, among them 3.1, 4.2, and the stable theorem [S, 8.1] proven by Stanley. But mainly, 5.1 gives the first real means for computing the $F(V^n_{\alpha,\beta})$.

Corollary 5.2. For some polynomial $g_{\alpha,\beta}(q,z)$ over $\mathbb{Z}$,

$$F(V^n_{\alpha,\beta}) = \frac{g_{\alpha,\beta}(q,q^{n-r+1})}{(1-q)\cdots(1-q^r)}$$

Moreover, $g_{\alpha,\beta}(q,z)(1-q)^{-1}\cdots(1-q^r)^{-1}$ is a linear combination, over $\mathbb{Z}[q]$, of the functions $(1-q^{r-1}z)\cdots(1-q^{r-i}z)(1-q)^{-1}\cdots(1-q^i)^{-1}$, $i=1,\ldots,r$.

We have some conjectures on the form of the $g_{\alpha,\beta}(q,z)$. The examples below, done by hand, are new, though the first is an old conjecture. For integers c_i, d_i, we set

$$\begin{bmatrix} c_1 & \cdots & c_r \\ d_1 & \cdots & d_r \end{bmatrix}_q := (1-q^{c_1})\cdots(1-q^{c_r})(1-q^{d_1})^{-1}\cdots(1-q^{d_r})^{-1}.$$

But we refrain from thinking about these unless they are polynomials in q.

Example 5.3. If $\alpha=\beta=(2,1)$, then 5.1 yields

$$F(V^n_{\alpha,\beta}) = q^3\begin{bmatrix} n+1 & n-1 & n-3 \\ 1 & 1 & 3 \end{bmatrix}_q + q^5\begin{bmatrix} n-1 & n-2 & n-3 \\ 1 & 1 & 3 \end{bmatrix}_q$$

Example 5.4. Let us find $F(V^6_\pi)$ when $\pi=(6,4,1,1)$. Then $\pi=\mathrm{prt}_6(\alpha,\beta)$ for $\alpha=(4,2)$ and $\beta=(2,2,1,1)$. 5.1 gives

$$F(V^n_{\alpha,\beta}) = q^9\begin{bmatrix} n+2 & n+1 & n-1 & n-2 & n-4 & n-5 \\ 1 & 1 & 2 & 2 & 4 & 5 \end{bmatrix}_q + q^{12}\begin{bmatrix} n+2 & n-1 & n-2 & n-3 & n-4 & n- \\ 1 & 1 & 2 & 2 & 4 & 5 \end{bmatrix}$$

$$+ q^{15}\begin{bmatrix} n-1 & n-2 & n-2 & n-3 & n-4 & n-5 \\ 1 & 1 & 2 & 2 & 4 & 5 \end{bmatrix}_q + q^9(1+q+q^2)\begin{bmatrix} n & n-1 & n-2 & n-3 & n-4 & n- \\ 1 & 1 & 2 & 2 & 2 & 5 \end{bmatrix}$$

So, at n=6, $F(V_\pi^6) = 2q^9+3q^{10}+7q^{11}+9q^{12}+13q^{13}+13q^{14}+15q^{15}+12q^{16}+11q^{17}+7q^{18}+5q^{19}+2q^{20}+q^{21}$.

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