RIGGED CONFIGURATIONS AND THE BETHE ANSATZ

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This note is a review of rigged configurations and the Bethe Ansatz. In the first part, we focus on the algebraic Bethe Ansatz for the spin 1/2 XXX model and explain how rigged configurations label the solutions of the Bethe equations. This yields the bijection between rigged configurations and crystal paths/Young tableaux of Kerov, Kirillov and Reshetikhin. In the second part, we discuss a generalization of this bijection for the symmetry algebra $D_n^{(1)}$, based on work in collaboration with Okado and Shimozono.

1. Introduction

These notes arose from three lectures presented at the Summer School on Theoretical Physics "Symmetry and Structural Properties of Condensed Matter" held in Myczkowce, Poland, on September 11-18, 2002. We review the algebraic Bethe Ansatz in the simple setting of the spin 1/2 XXX model, explain the physical meaning of rigged configurations and give a bijection between rigged configurations and crystal bases for type $D_n^{(1)}$ [20] generalizing the bijection of Kerov, Kirillov and Reshetikhin [15,16] for type $A_n^{(1)}$.

The Bethe Ansatz originated in a paper by Bethe [2] in 1931 in which he studied the eigenvectors and eigenfunctions of the Hamiltonian of the Heisenberg antiferromagnet. The method he used is today often called the coordinate Bethe Ansatz, distinguishing it from the algebraic Bethe Ansatz that will be presented here. The algebraic Bethe Ansatz is a generalization of the coordinate Bethe Ansatz and is one of the most important outcomes of the quantum inverse scattering method introduced in [23,24,25]. The quantum inverse scattering method has unified the treatment of quantum integrable systems considering each model as a representation of the quantum monodromy matrix which satisfies certain commutation relations. The algebraic Bethe Ansatz is based on the idea of constructing eigenvectors of the Hamiltonian (resp. trace of the monodromy matrix) by creation and annihilation operators on a vacuum; the elements of the monodromy matrix play the role of these operators. The eigenvectors are parametrized by solutions of a system of algebraic equations, called the Bethe equations. The solutions in turn are labeled by combinatorial objects called rigged configurations.

We consider $\mathfrak{g}$-invariant models where $\mathfrak{g}$ is the symmetry algebra. The Hilbert space is the tensor product of irreducible representations of $\mathfrak{g}$ denoted $\mathcal{H} = h_1 \otimes \cdots \otimes h_N$. The Bethe vectors are the highest weight vectors in the decomposition into irreducible components of $\mathcal{H}$. It is known from representation theory that the highest weight vectors are also labeled by Young tableaux (see for example [5]) or certain paths in crystal theory (see for example [7,8,18]). Assuming the completeness of the Bethe vectors, this suggests a bijection between rigged configurations and Young tableaux/crystal paths. For $\mathfrak{g} = \mathfrak{gl}_n$ such a bijection was given by Kirillov and Reshetikhin [16] and generalized in [17].

Analogous bijections for all $\mathfrak{g}$ of nonexceptional affine type were recently proven in [20] for tensor products of the fundamental representation. An important property of all these bijections is that they preserve statistics that can be defined on the set of rigged configurations and paths, respectively. As a corollary it follows that one-dimensional configuration sums defined in terms of crystal paths have fermionic formulas. Fermionic formulas reflect the quasiparticle structure of the underlying model and also reveal the statistics of the quasiparticles. For general affine Kac-Moody algebras fermionic formulas were conjectured by Hatayama, Kuniba, Okado, Takagi, Tsuboi and Yamada [7,8]. For type $A_n^{(1)}$ they were proven in [17] and for nonexceptional types in special cases in [20].

The paper is organized as follows. In section 2 we review the algebraic Bethe Ansatz for the spin 1/2 XXX model and derive the Bethe equations. In section 3 we present the solutions of the Bethe equations parametrized by rigged configurations and discuss the bijection between rigged configurations and paths in section 4. Sections 2 and 3 follow the presentation of Faddeev [4]. In sections 5-8 the bijection between paths and rigged configurations is generalized to types $A_n^{(1)}$ and $D_n^{(1)}$ based on work in collaboration with Okado and Shimozono [20]. Crystal bases are introduced in section 6 and section 7 states the fermionic formula and rigged configurations in the generalized set-up. The bijection is given explicitly in section 8.

2. Bethe Ansatz for the XXX model

In this section we discuss the algebraic Bethe Ansatz for the example of the spin 1/2 XXX Heisenberg chain. This is a one-dimensional quantum spin chain on N sites with periodic boundary conditions. It is defined on the Hilbert space $\mathcal{H}_N = \bigotimes_{n=1}^{N} h_n$ where in this case $h_n = \mathbb{C}^2$ for all n. Associated to each site is a local spin variable $\vec{s} = \frac{1}{2}\vec{\sigma}$ where

$$\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$

are the Pauli matrices. The spin variable acting on the n-th site is given by

$$\vec{s}_n = I \otimes \cdots \otimes I \otimes \vec{s} \otimes I \otimes \cdots \otimes I$$

where I is the identity operator and $\vec{s}$ is in the n-th tensor factor. We impose periodic boundary conditions $\vec{s}_n = \vec{s}_{n+N}$.

The Hamiltonian of the spin 1/2 XXX model is

$$H_N = J \sum_{n=1}^{N} \left(\vec{s}_n \cdot \vec{s}_{n+1} - \frac{1}{4} \right).$$

Our goal is to determine the eigenvectors and eigenvalues of H_N in the antiferromagnetic regime $J > 0$ in the limit when $N \to \infty$.

The main tool will be the Lax operator $L_{n,a}(\lambda)$, also called the local transition matrix. It acts on $h_n \otimes \mathbb{C}^2$ where $\mathbb{C}^2$ is an auxiliary space and is defined as

$$L_{n,a}(\lambda) = \lambda I_n \otimes I_a + i\vec{s}_n \otimes \vec{\sigma}_a.$$

Here I_n and I_a are unit operators acting on h_n and the auxiliary space $\mathbb{C}^2$, respectively; λ is a complex parameter, called the spectral parameter. Writing the action on the auxiliary space as a 2×2 matrix, we have

$$L_n(\lambda) = \begin{pmatrix} \lambda + is_n^3 & is_n^- \\ is_n^+ & \lambda - is_n^3 \end{pmatrix} \tag{5}$$

where $s_n^{\pm} = s_n^1 \pm is_n^2$.

The crucial fact is that the Lax operator satisfies commutation relations in the auxiliary space $V = \mathbb{C}^2$. Altogether there are 16 relations which can be written compactly in tensor notation. Given two Lax operators $L_{n,a_1}(\lambda)$ and $L_{n,a_2}(\mu)$ defined in the same quantum space h_n, but different auxiliary spaces V_1 and V_2, the products $L_{n,a_1}(\lambda)L_{n,a_2}(\mu)$ and $L_{n,a_2}(\mu)L_{n,a_1}(\lambda)$ are defined on the triple tensor product $h_n \otimes V_1 \otimes V_2$. There exists an operator $R_{a_1,a_2}(\lambda-\mu)$ defined on $V_1 \otimes V_2$ such that

$$R_{a_1,a_2}(\lambda-\mu)L_{n,a_1}(\lambda)L_{n,a_2}(\mu) = L_{n,a_2}(\mu)L_{n,a_1}(\lambda)R_{a_1,a_2}(\lambda-\mu). \tag{6}$$

Explicitly, the R-matrix $R_{a_1,a_2}(\lambda)$ is given by

$$R_{a_1,a_2}(\lambda) = \left(\lambda + \frac{i}{2}\right) I_{a_1} \otimes I_{a_2} + \frac{i}{2}\vec{\sigma}_{a_1} \otimes \vec{\sigma}_{a_2}.$$

To deduce the 16 relations explicitly, one may write (6) as matrices in the auxiliary space $V_1 \otimes V_2$ using the convention $(A \otimes B)^{ij}_{k\ell} = A_{ij}B_{k\ell}$ where

$$M^{ij}_{k\ell} = \begin{pmatrix} M^{11}_{11} & M^{11}_{12} & M^{12}_{11} & M^{12}_{12} \\ M^{11}_{21} & M^{11}_{22} & M^{12}_{21} & M^{12}_{22} \\ M^{21}_{11} & M^{21}_{12} & M^{22}_{11} & M^{22}_{12} \\ M^{21}_{21} & M^{21}_{22} & M^{22}_{21} & M^{22}_{22} \end{pmatrix}.$$

In this notation the R-matrix reads

$$R(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}$$

where $a(\lambda) = \lambda + i$, $b(\lambda) = \lambda$ and $c(\lambda) = i$.

Geometrically, the Lax operator $L_{n,a}(\lambda)$ can be interpreted as the transport between sites n and $n+1$ of the quantum spin chain. Hence

$$T_{N,a}(\lambda) = L_{N,a}(\lambda)\cdots L_{1,a}(\lambda)$$

is the monodromy around the circle (recall that we assume periodic boundary conditions). In the auxiliary space write

$$T_N(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

with entries in the full Hilbert space $\mathcal{H}_N$. From (6) it is clear that the monodromy matrix satisfies the following commutation relation

$$R_{a_1,a_2}(\lambda-\mu)T_{N,a_1}(\lambda)T_{N,a_2}(\mu) = T_{N,a_2}(\mu)T_{N,a_1}(\lambda)R_{a_1,a_2}(\lambda-\mu). \tag{12}$$

Explicitly, some of the relations contained in (12) are

$$\begin{aligned} [B(\lambda), B(\mu)] &= 0 \\ A(\lambda)B(\mu) &= f(\lambda-\mu)B(\mu)A(\lambda) + g(\lambda-\mu)B(\lambda)A(\mu) \\ D(\lambda)B(\mu) &= h(\lambda-\mu)B(\mu)D(\lambda) + k(\lambda-\mu)B(\lambda)D(\mu) \end{aligned} \tag{13}$$

where

$$\begin{array}{ll} f(\lambda) = \frac{\lambda - i}{\lambda} & \qquad g(\lambda) = \frac{i}{\lambda} \\ h(\lambda) = \frac{\lambda + i}{\lambda} & \qquad k(\lambda) = -\frac{i}{\lambda}. \end{array}$$

It is well-known [4,6,13] that the Hamiltonian is given in terms of the monodromy matrix as

$$H_N = \frac{i}{2} \frac{d}{d\lambda} \ln t_N(\lambda)|_{\lambda = i/2} - \frac{N}{2}$$

where $t_N(\lambda) = \mathrm{tr} T_N(\lambda) = A(\lambda) + D(\lambda)$.

Let $\omega_n = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In the auxiliary space the Lax operator is triangular on ω_n

$$L_n(\lambda)\omega_n = \begin{pmatrix} \lambda + \frac{i}{2} & * \\ 0 & \lambda - \frac{i}{2} \end{pmatrix} \omega_n$$

where $*$ stands for an for us irrelevant quantity. This follows directly from (5). On the Hilbert space $\mathcal{H}_N$ we define $\Omega = \bigotimes_n \omega_n$ so that

$$T_N(\lambda)\Omega = \begin{pmatrix} \alpha^N(\lambda) & * \\ 0 & \delta^N(\lambda) \end{pmatrix} \Omega$$

where $\alpha(\lambda) = \lambda + \frac{i}{2}$ and $\delta(\lambda) = \lambda - \frac{i}{2}$. Equivalently this means that

$$\begin{aligned} C(\lambda)\Omega &= 0 \\ A(\lambda)\Omega &= \alpha^N(\lambda)\Omega \\ D(\lambda)\Omega &= \delta^N(\lambda)\Omega \end{aligned}$$

so that Ω is an eigenstate of $A(\lambda)$ and $D(\lambda)$ and hence also of $t_N(\lambda) = A(\lambda) + D(\lambda)$.

The claim is that the other eigenvectors of $t_N(\lambda)$ are of the form

$$\Phi(\lambda, \Lambda) = B(\lambda_1) \cdots B(\lambda_n)\Omega$$

where the lambdas $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ satisfy a set of algebraic relations, called the Bethe equations. We will derive these now.

From the commutation relations (13) we find that

$$\begin{aligned} A(\lambda)B(\lambda_1) \cdots B(\lambda_n)\Omega = &\prod_{k=1}^{n} f(\lambda - \lambda_k)\alpha^N(\lambda)B(\lambda_1) \cdots B(\lambda_n)\Omega \\ &+ \sum_{k=1}^{n} M_k(\lambda, \Lambda)B(\lambda_1) \cdots \hat{B}(\lambda_k) \cdots B(\lambda_n)B(\lambda)\Omega. \end{aligned}$$

The first term on the right hand side is obtained by using only the first term on the right hand side of (13). The other terms come from a combination of the application of the first and second term when moving A past the B's. In general the coefficients $M_k(\lambda, \Lambda)$ are quite involved using the explicit formulas. However, $M_1(\lambda, \Lambda)$ is obtained by using the second term in (13) moving $A(\lambda)$ past $B(\lambda_1)$ followed by applications of the first term in (13) only. This yields

$$M_1(\lambda, \Lambda) = g(\lambda - \lambda_1) \prod_{k=2}^{n} f(\lambda_1 - \lambda_k) \alpha^N(\lambda_1).$$

Note that the B's commute with each other by (13). Hence $M_j(\lambda, \Lambda)$ can be obtained from $M_1(\lambda, \Lambda)$ by replacing λ_1 by λ_j so that

$$M_j(\lambda, \Lambda) = g(\lambda - \lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^{n} f(\lambda_j - \lambda_k) \alpha^N(\lambda_j).$$

Similarly,

$$\begin{aligned} D(\lambda)B(\lambda_1)\cdots B(\lambda_n)\Omega &= \prod_{k=1}^{n} h(\lambda - \lambda_k) \delta^N(\lambda) B(\lambda_1)\cdots B(\lambda_n)\Omega \\ &+ \sum_{k=1}^{n} N_k(\lambda, \Lambda) B(\lambda_1)\cdots \hat{B}(\lambda_k)\cdots B(\lambda_n)B(\lambda)\Omega \end{aligned}$$

where

$$N_j(\lambda, \Lambda) = k(\lambda - \lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^{n} h(\lambda_j - \lambda_k) \delta^N(\lambda_j).$$

For $\Phi(\lambda, \Lambda)$ to be an eigenvector of $t_N(\lambda) = A(\lambda) + D(\lambda)$ the terms

$$\begin{aligned} &\sum_{k=1}^{n} M_k(\lambda, \Lambda) B(\lambda_1)\cdots \hat{B}(\lambda_k)\cdots B(\lambda_n)B(\lambda)\Omega \\ + &\sum_{k=1}^{n} N_k(\lambda, \Lambda) B(\lambda_1)\cdots \hat{B}(\lambda_k)\cdots B(\lambda_n)B(\lambda)\Omega \end{aligned}$$

need to cancel. Since $g(\lambda - \lambda_j) = -k(\lambda - \lambda_j)$ this happens if the set of lambda's Λ satisfy the following set of equations

$$\prod_{\substack{k=1 \\ k \neq j}}^{n} f(\lambda_j - \lambda_k) \alpha^N(\lambda_j) = \prod_{\substack{k=1 \\ k \neq j}}^{n} h(\lambda_j - \lambda_k) \delta^N(\lambda_j)$$

for all $j = 1, 2, \ldots, n$. Explicitly this reads

$$\left(\frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}} \right)^N = \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i} \tag{23}$$

called the Bethe equations. In this case the eigenvalues of $\Phi(\lambda,\Lambda)$ are

$$\alpha^N(\lambda)\prod_{k=1}^{n} f(\lambda-\lambda_k)+\delta^N(\lambda)\prod_{k=1}^{n} h(\lambda-\lambda_k).$$

In the next section we will study solutions to (23) in the limit $N\to\infty$.

3. Solutions to the Bethe equations

Let us rewrite (23) in the following way

$$\left(\frac{\lambda+\frac{i}{2}}{\lambda-\frac{i}{2}}\right)^N=\prod_{\substack{\lambda'\in\Lambda\\ \lambda'\neq\lambda}}\frac{\lambda-\lambda'+i}{\lambda-\lambda'-i} \tag{25}$$

where $\lambda\in\Lambda=\{\lambda_1,\ldots,\lambda_n\}$.

Suggested by numerical analysis, it is assumed that in the limit $N\to\infty$ the λ's form strings. This hypothesis is called the string hypothesis. A string of length $\ell=2M+1$, where M is an integer or half-integer depending on the parity of ℓ, is a set of λ's of the form

$$\lambda_{jm}^M=\lambda_j^M+im$$

where $\lambda_j^M\in\mathbb{R}$ and $-M\leq m\leq M$ is integer or half-integer depending on M. The index j satisfies $1\leq j\leq m_\ell$ where m_ℓ is the number of strings of length ℓ. A decomposition of $\{\lambda_1,\ldots,\lambda_n\}$ into strings is called a configuration. Each configuration is parametrized by $\{m_\ell\}$. It follows that

$$\sum_\ell \ell m_\ell=n.$$

Now take (25) and multiply over a string

$$\prod_{m=-M}^{M}\left(\frac{\lambda_j^M+i(m+\frac{1}{2})}{\lambda_j^M+i(m-\frac{1}{2})}\right)^N =\prod_{m=-M}^{M}\prod_{\substack{M',j',m'\\ (M',j',m')\neq(M,j,m)}}\frac{\lambda_j^M-\lambda_{j'}^{M'}+i(m-m'+1)}{\lambda_j^M-\lambda_{j'}^{M'}+i(m-m'-1)}. \tag{28}$$

Many of the terms on the left and right cancel so that this equation can be rewritten as

$$e^{iNp_M(\lambda_j^M)}=\prod_{\substack{M',j'\\ (M',j')\neq(M,j)}} e^{iS_{MM'}(\lambda_j^M-\lambda_{j'}^{M'})}, \tag{29}$$

in terms of the momentum and scattering matrix

$$e^{ip_M(\lambda)} = \frac{\lambda + i(M+\frac{1}{2})}{\lambda - i(M+\frac{1}{2})}$$

$$e^{iS_{MM'}(\lambda)} = \prod_{m=|M-M'|}^{M+M'} \frac{\lambda + im}{\lambda - im} \cdot \frac{\lambda + i(m+1)}{\lambda - i(m+1)}.$$

Taking the logarithm of (29) using the branch cut

$$\frac{1}{i} \ln \frac{\lambda + ia}{\lambda - ia} = \pi - 2 \arctan \frac{\lambda}{a}$$

we obtain

$$2N \arctan \frac{\lambda_j^M}{M+\frac{1}{2}} = 2\pi Q_j^M + \sum_{\substack{M',j' \\ (M',j') \neq (M,j)}} \Phi_{MM'}(\lambda_j^M - \lambda_{j'}^{M'}), \qquad (31)$$

where

$$\Phi_{MM'}(\lambda) = 2 \sum_{m=|M-M'|}^{M+M'} \left(\arctan \frac{\lambda}{m} + \arctan \frac{\lambda}{m+1} \right).$$

The first term on the right is absent for $m = 0$. Here Q_j^M is an integer or half-integer depending on the configuration.

In addition to the string hypothesis, we assume that the Q_j^M classify the λ's uniquely: λ_j^M increases if Q_j^M increases and in a given string no Q_j^M coincide. As we will see shortly with this assumption one obtains the correct number of solutions to the Bethe equations (25).

Using $\arctan \pm\infty = \pm\frac{\pi}{2}$ we obtain from (31) putting $\lambda_j^M = \infty$

$$Q_\infty^M = \frac{N}{2} - \big(2M + \frac{1}{2}\big)\big(m_{2M+1} - 1\big) - \sum_{M' \neq M} \big(2\min(M, M') + 1\big) m_{2M'+1}.$$

Since there are $2M+1$ strings in a given string of length $2M+1$, the maximal admissible $Q_{\max}^M$ is

$$Q_{\max}^M = Q_\infty^M - (2M+1)$$

where we assume that if Q_j^M is bigger than $Q_{\max}^M$ then at least one root in the string is infinite and hence all are infinite which would imply $Q_j^M = Q_\infty^M$.

With the already mentioned assumption that each admissible set of quantum number Q_j^M corresponds uniquely to a solution of the Bethe equations we may now count the number of Bethe vectors. Since arctan is an odd function and by the assumption about the monotonicity we have

$$-Q_{\max}^M \leq Q_1^M < \cdots < Q_{m_{2M+1}}^M \leq Q_{\max}^M.$$

Hence defining P_ℓ as

$$P_\ell = N - 2\sum_{\ell'} \min(\ell,\ell') m_{\ell'}$$

so that

$$P_\ell + m_\ell = 2Q^M_{\max} + 1 \qquad \text{with } \ell = 2M+1.$$

With this the number of Bethe vectors with configuration $\{m_\ell\}$ is given by

$$Z(N,n|\{m_\ell\}) = \prod_{\ell\geq 1} \binom{P_\ell + m_\ell}{m_\ell}$$

where $\binom{p+m}{m} = (p+m)!/p!m!$ is the binomial coefficient. The total number of Bethe vectors is

$$Z(N,n) = \sum_{\substack{\{m_\ell\} \\ \sum_\ell \ell m_\ell = n}} \prod_{\ell\geq 1} \binom{P_\ell + m_\ell}{m_\ell}. \tag{39}$$

It should be emphasized that the derivation of (39) given here is not mathematically rigorous. Besides the various assumptions that were made we also did not worry about possible singularities of (28). However, as we shall see in the next section, (39) indeed yields the correct number of Bethe vectors.

4. Rigged configurations

In the last section we parametrized the Bethe vectors by solutions to the Bethe equations. As we have seen in section 2 the state space is the tensor product of irreducible representations of the underlying algebra, in our case the tensor product of $\mathbb{C}^2$ with underlying algebra being $\mathfrak{su}(2)$. The Bethe vectors are the highest weight vectors in the irreducible components in this tensor product.

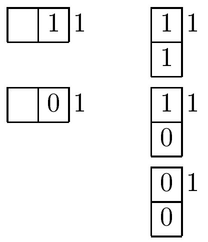
In this section we will interpret (39) combinatorially in terms of rigged configurations. Since the Bethe vectors are also the irreducible components of the underlying tensor product which can be labeled by Young tableaux or crystal elements, one may expect a bijection between the rigged configurations and crystal elements. For the case A_n such a bijection is indeed known to exist [15,16,17]. For other types it was recently given in special cases in [20].

To interpret (39) combinatorially let us view the set $\{m_\ell\}$ as a partition ν. A partition is a set of numbers $\nu = (\nu_1, \nu_2, \ldots)$ such that $\nu_i \geq \nu_{i+1}$ and only finitely many ν_i are nonzero. The partition has part i if $\nu_k = i$ for some k. The size of partition ν is $|\nu| := \nu_1 + \nu_2 + \cdots$. In the correspondence between $\{m_\ell\}$ and ν, m_ℓ specifies the number of parts of size ℓ in ν. For example, if $m_1 = 1$, $m_2 = 3$, $m_4 = 1$ and all other $m_\ell = 0$ then $\nu = (4,2,2,2,1)$.

It is well-known (see e.g. [1]) that $\binom{p+m}{m}$ is the number of partitions in a box of size $p \times m$, meaning, that the partition cannot have more than m parts and no part exceeds p. Let $\mathrm{RC}(N,n)$ be the set of all rigged configurations (ν, J) defined as follows. ν is a partition of size $|\nu| = n$ and J is a set of partition where J_ℓ is a partition in a box of size $P_\ell \times m_\ell$. Then (39) can be rewritten as

$$Z(N,n) = \sum_{(\nu,J)\in \mathrm{RC}(N,n)} 1.$$

Example 4.1. Let $N = 5$ and $n = 2$. Then the following is the set of rigged configuration $\mathrm{RC}(5,2)$



The underlying partition on the left is (2) and on the right (1,1). The partitions J_ℓ attached to part length ℓ is specified by the numbers in each part. For example, the partition J_1 for the top rigged configuration on the right is (1,1) whereas for the one in the middle and bottom is $J_1 = (1)$ and $J_1 = \emptyset$, respectively. The numbers to the right of part ℓ is P_ℓ.

There exists a statistics on $\mathrm{RC}(N,n)$, called cocharge. It is given by

$$\mathrm{cc}(\nu, J) = \mathrm{cc}(\nu) + \sum_\ell |J_\ell|$$

where

$$\mathrm{cc}(\nu) = \sum_{j,k} \min(j,k) m_j m_k.$$

For example, the cocharge for the rigged configurations in Example 4.1 from top to bottom, left to right is 3, 2, 6, 5, 4, respectively.

As mentioned before, rigged configurations are in bijection with crystal elements. For our $\mathfrak{su}(2)$ example these are all sequences of 1's and 2's of length N such that the number of 2's never exceeds the number of 1's reading the sequence from right to left. The last condition is that of Yamanouchi words. The number n fixes the number of 2's in the sequence. Denote the set of all such sequences by $\mathcal{P}(N,n)$. For a path $p = p_N \cdots p_1 \in \mathcal{P}(N,n)$ define the energy as

$$E(p) = \sum_{j=1}^{N-1} (N-j)\chi(p_{j+1} > p_j) \tag{43}$$

where $\chi(\text{True}) = 1$ and $\chi(\text{False}) = 0$. The generating function of paths is given by

$$X(N,n) = \sum_{p\in\mathcal{P}(N,n)} q^{E(p)}.$$

Example 4.2. The set $\mathcal{P}(5,2)$ is given by

$$\mathcal{P}(5,2) = \{22111, 21211, 12211, 21121, 12121\}.$$

The energies are 2, 4, 3, 5 and 6, respectively. Hence $X(5,2) = q^2 + q^3 + q^4 + q^5 + q^6$.

The bijection between $\mathcal{P}(N,n)$ and $\mathrm{RC}(N,n)$ is defined recursively. A path $p = p_N \cdots p_1 \in \mathcal{P}(N,n)$ is built up successively from right to left. The empty path is mapped to the empty rigged configuration. Assume that $p_{i-1} \cdots p_1$ corresponds to (ν^{i-1}, J^{i-1}). If $p_i = 1$, $(\nu^i, J^i) = (\nu^{i-1}, J^{i-1})$. If $p_i = 2$, then add a box to the largest singular string in (ν^{i-1}, J^{i-1}) and make it singular again. A string is singular if its label is equal to the vacancy number, in other words, if J_ℓ has a part of size P_ℓ. In the final rigged configuration (ν^N, J^N) take the complement of the partitions J_ℓ^N in the box $P_\ell^N \times m_\ell^N$. Let us call this map $\Psi : \mathcal{P}(N,n) \to \mathrm{RC}(N,n)$. We have the following theorem [15,16,17].

Theorem 4.1. *The map* $\Psi : \mathcal{P}(N,n) \to \mathrm{RC}(N,n)$ *is a bijection and* $E(p) = \mathrm{cc}(\Psi(p))$ *for all* $p \in \mathcal{P}(N,n)$.

Example 4.3. Take $p = 21121$. We get successively

p	(ν, J)
$\emptyset$	$\emptyset$
1	$\emptyset$
21	$\boxed{0}\,0$
121	$\boxed{0}\,1$
1121	$\boxed{0}\,2$
21121	$\begin{array}{l}\boxed{1}\,1\\ \boxed{0}\end{array}$

Hence $\Psi(21121) = \begin{array}{l}\boxed{1}\,1\\ \boxed{0}\end{array}$. Similarly,

$$\Psi(22111) = \boxed{}\boxed{0}\,1$$

$$\Psi(21211) = \begin{array}{l}\boxed{0}\,1\\ \boxed{0}\end{array}$$

$$\Psi(12211) = \boxed{}\boxed{1}\,1$$

$$\Psi(12121) = \begin{array}{l}\boxed{1}\,1\\ \boxed{1}\end{array}$$

Comparing with examples 4.1 and 4.2, the statistics match.

It follows immediately from Theorem 4.1 that

$$X(N, n) = \sum_{(\nu,J)\in \mathrm{RC}(N,n)} q^{\mathrm{cc}(\nu,J)}.$$

The q-binomial coefficient

$$\begin{bmatrix} p+m \\ m \end{bmatrix} = \frac{(q)_{p+m}}{(q)_p (q)_m},$$

where $(q)_m = \prod_{i=1}^{m}(1-q^i)$, is the generating function of partitions in a box of size $p \times m$ [1]. Hence, defining $\mathrm{C}(N, n)$ to be the set of all partitions ν of n such that $P_\ell \geq 0$ for all ℓ the following corollary holds. The right-hand side is called fermionic formula.

Corollary 4.1.

$$X(N, n) = \sum_{\nu \in \mathrm{C}(N,n)} q^{\mathrm{cc}(\nu)} \prod_{\ell} \begin{bmatrix} P_\ell + m_\ell \\ m_\ell \end{bmatrix}.$$

5. Generalizations

So far we have only considered the spin 1/2 XXX model and its counting. This model is based on the fundamental representation of $\mathfrak{su}(2)$. It turns out that the q-counting of Corollary 4.1 is associated with the Kac–Moody Lie algebra $A_1^{(1)}$. In the remainder of this note we will indicate how to generalize the q-counting that arises from the Bethe Ansatz.

The set of paths $\mathcal{P}(N,n)$, which is the set of Yamanouchi words in the letters 1 and 2 of length N with n twos, will be generalized to the set of highest weight elements in a tensor product of crystals of a given weight; the Yamanouchi condition is replaced by the highest weight condition and the condition on the number of twos becomes the requirement on the weight. Crystal bases were first introduced by Kashiwara [11] in connection with quantized universal enveloping algebras. The quantized universal enveloping algebra $U_q(\mathfrak{g})$ associated with a symmetrizable Kac–Moody Lie algebra $\mathfrak{g}$ was discovered independently by Drinfeld [3] and Jimbo [9] in their study of two dimensional solvable lattice models in statistical mechanics. The parameter q corresponds to the temperature of the underlying model. Kashiwara [11] showed that at zero temperature or $q=0$ the representations of $U_q(\mathfrak{g})$ have bases, which he coined crystal bases, with a beautiful combinatorial structure and favorable properties such as uniqueness and stability under tensor products.

In the generalization from $\mathfrak{su}(2)$ to other types, rigged configurations become sequences of partitions with riggings. The number of partitions depends on the rank of the underlying algebra.

The generalization of the bijection from paths to rigged configurations to type $A_n^{(1)}$ is given in [16,17] and to other nonexceptional types in [20] in special cases. It was shown in [19,21] that all crystals can be realized as crystals of simply-laced type A,D,E. Hence the bijections for these types can be viewed as fundamental.

In the next section we will introduce crystal bases. The bijection algorithm for type $A_n^{(1)}$ and $D_n^{(1)}$ is presented in section 8.

6. Crystals

6.1. *Axiomatic definition of crystals*

Let $\mathfrak{g}$ be an affine Lie algebra and I the index set of its Dynkin diagram. Let α_i, h_i, Λ_i $(i \in I)$ be the simple roots, simple coroots, and fundamental weights for $\mathfrak{g}$. Let $\delta = \sum_{i\in I} a_i\alpha_i$ denote the standard null root and $c = \sum_{i\in I} a_i^\vee h_i$ the canonical central element, where $a_i, a_i^\vee$ are the positive integers given in [10]. Let $P = \bigoplus_{i\in I} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$ be the weight lattice and

$P^+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\Lambda_i \oplus \mathbb{Z}\delta$ the dominant weights.

A crystal B is a set $B = \sqcup_{\lambda\in P} B_\lambda$ ($\mathrm{wt}\, b = \lambda$ if $b \in B_\lambda$) with the maps

$$e_i : B_\lambda \longrightarrow B_{\lambda+\alpha_i} \sqcup \{0\}, \quad f_i : B_\lambda \longrightarrow B_{\lambda-\alpha_i} \sqcup \{0\},$$
$$\varepsilon_i : B \longrightarrow \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_i : B \longrightarrow \mathbb{Z} \sqcup \{-\infty\}$$

for all $i \in I$ such that

$$\begin{aligned}
&\text{for } b \in B_\lambda,\ \varphi_i(b) = \langle h_i, \lambda\rangle + \varepsilon_i(b),\\
&\text{for } b \in B, \text{ we have}\\
&\qquad\qquad \varepsilon_i(b) = \varepsilon_i(e_i b) + 1 \text{ if } e_i b \neq 0,\\
&\qquad\qquad\qquad\ \ = \varepsilon_i(f_i b) - 1 \text{ if } f_i b \neq 0,\\
&\qquad\qquad \varphi_i(b) = \varphi_i(e_i b) - 1 \text{ if } e_i b \neq 0,\\
&\qquad\qquad\qquad\ \ = \varphi_i(f_i b) + 1 \text{ if } f_i b \neq 0,\\
&\text{for } b, b' \in B,\ e_i b' = b \text{ if and only if } b' = f_i b,\\
&\text{for } b \in B,\ \varepsilon_i(b) = \varphi_i(b) = -\infty \text{ implies } e_i b = f_i b = 0.
\end{aligned}$$

A crystal B can be regarded as a colored oriented graph by defining

$$b \xrightarrow{i} b' \quad \Longleftrightarrow \quad f_i b = b'.$$

If we want to emphasize I, B is called an I-crystal.

If B_1 and B_2 are crystals, then for $b_1 \otimes b_2 \in B_1 \otimes B_2$ the action of e_i is defined as

$$e_i(b_1 \otimes b_2) = \begin{cases} e_i b_1 \otimes b_2 \text{ if } \varepsilon_i(b_1) > \varphi_i(b_2),\\ b_1 \otimes e_i b_2 \text{ else,}\end{cases}$$

where $\varepsilon_i(b) = \max\{k \mid e_i^k b \text{ is defined}\}$ and $\varphi_i(b) = \max\{k \mid f_i^k b \text{ is defined}\}$. This is the opposite of the notation used by Kashiwara [11].

An element $b \in B$ is classically highest weight if $e_i b = 0$ for all $i = 1, 2, \ldots, n$. For $B = B_L \otimes \cdots \otimes B_1$ and $\Lambda \in P^+$, the set of paths is defined as follows

$$\mathcal{P}(B, \Lambda) = \{b \in B \mid e_i b = 0 \text{ for all } i = 1, 2, \ldots, n,\ \mathrm{wt} b = \Lambda\}.$$

In the following we will discuss the crystals of type $A_n^{(1)}$ and $D_n^{(1)}$ more explicitly.

6.2. *Dynkin data of type A_n and D_n*

Let ϵ_i be the i-th standard unit vector in $\mathbb{Z}^n$. Then for type A_{n-1}, the simple roots are

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \qquad \text{for } 1 \leq i < n$$

and the fundamental weights are

$$\Lambda_i = \epsilon_1 + \cdots + \epsilon_i \qquad \text{for } 1 \le i < n.$$

For type D_n, the simple roots are

$$\begin{aligned} \alpha_i &= \epsilon_i - \epsilon_{i+1} \qquad \text{for } 1 \le i < n \\ \alpha_n &= \epsilon_{n-1} + \epsilon_n \end{aligned}$$

and the fundamental weights are

$$\begin{aligned} \Lambda_i &= \epsilon_1 + \cdots + \epsilon_i \qquad\qquad \text{for } 1 \le i \le n-2 \\ \Lambda_{n-1} &= (\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n)/2 \\ \Lambda_n &= (\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n)/2. \end{aligned}$$

6.3. *Affine crystals of type $A_n^{(1)}$ and $D_n^{(1)}$*

In [8] it is conjectured that there is a family of finite-dimensional irreducible $U_q'(\mathfrak{g})$-modules $\{W_i^{(a)} \mid a \in J, i \in \mathbb{Z}_{\ge 0}\}$ which, unlike most finite-dimensional $U_q'(\mathfrak{g})$-modules, have crystal bases $B^{a,i}$. Here $U_q'(\mathfrak{g})$ is the quantum universal enveloping algebra of the derived subalgebra of $\mathfrak{g}$, obtained by omitting the degree operator, and $J = I \setminus \{0\}$.

Here we will restrict our attention to the simplest affine crystals $B^{1,1}$ of type $A_n^{(1)}$ and $D_n^{(1)}$. As a set $B^{1,1}$ is $\{1 < 2 < \cdots < n+1\}$ for type $A_n^{(1)}$ and $\{1 < 2 < \cdots < n-1 < \begin{smallmatrix} n \\ \overline{n} \end{smallmatrix} < \overline{n-1} < \cdots < \overline{1}\}$ for type $D_n^{(1)}$. The crystal graphs are given in Figure 1.

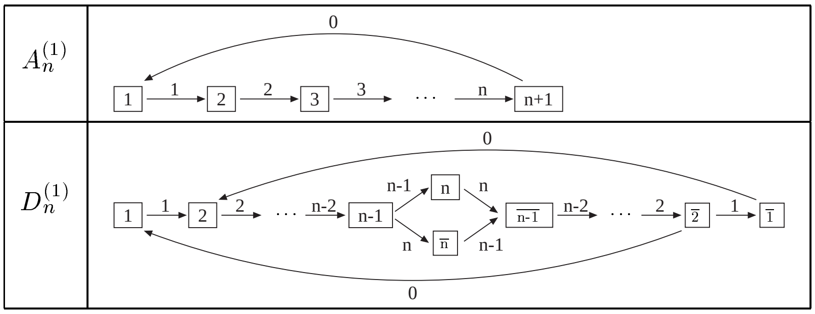


Figure 1. Crystals $B^{1,1}$

6.4. *One-dimensional sums*

The energy function (43) can be generalized to the crystal setting. In the case $B = (B^{1,1})^{\otimes L}$ it takes a simple form. There is [14] a unique (up to global additive constant) function $H : B^{1,1} \otimes B^{1,1} \to \mathbb{Z}$ called the local energy function, such that

$$H(e_i(b \otimes b')) = H(b \otimes b') + \begin{cases} -1 \text{ if } i = 0 \text{ and } e_0(b \otimes b') = b \otimes e_0 b' \\ 1 \quad \text{if } i = 0 \text{ and } e_0(b \otimes b') = e_0 b \otimes b' \\ 0 \quad \text{otherwise.} \end{cases}$$

We normalize H by the condition $H(1 \otimes 1) = 0$.

Example 6.1. Let $b \otimes b' \in B^{1,1} \otimes B^{1,1}$. Explicitly, the local energy function is given as follows. For type $A_n^{(1)}$, $H(b \otimes b') = -\chi(b > b')$. For type $D_n^{(1)}$, $H(b \otimes b') = 0$ if $b \leq b'$, $H(b \otimes b') = -1$ if $b \otimes b' = n \otimes \overline{n}, \overline{n} \otimes n$ or $b > b'$ where $b \otimes b' \neq \overline{1} \otimes 1$, and $H(\overline{1} \otimes 1) = -2$.

For $b_L \otimes \cdots \otimes b_1 \in B = (B^{1,1})^{\otimes L}$

$$E(b_L \otimes \ldots \otimes b_1) = \sum_{j=1}^{L-1} (L-j)\ H(b_{j+1} \otimes b_j).$$

Define the one-dimensional sum $X(B, \lambda; q) \in \mathbb{Z}[q, q^{-1}]$ by

$$X(B, \lambda; q) = \sum_{b \in \mathcal{P}(B,\lambda)} q^{E(b)}.$$

Example 6.2. In the crystal language the set of paths of Example 4.2 corresponds to $B = (B^{1,1})^{\otimes 5}$ of type $A_1^{(1)}$ of weight $\lambda = \Lambda_1 + 2\Lambda_2$.

7. Fermionic formula and rigged configurations

Fermionic formulas associated to a Kac-Moody algebra $\mathfrak{g}$ were conjectured in [7,8]. We review the fermionic formulas for type $A_n^{(1)}$ and $D_n^{(1)}$.

Let $L_i^{(a)}$ with $a \in J$ and $i \in \mathbb{Z}_{\geq 0}$ denote the number of tensor factors $B^{a,i}$ in B and let λ be a dominant integral weight. Say that $\nu^\bullet = (m_i^{(a)})$ is a (B, λ)-configuration if

$$\sum_{\substack{a \in J \\ i \in \mathbb{Z}_{\geq 0}}} i\, m_i^{(a)} \alpha_a = \sum_{\substack{a \in J \\ i \in \mathbb{Z}_{\geq 0}}} i\, L_i^{(a)} \Lambda_a - \lambda. \tag{58}$$

The configuration $\nu^\bullet$ is admissible if all vacancy numbers are nonnegative

$$p_i^{(a)} \geq 0 \qquad \text{for all } a \in J \text{ and } i \in \mathbb{Z}_{\geq 0},$$

where

$$p_i^{(a)} = \sum_{k\in\mathbb{Z}_{\geq 0}} \left(L_k^{(a)} \min(i,k) - \sum_{b\in J} (\alpha_a|\alpha_b) \min(i,k)\, m_k^{(b)} \right). \tag{60}$$

Write $C(B,\lambda)$ for the set of admissible (B,λ)-configurations. Define

$$cc(\nu^\bullet) = \frac{1}{2} \sum_{a,b\in J} \sum_{j,k\in\mathbb{Z}_{\geq 0}} (\alpha_a|\alpha_b) \min(j,k) m_j^{(a)} m_k^{(b)}.$$

The fermionic formula is defined by

$$M(B,\lambda;q) = \sum_{\nu^\bullet\in C(B,\lambda)} q^{cc(\nu^\bullet)} \prod_{a\in J} \prod_{i\in\mathbb{Z}_{\geq 0}} \begin{bmatrix} p_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{bmatrix}. \tag{62}$$

The $X = M$ conjecture of [7,8] states that

$$X(B,\lambda;q^{-1}) = M(B,\lambda;q).$$

The fermionic formula $M(B,\lambda)$ can be interpreted using rigged configurations. Denote by $(\nu^\bullet, J^\bullet)$ a pair where $\nu^\bullet = (m_i^{(a)})$ is a matrix and $J^\bullet = (J^{(a,i)})$ is a matrix of partitions with $a \in J$ and $i \in \mathbb{Z}_{\geq 0}$. Then a rigged configuration is a pair $(\nu^\bullet, J^\bullet)$ such that $\nu^\bullet \in C(B,\lambda)$ and the partition $J^{(a,i)}$ is contained in a $m_i^{(a)} \times p_i^{(a)}$ rectangle for all a, i. The set of rigged (B,λ)-configurations for fixed λ and B is denoted by $\mathrm{RC}(B,\lambda)$. Then (62) is equivalent to

$$M(B,\lambda) = \sum_{(\nu^\bullet,J^\bullet)\in\mathrm{RC}(B,\lambda)} q^{\mathrm{cc}(\nu^\bullet,J^\bullet)}$$

where $\mathrm{cc}(\nu^\bullet, J^\bullet) = \mathrm{cc}(\nu^\bullet) + |J^\bullet|$ and $|J^\bullet| = \sum_{(a,i)} |J^{(a,i)}|$. To emphasize the dependence on $\nu^\bullet$ we also write $m_i^{(a)}(\nu^\bullet)$ and $P_i^{(a)}(\nu^\bullet)$ for $m_i^{(a)}$ and $p_i^{(a)}$, respectively.

8. Bijection between rigged configurations and paths

In this section we give the description of the bijection $\Phi : \mathrm{RC}(B,\lambda) \to \mathcal{P}(B,\lambda)$ for types $A_n^{(1)}$ and $D_n^{(1)}$ when $B = (B^{1,1})^{\otimes L}$.

Let $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(B,\lambda)$. We shall define a map $\mathrm{rk} : \mathrm{RC}(B,\lambda) \to B^{1,1}$ which associates to $(\nu^\bullet, J^\bullet)$ an element of $B^{1,1}$ called its rank. Denote by $\mathrm{RC}_b(B,\lambda)$ the elements of $\mathrm{RC}(B,\lambda)$ of rank b. We shall define a bijection $\delta : \mathrm{RC}_b(B,\lambda) \to \mathrm{RC}(\tilde{B}, \lambda - \mathrm{wt}(b))$ where $\tilde{B} = (B^{1,1})^{\otimes(L-1)}$. The disjoint union of these bijections then defines a bijection $\delta : \mathrm{RC}(B,\lambda) \to \bigcup_{b\in B^{1,1}} \mathrm{RC}(\tilde{B}, \lambda - \mathrm{wt}(b))$.

The bijection Φ is defined recursively as follows. For $b \in B^{1,1}$ let $\mathcal{P}_b(B,\lambda)$ be the set of paths in B that have b as leftmost tensor factor. For $L=0$ the bijection Φ sends the empty rigged configuration (the only element of the set $\mathrm{RC}(B,\lambda)$) to the empty path (the only element of $\mathcal{P}(B,\lambda)$). Otherwise assume that Φ has been defined for $\tilde{B}$ and define it for B by the commutative diagram

$$\begin{array}{ccc}
\mathrm{RC}_b(B,\lambda) & \xrightarrow{\Phi} & \mathcal{P}_b(B,\lambda) \\
{\scriptstyle\delta}\big\downarrow & & \big\downarrow \\
\mathrm{RC}(\tilde{B},\lambda-\mathrm{wt}(b)) & \xrightarrow{\Phi} & \mathcal{P}(\tilde{B},\lambda-\mathrm{wt}(b))
\end{array}$$

where the right hand vertical map removes the leftmost tensor factor b. In short,

$$\Phi(\nu^\bullet,J^\bullet)=\mathrm{rk}(\nu^\bullet,J^\bullet)\otimes\Phi(\delta(\nu^\bullet,J^\bullet)).$$

We also require the bijection $\tilde{\Phi}:\mathrm{RC}(B,\lambda)\to\mathcal{P}(B,\lambda)$ given by $\tilde{\Phi}=\Phi\circ\mathrm{comp}$ where $\mathrm{comp}:\mathrm{RC}(B,\lambda)\to\mathrm{RC}(B,\lambda)$ with $\mathrm{comp}(\nu^\bullet,J^\bullet)=(\nu^\bullet,\tilde{J}^\bullet)$ is the function which complements the riggings, meaning that $\tilde{J}^\bullet$ is obtained from $J^\bullet$ by complementing all partitions $J^{(a,i)}$ in the $m_i^{(a)}(\nu^\bullet)\times P_i^{(a)}(\nu^\bullet)$ rectangle.

Remark 8.1. The bijection Ψ of section 4 is the inverse of $\tilde{\Phi}$ for type $A_1^{(1)}$.

Theorem 8.1. $\Phi:\mathrm{RC}(B,\lambda)\to\mathcal{P}(B,\lambda)$ *is a bijection such that*

$$cc(\nu^\bullet,J^\bullet)=-E(\tilde{\Phi}(\nu^\bullet,J^\bullet))\qquad \textit{for all } (\nu^\bullet,J^\bullet)\in\mathrm{RC}(B,\lambda).$$

For type $A_n^{(1)}$ a generalization of this theorem for $B=B^{a_L,i_L}\otimes\cdots\otimes B^{a_1,i_1}$ was proven in [17]. For other types Theorem 8.1 is proved in [20].

To describe the bijection explicitly for types $A_n^{(1)}$ and $D_n^{(1)}$, the following notation is needed. The matrix $\nu^\bullet=(m_i^{(a)})$ can be viewed as a sequence of partitions $\nu^\bullet=(\nu^{(1)},\nu^{(2)},\dots,\nu^{(n)})$ where $m_i^{(a)}$ is the number of parts of size i in the partition $\nu^{(a)}$. Denote by $Q_i(\rho)$ the number of boxes in the first i columns of the partition ρ. Finally the partition $J^{(a,i)}$ is called singular if it has a part of size $p_i^{(a)}$.

8.1. *Bijection for type $A_n^{(1)}$*

Using the Dynkin data for type A_n the vacancy numbers (60) and the constraints (58) can be rewritten in the following explicit way

$$P_i^{(a)}(\nu^\bullet)=Q_i(\nu^{(a-1)})-2Q_i(\nu^{(a)})+Q_i(\nu^{(a+1)})+L\delta_{a,1}\qquad\text{for } 1\le a\le n$$

and

$$|\nu^{(a)}| = L - \sum_{b=1}^{a} \lambda_b \qquad \text{for } 1 \le a \le n.$$

The algorithm δ is given as follows. Set $\ell^{(0)} = 0$ and repeat the following process for $a = 1, 2, \ldots, n$ or until stopped. Find the minimal index $i \ge \ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $b = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue with $a + 1$. If the process did not stop, set $b = n + 1$. Set all undefined $\ell^{(a)}$ to ∞.

The new rigged configuration is defined by

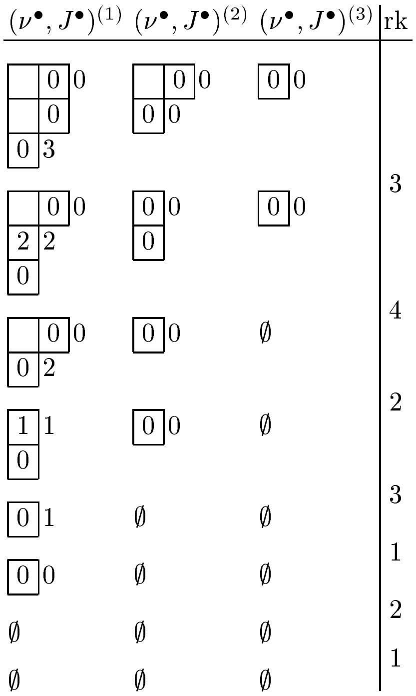
$$m_i^{(a)}(\tilde{\nu}^\bullet) = m_i^{(a)}(\nu^\bullet) + \begin{cases} 1 & \text{if } i = \ell^{(a)} - 1 \\ -1 & \text{if } i = \ell^{(a)} \\ 0 & \text{otherwise.} \end{cases}$$

The partition $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $P_i^{(a)}(\nu^\bullet)$ for $i = \ell^{(a)}$, adding a part of size $P_i^{(a)}(\tilde{\nu}^\bullet)$ for $i = \ell^{(a)} - 1$, and leaving it unchanged otherwise.

Example 8.1. Take $B = (B^{1,1})^{\otimes 7}$, $\lambda = \Lambda_3 + \Lambda_4$ and $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(B, \lambda)$ as

$$\left(\begin{array}[t]{|c|c|l} \cline{1-2} & 0 & 0 \\ \cline{1-2} & 0 & \\ \cline{1-2} 3 & \multicolumn{2}{l}{3} \\ \cline{1-1} \end{array}, \begin{array}[t]{|c|c|l} \cline{1-2} & 0 & 0 \\ \cline{1-2} 0 & \multicolumn{2}{l}{0} \\ \cline{1-1} \end{array}, \begin{array}[t]{|c|l} \cline{1-1} 0 & 0 \\ \cline{1-1} \end{array} \right).$$

The algorithm for Φ on $\mathrm{comp}(\nu^\bullet, J^\bullet)$ yields



Hence $\tilde{\Phi}(\nu^\bullet, J^\bullet) = b = 3 \otimes 4 \otimes 2 \otimes 3 \otimes 1 \otimes 2 \otimes 1$ and $E(b) = \mathrm{cc}(\nu^\bullet, J^\bullet) = 12$.

8.2. *Bijection for type $D_n^{(1)}$*

Using the Dynkin data for type D_n the vacancy numbers (60) and the constraints (58) can be rewritten in the following explicit way

$$
\begin{aligned}
P_i^{(a)}(\nu^\bullet) &= Q_i(\nu^{(a-1)}) - 2Q_i(\nu^{(a)}) + Q_i(\nu^{(a+1)}) + L\delta_{a,1} \qquad \text{for } 1 \le a < n-2 \\
P_i^{(n-2)}(\nu^\bullet) &= Q_i(\nu^{(n-3)}) - 2Q_i(\nu^{(n-2)}) + Q_i(\nu^{(n-1)}) + Q_i(\nu^{(n)}) \\
P_i^{(n-1)}(\nu^\bullet) &= Q_i(\nu^{(n-2)}) - 2Q_i(\nu^{(n-1)}) \\
P_i^{(n)}(\nu^\bullet) &= Q_i(\nu^{(n-2)}) - 2Q_i(\nu^{(n)})
\end{aligned}
$$

and

$$
\begin{aligned}
|\nu^{(a)}| &= L - \sum_{b=1}^{a} \lambda_b \qquad \text{for } 1 \le a \le n-2 \\
|\nu^{(n-1)}| &= \frac{1}{2}(L - \sum_{b=1}^{n-1} \lambda_b + \lambda_n)
\end{aligned}
$$

$$|\nu^{(n)}| = \frac{1}{2}(L - \sum_{b=1}^{n} \lambda_b).$$

The algorithm δ is given as follows. Set $\ell^{(0)} = 0$ and repeat the following process for $a = 1, 2, \ldots, n-2$ or until stopped. Find the minimal index $i \geq \ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $b = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue with $a + 1$.

If the process has not stopped at $a = n-2$ continue as follows. Find the minimal indices $i, j \geq \ell^{(n-2)}$ such that $J^{(n-1,i)}$ and $J^{(n,j)}$ are singular. If neither i nor j exist, set $b = n-1$ and stop. If i exists, but not j, set $\ell^{(n-1)} = i$, $b = n$ and stop. If j exists, but not i, set $\ell^{(n)} = j$, $b = \overline{n}$ and stop. If both i and j exist, set $\ell^{(n-1)} = i$, $\ell^{(n)} = j$ and continue with $a = n-2$.

Now continue for $a = n-2, n-3, \ldots, 1$ or until stopped. Find the minimal index $i \geq \overline{\ell}^{(a+1)}$ where $\overline{\ell}^{(n-1)} = \max(\ell^{(n-1)}, \ell^{(n)})$ such that $J^{(a,i)}$ is singular (if $i = \ell^{(a)}$ then there need to be two parts of size $P_i^{(a)}(\nu^\bullet)$ in $J^{(a,i)}$). If no such i exists, set $b = \overline{a+1}$ and stop. If the process did not stop, set $b = \overline{1}$.

Set all yet undefined $\ell^{(a)}$ and $\overline{\ell}^{(a)}$ to ∞.

The new rigged configuration is defined by

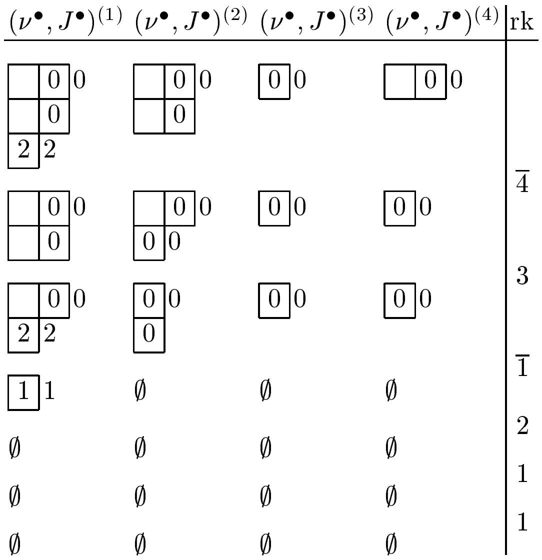
$$m_i^{(a)}(\tilde{\nu}^\bullet) = m_i^{(a)}(\nu^\bullet) + \begin{cases} 1 & \text{if } i = \ell^{(a)} - 1 \\ -1 & \text{if } i = \ell^{(a)} \\ 1 & \text{if } i = \overline{\ell}^{(a)} - 1 \text{ and } 1 \leq a \leq n-2 \\ -1 & \text{if } i = \overline{\ell}^{(a)} \text{ and } 1 \leq a \leq n-2 \\ 0 & \text{otherwise} \end{cases}$$

The partition $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $P_i^{(a)}(\nu^\bullet)$ for $i = \ell^{(a)}$ and $i = \overline{\ell}^{(a)}$, adding a part of size $P_i^{(a)}(\tilde{\nu}^\bullet)$ for $i = \ell^{(a)} - 1$ and $i = \overline{\ell}^{(a)} - 1$, and leaving it unchanged otherwise.

Example 8.2. Take $B = (B^{1,1})^{\otimes 6}$, $\lambda = 2\Lambda_3$ and $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(B, \lambda)$ as

$$\left(\begin{array}{|c|c|l} \hline & 0 & \multicolumn{1}{l}{0} \\ \cline{1-2} & 0 & \\ \cline{1-2} 0 & \multicolumn{1}{l}{2} & \\ \cline{1-1} \end{array}, \begin{array}{|c|c|l} \hline & 0 & \multicolumn{1}{l}{0} \\ \cline{1-2} & 0 & \\ \cline{1-2} \end{array}, \begin{array}{|c|l} \hline 0 & \multicolumn{1}{l}{0} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|l} \hline & 0 & \multicolumn{1}{l}{0} \\ \cline{1-2} \end{array} \right).$$

Then the algorithm for Φ on $\mathrm{comp}(\nu^\bullet, J^\bullet)$ gives the following intermediate steps



so that $\tilde{\Phi}(\nu^\bullet, J^\bullet) = b = \overline{4} \otimes 3 \otimes \overline{1} \otimes 2 \otimes 1 \otimes 1$. The statistics in this case are $E(b) = \mathrm{cc}(\nu^\bullet, J^\bullet) = 8$.

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