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Abstract

This paper develops a method to carry out the large-N asymptotic analysis of a class of N-dimensional integrals arising in the context of the so-called quantum separation of variables method. We push further ideas developed in the context of random matrices of size N, but in the present problem, two scales $1/N^{\alpha}$ and $1/N$ naturally occur. In our case, the equilibrium measure is N^{α}-dependent and characterised by means of the solution to a 2×2 Riemann–Hilbert problem, whose large-N behavior is analysed in detail. Combining these results with techniques of concentration of measures and an asymptotic analysis of the Schwinger-Dyson equations at the distributional level, we obtain the large-N behavior of the free energy explicitly up to $o(1)$. The use of distributional Schwinger-Dyson is a novelty that allows us treating sufficiently differentiable interactions and the mixing of scales $1/N^{\alpha}$ and $1/N$, thus waiving the analyticity assumptions often used in random matrix theory.

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An opening discussion

The present paper develops techniques enabling one to carry out the large-N asymptotic analysis of a class of multiple integrals that arise as representations for the correlation functions in quantum integrable systems solvable by the quantum separation of variables. We shall refer to the general class of such integrals as the sinh model:

$$\mathfrak{z}_N[W] = \int_{\mathbb{R}^N} \prod_{a<b}^{N} \Big\{ \sinh[\pi\omega_1(y_a - y_b)] \sinh[\pi\omega_2(y_a - y_b)] \Big\}^{\beta} \cdot \prod_{a=1}^{N} \mathrm{e}^{-W(y_a)} \cdot \mathrm{d}^N \boldsymbol{y} \,.$$

When $\beta = 1$ and for specific choices of the constants $\omega_1, \omega_2 > 0$ and of the confining potential W, $\mathfrak{z}_N$ represents norms or arises as a fundamental building block of certain classes of correlation functions in quantum integrable models that are solvable by the quantum separation of variable method. This method takes its roots in the works of Gutzwiller [54, 55] on the quantum Toda chain and has been developed in the mid '80s by Sklyanin [79, 80] as a way of circumventing certain limitations inherent to the algebraic Bethe Ansatz. Expressions for the norms or correlation functions for various models solvable by the quantum separation of variables method have been established, *e.g.* in the works [6, 38, 39, 51, 66, 67, 81, 84]. The expressions obtained there are either directly of the form (1.9) or are amenable to this form (with, possibly, a change of the integration contour from $\mathbb{R}^N$ to $\mathscr{C}^N$, with $\mathscr{C}$ a curve in $\mathbb{C}$) upon elementary manipulations. Furthermore, a degeneration of $\mathfrak{z}_N$ arises as a multiple integral representation for the partition function of the six-vertex model subject to domain wall boundary conditions [62]. In the context of quantum integrable systems, the number N of integrals defining $\mathfrak{z}_N$ is related to the number of sites in a model (as, *e.g.* in the case of the compact or non-compact XXZ chains or the lattice regularisations of the Sinh or Sine-Gordon models) or the number of particles (as, *e.g.* in the case of the quantum Toda chain). From the point of view of applications, one is mainly interested in the thermodynamic limit of the model, which is attained by sending N to $+\infty$. For instance, in the case of integrable lattice discretisations of some quantum field theory, one obtains in this way an exact and non-perturbative description of a quantum field theory in $1 + 1$ dimensions and in finite volume. This limit, at the level of $\mathfrak{z}_N$, translates itself in the need to extract the large N-asymptotic expansion of $\ln \mathfrak{z}_N$ up to $\mathrm{o}(1)$. It is, in fact, the constant term in the expansion on $\ln\big(\mathfrak{z}_N[W']/\mathfrak{z}_N[W]\big)$ with W' some deformation of W that provides one with the correlation functions of the underlying quantum field theory in finite volume. These applications to physics constitute the first motivation for our analysis. From the purely mathematical side, the motivation of our works stems from the desire to understand better the structure of the large-N asymptotic expansion of multiple integrals whose analysis demands to go out of the scheme of the β-ensembles.

As we shall argue in § 2.1, it is possible to understand the large-N asymptotic analysis of the multiple integral $\mathfrak{z}_N[W]$ from the one of the re-scaled multiple integral

$$\mathcal{Z}_N[V_N] \;=\; \int\limits_{\mathbb{R}^N} \prod_{a<b}^{N} \Big\{ \sinh[\pi\omega_1 T_N(\lambda_a-\lambda_b)] \sinh[\pi\omega_2 T_N(\lambda_a-\lambda_b)] \Big\}^{\beta} \cdot \prod_{a=1}^{N} \mathrm{e}^{-NT_N V_N(\lambda_a)} \cdot \mathrm{d}^N \boldsymbol{\lambda} \, .$$

There T_N is a sequence going to infinity with N whose form is fixed by the behaviour of $W(x)$ at large x, and: $V_N(\xi) \;=\; T_N^{-1} \cdot W(T_N \xi)$.

The main task of the paper is to develop an effective method of asymptotic analysis of the rescaled multiple integral $\mathcal{Z}_N[V]$ in the case when $T_N = N^{\alpha}$, $0 < \alpha < 1/6$ and V is a given N-independent strictly convex smooth potential V satisfying to a few additional technical hypothesis.

The treatment of the class of N-dependent potentials V_N which would enable one to deduce the large-N asymptotic expansion of $\mathfrak{z}_N[W]$ will be the matter of a future work.

Prior to discussing in more details the results obtained in this paper, we would like to provide a brief overview of the developments that took place, over the years, in the field of large-N asymptotic analysis of N-fold multiple integrals. This discussion serves as an introduction to various ideas that appeared fruitful in such an asymptotic analysis. More importantly, it will put these techniques in contrast with what happens in the case of the sinh model under study. In particular, we will to point out the technical aspects which complicate the large-N asymptotic analysis of $\mathfrak{z}_N[W]$ and thus highlight the features and techniques that are new in our analysis. Finally, such an organisation will permit us to emphasise the main differences occurring in the structure of the large N-asymptotic expansion of integrals related to the sinh-model as compared to the β-ensemble like multiple integrals.

The paper is organised as follows. Section 1 is the introduction where we attempt to give an overview of the various methods used and results obtained in respect to extracting the large number of integration asymptotics of integrals occurring to random matrix theory. Since we heavily rely on tools from potential theory, large deviations, Schwinger-Dyson equations, and Riemann-Hilbert techniques, which are often known separately in several communities but scarcely combined together, we thought useful to give a detailed introduction for readers with various backgrounds. In Section 2, we state and describe the results obtained in this paper. In Section 3 appears the *first part of the proof*: we carry out the *asymptotic analysis of the system of Schwinger-Dyson equations* subordinate to the sinh-model. It relies on results concerning the inversion of the master operator related with our problem. It is a singular integral operator whose inversion enables one, among other, to construct an N-dependent equilibrium measure. The *second part of the proof* is precisely the *construction of this inverse operator*: it is carried out in Section 4 by solving, for N large enough, an auxiliary 2×2 Riemann-Hilbert problem. The inverse operator itself and its main properties are described in Section 5. The *third part of the proof* consists in obtaining *fine information concerning the large N-behaviour of the inverse operator*: Section 6 is devoted to deriving uniform large-N local behaviour for the inverse operator. In Section 7 we build on the results established so far to carry out the large-N asymptotic analysis of single integrals involving the inverse operator. Finally, in Section 9 we establish the large-N asymptotic expansion of certain bi-dimensional integrals, a result that is needed so as to obtain the final answer for the expansion of the partition function. The paper contains four appendices. In Appendix A we remind a few useful results of functional analysis. In Appendix B, we establish the asymptotic analysis for the leading order $\ln \mathfrak{z}_N[W]$ by adapting known large deviation techniques. Then, in Appendix D, we derive an exact expression for the partition function $\mathcal{Z}_N[V_G]$ when $\beta = 1$ and V_G is a Gaussian potential. We also obtain there the large-N asymptotics of $\mathcal{Z}_N[V_G]$. This result is instrumental in deriving the asymptotic expansion of $\mathcal{Z}_N[V]$ for more general potential, since the Gaussian partition function always appears as a factor of the latter. Finally, Appendix E recapitulates all the symbols used in the paper. Some basic notations are also collected in § 1.4.

1 Introduction

1.1 Beta ensembles with varying weights

One of the simplest and yet non-trivial examples of an N-fold multiple integral are provided by β-ensembles with varying weights:

$$\mathcal{Z}_N^{(\beta)} \;=\; \int_{\mathbb{R}^N} \prod_{a<b}^{N} |\lambda_a - \lambda_b|^{\beta} \cdot \prod_{a=1}^{N} \mathrm{e}^{-NV(\lambda_a)} \cdot \mathrm{d}^N \boldsymbol{\lambda}\;. \tag{1.1}$$

$\beta > 0$ is a positive parameter and V is a potential growing sufficiently fast at infinity for the integral (1.1) to be convergent. This partition function arises when integrating over the spectra of random matrices drawn from the so-called orthogonal ($\beta = 1$), unitary ($\beta = 2$) or symplectic ($\beta = 4$) ensembles. The aforementioned three cases are very special, since they feature a determinantal or Pfaffian structure unknown for general β, and they can be solved in terms of orthogonal or skew-orthogonal polynomials [72]. For general $\beta > 0$ and polynomial V, the partition function (1.1) can be interpreted as the integral over the spectrum of a well-tailored family of random tri-diagonal matrices [40, 68]. Independently of these interpretations, $\mathcal{Z}_N^{(\beta)}$ can also be thought of as the partition function of a classical system of N particles at temperature β^{-1} that interact through a a two-body repulsive logarithmic interaction and are placed in an overall confining potential V.

Universality

The β-ensembles have been extensively studied for more than 20 years, see *e.g.* the books [4, 29, 72, 76]. The statistical-mechanics interpretation of β-ensembles makes $\mathcal{Z}_N^{(\beta)}$ and its associated probability distribution a good playground for testing the local universality of the distribution of repulsive particles [48]. The physical idea behind universality is that the logarithmic repulsion dictates the local behaviour of the particles[4]. The universality classes should only depend on β and the local environment of the chosen position on $\mathbb{R}$. First results of local universality in the bulk where obtained by Shcherbina and Pastur [75] at $\beta = 2$. Then, at $\beta = 2$ and for polynomial V, Deift, Kriechenbauer, McLaughlin, Venakides and Zhou [36] established the local universality in the bulk within the Riemann-Hilbert approach to orthogonal polynomials with orthogonality weight $\mathrm{e}^{-NV(x)}$ on the real line. These results were then extended by Deift and Gioev to $\beta \in \{1, 2, 4\}$ for the bulk [32] and then for the edge [31] universality. The bulk and edge universality for general $\beta > 0$ were recently established by various methods and under weaker assumptions. Bourgade, Erdös and Yau built on relaxation methods so as to establish the bulk [18, 20] and the edge [19] universality in the presence of generic C^k potentials. Krishnapur, Rider and Virág [68] proved both universalities by means of stochastic operator methods and in the presence of convex polynomial potentials. Finally, the bulk universality was also established on the basis of measure transport techniques by Shcherbina [78] in the presence of real-anaytic potentials while universality both at the bulk and edge was derived by Bekerman, Figalli and Guionnet [8] for C^k potentials with k large enough.

Leading order of $\mathcal{Z}_N^{(\beta)}$: the equilibrium measure and large deviations

The leading asymptotic behaviour of the partition function $\mathcal{Z}_N^{(\beta)}$ takes the form :

$$\ln \mathcal{Z}_N^{(\beta)} \;=\; -N^2\Big(\mathcal{E}^{(\beta)}[\mu_{\mathrm{eq}}] \;+\; \mathrm{o}(1)\Big) \quad \text{with} \quad \mathcal{E}^{(\beta)}[\mu] \;=\; \int V(x)\,\mathrm{d}\mu(x) \;-\; \beta \int_{x<y} \ln|x-y|\,\mathrm{d}\mu(x)\mathrm{d}\mu(y)\;. \tag{1.2}$$

[4] By local we understand looking at intervals shrinking with N so that these contain typically only a finite number of particles in the $N \to \infty$ limit

In these leading asymptotics, the functional $\mathcal{E}^{(\beta)}$ is evaluated at the so-called equilibrium measure μ_{eq}, a probability measure on $\mathbb{R}$ that minimises the functional $\mathcal{E}^{(\beta)}$. This minimiser can be characterised within the framework of potential theory [69] and arises in numerous other branches of mathematical physics. In particular, it exists and is unique. We stress that the leading order (1.2) depends on β only via a rescaling of the potential.

We shall begin the present discussion by describing, on a heuristic level, the mechanism which gives rise to (1.2). For this purpose, observe that the integrand of $\mathcal{Z}_N^{(\beta)}$ can be recast as

$$\exp\Big\{-N^2\mathcal{E}^{(\beta)}[L_N^{(\lambda)}]\Big\} \qquad \text{where} \qquad L_N^{(\lambda)} = \frac{1}{N}\sum_{a=1}^{N}\delta_{\lambda_a} \tag{1.3}$$

is the empirical measure while δ_x refers to the Dirac mass at x. For finite but large N, $\boldsymbol{\lambda} \mapsto \mathcal{E}^{(\beta)}[L_N^{(\lambda)}]$ attains its minimum at a point $\boldsymbol{\gamma}_{\mathrm{eq}} = (\gamma_{\mathrm{eq};1}, \ldots, \gamma_{\mathrm{eq};N})$ whose coordinates $\gamma_{\mathrm{eq};1} < \cdots < \gamma_{\mathrm{eq};N}$ are bounded, uniformly in N, from above and below. This minimum results from a balance[5] between the repulsion of the integration variables induced by the logarithmic interaction and the confining nature of the potential V. It seems reasonable that the main contribution to the integral, namely the one not including exponentially small corrections, will issue from a small neighbourhood of the point $\boldsymbol{\gamma}_{\mathrm{eq}}$ (or those issuing from permutations of its coordinates) and hence yield, to the leading order in N, $\ln \mathcal{Z}_N^{(\beta)} = -N^2(\mathcal{E}^{(\beta)}[L_N^{(\gamma_{\mathrm{eq}})}] + \mathrm{o}(1))$. As a matter of fact, the $\gamma_{\mathrm{eq};a}$ are distributed in such a way that they densify on some compact subset of $\mathbb{R}$ and in such a way that, in fact, $L_N^{(\gamma_{\mathrm{eq}})}$ converges to the probability measure μ_{eq}.

This reasoning thus indicates that the leading asymptotics of $\ln \mathcal{Z}_N^{(\beta)}$ issue from a saddle-point like estimation of the integral (1.1). This statement can be made precise within the framework of large deviations. Ben Arous and Guionnet [5] showed that the sequence of probability measures associated with $\mathcal{Z}_N^{(\beta)}$ satisfies a large deviation principle with good rate function $\mathcal{E}^{(\beta)}[\mu]$. Their framework shows that, in fact, $L_N^{(\lambda)}$ converges almost surely and in expectation towards the equilibrium measure μ_{eq}.

The properties of the equilibrium measure μ_{eq} have been extensively studied [34, 69, 77]. One can prove that if V is C^k for $k \geq 2$, then μ_{eq} is Lebesgue continuous with a C^{k-2} density. Besides, if V is real-analytic, the density is the square-root of an analytic function, hence its support consists of a finite number of segments, called *cuts*. Critical points of the model occur when the topology of the support is not stable under small perturbations of the potential, i.e. one component of the support splits in two, two cuts merge, or a new cut appears. When this is not the case, we say that the potential is *off-critical*.

A remarkable feature of this model is that the density of μ_{eq} can be built in terms of the solution to a *scalar* Riemann–Hilbert problem for a piecewise holomorphic functions having jumps on the support of μ_{eq}. Such Riemann–Hilbert problems can be solved explicitly leading to a *one-fold* integral representation for the density. These manipulations originate in the work of Carleman [23], and some aspects have also been treated in the book of Tricomi [82]. The endpoints of the support, however, have to be determined by non-linear (and sometimes transcendental) consistency relations. We stress that the very existence of a *one-fold* integral representation with *fully explicit* integrand tremendously simplifies the analysis, be it in what concerns the description of the properties of μ_{eq}, or any handling that actually involves the equilibrium measure.

The all-order large-N expansion

The motivation to study all-order asymptotic expansions of $\ln \mathcal{Z}_N^{(\beta)}$ when $N \to \infty$ initially came from physics and the study of $2d$-quantum gravity [21, 49] partly since the coefficients in the all order asymptotic expansion of

[5] The Lebesgue measure does not participate to the setting of this equilibrium: the aforementioned terms induce a $\mathrm{e}^{\mathrm{O}(N^2)}$ behaviour in the light of (1.2), while on compact subsets of $\mathbb{R}^N$, the Lebesgue measure produces at most a $\mathrm{O}(\mathrm{e}^{cN})$ contribution, with c depending on the size of the compact set.

$\mathcal{Z}_N^{(\beta)}$ provide solutions to many problems in enumerative geometry or topological strings, and also because of the richness of the algebraic structures in which those expansions fit. Going beyond the leading order demands taking into account the effect of fluctuations of the integration variables around their large-N equilibrium distribution. The most effective way of doing so consists in studying the so-called Schwinger-Dyson equations associated with $\mathcal{Z}_N^{(\beta)}$, which are, in fact, sometimes referred to as "loop equations". The Schwinger-Dyson equations consist of a tower of equations which relate multi-point expectation values of test functions versus the probability measure induced by $\mathcal{Z}_N^{(\beta)}$. In the case of analytic interactions, it is possible to introduce a collection of fundamental objects, the n-point correlators, $n = 1, 2, \ldots$. These are specific expectation values whose knowledge, in the analytic setting, is enough for computing all the expectation values related with the given model. Their use constitutes an important technical simplification of the intermediate analysis.

The calculation of the first sub-leading correction to (1.2) based on the use of Schwinger-Dyson equations for correlators was first carried out in the seminal papers of Ambjørn, Chekhov and Makeenko [3] and of these authors with Kristjansen [2]. The approach developed in these papers allowed, in principle, for a formal[6], order-by-order computation of the large-N asymptotic behaviour of $\mathcal{Z}_N^{(2)}$. However due to its combinatorial intricacy, the approach was quite complicated to set in practice. In [43], Eynard proposed a rewriting of the solutions of Schwinger-Dyson equations in a geometrically intrinsic form that strongly simplified the structure and intermediate calculations. Chekhov and Eynard then described the corresponding diagrammatics [24], and it led to the emergence of the so-called topological recursion fully developed by Eynard and Orantin in [45, 46]. It allows, in its present setting, for a formal yet quite systematic order-by-order calculation of the coefficients arising in the large-N asymptotic behaviour of the β-ensemble partition functions, just as numerous other instances of multiple integrals, see *e.g.* the work of Borot, Eynard and Orantin [14].

We have not yet discussed the problem of actually proving the existence of an asymptotic expansion of $\ln \mathcal{Z}_N^{(\beta)}$ to all algebraic orders in N, namely the fact that

$$\ln \mathcal{Z}_N^{(\beta)} = \sum_{k\geq 0}^{K} N^{2-k} F_k^{(\beta)}[V] + O(N^{-K}) \tag{1.4}$$

for any $K \geq 0$ and with coefficients being some β-dependent functionals of the potential V. The existence and form of the expansion up to o(1) when $\beta = 2$ was proven by Johansson [61] for polynomial V under the one-cut hypothesis, this by using the machinery of Schwinger-Dyson equations and *a priori* bounds for the correlators first obtained by Boutet de Monvel, Pastur et Shcherbina [28]. Then, the existence of the all-order asymptotic expansion at $\beta = 2$ was proven by Albeverio, Pastur and Shcherbina [1] by combining Schwinger-Dyson equations and the bounds derived in [28]. Later, within the Riemann–Hilbert problem approach, Ercolani and McLaughlin [42] established the existence of the all order asymptotic expansion at $\beta = 2$ in the case of potentials that are a perturbation of the Gaussian interaction. In particular, this work proved that the coefficients of the asymptotic expansion coincide with the formal generating series enumerating ribbon graphs of [21] – also known under the name of "maps". Finally, Borot and Guionnet [16] systematised and extended to all $\beta > 0$ the approach of [1], hence establishing the existence of the all-order large-N asymptotic expansion of $\mathcal{Z}_N^{(\beta)}$ at arbitrary β and for convex real analytic potentials. Though this phenomenon will not occur in the present article, let us mention for completeness that, when μ_{eq} is supported on several cuts, the form (1.4) of the asymptotic expansion is not valid anymore, and oscillatory terms in N have to be included. When adopting the physical picture, this effect takes its roots in the possibility the particles have to tunnel from one cut to another [11, 44]. For real-analytic off-critical potentials and general $\beta > 0$, the all-order asymptotic expansion was conjectured in [44] and established in [15]. We refer the reader to the latter reference for a deeper discussion relative to the history of this problem.

[6]Namely based, among other things, on the assumption of the very existence of the asymptotic expansion.

Generalisations

It is fair to say that presently, there exists a pretty good understanding of the large-N asymptotic expansions in β ensembles. The main remaining open questions concern the description of the asymptotic expansion uniformly around critical points (*viz.* when the number of cuts changes) and the possibility to relax the regularity of the potential, for instance by allowing the existence of Fisher–Hartwig singularities[7]. What we would like to stress is that the techniques of asymptotic analysis described so far are effective in the sense that they allow, upon certain more or less obvious generalisations of technical details, treating various instances of other multiple integrals.

The framework of small enough perturbation of the Gaussian potential is, in general, the easiest to deal with. Asymptotic expansions for multi-matrix models have been obtained in such a setting. For instance, the expansion including the first sub-leading order was derived for a two-matrix model by Guionnet and Maurel-Segala [52], the one to all orders for multi-matrix models by Maurel-Segala [71] and to all orders for unitary random matrices in external fields in [53] was then obtained by an appropriate adaptation of the analysis of Schwinger-Dyson equations.

Another natural generalisation of β-ensembles consists in replacing the one-particle varying potential $N \cdot V$ by a regular and varying multi-particle potential

$$N \sum_{a=1}^{N} V(\lambda_a) \hookrightarrow \sum_{p=1}^{r} \frac{N^{2-p}}{p!} \sum_{\substack{i_1 < \cdots < i_p \\ i_a = 1, \ldots, N}} V_p(\lambda_{i_1}, \ldots, \lambda_{i_p}) . \tag{1.5}$$

When $r = 2$, such interactions were studied by Götze, Venker [50] and Venker [83] where it was shown that their bulk universality corresponds to the universality class of β-ensembles. In fact, at $r = 2$ and when $\beta = 2$, the structure of such models becomes determinantal in the special cases where the two-body interaction takes the form:

$$V_2(\lambda_1, \lambda_2) = \ln\left(\frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1}\right) . \tag{1.6}$$

It is well known that, then, the associated multiple integrals can be fully characterised in terms of appropriate systems of biorthogonal polynomials in the sense of [65]. It is for this reason that such multiple integrals are referred to as biorthogonal ensembles. The case $f(\lambda) = \lambda^\theta$ for $\beta = 2$ is of special interest in that such a setting allows one to push the calculations even further. Borodin [12] was able to establish certain universality results for specific examples of confining potentials V. Furthermore, it was observed, first on a specific example by Claeys and Wang [27] and then in full generality by Claeys and Romano [26] that the biorthogonal polynomials can be characterised by means of a Riemann-Hilbert problem. However, for the moment, the Riemann–Hilbert problem-based machinery still did not lead to the asymptotic evaluation of the associated partition functions

For general r, Borot [13] has shown that the formal asymptotic expansion of the partition function subordinate to multi-particle potentials is captured by a generalisation of the topological recursion. The existence of the all-order asymptotic expansion was established by the authors in [17] under certain regularity assumptions on the multi-particle interactions. Note that for perturbations of the Gaussian potential of the form (1.5) the hypothesis of [17] are indeed satisfied.

1.2 β-ensembles with non-varying weights

In all the examples of the multiple integrals discussed so far, the interaction potential V is preceded by a power of N. This scaling ensures that, for typical configurations of the λ_a's, the logarithmic repulsion is of the same order

[7] Although, even in these two cases, some partial progress has been achieved at $\beta = 2$ where one can build on the Riemann–Hilbert approach [9, 25, 33]

of magnitude in N than the confining potential. As a consequence, with overwhelming probability when $N \to \infty$, the integration variables remain in a bounded region and exhibit at typical spacing $1/N$. The scheme developed in [1, 16, 17, 36] for the asymptotic analysis was adapted to this particular tuning of the interactions with N and, in general, breaks down if the nature of the balance between the interactions changes.

Serious problems relative to extracting the large-N asymptotic behaviour already start to arise in the case of *non-varying* weights, *i.e.* for multiple integrals:

$$\int\limits_{\mathbb{R}^N} \prod_{a<b}^{N} |y_a - y_b|^\beta \prod_{a=1}^{N} \mathrm{e}^{-W(y_a)} \cdot \mathrm{d}^N \boldsymbol{y} \,. \tag{1.7}$$

Indeed, consider the integral (1.7) for N-large and focus on the contribution of a bounded domain of $\mathbb{R}^N$. In this case, the logarithmic interactions are dominant in respect to the confinement (and this by one order in N): the dominant contribution of such a region is obtained by spacing the y_a's as far apart as possible. Increasing the size of such a bounded region will increase the value of the dominant contribution, at least until the confining nature of the potential kicks in. Hence, to identify the configuration maximising the value of the integral, one should rescale the integration variables as $y_a = T_N \lambda_a$ with $T_N \to \infty$. The sequence T_N is chosen in such a way that the 2-body interaction and the confinement ensured by the potential have the same order of magnitude in N, *viz.*:

$$W(T_N \lambda) \;=\; N V_N(\lambda) \qquad \text{with} \qquad V_N(\lambda) \;=\; V_\infty(\lambda) \cdot (1 + \mathrm{o}(1)) \tag{1.8}$$

for some potential V_∞ and pointwise almost-everywhere in λ. These new variables λ are typically distributed in a bounded region and have a typical spacing $1/N$.

The simplest illustration of such a mechanism issues from the case of a polynomial potential $V(\lambda) = \sum_{a=1}^{2\ell} c_a \lambda^a$, $c_{2\ell} > 0$. In this case, the sequence T_N takes the form $T_N = N^{1/(2\ell)}$. Note that, up to a trivial prefactor, the two-body interaction $\lambda \mapsto |\lambda|^\beta$ is invariant under dilatations. As a consequence, for polynomial potentials, the asymptotic analysis can still be carried out by means of the previously described methods [35], with minor technical complications due to the handling of a N-dependent potential. Although illustrative, the polynomial case is by far not representative of the complexity represented by working with non-varying weights. Indeed, the genuinely hard part of the analysis stems form the fact that, in principle, in the expansion (1.8):

- the remainder may not be "sufficiently" uniform ;
- the non-varying potential W may have singularities in the complex plane. This last scenario means that the singularities of the rescaled potential V_N given in (1.8) will collapse, with a N-dependent rate, on the integration domain.

In this situation, the usual scheme for obtaining sub-leading corrections breaks down. So far, the large-N asymptotic analysis of a "non-trivial" multiple integral of the type (1.7) were carried out only when $\beta = 2$ and this for only a handful of examples. Zinn-Justin [85] proposed an N-fold multiple integral representation of the type (1.8) for the partition function of the six-vertex model in its massless phase and subject to domain wall boundary conditions. By using a proper rescaling of the variables suggested in [85], Bleher and Fokin [10] carried out the large-N asymptotic analysis of the associated multiple integral within the Riemann–Hilbert problem approach to orthogonal polynomials. The most delicate point of their analysis was to absorb the contribution of the sequence of poles ζ_n/N, $n = 1, 2, \ldots$, of the rescaled potential that were collapsing on $\mathbb{R}$. *In fine*, they obtained the asymptotic expansion of the logarithm of the integral up to $\mathrm{o}(1)$ corrections.

To conclude, it seems fair to state that despite the considerable developments that took place over the last 20 years in the field of large-N asymptotic expansion of N-dimensional integrals, the techniques of asymptotic analysis are still far from enabling one to grasp the large-N asymptotic behaviour of multiple integrals lacking the presence of a scaling of interactions. Such integral arise quite naturally in concrete applications. For instance, it is well known that correlation functions in quantum integrable models are described by N-fold multiple integrals [58, 59, 60, 64] or series thereof [63]. Usually, for reasons stemming from the physics of the underlying model, one is interested in the large-N behaviour of these integrals and, in particular, in the constant term arising in their asymptotics. However, for most cases of interest, the given N-fold integrals have a much too complicated integrand in order to apply any of the existing methods of analysis.

1.3 The integrals issued from the method of quantum separation of variables

In the present paper, we develop the main features of a theory that would enable one to extract the large-N asymptotic behaviour out of the class of multiple integrals that naturally arises in the context of the so-called quantum separation of variables method:

$$\mathfrak{z}_N[W] \; = \; \int\limits_{\mathbb{R}^N} \prod_{a<b}^{N} \Big\{ \sinh[\pi\omega_1(y_a - y_b)] \sinh[\pi\omega_2(y_a - y_b)] \Big\}^{\beta} \cdot \prod_{a=1}^{N} \mathrm{e}^{-W(y_a)} \cdot \mathrm{d}^N \boldsymbol{y} \, . \tag{1.9}$$

Independently of its numerous potential applications to physics, should one only have in mind characterising the large-N behaviour of N-fold multiple integrals, it is precisely the class of integrals described by (1.9) that constitutes naturally the next one to investigate and understand after the β-ensembles issued ones (1.1), (1.5). Indeed, on the one hand the integrand in (1.9) bears certain structural similarities with the one arising in β-ensembles. On the other hand, it brings two new features into the game. Therefore, $\mathfrak{z}_N[W]$ provide ones with a good playground for pushing forward the methods of asymptotic analysis of N-fold integrals and learning how to circumvent or deal with certain of the problematic features mentioned above. To be more precise, the main features of the integrand in $\mathfrak{z}_N[W]$ being an obstruction to applying the already established methods stem from the presence of

- a non-varying confining one-body potential W;
- a two-body interaction that has the same local (*viz.* when $\lambda_a \to \lambda_b$) singularity structure as in the β-ensemble case, while breaking other properties of the Van-der-Monde interaction such as the invariance under a re-scaling of all the integration variables.

Although, the tools of asymptotic analysis discussed previously break down or have to be altered in a significant way, a certain analogy with matrix models and β-ensembles persists. Indeed, upon a proper rescaling in the spirit of Sub-Section 1.2, one can show that the integral localises at a configuration of the integration variables in such a way that these condensate, in the large-N limit, with a density ρ_{eq}. In fact, we show in Appendix B that it is possible to repeat, with some modifications, the large-deviation approach to β-ensemble integrals so as to obtain the leading asymptotic behaviour of $\ln \mathfrak{z}_N$. However, in order to go beyond the leading asymptotic behaviour of the logarithm, one has to alter the picture and work directly at the level of the rescaled model

$$Z_N[V] \; = \; \int\limits_{\mathbb{R}^N} \prod_{a<b}^{N} \Big\{ \sinh\left[\pi\omega_1 N^{\alpha}(\lambda_a - \lambda_b)\right] \sinh\left[\pi\omega_2 N^{\alpha}(\lambda_a - \lambda_b)\right] \Big\}^{\beta} \cdot \prod_{a=1}^{N} \Big\{ \mathrm{e}^{-N^{1+\alpha} V(\lambda_a)} \Big\} \cdot \prod_{a=1}^{N} \mathrm{d}\lambda_a \, . \tag{1.10}$$

This integral is related to $\mathfrak{z}_N[W]$ by a rescaling of the integration variables. The exponent α is fixed by the growth of the original potential W at infinity. Finally, the potential V should depend on N and correspond to some rescaling of the original potential W. In fact, the main result obtained in the present paper deals with the large-N asymptotic expansion of the rescaled partition function $Z_N[V]$ and this in the case where

- $0 < \alpha < 1/6$;
- the potential V is smooth, strictly convex, has sub-exponential growth and is N-independent.

Per se, the application of our technique and results to computing the asymptotics of the original integral $\mathfrak{z}_N[W]$ would demand to take a N dependent potential and study $Z_N[V_N]$, which is technically more involved. However, this problem is *not* conceptually different from the one studied in this paper. Therefore, the setting we shall discuss is more fit for developing the method of asymptotic analysis of this class of integrals. We shall address the question of N-dependent potentials V_N related to specific applications to quantum integrable models in a separate publication.

Within our setting, in order to grasp sub-leading corrections to $\ln Z_N[V]$, one faces several difficulties:

(*i*) owing to the scaling N^{α}, the nature of the repulsive interaction between the λ_a's changes drastically between $N = \infty$ and N finite. Therefore, one has to keep track of the transition of scales between the *per-se* leading contribution – which feels, effectively, only the brute $N = \infty$ behaviour of the two body interaction – and the sub-leading corrections which experience the two-body interactions at all scales.

(*ii*) The presence of two scales N and N^{α} weakens a naive approach to the concentration of measures.

(*iii*) The derivative of the two-body interaction possesses a tower of poles that collapse down to the integration line, hence making the use of correlators and complex variables methods to study Schwinger-Dyson equations completely ineffective.

(*iv*) The master operator arising in the Schwinger-Dyson equations is an N-dependent singular integral operator of truncated Wiener–Hopf type. One has to invert this operator effectively and derive the fine, N-dependent bounds on its continuity constant as an operator between spaces of sufficiently differentiable functions.

(*v*) The large-N behaviour of one point functions, as fixed by a successful large-N analysis of the Schwinger-Dyson equations, is expressed in terms of one and two dimensional integrals involving the inverse of the master operator. One has to extract the large-N asymptotic behaviour of such integrals.

The setting of methods enabling one to overcome these problems constitutes the main contributions of this work.

First, in order to strengthen the concentration of measures and, in fact, effectively absorb part of the asymptotic expansion into a single expression, one should work with N-dependent equilibrium measures, that is to say equilibrium measures associated with a minimisation problem of a quadratic N-dependent functional on the space of probability measures on $\mathbb{R}$. The density of such an N-dependent measure can be expressed as an integral transform whose kernel is given by a double integral involving the solution to a matrix 2×2 Riemann–Hilbert problem. This very fact constitutes a crucial difference with the matrix model case in that, in the latter case, the density of equilibrium measure solves a scalar Riemann–Hilbert problem, hence admitting an explicit, one dimensional integral representation. On top of improving numerous bounds, the use of such N-dependent equilibrium measures turns out to be crucial in order to push the asymptotic expansion of $\ln \mathfrak{z}_N[W]$ up to $o(1)$

Second, the *per se* machinery of topological recursion mentioned earlier breaks down for this class of multiple integrals. In order to circumvent dealing with the collapsing of poles, we develop a distributional approach to the asymptotic analysis of Schwinger-Dyson equations. The latter demands, in particular, to have a much more precise control on its constituents.

Third, the inversion of the master operator is based on handlings of the inverse of the operator driving the singular integral equation for the density of equilibrium measure. Obtaining fine, N dependent bounds for this operator demands to go deep into the details of the solution of the 2×2 Riemann–Hilbert problem which arises as the building block of this inverse kernel. We develop techniques enabling one to do so.

Finally, the precise control on the objects issuing from Schwinger-Dyson equations yield, through usual interpolation by means of t-varying potentials, an N-dependent functional of the density of equilibrium measure – itself also depending on N – as an answer for the large-N asymptotics of $\ln Z_N[V]$. Setting forth methods for the asymptotic analysis of this functional demands, again, a very fine control of the inverse build through the Riemann–Hilbert problem approach. We develop such methods, in particular, by describing the new class of special functions related to our problem.

Putting in perspective the bi-orthogonal ensembles.

At this point, we shall make several comments in respect to the existing literature on bi-orthogonal ensembles. Indeed, the applications to the quantum separation of variables correspond to setting β to 1 in $\mathfrak{z}_N[W]$ and hence $Z_N[V]$. In this case, these multiple integral corresponds to a bi-orthogonal ensemble. As such, they can be explicitly computed, at least in principle, by means of the system of bi-orthogonal polynomials associated with the bi-periodic functions $\mathrm{e}^{\pi\omega_1 y}$, $\mathrm{e}^{\pi\omega_2 y}$ and in respect to the weight $\mathrm{e}^{-W(y)}$ supported on $\mathbb{R}$. As shown by Claeys and Wang [27] for a specific degeneration (which corresponds basically to sending one of the ω's in (1.9) to zero) and then in full extent by Claeys and Romano [26], such a systems of bi-orthogonal polynomials solves a vector Riemann-Hilbert problem. Furthermore, the non-linear steepest descent approach [35, 36] to the uniform in the variable large degree-N asymptotics of orthogonal polynomials can be generalised to such a bi-orthogonal setting, leading to Plancherel-Rotach like asymptotics. In principle, by adapting the steps of [42], one should be able to derive the large N asymptotic expansion of the integral $\mathfrak{z}_N$ in presence of *varying* weights, *viz.* provided the replacement $W \hookrightarrow NV$ is made. However, such a results would by no means allow one for any easy generalisation to non-varying weights. Indeed, as we have argued, in the non-varying case, one rather needs to carry out the large-N analysis of the rescaled model $Z_N[V]$. However, starting from such a multiple integral would imply that one should study the system of bi-orthogonal polynomials associated with the functions $\mathrm{e}^{\pi N^{\alpha}\omega_1 y}$, $\mathrm{e}^{\pi N^{\alpha}\omega_2 y}$. This presence of N^{α} introduces a new scale in N to the Riemann–Hilbert analysis, what would probably demand a quite non-trivial modification of the non-linear steepest descent method.

On top of all this, one needs to construct the equilibrium measure. For similar reasons of absorbing part of the asymptotic expansion, this measure will have to issue from the same N-dependent minimisation problem and hence correspond to the N-dependent equilibrium measure that we construct in the present paper. However, if one goes into the details of the work [26], one observes that these authors provide a one-fold integral representation for the density of the one-cut equilibrium measure arising in bi-orthogonal ensembles. The kernel of this transform involves the inverse of an explicit and basic transcendental function. Although extremely effective in the varying case, such an integral representation appears ineffective in the analysis of $Z_N[V]$. Indeed, then, one would have to manipulate N-dependent versions of this inverse and, in particular, obtain uniform in N local behaviours thereof. *A priori*, since this inverse does not seem to admit an explicit series expansion or a manageable integral representation, such a characterisation appears to be quite complicated. Furthermore, the transform constructed in [26] does not exhibit explicitly the factorisation of square root singularities at the edges - in contrast to the case of the one-fold integral representation arising in β ensembles. This means that, just as in our setting, one would have to extract the square root behaviour by hand. Therefore, although one dimensional, we believe that this transform, in the present state of the art, is much less effective then ours, at least from the point of view of our perspective of asymptotic analysis. In fact, when specialised to the construction of the equilibrium measure, the 2×2 Riemann–Hilbert analysis we use enable us, among other things, to provide the leading, up to exponentially small corrections in N, behaviour of the inverse of the N-rescaled map built in [26]. Thus, indirectly, our approach solves such a problem.

1.4 Notations and basic definitions

In this section, we introduce basic notations that we shall use throughout the paper.

General symbols

- o and O refer to standard domination relations between functions. In the case of matrix function $M(z)$ and $N(z)$, the relation $M(z) = \mathrm{O}(N(z))$ is to be understood entry-wise, *viz.* $M_{jk}(z) = \mathrm{O}(N_{jk}(z))$.
- $\mathrm{O}(N^{-\infty})$ means $\mathrm{O}(N^{-K})$ for arbitrarily large K's.
- Given a set $A \subseteq X$, $\mathbf{1}_A$ stands for the indicator function of A, and A^{c} denotes its complement in X.
- A Greek letter appearing in bold, *e.g.* $\boldsymbol{\lambda}$, will always denote an N-dimensional vector:

$$\boldsymbol{\lambda} \;=\; (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N \;. \tag{1.11}$$

  and $\mathrm{d}^N \boldsymbol{\lambda}$ denotes the product of Lebesgue measures $\prod_{a=1}^{N} \mathrm{d}\lambda_a$.
- given $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the integer satisfying $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$
- Throughout the file, the curve $\mathscr{C}^{+}_{\mathrm{reg}}$ will denote the curve depicted in Figure 4 appearing in § 6.1. This curve is such that $2\varsigma = \mathrm{dist}(\mathbb{R}, \mathscr{C}^{+}_{\mathrm{reg}}) > 0$. Throughout the text, this distance will always be denoted by 2ς.
- I_2 is the 2×2 identity matrix while $\sigma^{\pm}$ and σ_3 stand for the Pauli matrices:

$$\sigma^{+} \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad \sigma^{-} \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma_3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;. \tag{1.12}$$

Functional spaces

- $\mathcal{M}^1(\mathbb{R})$ denotes the space of probability measures on $\mathbb{R}$. The weak topology on $\mathcal{M}^1(\mathbb{R})$ is metrized by the Vasershtein distance, defined for any two probability measure μ_1 and μ_2 by:

$$D_V[\mu, \nu] = \sup_{f \in \mathrm{Lip}_{1,1}(\mathbb{R})} \int_{\mathbb{R}} f(\xi)\, \mathrm{d}(\mu_1 - \mu_2)(\xi) \;, \tag{1.13}$$

  where $\mathrm{Lip}_{1,1}(\mathbb{R})$ is the set of Lipschitz functions bounded by 1 and with Lipschitz constant bounded by 1. If f is a bounded, Lipschitz function, its bounded Lipschitz norm is:

$$||f||_{\mathrm{BL}} = ||f||_{L^{\infty}(\mathbb{R})} + \sup_{\xi \neq \eta \in \mathbb{R}} \left| \frac{f(\xi) - f(\eta)}{\xi - \eta} \right| . \tag{1.14}$$

- Given an open subset U of $\mathbb{C}^n$, $\mathcal{O}(U)$ refers to the ring of holomorphic functions on U. If f is a matrix of vector valued function, the notation $f \in \mathcal{O}(U)$ is to be understood entrywise, *viz.* $\forall\ a, b$ one has $f_{ab} \in \mathcal{O}(U)$.
- $C^k(A)$ refers to the space of function of class k on the manifold A. $C^k_{\mathrm{c}}(A)$ refers to the spaces built out of functions in $C^k(A)$ that have a compact support.

- $L^p(A, \mathrm{d}\mu)$ refers to the space of p^{th}-power integrable functions on a set A in respect to the measure μ. $L^p(A, \mathrm{d}\mu)$ is endowed with the norm

$$||f||_{L^p(A,\mathrm{d}\mu)} \;=\; \Big\{ \int_A |f(x)|^p \, \mathrm{d}\mu(x) \Big\}^{\frac{1}{p}} \; . \tag{1.15}$$

- More generally, given an n-dimensional manifold A, $W_k^p(A, \mathrm{d}\mu)$ refers to the p^{th} Sobolev space of order k defined as

$$W_k^p(A, \mathrm{d}\mu) \;=\; \Big\{ f \in L^p(A, \mathrm{d}\mu) \;:\; \partial_{x_1}^{a_1} \dots \partial_{x_n}^{a_n} f \in L^p(A, \mathrm{d}\mu) \; , \; \sum_{\ell=1}^{n} a_\ell \leq k \quad \text{with} \quad a_\ell \in \mathbb{N} \Big\} \; . \tag{1.16}$$

  This space is endowed with the norm

$$||f||_{W_k^p(A,\mathrm{d}\mu)} \;=\; \max \Big\{ ||\partial_{x_1}^{a_1} \dots \partial_{x_n}^{a_n} f||_{L^p(A,\mathrm{d}\mu)} \;:\; a_\ell \in \mathbb{N}, \;\; \ell = 1, \dots, n, \;\; \text{and satisfying} \; \sum_{\ell=1}^{n} a_\ell \leq k \Big\} \; . \tag{1.17}$$

  In the following, we shall simply write $L^p(A)$, $W_k^p(A)$ unless there will arise some ambiguity on the measure chosen on A.

- We shall also need the N-weighted norms of order ℓ for a function $f \in W_\ell^\infty(\mathbb{R}^n)$, which are defined as

$$\mathcal{N}_N^{(\ell)}[f] \;=\; \sum_{p=0}^{\ell} \frac{||f||_{W_p^\infty(\mathbb{R}^n)}}{N^{p\alpha}} \; . \tag{1.18}$$

  In particular, we have the trivial bound $\mathcal{N}_N^{(\ell)}[f] \leq \ell ||f||_{W_k^\infty(\mathbb{R}^n)}$. Also, the number of variables of f is implicit in this notation.

- The symbol $\mathcal{F}$ denotes the Fourier transform on $L^2(\mathbb{R})$ whose expression, versus $L^1 \cap L^2(\mathbb{R})$ functions, takes the form

$$\mathcal{F}[\varphi](\lambda) \;=\; \int_{\mathbb{R}} \varphi(\xi) \, \mathrm{e}^{\mathrm{i}\xi\lambda} \mathrm{d}\xi \; . \tag{1.19}$$

  Given $\mu \in \mathcal{M}^1(\mathbb{R})$, we shall use the same symbol for denoting its Fourier transform, *viz*. $\mathcal{F}[\mu]$. The Fourier transform on $L^2(\mathbb{R}^n)$ is defined with the same normalisation.

- The s^{th} Sobolev space on $\mathbb{R}^n$ is defined as

$$H_s(\mathbb{R}^n) \;=\; \Big\{ u \in \mathcal{S}'(\mathbb{R}^n) \;:\; ||u||^2_{H_s(\mathbb{R}^n)} \;=\; \int_{\mathbb{R}^n} \Big(1 + \Big| \sum_{a=1}^{n} t_a^2 \Big|^{\frac{1}{2}} \Big)^{2s} \Big| \mathcal{F}[u](t_1, \dots, t_n) \Big|^2 \cdot \mathrm{d}^n \boldsymbol{t} \;<\; +\infty \Big\} \, , \tag{1.20}$$

  in which $\mathcal{S}'$ refers to the space of tempered distributions. We remind that given a closed subset $F \subseteq \mathbb{R}^n$, $H_s(F)$ corresponds to the subspace of $H_s(\mathbb{R}^n)$ of functions whose support is contained in F.

- The subspace

$$\mathfrak{X}_s(A) \;=\; \Big\{ H \in H_s(A) \;:\; \int\limits_{\mathbb{R}+\mathrm{i}\epsilon} \chi_{11}(\mu)\mathcal{F}[H](N^{\alpha}\mu)\mathrm{e}^{-\mathrm{i}N^{\alpha}\mu b_N}\frac{\mathrm{d}\mu}{2i\pi} \;=\; 0 \Big\} \tag{1.21}$$

in which $A \subseteq \mathbb{R}$ is closed will play an important role in the analysis. It is defined in terms of χ_{11}, the $(1,1)$ entry of the unique solution χ to the 2×2 matrix valued Riemann–Hilbert problem given in Section 4.3.

- Given a smooth curve Σ in $\mathbb{C}$, the space $\mathcal{M}_{\ell}(L^2(\Sigma))$ refers to $\ell\times\ell$ matrices with coefficients belonging to $L^2(\Sigma)$. It is endowed with the norm

$$||M||_{\mathcal{M}_{\ell}(L^2(\Sigma))} \;=\; \Big\{ \int\limits_{\Sigma}\sum_{a,b}[M_{ab}(s)]^{*}M_{ab}(s)\,\mathrm{d}\mu(s) \Big\}^{\frac{1}{2}} . \tag{1.22}$$

and * denotes the complex conjugation.

Certain standard operators

- Given an oriented curve $\Sigma \subseteq \mathbb{C}$, $-\Sigma$ refers to the same curve but endowed with the opposite orientation.

- Given a function f defined on $\mathbb{C}\setminus\Sigma$, with Σ an oriented curve in $\mathbb{C}$, we denote -if these exists- by $f_{\pm}(s)$ the boundary values of $f(z)$ on Σ when the argument z approaches the point $s\in\Sigma$ non-tangentially and from the left $(+)$ or the right $(-)$ side of the curve. Furthermore, if one deals with vector or matrix-valued function, then this notation is to be understood entry-wise.

- $\mathbb{H}^{\pm} = \{z\in\mathbb{C} \;:\; \mathrm{Im}\,(\pm z) > 0\}$ is the upper/lower half-plane, and $\mathbb{R}^{\pm} = \{z\in\mathbb{R} \;:\; \pm z \geq 0\}$ is the closed positive/negative real axis.

- The symbol C refers to the Cauchy transform on $\mathbb{R}$:

$$C[f](\lambda) \;=\; \int\limits_{\mathbb{R}} \frac{f(s)}{s-\lambda}\cdot\frac{\mathrm{d}s}{2\mathrm{i}\pi} . \tag{1.23}$$

The $\pm$ boundary values $C_{\pm}$ define continuous operators on $H_s(\mathbb{R})$ and admit the expression

$$C_{\pm}[f](\lambda) \;=\; \frac{f(\lambda)}{2} \;+\; \frac{1}{2\mathrm{i}}\fint\limits_{\mathbb{R}}\frac{f(s)\,\mathrm{d}s}{\pi(s-\lambda)} . \tag{1.24}$$

- Given a function f supported on a compact set A of $\mathbb{R}^n$, we denote by $f_{\mathfrak{e}}$ an extension of f onto some compact set K such that $A\subseteq \mathrm{Int}(K)$. We do stress that the compact support is part of the data of the extension. As such, it can vary from one extension to another. However, the extension $f_{\mathfrak{e}}$ is always assumed to be of the same class as f. For instance, if f is $L^p(A)$, $W^p_k(A)$ or $C^k(A)$, then $f_{\mathfrak{e}}$ is $L^p(K)$, $W^p_k(K)$ or $C^k(K)$.

2 Main results and strategy of proof

2.1 A baby integral as a motivation

The purpose of this section is to provide an example, in Theorem 2.1, of the leading large-N asymptotic expansion of $\ln \mathfrak{z}_N[W]$ where $\mathfrak{z}_N[W]$ is the unscaled partition function defined by (1.9). We shall also argue that the large-N asymptotic behaviour of (1.9) – whose integrand does not depend explicitly on N – can be deduced from the one of the rescaled model (2.47) – whose integrand depends explicitly on N– that we propose to study.

Let $\mathcal{E}_{(\mathrm{ply})}$ the functional, defined in $\mathbb{R} \cup \{+\infty\}$ for any probability measure $\mu \in \mathcal{M}^1(\mathbb{R})$:

$$\mathcal{E}_{(\mathrm{ply})}[\mu] \;=\; \int c_q |\xi|^q \,\mathrm{d}\mu(\xi) \;-\; \frac{\beta\pi(\omega_1+\omega_2)}{2} \int |\xi-\eta|\,\mathrm{d}\mu(\xi)\mathrm{d}\mu(\eta)\;, \tag{2.1}$$

Theorem 2.1 *Let W be a potential such that*

$$\lim_{|\xi|\to+\infty} |\xi|^{-q}\, W(x) \;=\; c_q > 0 \quad \text{for some} \;\; q>1\;, \tag{2.2}$$

$\mathcal{E}_{(\mathrm{ply})}$ is a lower semi-continuous good rate function, and

$$\lim_{N\to+\infty} \frac{\ln \mathfrak{z}_N[W]}{N^{2+\frac{1}{q-1}}} \;=\; -\inf_{\mu\in\mathcal{M}^1(\mathbb{R})} \mathcal{E}_{(\mathrm{ply})}[\mu]\;. \tag{2.3}$$

This infimum is attained at a unique probability measure $\mu_{\mathrm{eq}}^{(\mathrm{ply})}$. This measure is continuous with respect to the Lebesgue measure and has density

$$\rho_{\mathrm{eq}}^{(\mathrm{ply})}(\xi) \;=\; \frac{q(q-1)|\xi|^{q-2}}{2\pi\beta(\omega_1+\omega_2)} \cdot \mathbf{1}_{[a\,;b]}(\xi)\;. \tag{2.4}$$

$\mu_{\mathrm{eq}}^{(\mathrm{ply})}$ is supported on the interval $[a\,;b]$, with (a,b) being the unique solution to the set of equations

$$|b|^{q-1} \;=\; -|a|^{q-1} \;=\; \frac{\pi\beta(\omega_1+\omega_2)}{q}\;. \tag{2.5}$$

We have, explicitly:

$$\lim_{N\to+\infty} \frac{\ln \mathfrak{z}_N[W]}{N^{2+\frac{1}{q-1}}} \;=\; (c_q)^{\frac{1}{q}} \cdot \left(\frac{\pi\beta}{q}(\omega_1+\omega_2)\right)^{\frac{q+1}{q}} \cdot \frac{2q^2-9q+6}{2(2q-1)}\;. \tag{2.6}$$

The proof of this proposition is postponed to Appendix B, and follows similar steps to, *e.g.*, [4]. We now provide heuristic arguments to justify the occurrence of scaling in N in this problem. Just as discussed in the introduction, the repulsive effect of the sinh-2 body interactions will dominate over the confining effect of the potential as long as the integration variables will be located in some bounded set. Furthermore, in the same situation, the Lebesgue measure should contribute to the integral at most as an exponential in N. We thus look for a rescaling of the variables $y_a = T_N\lambda_a$ where the effects of the confining potential and the sinh-2 body interactions will be of the same order of magnitude in N. This recasts the partition function as

$$\mathfrak{z}_N[W] \;=\; \big(T_N\big)^N \int_{\mathbb{R}^N} \prod_{a<b}^{N} \Big\{ \sinh\left[\pi\omega_1 T_N(\lambda_a-\lambda_b)\right] \sinh\left[\pi\omega_2 T_N(\lambda_a-\lambda_b)\right] \Big\}^{\beta} \prod_{a=1}^{N} \Big\{ \mathrm{e}^{-W(T_N\lambda_a)} \Big\}\, \mathrm{d}^N\boldsymbol{\lambda}\;, \tag{2.7}$$

Taking into account the large-variable asymptotics of the potential, we have:

$$\sum_{a=1}^{N} W(T_N \lambda_a) \ \sim \ T_N^q \, N \ , \tag{2.8}$$

where the symbol $\sim$ means that for a "typical" distribution of the variables $\{\lambda_a\}_1^N$, the leading in N asymptotic behaviour of the sum in the right-hand side should be of the order of the left-hand side. Similarly, assuming a typical distribution of the variables $\{\lambda_a\}_1^N$ such that most of the pairs $\{\lambda_a, \lambda_b\}$ satisfy $T_N|\lambda_a - \lambda_b| \gg 1$, one has

$$\sum_{a<b}^{N} \beta \ln\Big\{ \sinh\big[\pi\omega_1 T_N(\lambda_a - \lambda_b)\big] \sinh\big[\pi\omega_2 T_N(\lambda_a - \lambda_b)\big] \Big\} \ \sim \ C\, N^2 \, T_N \ . \tag{2.9}$$

Thus, the confining potential and the two-body interaction will generate a comparable order of magnitude in N as soon as $N^2 \cdot T_N \ = \ T_N^q \cdot N$, *i.e.*

$$T_N \ = \ N^{\frac{1}{q-1}} \ . \tag{2.10}$$

Theorem 2.1 indeed justifies that the empirical distribution $L_N^{(\lambda)}$ of $\lambda_a \ = \ N^{\frac{-1}{q-1}} y_a$ concentrates around the equilibrium measure, with a large deviation principle governed by the rate function (2.1).

This observation implies that, in fact, $Z_N[V_N]$ with $V_N(\lambda) = N^{-\frac{q}{q-1}} \cdot W(N^{\frac{1}{q-1}}\lambda)$ is the good object to study in that it involves interactions that are already tuned to the proper scale in N. Due to the relation $\mathfrak{z}_N[W] = N^{\frac{N}{q-1}} \cdot Z_N[V_N]$, one readily has access to the large-N asymptotic expansion of $\mathfrak{z}_N[W]$.

2.2 The model of interest and our assumptions

It follows from the arguments given in the previous section that, effectively, the analysis of the unrescaled model boils down to the one subordinate to the partition function

$$Z_N[V] \ = \ \int_{\mathbb{R}^N} \prod_{a<b}^{N} \Big\{ \sinh\big[\pi\omega_1 N^{\alpha}(\lambda_a - \lambda_b)\big] \sinh\big[\pi\omega_2 N^{\alpha}(\lambda_a - \lambda_b)\big] \Big\}^{\beta} \prod_{a=1}^{N} \mathrm{e}^{-N^{1+\alpha} V(\lambda_a)} \cdot \mathrm{d}^N \boldsymbol{\lambda} \ , \tag{2.11}$$

with α some parameter – equal to $1/(q-1)$ in the previous paragraph – and V a potential that possibly depends on N. Due to such an effective reduction, in this paper, we shall develop the general formalism to extract the large-N asymptotic behaviour. Therefore, we shall keep the complexity at minimum. In particular, we shall *not* consider the case of N-dependent potentials which would put the analysis of $Z_N[V]$ in complete correspondence with the one of $\mathfrak{z}_N[W]$. Indeed, this would lead to numerous technical complication in our arguments, without bringing more light on the underlying phenomena. By focusing on (2.11), we believe that the new features and ideas of our methods are better isolated and illustrated. We shall incorporate the peculiarities of the model $\mathfrak{z}_N[W]$ of (1.9) and investigate its asymptotic behaviour up to $o(1)$ in a future publication.

In the present paper we obtain the large-N asymptotic expansion of $\ln Z_N[V]$ up to o(1) under four hypothesis

- the potential V is confining, *viz.* there exists $\epsilon > 0$ such that

$$\limsup_{|\xi|\to+\infty} |\xi|^{-(1+\epsilon)} \, V(\xi) \ = \ +\infty \ ; \tag{2.12}$$

- the potential V is smooth and strictly convex on $\mathbb{R}$;

- the potential is sub-exponential, namely there exists $\epsilon > 0$ and $C_V > 0$ such that

$$\forall \xi \in \mathbb{R}, \qquad \sup_{\eta \in [0\,;\epsilon]} \left| V'(\xi + \eta) \right| \;\leq\; C_V (|V(\xi)| + 1) \;, \tag{2.13}$$

  and given any $\kappa > 0$ and $p \in \mathbb{N}$, there exists $C_{\kappa,p}$ such that

$$\forall \xi \in \mathbb{R}, \qquad \left| V^{(k)}(\xi) \right| \mathrm{e}^{-\kappa V(\xi)} \;\leq\; C_{\kappa,p} \;. \tag{2.14}$$

- the exponent α in N^{α} is neither too large nor too small:

$$0 < \alpha < 1/6 \;. \tag{2.15}$$

The first hypothesis guarantees that the integral (2.11) is well-defined, and that the λ's will typically remain in a compact region on $\mathbb{R}$ independent of N. It could be weakened to study weakly confining potentials, to the price of introducing more technicalities, similar to those already encountered for β ensembles – see *e.g.* [56].

In the second assumption, V could be assumed C^k for k large enough. The convexity assumption guarantees that the support of the equilibrium measure is a single segment[8]. In principle, the multi-cut regime that may arise when the potential is not strictly convex can be addressed by importing the ideas of [15] to the present framework. We expect that the analysis of the Riemann-Hilbert problem in the multi-cut regime is very similar to the present case, but with a larger range of degrees for the polynomial freedom appearing in the solution (5.14). Though it would certainly represent some amount of work, the ideas we develop here should also be applicable to derive the fine large N analysis of the solution of the Riemann-Hilbert problem in the bulk and in the vicinity of all the edges of the support of the equilibrium measure.

The third assumption is not essential, but allows some simplification of the intermediate proofs concerning the equilibrium measure and the large deviation estimates, *e.g.* Theorem 3.7 and Corollary 3.10. It is anyway satisfied in physically relevant problems.

In the fourth assumption, $\alpha = 0$ can already be addressed with existing methods [17]. The upper limit $\alpha < \alpha^* = 1/6$ has a purely technical origin. The value of α^* could be increased by entering deeper into the fine structure of the analysis of the Schwinger-Dyson equation, and by finding more precise local and global bounds for the large N behaviour of the inverse of the master operator $\mathcal{U}_N^{-1}$, in more cunning norms. Intuitively, the genuine upper limit should be $\alpha^* = 1$, since in the $\alpha > 1$ case, we reach a regime where the particles do not feel the local repulsion any more. However, obtaining microscopic estimates is usually a difficult question – for β ensembles, it has been addressed *e.g.* in [19, 20]. So, one can expect important technical difficulties to extend our result to values of α increasing up to 1.

This set of hypothesis offers a convenient framework for our purposes, enabling us to focus on the technical aspects $(i) - (vi)$ listed in § 1.3 without adding extra complications.

2.3 Main results: asymptotic expansion of $Z_N[V]$ at $\beta = 1$

We now state one of the main results of the paper, namely the large-N asymptotic expansion of the partition function $Z_N[V]$ which holds for any potential V satisfying the hypothesis stated above

Theorem 2.2 *The below asymptotic expansion holds*

$$\ln\left(\frac{Z_N[V]}{Z_N[V_{G;N}]} \right)_{|\beta=1} \;=\; -N^{2+\alpha} \sum_{p=0}^{\lfloor 2/\alpha \rfloor + 1} \frac{\mathfrak{F}_p[V]}{N^{\alpha p}} \;+\; N^{\alpha} \cdot \beth_0 \cdot \Big(\Omega[V, V_{G;N}](b_N) - \Omega[V, V_{G;N}](a_N) \Big)$$
$$+\; \aleph_0 \cdot \Big(\Omega[V, V_{G;N}]'(b_N) + \Omega[V, V_{G;N}]'(a_N) \Big) \;+\; \mathrm{o}(1) \;. \tag{2.16}$$

[8] See e.g. the expression of the $N = \infty$ equilibrium measure (2.29). Its proof is given in Appendix C .

The whole V-dependence of this expansion is encoded in the coefficients $\mathscr{Z}_p[V]$ and in the function $\Omega[V, V_{G;N}](\xi)$. $\gimel_0$ and $\aleph_0$ are numerical coefficients given, resp., in terms of a single and four-fold integral. Also, the answer involves the Gaussian potential

$$V_{G;N}(\xi) \;=\; \frac{\pi\beta(\omega_1+\omega_2)\cdot\Big[\xi^2 - (a_N + b_N \;+\; \mathrm{O}(N^{-\infty}))\xi\Big]}{b_N - a_N \;+\; \dfrac{1}{N^{\alpha}}\sum_{p=1}^{2}\dfrac{1}{\pi\omega_p}\ln\Big(\dfrac{\omega_1\omega_2}{\omega_p(\omega_1+\omega_2)}\Big) \;+\; \mathrm{O}(N^{-\infty})} \tag{2.17}$$

and sequences a_N and b_N that are given in Theorem 2.5. If we denote $V_N^{\pm} = V \pm V_{N;G}$, the coefficients $\mathscr{Z}_p[V]$ take the form

$$\mathscr{Z}_0[V] \;=\; \frac{-1}{4\pi(\omega_1+\omega_2)}\int_{a_N}^{b_N} V_N^{-}(\xi)\cdot(V_N^{-})''(\xi)\,\mathrm{d}\xi \tag{2.18}$$

when $p = 0$ and, for any $p \geq 1$:

$$\begin{aligned} \mathscr{Z}_p[V] \;&=\; u_{p+1}\int_{a_N}^{b_N} V_N^{-}(\xi)\cdot V^{(p+2)}(\xi)\,\mathrm{d}\xi \\ &+\sum_{\substack{s+\ell=p-1 \\ s,\ell\geq 0}} \frac{\daleth_{s,\ell}}{s!}\Big\{(-1)^{\ell}\,(V_N^{-})^{(\ell+1)}(a_N)\cdot(V_N^{+})^{(s+1)}(a_N) \;+\; (-1)^{s}\,(V_N^{-})^{(\ell+1)}(b_N)\cdot(V_N^{+})^{(s+1)}(b_N)\Big\}\,. \end{aligned} \tag{2.19}$$

The coefficients $\daleth_{s,\ell}$ are defined by:

$$\daleth_{s,\ell} \;=\; \frac{\mathrm{i}^{s+\ell+1}}{2\pi}\sum_{r=1}^{\ell+1}\frac{s!}{r!(s+\ell+1-r)!}\cdot\frac{\partial^r}{\partial\mu^r}\Big(\frac{\mu}{R_{\downarrow}(-\mu)}\Big)_{|\mu=0}\cdot\frac{\partial^{s+1+\ell-r}}{\partial\mu^{s+1+\ell-r}}\Big(\frac{1}{R_{\downarrow}(\mu)}\Big)_{|\mu=0}\,, \tag{2.20}$$

$R_{\downarrow}$ is the $\mathbb{H}^+$ Wiener–Hopf factor of $1/\mathcal{F}[S](\lambda)$, with S defined in (2.42), that reads

$$R_{\downarrow}(\lambda) \;=\; \frac{\lambda}{2\pi\sqrt{\omega_1+\omega_2}}\cdot\Big(\frac{\omega_2}{\omega_1+\omega_2}\Big)^{-\frac{\mathrm{i}\lambda}{2\pi\omega_1}}\cdot\Big(\frac{\omega_1}{\omega_1+\omega_2}\Big)^{-\frac{\mathrm{i}\lambda}{2\pi\omega_2}}\cdot\frac{\Gamma\Big(\dfrac{\mathrm{i}\lambda}{2\pi\omega_1}\Big)\cdot\Gamma\Big(\dfrac{\mathrm{i}\lambda}{2\pi\omega_1}\Big)}{\Gamma\Big(\dfrac{\mathrm{i}\lambda(\omega_1+\omega_2)}{2\pi\omega_1\omega_2}\Big)}\,. \tag{2.21}$$

The function Ω describing the constant term is defined as

$$\Omega[V, V_{G;N}](\xi) \;=\; \frac{V'(\xi) - V'_{G;N}(\xi)}{V''(\xi) - V''_{G;N}(\xi)}\,\ln\bigg(\frac{V''(\xi)}{V''_{G;N}(\xi)}\bigg)\,. \tag{2.22}$$

The V-independent coefficient $\gimel_0$ in front of the term N^{α} reads

$$\gimel_0 \;=\; \int_0^{+\infty}\frac{\mathrm{d}u\,J(u)}{2\pi}(uS'(u) \;+\; S(u)) \qquad \text{with} \quad J(u) \;=\; \int_{\mathscr{C}^{(+)}_{\mathrm{reg}}}\frac{2\sinh[\lambda/(2\omega_1)]\sinh[\lambda/(2\omega_2)]}{\sinh[\lambda(\omega_1+\omega_2)/(2\omega_1\omega_2)]}\cdot\frac{\mathrm{e}^{\mathrm{i}\lambda u}\,\mathrm{d}\lambda}{2\mathrm{i}\pi}\,. \tag{2.23}$$

Finally, the numerical prefactor $\aleph_0$ is expressed in terms of the four-fold integral

$$\aleph_0 \;=\; -\frac{\omega_1+\omega_2}{2}\int_{\mathbb{R}}\frac{\mathrm{d}u\,J(u)}{2\pi}\int_{|u|}^{+\infty}\mathrm{d}v\,\partial_u\Big\{S(u)\cdot\Big(\mathfrak{r}[\frac{v-u}{2}]-\mathfrak{r}[\frac{v+u}{2}]\Big)\Big\}$$
$$+\int_{\mathscr{C}_{\mathrm{reg}}^{(+)}}\frac{\mathrm{d}\lambda\,\mathrm{d}\mu}{(2\mathrm{i}\pi)^2}\,\frac{\mu(\omega_1+\omega_2)}{(\lambda+\mu)R_{\downarrow}(\lambda)R_{\downarrow}(\mu)}\int_0^{+\infty}\mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{\mathrm{i}\lambda x+\mathrm{i}\mu y}\,\partial_x\bigg\{S(x-y)\bigg(\mathfrak{r}(x)-\mathfrak{r}(y)-\frac{x-y}{2\pi(\omega_1+\omega_2)}\bigg)\bigg\}\,. \quad (2.24)$$

The integrand of $\aleph_0$ involves the function $\mathfrak{r}$ which is given by

$$\mathfrak{r}(x) \;=\; \frac{\mathfrak{c}_1(x)+\mathfrak{c}_0(x)\Big[\sum_{p=1}^{2}\frac{1}{2\pi\omega_p}\ln\Big(\frac{\omega_1\omega_2}{\omega_p(\omega_1+\omega_2)}\Big)\Big]}{1+2\pi\beta(\omega_1+\omega_2)\mathfrak{c}_0(x)} \quad (2.25)$$

with

$$\mathfrak{c}_p(x) \;=\; \frac{\mathrm{i}^p}{2\mathrm{i}\pi\sqrt{\omega_1+\omega_2}}\int_{\mathscr{C}_{\mathrm{reg}}^{(+)}}\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\lambda}\,\frac{\partial^p}{\partial\lambda^p}\Big(\frac{1}{R_{\downarrow}(\lambda)}\Big)\cdot\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\,. \quad (2.26)$$

The result for $\beta \neq 1$ contain two more terms, and is given in the body in the article, by Proposition 3.20, in terms of N-dependent simple and double integrals $\mathfrak{I}^{(2)}_{\mathrm{s};\beta}$. The final form for the asymptotics up to $o(1)$ of these extra terms can be worked out following the steps of Section 9, although we decided to leave it out of the scope of this article, since $\beta \neq 1$ does not seem to appear in quantum integrable systems.

2.4 Main results: the N-dependent equilibrium measure and the master operator

It is not hard to generalise the proof of Theorem 2.1 to the present setting so as to obtain the below characterisation of the leading in N asymptotic behaviour for $Z_N[V]$.

Theorem 2.3 *Let $\mathcal{E}_\infty$ be the lower semi-continuous good rate function*

$$\mathcal{E}_\infty[\mu] \;=\; \frac{1}{2}\int\big(V(\eta)+V(\xi)\;-\;\pi\beta(\omega_1+\omega_2)|\xi-\eta|\big)\,\mathrm{d}\mu(\xi)\mathrm{d}\mu(\eta)\,. \quad (2.27)$$

Then, one has that

$$\lim_{N\to+\infty}\frac{\ln Z_N[V]}{N^{2+\alpha}} \;=\; -\inf_{\mu\in\mathcal{M}^1(\mathbb{R})}\mathcal{E}_\infty[\mu]\,. \quad (2.28)$$

The infimum is attained at a unique probability measure μ_{eq}. This measure is continuous with respect to the Lebesgue measure, and has density

$$\rho_{\mathrm{eq}}(x) \;=\; \frac{V''(\xi)}{2\pi\beta(\omega_1+\omega_2)}\cdot\mathbf{1}_{[a\,;b]}(\xi) \quad (2.29)$$

supported on the interval $[a\,;b]$, with (a,b) being the unique solution to the set of equations

$$V'(b) \;=\; -V'(a) \;=\; \pi\beta(\omega_1+\omega_2)\,. \quad (2.30)$$

One has, explicitly,

$$\lim_{N\to+\infty} \frac{\ln Z_N[V]}{N^{2+\alpha}} \;=\; -\frac{V(a)+V(b)}{2} \;+\; \frac{(V'(b))^2(b-a) \;+\; \int_a^b (V'(\xi))^2\,\mathrm{d}\xi}{4\pi\beta(\omega_1+\omega_2)} \;. \tag{2.31}$$

The strict convexity of V guarantees that the density (2.29) is positive and that it reaches a non-zero limit at the endpoints of the support. This behaviour differs from the situation usually studied in β ensembles with analytic potentials which leads to a generic square root (or inverse square root) vanishing (or divergence) of the equilibrium density at the edges.

Note that the function $\mathcal{E}_\infty$ defined in (2.27) arises as a good rate function in the large deviation estimates for the empirical measure $L_N^{(\lambda)}$, *c.f.* (1.3). In fact, a refinement of Theorem 2.3 would lead to the more precise estimates

$$\ln Z_N[V] \;=\; -N^{2+\alpha}\mathcal{E}_\infty[\mu_{\mathrm{eq}}] \;+\; \mathrm{O}(N^2) \;. \tag{2.32}$$

Thus, in respect to the usual varying weight β-ensemble case, there is a loss of precision by a $N^{1-\alpha}$ factor. This, in fact, takes its origin in that the purely asymptotic rate function $\mathcal{E}_\infty[\mu_{\mathrm{eq}}]$ does not absorb enough of the fine structure of the saddle-point. As a consequence, the remainder $\mathrm{O}(N^2)$ mixes both types of contributions: the deviation of the saddle-point in respect to its asymptotic position and the fluctuation of the integration variables around the saddle-point.

The fine, N-dependent, structure of the saddle-point is much better captured by the N-dependent deformation[9] of the rate functions $\mathcal{E}_\infty$:

$$\mathcal{E}_N[\mu] \;=\frac{1}{2}\;\int\left(V(\xi) \;+\; V(\eta) \;-\; \frac{\beta}{N^\alpha}\ln\left\{\prod_{p=1}^{2}\sinh\left[\pi N^\alpha\omega_p(\xi-\eta)\right]\right\}\right)\mathrm{d}\mu(\xi)\mathrm{d}\mu(\eta) \;. \tag{2.33}$$

This N-dependent rate functions appear extremely effective for the purpose of our analysis. Namely, it allows us re-summing a whole tower of contributions into a single term. The use of $\mathcal{E}_N$ should not be considered as a mere technical simplification of the intermediate steps; it is, in fact, of prime importance. The use of the more classical object $\mathcal{E}_\infty$ would render the analysis of the Schwinger-Dyson equations impossible. This fact will become apparent in the core of the file. Here, we only state the improvement provided by the use of the finite-N minimiser of $\mathcal{E}_N$:

$$\ln Z_N[V] \;=\; -N^{2+\alpha}\inf_{\mu\in\mathcal{M}^1(\mathbb{R})}\mathcal{E}_N[\mu] \;+\; \mathrm{O}(N^{1+\alpha}) \;. \tag{2.34}$$

As usual, this minimiser admits a characterisation in terms of a variational problem:

Theorem 2.4 *For any strictly convex potential V, the N-dependent rate function $\mathcal{E}_N$ admits its minimum on $\mathcal{M}^1(\mathbb{R})$ at a unique probability measure $\mu_{\mathrm{eq}}^{(N)}$. This equilibrium measure is supported on a segment $[a_N\,;b_N]$ and corresponds to the unique solution to the integral equations*

$$V(\xi) \;-\; \frac{\beta}{N^\alpha}\int\ln\left\{\prod_{p=1}^{2}\sinh\left[\pi N^\alpha\omega_p(\xi-\eta)\right]\right\}\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(\eta) \;=\; C_{\mathrm{eq}}^{(N)} \qquad \text{on } [a_N\,;b_N] \tag{2.35}$$

$$V(\xi) \;-\; \frac{\beta}{N^\alpha}\int\ln\left\{\prod_{p=1}^{2}\sinh\left[\pi N^\alpha\omega_p(\xi-\eta)\right]\right\}\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(\eta) \;>\; C_{\mathrm{eq}}^{(N)} \qquad \text{on } \mathbb{R}\setminus[a_N\,;b_N]\;, \tag{2.36}$$

[9]The property of lower semi-continuity along with the fact that $\mathcal{E}_N$ has compact level sets is verified exactly as in the case of β-ensembles, so we do not repeat the proof here.

with $C^{(N)}_{\mathrm{eq}}$ a constant whose determination is part of the problem (2.35)-(2.36). *The equilibrium measure admits a density $\rho^{(N)}_{\mathrm{eq}}$, which is C^{k-2} in the interior $]a_N;b_N[$ if V is C^k. Finally, one has the behaviour at the edges:*

$$\rho^{(N)}_{\mathrm{eq}}(\xi) \underset{\xi\to a_N^+}{=} \mathrm{O}\big(\sqrt{\xi-a_N}\big)\;, \qquad\qquad \rho^{(N)}_{\mathrm{eq}}(\xi) \underset{\xi\to b_N^-}{=} \mathrm{O}\big(\sqrt{b_N-\xi}\big)\;. \tag{2.37}$$

The proof of the proposition above is rather classical. It follows, for instance, from [17, Section 2.3] in what concerns the regularity, and from a convexity argument already used in [14, Lemma 6.2] in what concerns connectedness of the support and the strict inequality in (2.36). Elements of proof are nevertheless gathered in Appendix C. In fact, regarding to the equilibrium measure, we can be much more precise when N is large enough:

Theorem 2.5 *In the $N\to\infty$ regime, the equilibrium measure $\mu^{(N)}_{\mathrm{eq}}$:*

- *is supported on the single interval $[a_N\,;b_N]$ whose endpoints admit the asymptotic expansion*

$$a_N \;=\; a \,+\, \sum_{\ell=1}^{k}\frac{a_{N;\ell}}{N^{\ell\alpha}} \,+\, \mathrm{O}\Big(\frac{1}{N^{(k+1)\alpha}}\Big) \quad \text{and} \quad b_N \;=\; b \,+\, \sum_{\ell=1}^{k}\frac{b_{N;\ell}}{N^{\ell\alpha}} \,+\, \mathrm{O}\Big(\frac{1}{N^{(k+1)\alpha}}\Big)\,, \tag{2.38}$$

  where $k\in\mathbb{N}^*$ is arbitrary, (a,b) are as defined in (2.30) *while*

$$\begin{pmatrix} b_{N;1} \\ a_{N;1}\end{pmatrix} \;=\; \Big\{\sum_{p=1}^{2}\frac{1}{2\pi\omega_p}\ln\Big(\frac{\omega_1\omega_2}{\omega_p(\omega_1+\omega_2)}\Big)\Big\}\cdot\begin{pmatrix} V''(a)\cdot\{V''(b)\}^{-1} \\ -V''(b)\cdot\{V''(a)\}^{-1}\end{pmatrix}\;; \tag{2.39}$$

- *is continuous in respect to Lebesgue. Its density is $\rho^{(N)}_{\mathrm{eq}}$ vanishes like a square-root at the edges:*

$$\rho^{(N)}_{\mathrm{eq}}(\xi) \underset{\xi\to a_N^+}{\sim} \Big(\frac{V''(a_N)\,+\,\mathrm{O}(N^{-\alpha})}{\pi\beta\sqrt{\pi(\omega_1+\omega_2)}}\Big)\sqrt{\xi-a_N}\;, \qquad \rho^{(N)}_{\mathrm{eq}}(\xi) \underset{\xi\to b_N^-}{\sim} \Big(\frac{V''(b_N)\,+\,\mathrm{O}(N^{-\alpha})}{\pi\beta\sqrt{\pi(\omega_1+\omega_2)}}\Big)\sqrt{b_N-\xi}\;, \tag{2.40}$$

  and there exists a constant $C>0$ independent of N such that:

$$\|\rho^{(N)}_{\mathrm{eq}}\|_{L^\infty([a_N\,;b_N])} \;\leq\; C\,\|V''\|_{L^\infty([a_N\,;b_N])}\;. \tag{2.41}$$

  This density takes the form $\rho^{(N)}_{\mathrm{eq}} = \mathcal{W}_N[V']$, with $\mathcal{W}_N$ as defined in (2.44).

If the potential V defining the equilibrium measures satisfies $V\in C^k([a_N\,;b_N])$, then the density is of class C^{k-2} on $]a_N\,;b_N[$.

Note that the characterisation of $\rho^{(N)}_{\mathrm{eq}}$ in the theorem above comes from the fact that it is solution to the singular integral equation $\mathcal{S}_N[\rho^{(N)}_{\mathrm{eq}}](\xi) = V'(\xi)$ on $[a_N\,;b_N]$, where

$$\mathcal{S}_N[\phi](\xi) \;=\; \fint_{a_N}^{b_N} S\,[N^\alpha(\xi-\eta)]\phi(\eta)\,\mathrm{d}\eta \qquad \text{and} \qquad S\,(\xi) \;=\; \sum_{p=1}^{2}\beta\pi\omega_p\,\mathrm{cotanh}\,[\pi\omega_p\xi]\;. \tag{2.42}$$

The unknowns in this equation $(\rho^{(N)}_{\mathrm{eq}}, a_N, b_N)$ should be picked in such a way that $\rho^{(N)}_{\mathrm{eq}}$ has mass 1 on $[a_N\,;b_N]$ and is regular at the endpoints a_N, b_N. Thus, determining the equilibrium measure boils down to an inversion of the singular integral operator $\mathcal{S}_N$. In fact, the singular integral operator $\mathcal{S}_N$ also intervenes in the Schwinger-Dyson equations. The precise control on its inverse $\mathcal{W}_N$ – defined between appropriate functional spaces – plays a crucial role in the whole asymptotic analysis.

These information can be obtained by exploiting the fact that the operator $\mathcal{S}_N$ is of truncated Wiener–Hopf type. As such, its inversion is equivalent to solving a 2×2 matrix valued Riemann–Hilbert problem. This Riemann–Hilbert problem admits a solution for N large enough that can be constructed by means of a variant of the non-linear steepest descent method. By doing so, we are able to describe, quite explicitly, the inverse $\mathcal{W}_N$ by means of the unique solution χ to the 2×2 matrix valued Riemann–Hilbert problem given in Section 4.3. We will not discuss the structure of this solution here and, rather, refer the reader to the relevant section. We will, however, provide the main consequence of this analysis, *viz.* an explicit representation for the operator $\mathcal{W}_N$. For this purpose, we need to announce that χ_{11}, the $(1,1)$ matrix entry of χ, is such that $\mu \mapsto \mu^{1/2}\cdot\chi_{11}(\mu) \in L^{\infty}(\mathbb{R})$.

Theorem 2.6 *Let* $0 < s < 1/2$*. The operator* $\mathcal{S}_N \;:\; H_s([a_N\,;b_N]) \to \mathfrak{X}_s(\mathbb{R})$ *is continuous and invertible where, for any closed* $A \subseteq \mathbb{R}$*,*

$$\mathfrak{X}_s(A) \;=\; \Big\{ H \in H_s(A) \;:\; \int\limits_{\mathbb{R}+\mathrm{i}\epsilon} \chi_{11}(\mu)\mathcal{F}[H](N^{\alpha}\mu)\mathrm{e}^{-\mathrm{i}N^{\alpha}\mu b_N}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \;=\; 0 \Big\} \tag{2.43}$$

is a closed subspace of $H_s(A)$ *such that* $\mathcal{S}_N(H_s([a_N\,;b_N])) \;=\; \mathfrak{X}_s(\mathbb{R})$*. The inverse is given by the integral transform* $\mathcal{W}_N$ *which takes, for* $H \in C^1([a_N\,;b_N]) \cap \mathfrak{X}_s(\mathbb{R})$*, the form*

$$\mathcal{W}_N[H](\xi) \;=\; \frac{N^{2\alpha}}{2\pi\beta}\int\limits_{\mathbb{R}+2\mathrm{i}\epsilon}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\int\limits_{\mathbb{R}+\mathrm{i}\epsilon}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\,\frac{\mathrm{e}^{-\mathrm{i}N^{\alpha}(\xi-a_N)\lambda}}{\mu-\lambda}\Big\{\chi_{11}(\lambda)\chi_{12}(\mu)-\frac{\mu}{\lambda}\cdot\chi_{11}(\mu)\chi_{12}(\lambda)\Big\}\cdot\int\limits_{a_N}^{b_N}\mathrm{d}\eta\,\mathrm{e}^{\mathrm{i}N^{\alpha}\mu(\eta-b_N)}H(\eta)\;. \tag{2.44}$$

In the above integral representations the parameter $\epsilon > 0$ *is small enough but arbitrary. Furthermore, for any* $H \in C^1([a_N\,;b_N])$*, the transform* $\mathcal{W}_N$ *exhibits the local behaviour*

$$\mathcal{W}_N[H](\xi) \underset{\xi\to a_N^+}{\sim} C_L H'(a_N)\sqrt{\xi-a_N} \qquad \text{and} \qquad \mathcal{W}_N[H](\xi) \underset{\xi\to b_N^-}{\sim} C_R H'(b_N)\sqrt{b_N-\xi}\;. \tag{2.45}$$

where $C_{L/R}$ *are some* H*-independent constants.*

Note that, within such a framework, the density of the equilibrium measure $\mu_{\mathrm{eq}}^{(N)}$ is expressed in terms of the inverse as $\rho_{\mathrm{eq}}^{(N)} \;=\; \mathcal{W}_N[V']$. In this case, the pair of endpoints (a_N, b_N) of the support of $\mu_{\mathrm{eq}}^{(N)}$ corresponds to the unique solution to the system of equations

$$\int\limits_{a_N}^{b_N}\mathcal{W}_N[V'](\xi)\,\mathrm{d}\xi \;=\; 1 \qquad \text{and} \qquad \int\limits_{\mathbb{R}+\mathrm{i}\epsilon}\frac{\mathrm{d}\mu\,\chi_{11}(\mu)}{2\mathrm{i}\pi}\int\limits_{a_N}^{b_N}\mathrm{e}^{\mathrm{i}\mu N^{\alpha}(\eta-b_N)}V'(\eta)\,\mathrm{d}\eta \;=\; 0\;. \tag{2.46}$$

The first condition guarantees that $\mu_{\mathrm{eq}}^{(N)}$ has indeed mass 1, while the second one ensures that its density vanishes as a square root at the edges a_N, b_N. Using fine properties of the inverse, these conditions can be estimated more precisely in the large-N limit, hence enabling one to fix the large-N asymptotic expansion of the endpoints a_N, b_N as announced in (2.38)-(2.39).

2.5 The overall strategy of the proof

In the following, we shall denote by $p_N(\boldsymbol{\lambda})$ the probability density on $\mathbb{R}^N$ associated with the partition function $Z_N[V]$ defined in (2.11):

$$p_N(\boldsymbol{\lambda}) \;=\; \frac{1}{Z_N[V]}\prod_{a<b}^{N}\Big\{\sinh\left[\pi\omega_1 N^{\alpha}(\lambda_a-\lambda_b)\right]\sinh\left[\pi\omega_2 N^{\alpha}(\lambda_a-\lambda_b)\right]\Big\}^{\beta}\prod_{a=1}^{N}\mathrm{e}^{-N^{1+\alpha}V(\lambda_a)}\;. \tag{2.47}$$

$p_N(\boldsymbol{\lambda})$ gives rise to a probability measure $\mathbb{P}_N$ on $\mathbb{R}^N$. We also agree that, throughout the file, $L_N^{(\lambda)}$ refers to the empirical measure

$$L_N^{(\lambda)} \; = \; \frac{1}{N}\sum_{a=1}^{N}\delta_{\lambda_a} \tag{2.48}$$

associated with the stochastic vector $\boldsymbol{\lambda}$.

Definition 2.7 *Let $\nu_1, \ldots, \nu_\ell$ be any (possibly depending on the stochastic vector λ) measures and ψ a function in ℓ variables. Then we agree upon*

$$\big\langle \psi \big\rangle_{\nu_1\otimes\cdots\otimes\nu_\ell} \equiv \big\langle \psi(\xi_1,\ldots,\xi_\ell)\big\rangle_{\nu_1\otimes\cdots\otimes\nu_\ell} \equiv \mathbb{P}_N\Big[\int_{\mathbb{R}^\ell}\psi(\xi_1,\ldots,\xi_\ell)\,\mathrm{d}\nu_1\otimes\cdots\otimes\mathrm{d}\nu_\ell\Big] \tag{2.49}$$

whenever it makes sense. We shall add the superscript V whenever the functional dependence of the probability measure on the potential V needs to be made clear.

Note that if none of the measures $\nu_1, \ldots, \nu_\ell$ is stochastic, then the expectation versus $\mathbb{P}_N$ in (2.49) can be omitted.

The Schwinger-Dyson equations constitute a tower of equations which relate expectation values of functions in many, non necessarily fixed, variables that are integrated versus the empirical measure (2.48). More precisely, the Schwinger-Dyson equations at level k ($k \geq 1$) yield exact relations between various expectation values of a function in k variables and its transforms, this versus the empirical measure. The knowledge of these expectation values, yields an access to the derivatives of the partition function with respect of external parameters. For instance, if $\{V_t\}_t$ is a smooth one parameter family of potentials, then

$$\partial_t \ln Z_N[V_t] \; = \; -N^{2+\alpha}\langle\partial_t V_t\rangle^{V_t}_{L_N^{(\lambda)}} \; . \tag{2.50}$$

The exponent V_t appearing in the right-hand side is there so as to emphasise that the expectation value is computed in respect to the probability measure subordinate to the t-dependent potential V_t.

Thus the problem boils down to obtaining a sufficiently precise control on the behaviour in N of the one-point expectation values. This can be achieved on the basis of a careful analysis of the system of Schwinger-Dyson equations associated with the present model. Since this machinery does not simplify much in the $\beta = 1$ case, we do this for general β. The result for some sufficiently regular function H and potentials V satisfying to the general hypothesis, is our Proposition 3.19.

In the $\beta = 1$ case, Proposition 3.19 reads:

$$-N^{2+\alpha}\langle H\rangle^{V}_{L_N^{(\lambda)}} \; = \; -N^{2+\alpha}\int_{a_N}^{b_N} H(\xi)\cdot\mathcal{W}_N[V'](\xi)\,\mathrm{d}\xi \; + \; \frac{1}{2}\mathfrak{I}_{\mathrm{d}}[H,V] \; + \; \mathrm{o}(1)\; . \tag{2.51}$$

and the proof shows that the remainder o(1) is uniform in H and V provided that H is regular enough and that V satisfies to the hypothesis given in (2.12)-(2.14). Furthermore, the expansion (2.51) involves

$$\mathfrak{I}_{\mathrm{d}}[H,V] \; = \; \int_{a_N}^{b_N}\mathcal{W}_N\Big[\partial_\xi\{S\left(N^\alpha(\xi-*)\right)\cdot\mathcal{G}_N[H,V](\xi,*)\}\Big](\xi)\,\mathrm{d}\xi\;, \tag{2.52}$$

with

$$\mathcal{G}_N[H,V](\xi,\eta) \;=\; \frac{\mathcal{W}_N[H](\xi)}{\mathcal{W}_N[V'](\xi)} \;-\; \frac{\mathcal{W}_N[H](\eta)}{\mathcal{W}_N[V'](\eta)} \; . \tag{2.53}$$

Note that, in (2.52), the $*$ indicates the variable of the function on which the operator $\mathcal{W}_N$ acts. Given sufficiently regular functions H, V, we obtain in Section 9 and more precisely in Proposition 9.10 the large-N asymptotic behaviour of $\mathfrak{I}_{\mathrm{d}}[H,V]$. We then have all the elements to calculate the large-N asymptotic behaviour of the partition function $Z_N[V]$. For this purpose, we observe that, when $\beta = 1$, the partition function associated to a quadratic potential can be explicitly evaluated as shown in Proposition D.2. One can also show (*cf.* Lemma D.1) that there exists a unique, up to a constant, quadratic potential $V_{G;N}$ such that its associated equilibrium measure has the same support $[a_N, b_N]$ as the one associated with V. Then $V_t = (1-t)V_{G;N} + tV$ is a one parameter t smooth family of strictly convex potentials, and $\mu^{(N)}_{\mathrm{eq};V_t} = (1-t)\mu^{(N)}_{\mathrm{eq};V_{G;N}} + t\mu^{(N)}_{\mathrm{eq};V}$. Furthermore, if follows from the details of the analysis that led to (2.51) that the remainder o(1) will be uniform in $t \in [0\,;1]$. As a consequence, by combining all of the above results and integrating equation (2.50) over t, we get that, in the asymptotic regime,

$$\ln\biggl(\frac{Z_N[V]}{Z_N[V_{G;N}]}\biggr) \;=\; -N^{2+\alpha}\int\limits_0^1 \mathrm{d}t \int \partial_t V_t(\xi)\,\mathrm{d}\mu^{(N)}_{\mathrm{eq};V_t}(\xi) \;+\; N^{\alpha}\cdot \beth_0 \cdot \Bigl(\mho[V,V_{G;N}](b_N) - \mho[V,V_{G;N}](a_N)\Bigr)$$
$$+\;\aleph_0 \cdot \Bigl(\mho'[V,V_{G;N}](b_N) + \mho'[V,V_{G;N}](a_N)\Bigr) \;+\; \mathrm{o}(1)\; . \tag{2.54}$$

The constants $\beth_0$ and $\aleph_0$ were defined respectively in (2.23) and (2.24), while $\mho$ is as given by (2.22).

Note also that the first integral can be readily evaluated (integration of rational functions in t) on the asymptotic level by means of Proposition 7.6. It produces an expansion into inverse powers of N^{α} and, as such, does not contribute to the constant term unless α is of the form $2/n$ for some integer n. Note that it is this integral that gives rise to the functional $\mathfrak{S}_p[V]$ in (2.16). Finally, the answer for the large-N asymptotic behaviour of the partition function $Z_N[V_{G;N}]_{|\beta=1}$ can be found in Proposition D.2.

For $\beta \neq 1$, (2.51) is modified by the addition of two more terms $\mathfrak{I}^{(2)}_{\mathrm{s};\beta}$ and $\mathfrak{I}_{\mathrm{d};\beta}$. Their large N behaviour can be determined without difficulty – but with some algebra – along the lines of Section 7.2 and § 9. Then, to arrive to a final answer for $Z_N[V]_{\beta\neq 1}$ similar to (2.54), we would need to compute exactly the partition function for the Gaussian potential $Z_N[V_{G;N}]_{\beta\neq 1}$. We do not know at present how to perform such a calculation. Thus, we would be able to derive the asymptotic behaviour of the partition function at $\beta \neq 1$ up to a universal, *i.e.* not depending on the potential V, function of β. However, since the values $\beta \neq 1$ do not seem to appear in quantum integrable systems, we shall limit ourselves in this article to the result of Proposition 3.20 for the case $\beta \neq 1$.

3 Asymptotic expansion of $\ln Z_N[V]$ - the Schwinger-Dyson equation approach

In the present section we develop all the necessary tools to prove the large-N asymptotic expansion for $\ln Z_N[V]$ up to o(1) terms, in the form described in (2.51). The asymptotic expansion we obtain contains N-dependent functionals of the equilibrium measure whose large-N asymptotic analysis will be carried out in Sections 7-9. We shall first obtain some *a priori* bounds on the fluctuations of linear statistics around their means computed *vs.* the N-dependent equilibrium measure $\mu^{(N)}_{\mathrm{eq}}$. In other words, we consider observables given by integration against products of the centred measure:

Definition 3.1 *We define the centred empirical measure as:*

$$\mathcal{L}^{(\lambda)}_N = L^{(\lambda)}_N - \mu^{(N)}_{\mathrm{eq}} \; . \tag{3.1}$$

Then we shall build on a bootstrap approach to the Schwinger-Dyson equations so as to improve these *a priori* bounds. We shall use these improved bounds so as to identify the leading and sub-leading terms in the Schwinger-Dyson equations what, eventually, leads to an analogue, at $\beta \neq 1$, of the representation (2.51) which will be given in Proposition 3.19. Finally, upon integrating the relation (2.50) so as to to interpolate the partition function between a Gaussian and a general potential, we will get the N-dependent large-N asymptotic expansion of $\ln Z_N[V]$ in Proposition 3.20.

For simplification, we use the notation:

$$s_N(\xi) \;=\; \frac{\beta}{2N^{\alpha}} \ln\Big[\sinh\left(\pi\omega_1 N^{\alpha}\xi\right)\, \sinh\left(\pi\omega_2 N^{\alpha}\xi\right)\Big] \tag{3.2}$$

for the two-body interaction kernel, and we introduce the effective potential associated to the N-dependent equilibrium measure:

$$V_{N;\mathrm{eff}}(\xi) \;=\; V(\xi) \;-\; 2\int s_N(\xi-\eta)\, \mathrm{d}\mu_{\mathrm{eq}}^{(N)}(\eta) \;-\; C_{\mathrm{eq}}^{(N)} \;. \tag{3.3}$$

By the characterisation of the equilibrium measure (Theorem 2.4), $V_{N;\mathrm{eff}} \;=\; 0$ in the support $[a_N\,; b_N]$, while $V_{N;\mathrm{eff}} > 0$ outside $[a_N\,; b_N]$.

3.1 A priori estimates for the fluctuations around $\mu_{\mathrm{eq}}^{(N)}$

The model provides a natural way of comparing two probability measures:

Definition 3.2 *If $\mu, \nu \in \mathcal{M}_1(\mathbb{R})$, we set:*

$$\mathfrak{D}^2[\mu,\nu] \;\equiv\; -\int s_N(\xi-\eta)\, \mathrm{d}(\mu-\nu)(\xi)\, \mathrm{d}(\mu-\nu)(\eta)\;, \tag{3.4}$$

with s_N as given in (3.2). *$\mathfrak{D}^2[\mu,\nu]$ is a well-defined number in $\mathbb{R}\cup\{+\infty\}$.*

The notation is justified by the property $\mathfrak{D}^2 \geq 0$ following from:

Lemma 3.3 *We have the representation:*

$$\mathfrak{D}^2[\mu,\nu] \;=\; \int \bigg\{ \frac{\pi\beta}{2N^{\alpha}\varphi} \sum_{p=1}^{2} \mathrm{cotanh}\Big[\frac{\varphi}{2\omega_p N^{\alpha}}\Big]\bigg\} \cdot \big|\mathcal{F}[\mu-\nu](\varphi)\big|^2 \cdot \frac{\mathrm{d}\varphi}{2\pi}\;, \tag{3.5}$$

where $\mathcal{F}[\mu](\xi)$ is the Fourier transform of the measure μ.

Proof — Direct given the formula $\mathcal{F}[\,f_t](\varphi) \;=\; -(\pi/\varphi)(\,\mathrm{cotanh}[\pi\varphi/2t] - 2t/\pi\varphi)$ with $f_t(x) \;=\; \ln|\sinh(tx)| - t|x| + \ln 2$. ■

Definition 3.4 *The classical positions x_i^N for the measure $\mu_{\mathrm{eq}}^{(N)}$ are defined by*

$$\frac{i}{N} \;=\; \int_{-\infty}^{x_i^N} \mathrm{d}\mu_{\mathrm{eq}}^{(N)}(y) \qquad for \qquad i \in [\![\, 1\,;\, N\,]\!] \qquad and \qquad x_0^N = a_N \quad,\quad x_N^N = b_N\;. \tag{3.6}$$

Our first task is to derive a lower bound for the partition function (2.11), by restricting to configurations of points close to their classical positions:

Lemma 3.5 $Z_N[V] \;\geq\; \exp\Big\{-N^{2+\alpha}\mathcal{E}_N[\mu_{\mathrm{eq}}^{(N)}] \;+\; \mathrm{O}(N^{1+\alpha})\Big\}\,.$

We stress on this occasion that using the N-dependent rate function $\mathcal{E}_N$ allows the gain of a factor $1/N$ in the remainder with respect to the leading term, while using $\mathcal{E}_\infty$ would lead to a weaker estimate $\mathrm{O}(N^2)$ for the remainder. This is of particular importance to simplify the analysis of Schwinger-Dyson equations that will follow.

Proof — It follows from the local expressions obtained in Section 6 that $\mu_{\mathrm{eq}}^{(N)}$ is continuous with respect to Lebesgue measure with density bounded by a constant M independent of N, as shown in (2.41). This ensures that

$$\big|x_{i+1}^N - x_i^N\big| \geq \frac{1}{MN}\;, \qquad i \in [\![\,0\,;\,N-1\,]\!]\;. \tag{3.7}$$

We obtain our lower bound by keeping only configurations in

$$\Omega \;=\; \{\boldsymbol{\lambda} \in \mathbb{R}^N \;:\; \sup_a |\lambda_a - x_a^N| \;\leq\; \frac{1}{4MN}\}\;.$$

Let σ_ϵ be some N-independent ϵ-neighbourhood of $[a_N\,;b_N]$. Since $V \in C^1(\mathbb{R})$, it follows that

$$\big|V(\lambda_a) - V(x_a^N)\big| \;\leq\; \frac{||V'||_{L^\infty(\sigma_\epsilon)}}{4MN} \qquad \textit{viz.} \qquad -V(x_a^N) - \frac{||V'||_{L^\infty(\sigma_\epsilon)}}{4MN} \;\leq\; -V(\lambda_a) \tag{3.8}$$

for $a \in [\![\,1\,;\,N\,]\!]$ and for any $\boldsymbol{\lambda} \in \Omega$. Thus, upon a re-centring at x_a^N of the integration in respect to λ_a, we get

$$\begin{aligned} Z_N[V] \geq & \prod_{a=1}^{N}\Big\{\mathrm{e}^{-N^{1+\alpha}V(x_a^N)}\Big\}\cdot \mathrm{e}^{-\frac{N^{1+\alpha}}{4M}||V'||_{L^\infty(\sigma_\epsilon)}} \times \int\limits_{[-1/(4MN),1/(4MN)]^N} \mathrm{d}^N\nu \cdot \prod_{a<b}^{N}\Big\{\prod_{p=1}^{2}\sinh\big[\pi\omega_p N^\alpha(\nu_a-\nu_b+x_a^N-x_b^N)\big]\Big\}^\beta \\ \geq\; & \prod_{a=1}^{N}\Big\{\mathrm{e}^{-N^{1+\alpha}V(x_a^N)}\Big\}\cdot \mathrm{e}^{-\frac{N^{1+\alpha}}{4M}||V'||_{L^\infty(\sigma_\epsilon)}} \times \prod_{a<b}^{N}\mathrm{e}^{2N^\alpha s_N(x_b^N-x_a^N)} \times \int\limits_{\nu_1<\cdots<\nu_N}\mathrm{d}^N\nu\prod_{a=1}^{N}\Big\{\mathbf{1}_{|\xi|<\frac{1}{4MN}}(\nu_a)\Big\}\,. \end{aligned} \tag{3.9}$$

We remind that s_N has been defined in (3.2). The second line is obtained by keeping only the configurations where $i \mapsto \nu_i$ is increasing, and then using that sinh is an increasing function. Finally:

$$Z_N[V] \;\geq\; \prod_{a=1}^{N}\Big\{\mathrm{e}^{-N^{1+\alpha}V(x_a^N)}\Big\}\cdot \mathrm{e}^{-\frac{N^{1+\alpha}}{4M}||V'||_{L^\infty(\sigma_\epsilon)}}\cdot\prod_{a<b}^{N}\mathrm{e}^{2N^\alpha s_N(x_b^N-x_a^N)}\cdot\frac{1}{N!}\Big(\frac{1}{4NM}\Big)^N\,. \tag{3.10}$$

We rewrite the first product involving the potential by comparison between the Riemann sum and the integral:

$$\frac{1}{N}\sum_{a=1}^{N}V(x_a^N) \;=\; \int_{\mathbb{R}} V(\xi)\,\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(\xi) \;+\; \delta_N, \qquad |\delta_N| \leq \frac{||V'||_{L^\infty(\sigma_\epsilon)}}{N}\cdot(b_N-a_N)\,. \tag{3.11}$$

It thus remains to bound from below the β-exponent part. Using that s_N is increasing on $\mathbb{R}^+$, we get:

$$\begin{aligned} \int_{x<y} s_N(y-x)\,\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(x)\,\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(y) \;=\; & \sum_{a,b=0}^{N-1}\int_{x_a^N}^{x_{a+1}^N}\int_{x_b^N}^{x_{b+1}^N}\mathbf{1}_{x<y}(x,y)s_N(y-x)\,\mathrm{d}\mu_{\mathrm{eq}}(x)\,\mathrm{d}\mu_{\mathrm{eq}}(y) \\ \leq\; & \frac{1}{N^2}\sum_{a=0}^{N-1}\sum_{b=a+1}^{N-1}s_N(x_{b+1}^N-x_a^N) \;+\; \sum_{a=0}^{N-1}s_N(x_{a+1}^N-x_a^N)\cdot\frac{1}{2N^2}\,. \end{aligned} \tag{3.12}$$

The first sum can be recast as

$$\sum_{a=0}^{N-1}\sum_{b=a+2}^{N} s_N(x_b^N - x_a^N) \;=\; \sum_{a=1}^{N-1}\sum_{b=a+1}^{N} s_N(x_b^N - x_a^N) \;+\; \sum_{b=1}^{N} s_N(x_b^N - x_0^N) \;-\; \sum_{a=0}^{N-1} s_N(x_{a+1}^N - x_a^N)\,. \tag{3.13}$$

It follows from (3.7) and from $|x_a^N - x_b^N| < |b_N - a_N| < C$ for some $C > 0$ independent of N, that:

$$\max_{0\le a\le N-1} |s_N(x_{a+1}^N - x_a^N)| \;=\; N^{-\alpha}\mathrm{O}(\ln N \;+\; N^{\alpha}) \qquad \text{and} \qquad \max_{1\le a\le N} |s_N(x_a^N - x_0^N)| \;=\; \mathrm{O}(1)\,. \tag{3.14}$$

Hence, it follows that

$$N^2 \int_{x<y} s_N(y-x)\,\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(x)\,\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(y) \;\le\; \mathrm{O}(N) + \sum_{a<b} s_N(x_b^N - x_a^N)\,, \tag{3.15}$$

thus leading to the claim. ■

We now estimate the fluctuations of linear statistics by using an idea introduced in [70].

Definition 3.6 *Given a configuration of points $\lambda_1 \le \cdots \le \lambda_N$, we build a sequence $\widetilde{\lambda}_1 < \cdots < \widetilde{\lambda}_N$ defined as*

$$\widetilde{\lambda}_1 = \lambda_1 \quad \textit{and} \quad \widetilde{\lambda}_{k+1} = \widetilde{\lambda}_k \;+\; \max\left(\lambda_{k+1} - \lambda_k, \mathrm{e}^{-(\ln N)^2}\right). \tag{3.16}$$

Further, for any $\boldsymbol{\lambda} \in \mathbb{R}^N$, we associate a vector $\widetilde{\boldsymbol{\lambda}} \in \mathbb{R}^N$ by ordering the λ's with a permutation σ, apply the previous construction to obtain a N-uple $\widetilde{\boldsymbol{\lambda}}$, and put them in original order with the permutation σ^{-1}. The corresponding empirical measure is:

$$L_N^{(\widetilde{\lambda})} = \frac{1}{N}\sum_{a=1}^{N} \delta_{\widetilde{\lambda}_a}$$

and we denote $L_{N;u}^{(\widetilde{\lambda})}$ the convolution of $L_N^{(\widetilde{\lambda})}$ with the uniform probability measure on $[0\,;\mathrm{e}^{-(\ln N)^2}/N]$.

The new configuration has been constructed such that, for $\ell \neq k$,

$$\left|\widetilde{\lambda}_k - \widetilde{\lambda}_\ell\right| \;\ge\; \mathrm{e}^{-(\ln N)^2} \qquad , \quad \left|\lambda_k - \lambda_\ell\right| \;\le\; \left|\widetilde{\lambda}_k - \widetilde{\lambda}_\ell\right| \quad \text{and} \quad \left|\lambda_k - \widetilde{\lambda}_k\right| \le (k-1)\cdot \mathrm{e}^{-(\ln N)^2}\,. \tag{3.17}$$

The advantage of working with $L_{N;u}^{(\widetilde{\lambda})}$ is that it is Lebesgue continuous; as such it can appear in the argument of $\mathcal{E}_N$ or $\mathfrak{D}^2$ and yield finite results. The scale of regularisation $e^{-(\ln N)^2} = N^{-\ln N}$ is somewhat arbitrary, but in any case negligible compared to $N^{-\alpha}$.

Proposition 3.7 *Assume that*

- *the partition function $Z_N[V]$ satisfies a lower-bound of the form*

$$Z_N[V] \;\ge\; \exp\Big\{-N^{2+\alpha}\mathcal{E}_N[\mu_{\mathrm{eq}}^{(N)}] \;+\; \delta_N\Big\}\,, \qquad \delta_N = \mathrm{o}(N^{2+\alpha})\,; \tag{3.18}$$

- *the potential is sub-exponential, viz. there exists $\epsilon > 0$ and $C_V > 0$ such that*

$$\forall x \in \mathbb{R}\,, \qquad \sup_{t\in[0\,;\epsilon]} \left|V'(x+t)\right| \;\le\; C_V\Big(\left|V(x)\right| + 1\Big)\,. \tag{3.19}$$

Then, given any $0 < \eta < 1$, *we have for all* $\boldsymbol{\lambda} \in \mathbb{R}^N$ *that*

$$p_N(\boldsymbol{\lambda}) \;\leq\; \exp\bigg\{ - N^{2+\alpha}\mathfrak{D}^2[L^{(\widetilde{\lambda})}_{N;u}, \mu^{(N)}_{\mathrm{eq}}] - \delta_N - N^{2+\alpha}(1-\eta) \int\limits_{\mathbb{R}} V_{N;\mathrm{eff}}(\xi)\, \mathrm{d}L^{(\widetilde{\lambda})}_{N;u}(\xi) \;+\; \mathrm{O}(N \ln N) \bigg\}\;. \qquad (3.20)$$

The effective potential $V_{N;\mathrm{eff}}$ *has been defined in* (3.3) *while* $\mathfrak{D}^2[\mu, \nu]$ *is as given in* (3.2).

Proof — The partition function takes the form:

$$Z_N[V] = \int\limits_{\mathbb{R}^N} \mathrm{d}^N\lambda\, \exp\bigg\{ - N^{2+\alpha}\Big(\int V(x)\, \mathrm{d}L^{(\lambda)}_N(x) - \Sigma_{\mathrm{diag}}[L^{(\lambda)}_N]\Big)\bigg\}, \qquad \Sigma_{\mathrm{diag}}[\mu] \;=\; \int\limits_{x \neq y} s_N(x-y)\, \mathrm{d}\mu(x)\mathrm{d}\mu(y)\;.$$

where s_N defined in (3.2). We are going to estimate the cost of replacing $L^{(\lambda)}_N$ by $L^{(\widetilde{\lambda})}_{N;u}$ in the above integration. We start with the term involving the potential. Since we assumed V sub-exponential, we have:

$$\begin{aligned}\bigg| \int V(x)\, \mathrm{d}L^{(\lambda)}_N(x) \;-\; \int V(x)\, \mathrm{d}L^{(\widetilde{\lambda})}_N(x) \bigg| \;&\leq\; \frac{1}{N} \sum_{a=1}^{N} \frac{(a-1)}{\mathrm{e}^{(\ln N)^2}} \cdot \sup\Big\{ |V'(\widetilde{\lambda}_a + t)| \;:\; t \in \Big[0; \frac{(a-1)}{\mathrm{e}^{(\ln N)^2}}\Big]\Big\} \\ &\leq\; \frac{N C_V}{\mathrm{e}^{(\ln N)^2}} \Big(\int |V(x)|\, \mathrm{d}L^{(\widetilde{\lambda})}_N(x) \;+\; 1\Big)\;. \end{aligned} \qquad (3.21)$$

Further, since $V(x) \to +\infty$ when $|x| \to \infty$, there exists $C'_{\mathrm{eff}} > 0$ such that

$$\forall x \in \mathbb{R}, \qquad C'_{\mathrm{eff}}(1 + V_{N;\mathrm{eff}}(x)) \;\geq\; C_V(|V(x)| + 1)\;. \qquad (3.22)$$

As a consequence,

$$\exp\bigg\{ - N^{2+\alpha} \int V(x)\, \mathrm{d}L^{(\lambda)}_N(x) \bigg\} \;\leq\; \exp\bigg\{ \frac{N^{3+\alpha}\, C'_{\mathrm{eff}}}{e^{(\ln N)^2}} \Big[1 + \int V_{N;\mathrm{eff}}(x)\, \mathrm{d}L^{(\widetilde{\lambda}}_N(x)\Big] - N^{2+\alpha} \int V(x) \mathrm{d}L^{(\widetilde{\lambda})}_N(x) \bigg\}\;. \qquad (3.23)$$

Now, let us consider the term involving the sinh interaction. Since s_N is increasing on $\mathbb{R}^+$ and the spacings between $\widetilde{\lambda}_a$'s are larger than those between the λ_a's, it follows that $\Sigma_{\mathrm{diag}}[L^{(\lambda)}_N] \leq \Sigma_{\mathrm{diag}}[L^{(\widetilde{\lambda})}_N]$. Furthermore, we have:

$$\begin{aligned}\Sigma_{\mathrm{diag}}[L^{(\widetilde{\lambda})}_N] \;-\; \Sigma_{\mathrm{diag}}[L^{(\widetilde{\lambda})}_{N;u}] \;&=\; \int\limits_{x \neq y} \mathrm{d}L^{(\widetilde{\lambda})}_N(x)\, \mathrm{d}L^{(\widetilde{\lambda})}_N(y) \int\limits_{[0\,;1]^2} \mathrm{d}^2u \Big\{ s_N(x-y) \;-\; s_N(x - y + N^{-1}e^{-(\ln N)^2}(u_1 - u_2))\Big\} \\ &\quad -\; \frac{1}{N} \int\limits_{[0\,;1]^2} \mathrm{d}^2u\, s_N[N^{-1}e^{-(\ln N)^2}\,(u_1 - u_2)]\;. \end{aligned} \qquad (3.24)$$

When N is large enough, we can use the Lipschitz behaviour of s_N on $[e^{-(\ln N)^2}/2, +\infty[$ for the first term. Indeed:

$$|s'_N(x)| = \sum_{p=1}^{2} \frac{\beta\pi\omega_p}{2}\; \mathrm{cotanh}[\pi\omega_p N^{\alpha}|x|] \leq c' N^{-\alpha} e^{(\ln N)^2} \qquad (3.25)$$

for some $c' > 0$. Besides, we exploit that s_N is increasing to bound the second term. This leads to:

$$\Big|\Sigma_{\mathrm{diag}}[L_N^{(\widetilde{\lambda})}] \;-\; \Sigma_{\mathrm{diag}}[L_{N;u}^{(\widetilde{\lambda})}]\Big| \;\leq\; C\Big(N^{-\alpha-1}\Big) \;+\; C'\,N^{-(1+\alpha)}\,(\ln N)^2\;. \tag{3.26}$$

Since the measure $L_{N;u}^{(\widetilde{\lambda})}$ is continuous with respect to Lebesgue, it is not any more necessary to take care of the diagonal singularity in s_N, and we obtain:

$$\begin{aligned}\exp\bigg\{-N^{2+\alpha}\bigg(\int_{\mathbb{R}} V(x)\mathrm{d}L_N^{(\lambda)}(x) - \Sigma_{\mathrm{diag}}[L_N^{(\lambda)}]\bigg)\bigg\} \;\;\leq\;\; & \exp\bigg\{-N^{2+\alpha}\mathcal{E}_N[L_{N;u}^{(\widetilde{\lambda})}] \;+\; \mathrm{O}(N(\ln N)^2)\bigg\} \\ & \times\exp\bigg\{e^{-(\ln N)^2}\,N^{3+\alpha}\,C'_{\mathrm{eff}}\int V_{N;\mathrm{eff}}(x)\mathrm{d}L_{N;u}^{(\widetilde{\lambda})}(x)\bigg\}\;.\end{aligned} \tag{3.27}$$

Since $\mu_{\mathrm{eq}}^{(N)}$ is also continuous with respect to Lebesgue, $\mathcal{E}_N[\mu_{\mathrm{eq}}^{(N)}]$ is finite and we can expand the first term around $\mu_{\mathrm{eq}}^{(N)}$:

$$\mathcal{E}_N[L_{N;u}^{(\widetilde{\lambda})}] = \mathcal{E}_N[\mu_{\mathrm{eq}}^{(N)}] + \mathfrak{D}^2[L_{N;u}^{(\widetilde{\lambda})},\mu_{\mathrm{eq}}^{(N)}] + \int_{\mathbb{R}} \mathrm{d}(L_{N;u}^{(\widetilde{\lambda})} - \mu_{\mathrm{eq}}^{(N)})(x)\bigg\{V(x) - 2\int_{\mathbb{R}} \mathrm{d}\mu_{\mathrm{eq}}^{(N)}(y)\,s_N(x-y)\bigg\}\;.$$

We recognize in the last integral $V_{N;\mathrm{eff}}(x) + C_{\mathrm{eq}}^{(N)}$ integrated against a measure of mass 0. So, we can omit the constant C_{eq}, and since $V_{N;\mathrm{eff}} = 0$ on the support of $\mu_{\mathrm{eq}}^{(N)}$, we actually find:

$$\mathcal{E}_N[L_{N;u}^{(\widetilde{\lambda})}] = \mathcal{E}_N[\mu_{\mathrm{eq}}^{(N)}] + \mathfrak{D}^2[L_{N;u}^{(\widetilde{\lambda})},\mu_{\mathrm{eq}}^{(N)}] + \int_{\mathbb{R}} V_{N;\mathrm{eff}}(x)\,\mathrm{d}L_{N;u}^{(\widetilde{\lambda})}(x)$$

If we plug this relation in (3.28), we obtain a similar bound but now with $V_{N;\mathrm{eff}}$ having the prefactor $N^{2+\alpha} - e^{-(\ln N)^2}N^{3+\alpha}C'_{\mathrm{eff}} \leq (1-\eta)N^{2+\alpha}$, this for any $0 < \eta < 1$, provided that N is large enough. ■

In order to bound the one and multi-point expectation values and in particular the various terms arising in the Schwinger-Dyson equations, we introduce the exponential regularisation of a function:

Definition 3.8 *Given a function f in n variables, its exponential regularisation with growth κ is defined by*

$$\mathcal{K}_\kappa[f](\xi_1,\ldots,\xi_n) \;=\; \Big\{\prod_{a=1}^{n} \mathrm{e}^{-\kappa V(\xi_a)}\Big\}\cdot f(\xi_1,\ldots,\xi_n)\;. \tag{3.28}$$

We also denote $\mathbb{M}_{N;\kappa}^{(n)}$ the (un-normalised) measure on $\mathbb{R}^N$ whose density reads $p_N(\boldsymbol{\lambda})\prod_{a=1}^{n}\mathrm{e}^{\kappa V(\lambda_a)}$ (notice that the exponential factor only affects n variables out of N).

We will use repeatedly the transformation:

$$\mathbb{M}_{N;\kappa}^{(n)}[\mathcal{K}_\kappa[f]] = \langle f\rangle\;. \tag{3.29}$$

In this respect, prior to establishing the simplest *a priori* bounds on the multi-point expectation values, we need an easy bound on the total mass of $\mathbb{M}_{N;\kappa}^{(n)}$.

Lemma 3.9 *For any $\kappa \geq 0$ and positive integer n_0 there exists $c_{n_0}, C_{n_0} > 0$ such that, for any $n \leq n_0$ and any measurable set $\Omega \subset \mathbb{R}^N$ that is invariant under permutations of coordinates, it holds*

$$\Big|\mathbb{M}_{N;\kappa}^{(n)}[\Omega]\Big| \;\leq\; (C_{n_0})^n \cdot \mathbb{P}_N[\Omega] \;+\; \mathrm{O}\Big(\mathrm{e}^{-c_{n_0} N^{1+\alpha}}\Big) \cdot \exp\Big\{ - N^{2+\alpha} \inf_{\lambda \in \Omega} \mathfrak{D}^2[L_{N;u}^{(\widetilde{\lambda})}, \mu_{\mathrm{eq}}^{(N)}]\Big\}\;. \tag{3.30}$$

Proof — We first claim that the constant $C_{\mathrm{eq}}^{(N)}$ arising in the minimisation problem for the equilibrium measure (2.35) is bounded in N. Indeed, it follows from (2.35) that

$$C_{\mathrm{eq}}^{(N)} \;=\; \int_{a_N}^{b_N} V(\xi)\,\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(\xi) \;-\; \frac{\beta}{N^{\alpha}} \int\limits_{[a_N\,;b_N]^2} \ln\Big\{\prod_{p=1}^{2} \sinh[\pi\omega_p N^{\alpha}(\xi-\eta)]\Big\}\,\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(\xi)\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(\eta)\;. \tag{3.31}$$

Therefore, we have:

$$\Big|C_{\mathrm{eq}}^{(N)}\Big| \;\leq\; ||V||_{L^{\infty}([a_N\,;b_N])} \;+\; C\,||V''||^2_{L^{\infty}([a_N\,;b_N])} \int\limits_{[a_N\,;b_N]^2} \frac{1}{N^{\alpha}}\Big|\ln\Big\{\prod_{p=1}^{2}\sinh[\pi\omega_p N^{\alpha}(\xi-\eta)]\Big\}\Big|\,\mathrm{d}\xi\mathrm{d}\eta\;, \tag{3.32}$$

where we have used that $\mu_{\mathrm{eq}}^{(N)}$ is a probability measure and that its density is bounded by (2.41). The double integral remaining in (3.32) can be bounded by an N-independent constant. Such bounds are obtained by using that the function

$$g_N(\xi) \;=\; \frac{1}{N^{\alpha}}\Big|\ln\Big\{\prod_{a=1}^{2}\sinh[\pi\omega_a N^{\alpha}(\xi)]\Big\}\Big| - \pi(\omega_1+\omega_2)|\xi| \tag{3.33}$$

approaches 0 pointwise in $\xi \in [a_N - b_N\,; b_N - a_N] \setminus \{0\}$ and is bounded as $|g_N(\xi)| \leq C(1 + |\ln\xi|)$. Since the endpoints a_N and b_N are bounded in N in virtue of (2.38), we can apply the dominated convergence theorem to $(\xi,\eta) \mapsto g_N(\xi-\eta)$ on $[a_N\,;b_N]^2$.

The confinement hypothesis (2.12) on the potential implies the existence of $t > 0$ independent of N such that:

$$\forall \xi \in \mathbb{R}\setminus[-t,t], \qquad V_{N;\mathrm{eff}}(\xi) \;\geq\; \frac{V(\xi)}{2} \;\geq\; \frac{|\xi|}{2}\,. \tag{3.34}$$

where the effective potential is defined by (3.3).

The left hand side of (3.30) can be decomposed, by invoking the symmetry of the integrand, into pieces where the variables are either outside or inside the segment $[-t,t]$:

$$\int\limits_{\mathbb{R}^N} p_N(\boldsymbol{\lambda})\cdot\prod_{a=1}^{n}\mathrm{e}^{\kappa V(\lambda_a)}\mathbf{1}_{\Omega}(\boldsymbol{\lambda})\,\mathrm{d}^N\boldsymbol{\lambda} \;=\; \sum_{p=0}^{n}\binom{n}{p}\int\limits_{([-t\,;t]^c)^p}\prod_{a=1}^{p}\mathrm{d}\lambda_a \int\limits_{[-t,t]^{n-p}}\prod_{a=1+p}^{n}\mathrm{d}\lambda_a \int\limits_{\mathbb{R}^{N-n}}\prod_{a=1+n}^{N}\mathrm{d}\lambda_a\; p_N(\boldsymbol{\lambda})\,\mathbf{1}_{\Omega}(\boldsymbol{\lambda})\prod_{a=1}^{n}\mathrm{e}^{\kappa V(\lambda_a)}\;.$$

Since $p_N(\boldsymbol{\lambda})$ is the density of a probability measure on $\mathbb{R}^N$, the term $p = 0$ corresponding to all variables in $[-t,t]$ is bounded as:

$$\Big|\int\limits_{[-t,t]^n}\prod_{a=1}^{n}\mathrm{d}\lambda_a\int\limits_{\mathbb{R}^{N-n}}\prod_{a=1+n}^{N}\mathrm{d}\lambda_a\; p_N(\boldsymbol{\lambda})\,\mathbf{1}_{\Omega}(\boldsymbol{\lambda})\prod_{a=1}^{n}\mathrm{e}^{\kappa V(\lambda_a)}\Big| \;\leq\; \mathrm{e}^{n\kappa||V||_{L^{\infty}([-t\,;t])}}\mathbb{P}_N[\Omega]\;. \tag{3.35}$$

For the other terms $p \geq 1$, we rather take advantage of:

$$\big|p_N(\boldsymbol{\lambda})\,\mathbf{1}_{\Omega}(\boldsymbol{\lambda})\big| \;\leq\; \prod_{a=1}^{N} \mathrm{e}^{-\frac{1}{2}N^{1+\alpha}V_{N;\mathrm{eff}}(\lambda_a)} \cdot \exp\Big\{-N^{2+\alpha}\inf_{\lambda\in\Omega}\mathfrak{D}^2[L_{N;u}^{(\widetilde{\lambda})},\mu_{\mathrm{eq}}^{(N)}]\Big\} \tag{3.36}$$

which follows from (3.20) with $\eta = 1/2$ given in Proposition 3.7. Indeed, we have:

$$\Big|\int\limits_{([-t;t]^c)^p}\prod_{a=1}^{p}\mathrm{d}\lambda_a \int\limits_{[-t;t]^{n-p}}\prod_{a=1+p}^{n}\mathrm{d}\lambda_a \int\limits_{\mathbb{R}^{N-n}}\prod_{a=1+n}^{N}\mathrm{d}\lambda_a\; p_N(\boldsymbol{\lambda})\,\mathbf{1}_{\Omega}(\boldsymbol{\lambda})\prod_{a=1}^{n}\mathrm{e}^{\kappa V(\lambda_a)}\Big|$$
$$\leq\; \mathrm{e}^{(n-p)\kappa||V||_{L^\infty([-t;t])}-N^{2+\alpha}\inf_{\lambda\in\Omega}\mathfrak{D}^2\big[L_{N;u}^{(\widetilde{\lambda})},\mu_{\mathrm{eq}}^{(N)}\big]}\Big\{\int\limits_{[-t;t]^c}\mathrm{e}^{-\frac{1}{2}N^{1+\alpha}V_{N;\mathrm{eff}}(\xi)+\kappa V(\xi)}\mathrm{d}\xi\Big\}^p\cdot\Big\{\int\limits_{\mathbb{R}}\mathrm{e}^{-\frac{1}{2}N^{1+\alpha}V_{N;\mathrm{eff}}(\xi)}\mathrm{d}\xi\Big\}^{N-p}\,. \tag{3.37}$$

Further, in virtue of (3.34) we have, for N large enough,

$$\Big|\int\limits_{[-t;t]^c}\mathrm{e}^{-\frac{1}{2}N^{1+\alpha}V_{N;\mathrm{eff}}(\xi)+\kappa V(\xi)}\mathrm{d}\xi\Big| \;\leq\; \Big|\int\limits_{[-t;t]^c}\mathrm{e}^{-\frac{1}{8}N^{1+\alpha}V(\xi)}\mathrm{d}\xi\Big| \;\leq\; \Big|\int\limits_{[-t;t]^c}\mathrm{e}^{-\frac{1}{8}N^{1+\alpha}|\xi|}\mathrm{d}\xi\Big| \;=\; \mathrm{O}(\mathrm{e}^{-cN^{1+\alpha}})\,. \tag{3.38}$$

The integral over $\mathbb{R}$ in (3.37) is bounded uniformly by a constant A, since $V_{N;\mathrm{eff}} \geq 0$ and $V_{N;\mathrm{eff}}$ grows at least linearly at infinity. All-in-all, for any $p \geq 1$, (3.37) is bounded by $A^N e^{-cN^{1+\alpha}} = o(e^{-c'N^{1+\alpha}})$. Summing up over $p \in [\![\,0\,;\,n\,]\!]$, we see that the upper bound for $p = 0$ obtained in (3.35) dominates the sum, whence the result.

■

Corollary 3.10 *Let $\kappa \geq 0$. There exist constants $C_n > 0$ depending on n and κ such that the below bounds hold for any f satisfying $\mathcal{K}_\kappa[f] \in W_1^\infty(\mathbb{R}^n)$*

$$\Big|\Big\langle f(\xi_1,\ldots,\xi_n)\Big\rangle_{\bigotimes_1^n\mathcal{L}_N^{(\lambda)}}\Big| \;\leq\; C_n\Big\{N^{-n}\,||\mathcal{K}_\kappa[f]||_{W_1^\infty(\mathbb{R}^n)} \;+\; N^{(\alpha-1)n/2}\,||\mathcal{K}_\kappa[f]||_{W_n^\infty(\mathbb{R}^n)}^{1/2}\cdot||\mathcal{K}_\kappa[f]||_{W_0^\infty(\mathbb{R}^n)}^{1/2}\Big\}\,. \tag{3.39}$$

Proof — Using the trick (3.29) and decomposing $\mathcal{L}_N^{(\lambda)} = \mathcal{L}_{N;u}^{(\widetilde{\lambda})} + (L_N^{\lambda} - L_{N;u}^{(\widetilde{\lambda})})$, we can write:

$$\langle f\rangle_{\bigotimes_1^n\mathcal{L}_N^{(\lambda)}} \;=\; \sum_{\ell=1}^{n}\sum_{\substack{i_1<\cdots<i_\ell\\=1}}^{n}\mathbb{M}_{N;\kappa+\kappa'}^{(n)}\Big[\mathcal{K}_{\kappa+\kappa'}[f](\xi_1,\ldots,\xi_n)\prod_{a=1}^{\ell}\mathrm{d}\mathcal{L}_{N;u}^{(\widetilde{\lambda})}(\xi_{i_a})\prod_{\substack{a=1\\ \neq 1,\ldots,\ell}}^{n}\mathrm{d}\Big(L_{N;u}^{(\widetilde{\lambda})}-L_N^{(\lambda)}\Big)(\xi_{i_a})\Big] \;+\; \langle f\rangle_{\bigotimes_1^n\mathcal{L}_{N;u}^{(\widetilde{\lambda})}}\,. \tag{3.40}$$

Since $\widetilde{\lambda}_a$'s are not far from $\widetilde{\lambda}_a$'s according to (3.17), we can bound for any $\ell \leq n-1$,

$$\Big|\mathbb{M}_{N;\kappa+\kappa'}^{(n)}\Big[\mathcal{K}_{\kappa+\kappa'}[f](\xi_1,\ldots,\xi_n)\prod_{a=1}^{\ell}\mathrm{d}\mathcal{L}_{N;u}^{(\widetilde{\lambda})}(\xi_{i_a})\prod_{\substack{a=1\\ \neq 1,\ldots,\ell}}^{n}\mathrm{d}(L_{N;u}^{(\widetilde{\lambda})}-L_N^{(\lambda)})(\xi_{i_a})\Big]\Big|$$
$$\leq\; (2C)^n\cdot||\mathcal{K}_\kappa[f]||_{W_1^\infty(\mathbb{R}^n)}\frac{N(N-1)}{2}\cdot\frac{\mathrm{e}^{-(\ln N)^2}}{N}\,. \tag{3.41}$$

The first factor comes from the geometric bound (3.9) on the partition function of the measures $\mathbb{M}_{N;\kappa+\kappa'}$, while in the second factor, we used the sub-exponential hypothesis (2.14) to get rid of the operator $\mathcal{K}_{\kappa'}$.

As a consequence, the first sum in (3.40) will only give rise to $\|\mathcal{K}_\kappa[f]\|_{W_1^\infty(\mathbb{R}^n)}\cdot \mathrm{O}(N^{-\infty})$ corrections. This being settled, Proposition 3.7 ensures the existence of $M>0$ and a constant $C>0$ such that, for N large enough:

$$\mathbb{P}_N\Big[\Omega_{M;N}\Big] \;=\; \mathrm{O}\Big(\mathrm{e}^{-CM\,N^{1+\alpha}}\Big) \quad \text{with} \qquad \Omega_{M;N} \;=\; \Big\{\boldsymbol{\lambda}\in\mathbb{R}^N \;:\; \mathfrak{D}^2[L_{N;u}^{(\widetilde{\lambda})},\mu_{\mathrm{eq}}^{(N)}] \;>\; M/N\Big\}\,. \tag{3.42}$$

This ensures that

$$\Big|\langle f\rangle_{\bigotimes_1^n\mathcal{L}_N^{(\widetilde{\lambda})}}\Big| \;\leq\; C'\cdot C_{n_0}^n\,\|\mathcal{K}_\kappa[f]\|_{L^\infty(\mathbb{R}^n)}\,\mathrm{e}^{-C''M\,N^{1+\alpha}} \;+\; \Big|\langle f\cdot\mathbf{1}_{\Omega_{M;N}^c}\rangle_{\bigotimes_1^n\mathcal{L}_N^{(\widetilde{\lambda})}}\Big|\,. \tag{3.43}$$

Finally, using Cauchy-Schwarz inequality to make the distance $\mathfrak{D}$ appear:

$$\begin{aligned}\Big|\langle f\cdot\mathbf{1}_{\Omega_{M;N}^c}\rangle_{\bigotimes_1^n\mathcal{L}_N^{(\widetilde{\lambda})}}\Big| \;&=\; \Bigg|\mathbb{M}_{N;\kappa+\kappa'}^{(n)}\Bigg[\mathbf{1}_{\Omega_{M;N}^c}\int_{\mathbb{R}^n}\mathcal{F}[\mathcal{K}_{\kappa+\kappa'}[f]](\varphi_1,\dots,\varphi_n)\prod_{a=1}^n\mathcal{F}[\mathcal{L}_{N;u}^{(\widetilde{\lambda})}](-\varphi_a)\cdot\frac{\mathrm{d}^n\boldsymbol{\varphi}}{(2\pi)^n}\Bigg]\Bigg| \\ &\leq\; \Bigg\{\int_{\mathbb{R}^n}\frac{\big|\mathcal{F}[\mathcal{K}_{\kappa+\kappa'}[f]](\varphi_1,\dots,\varphi_n)\big|^2}{\prod_{i=1}^n\Big\{\frac{\pi\beta}{2N^\alpha\varphi_i}\sum_{p=1}^2\mathrm{cotanh}\Big[\frac{\varphi_i}{2\omega_pN^\alpha}\Big]\Big\}}\cdot\frac{\mathrm{d}^n\boldsymbol{\varphi}}{(2\pi)^n}\Bigg\}^{\frac12}\cdot C^n\cdot\mathbb{M}_{N;\kappa+\kappa'}^{(n)}\Big[\,\mathbf{1}_{\Omega_{M;N}^c}\,\mathfrak{D}^n[L_{N;u}^{(\widetilde{\lambda})},\mu_{\mathrm{eq}}^{(N)}]\Big]\,.\end{aligned} \tag{3.44}$$

The last factor, because it is evaluated on the complement on $\Omega_{M;N}$, is at most $\mathrm{O}(N^{-n/2})$. The Fourier transform part of the bound can be estimated with the bound:

$$\prod_{i=1}^n\Bigg|\frac{\pi\beta}{2N^\alpha\varphi_i}\sum_{p=1}^2\mathrm{cotanh}\Big[\frac{\varphi_i}{2\omega_pN^\alpha}\Big]\Bigg|^{-1} \;\leq\; \prod_{i=1}^n\big(C\,N^\alpha|\varphi_i|\big)\leq(CN^\alpha)^n\Bigg(1+\Big\{\sum_{i=1}^n\varphi_i^2\Big\}^{1/2}\Bigg)^n\,. \tag{3.45}$$

Hence, there exists a constant $C_n'>0$ such that:

$$\Big|\langle f\rangle_{\bigotimes_1^n\mathcal{L}_N^{(\widetilde{\lambda})}}\Big| \;\leq\; C_n'\,N^{(\alpha-1)n/2}\Big(\|\mathcal{K}_{\kappa+\kappa'}[f]\|_{H_{n/2}(\mathbb{R}^n)} \;+\; \|\mathcal{K}_\kappa[f]\|_{W_0^\infty(\mathbb{R}^n)}\Big)\,. \tag{3.46}$$

where the W_0^∞ norm is nothing but the L^∞ norm. In order to bound $\|\mathcal{K}_{\kappa+\kappa'}[f]\|_{H_{n/2}(\mathbb{R}^n)}$ by the W_n^∞ norms (c.f. their definition (1.17)), we observe that:

$$\|\mathcal{K}_{\kappa+\kappa'}[f]\|_{H_{n/2}(\mathbb{R}^n)}^2 \;\leq\; \|\mathcal{K}_{\kappa+\kappa'}[f]\|_{H_n(\mathbb{R}^n)}\cdot\|\mathcal{K}_{\kappa+\kappa'}[f]\|_{L^2(\mathbb{R}^n)}\,. \tag{3.47}$$

The $L^2(\mathbb{R}^n)$ norm is bounded directly as:

$$\|\mathcal{K}_{\kappa+\kappa'}[f]\|_{L^2(\mathbb{R}^n)} \;\leq\; \|\mathcal{K}_{\kappa'}[1]\|_{L^2(\mathbb{R}^n)}\cdot\|\mathcal{K}_\kappa[f]\|_{W_0^\infty(\mathbb{R}^n)}\,. \tag{3.48}$$

Finally, in order to bound $\|\mathcal{K}_{\kappa+\kappa'}[f]\|_{H_n(\mathbb{R}^n)}$, we remark that $(1+|t|)^{2n} \;\leq\; 4^n(1+t^2)^n$, so that:

$$\Bigg(1+\Big\{\sum_{a=1}^n\varphi_a^2\Big\}^{1/2}\Bigg)^{2n} \;\leq\; C\sum_{k=0}^n P_k(\varphi_1^2,\dots,\varphi_n^2) \tag{3.49}$$

for some symmetric homogeneous polynomial of degree k:

$$P_k(\varphi_1^2,\dots,\varphi_n^2) \;=\; \sum_{k_1+\cdots+k_n=k} p_{\{k_a\}}\cdot\varphi_1^{2k_1}\cdots\varphi_n^{2k_n} \qquad \text{with} \qquad p_{\{k_a\}}\geq 0\,. \tag{3.50}$$

This ensures that

$$\begin{aligned}||\mathcal{K}_{\kappa+\kappa'}[f]||^2_{H_n(\mathbb{R}^n)} &\leq C\sum_{k=0}^{n}\sum_{k_1+\cdots+k_n=k} p_{\{k_a\}}\int\Big|\prod_{a=1}^{n}\partial_{\xi_a}^{k_a}\cdot\mathcal{K}_{\kappa+\kappa'}[f](\xi_1,\ldots,\xi_n)\Big|^2\cdot\mathrm{d}^n\boldsymbol{\xi} \\ &\leq C'\cdot||\mathcal{K}_{\kappa'/2}[1]||^2_{L^2(\mathbb{R}^n)}\cdot||\mathcal{K}_{\kappa}[F]||^2_{W_n^\infty(\mathbb{R}^n)}\,. \end{aligned} \tag{3.51}$$

To get the last line, we have repeatedly used the sub-exponential hypothesis (2.14). As a consequence, for some constant C'

$$||\mathcal{K}_{\kappa+\kappa'}[f]||_{H_{n/2}(\mathbb{R}^n)} \leq C'\cdot||\mathcal{K}_{\kappa}[f]||^{\frac{1}{2}}_{W_n^\infty(\mathbb{R}^n)}\cdot||\mathcal{K}_{\kappa}[f]||^{\frac{1}{2}}_{W_0^\infty(\mathbb{R}^n)}\,. \tag{3.52}$$

Inserting the above bound in (3.46), we obtain

$$\big|\langle f\rangle_{\bigotimes_1^n\mathcal{L}_N^{(\bar\lambda)}}\big| \leq C''_n\,N^{(\alpha-1)n/2}\,||\mathcal{K}_{\kappa}[f]||^{\frac{1}{2}}_{W_n^\infty(\mathbb{R}^n)}\cdot||\mathcal{K}_{\kappa}[f]||^{\frac{1}{2}}_{W_0^\infty(\mathbb{R}^n)}\,, \tag{3.53}$$

what leads to the desired form of the bound on the average $\langle f\rangle_{\bigotimes_1^n\mathcal{L}_N^{(\lambda)}}$. ■

3.2 The Schwinger-Dyson equations

In the present section, we derive the system of Schwinger-Dyson equations in our model. The operator

$$\mathcal{U}_N[\phi](\xi) = \phi(\xi)\cdot\{V'(\xi) - \mathcal{S}_N[\rho_{\mathrm{eq}}^{(N)}](\xi)\} + \mathcal{S}_N[\phi\cdot\rho_{\mathrm{eq}}^{(N)}](\xi)\,, \tag{3.54}$$

with $\mathcal{S}_N$ defined in (2.42) will arise in their expression, and play a crucial role in the large-N analysis. $\mathcal{U}_N$ (and $\mathcal{S}_N$) are invertible and all the informations on the inverses are obtained later in Proposition 8.2 (the inverse of $\mathcal{S}_N$ is denoted $\mathcal{W}_N$, see § 5.4).

Since we will be dealing with operators initially defined on functions in one variable but acting on one of the variables of a function in many variables, it is useful to introduce the

Definition 3.11 *Given an operator $\mathcal{O} : W_p^\infty(\mathbb{R}) \to W_p^\infty(\mathbb{R}^\ell)$ acting on functions of one variable and $\phi \in W_p^\infty(\mathbb{R}^n)$, $\mathcal{O}_k[\phi]$ refers to the function*

$$\mathcal{O}_k[\phi](\xi_1,\ldots,\xi_{n+\ell-1}) = \mathcal{O}_k[\phi(\xi_1,\ldots,\xi_{k-1},*,\xi_{k+\ell},\ldots,\xi_{n+\ell-1})](\xi_k,\ldots,\xi_{k+\ell-1})\,, \tag{3.55}$$

in which $*$ denotes the variable of ϕ on which the operator $\mathcal{O}_k$ acts.

For instance, according to the above definition, we have $\mathcal{U}_{N;1}[\phi](\xi_1,\ldots,\xi_n) = \mathcal{U}_N[\phi(*,\xi_2,\ldots,\xi_n)](\xi_1)$.

Definition 3.12 *If ϕ is a function in $n \geq 1$ variables, we denote ∂_p the differentiation with respect to the p^{th} variable. We also define an operator $\Xi^{(p)} : W_\ell^\infty(\mathbb{R}^n) \to W_\ell^\infty(\mathbb{R}^{n-1})$ by:*

$$\Xi^{(p)}[\phi](\xi_1,\ldots,\xi_n) = \phi(\xi_1,\ldots,\xi_{p-1},\xi_1,\xi_p,\ldots,\xi_{n-1})\,.$$

Proposition 3.13 *Let ϕ_n be a function in n real variables such that $\mathcal{K}_\kappa[\phi_n] \in W_1^\infty(\mathbb{R}^n)$, cf. (3.28), for some $\kappa \geq 0$ that can depend on n. Then, all expectation values appearing below are well-defined. Furthermore, the level 1 Schwinger-Dyson equation takes the form:*

$$-\langle\phi_1\rangle_{\mathcal{L}_N^{(\lambda)}} + \frac{1}{2}\Big\langle\mathcal{D}_N\circ\mathcal{U}_N^{-1}[\phi_1]\Big\rangle_{\mathcal{L}_N^{(\lambda)}\otimes\mathcal{L}_N^{(\lambda)}} + \frac{(1-\beta)}{N^{1+\alpha}}\langle\partial_1\mathcal{U}_N^{-1}[\phi_1]\rangle_{\mu_{\mathrm{eq}}^{(N)}} + \frac{(1-\beta)}{N^{1+\alpha}}\langle\partial_1\mathcal{U}_N^{-1}[\phi_1]\rangle_{\mathcal{L}_N^{(\lambda)}} = 0\,. \tag{3.56}$$

There, $\mathcal{D}_N$ corresponds to the non-commutative derivative

$$\mathcal{D}_N[\phi](\xi,\eta) \;=\; \bigg\{ \sum_{p=1}^{2} \beta\pi\omega_p \operatorname{cotanh}\big[\pi\omega_p N^{\alpha}(\xi-\eta)\big] \bigg\} \cdot \big(\phi(\xi)-\phi(\eta)\big)\;. \tag{3.57}$$

In their turn, the Schwinger-Dyson equation at level n takes the form:

$$\begin{aligned}\big\langle \phi_n \big\rangle_{\bigotimes^{n} \mathcal{L}_N^{(\lambda)}} \;=\; & \frac{1}{N^{2+\alpha}} \sum_{p=2}^{n} \Big\langle \Xi^{(p)} \circ \mathcal{U}_{N;1}^{-1}[\partial_p \phi_n] \Big\rangle_{\bigotimes^{n-1} \mathcal{L}_N^{(\lambda)}} \;+\; \frac{1}{2} \Big\langle \mathcal{D}_{N;1} \circ \mathcal{U}_{N;1}^{-1}[\phi_n] \Big\rangle_{\bigotimes^{n+1} \mathcal{L}_N^{(\lambda)}} \\ & + \frac{(1-\beta)}{N^{1+\alpha}} \Big\langle \partial_1 \mathcal{U}_{N;1}^{-1}[\phi_n] \Big\rangle_{\mu_{\mathrm{eq}}^{(N)} \bigotimes^{n-1} \mathcal{L}_N^{(\lambda)}} + \frac{1}{N^{2+\alpha}} \sum_{p=2}^{n} \Big\langle \Xi^{(p)} \circ \mathcal{U}_{N;1}^{-1}[\partial_p \phi_n] \Big\rangle_{\mu_{\mathrm{eq}}^{(N)} \bigotimes^{n-2} \mathcal{L}_N^{(\lambda)}} + \frac{(1-\beta)}{N^{1+\alpha}} \Big\langle \partial_1 \mathcal{U}_{N;1}^{-1}[\phi_n] \Big\rangle_{\bigotimes^{n} \mathcal{L}_N^{(\lambda)}}\;.\end{aligned} \tag{3.58}$$

Proof — Schwinger-Dyson equations express the invariance of an integral under change of variables, or equivalently, integration by parts. Although the principle of derivation is well-known, we include the proof to be self-contained, following the route of infinitesimal change of variables. Let $\phi^{(a)}$, $a = 1, \dots, n+1$ be a collection of smooth and compactly supported functions. We introduce an ϵ-deformation of the probability density p_N given in (2.47) by setting:

$$p_N^{(\{\epsilon_a\}_1^n)}(\boldsymbol{\lambda}) \;=\; \frac{1}{Z_N(\{\epsilon_a\})} \prod_{a<b}^{N} \Big\{ \sinh\big[\pi\omega_1 N^{\alpha}(\lambda_a-\lambda_b)\big] \sinh\big[\pi\omega_2 N^{\alpha}(\lambda_a-\lambda_b)\big] \Big\}^{\beta} \prod_{a=1}^{N} \mathrm{e}^{-N^{1+\alpha} V_{(\{\epsilon_a\})}(\lambda_a)}\;, \tag{3.59}$$

where:

$$V_{(\{\epsilon_a\})}(\lambda) \;=\; V(\lambda) \;+\; \sum_{a=2}^{n+1} \epsilon_a \bigg(\phi^{(a)}(\xi) - \int \phi^{(a)}(\eta)\, \mathrm{d}\mu_{\mathrm{eq}}^{(N)}(\eta) \bigg)\,. \tag{3.60}$$

The new normalisation constant $Z_N(\{\epsilon_a\})$ in (3.59) is such that $p_N^{(\{\epsilon_a\})}$ is a still a probability density on $\mathbb{R}^N$.

We then define $G_t(\mu) = \mu + t\phi^{(1)}(\mu)$. Since $\partial_{\xi}\phi^{(1)}(\xi)$ is bounded from below, for t small enough G_t is a diffeomorphism of $\mathbb{R}$. Let us carry out the change of variables $\lambda_a = G_t(\mu_a)$ and translate the fact that $p_N^{(\{\epsilon_a\})}$ is a probability measure. This yields

$$1 \;=\; \int_{\mathbb{R}^N} p_N^{(\{\epsilon_a\})}(\boldsymbol{\lambda}) \prod_{a=1}^{N} \mathrm{d}\lambda_a \;=\; \int_{\mathbb{R}^N} p_N^{(\{\epsilon_a\})}\big(G_t(\lambda_1),\dots,G_t(\lambda_N)\big) \prod_{a=1}^{N} G_t'(\lambda_a)\, \mathrm{d}\lambda_a\;. \tag{3.61}$$

As a consequence, the change of variables yields, to the first order in t:

$$\begin{aligned} 1 \;=\; & \int_{\mathbb{R}^N} \mathrm{d}^N\boldsymbol{\lambda} \bigg\{ 1 + t \sum_{a=1}^{N} \partial_{\lambda_a} \phi^{(1)}(\lambda_a) \bigg\} \bigg\{ 1 \;-\; t\, N^{1+\alpha} \sum_{a=1}^{N} \big(V_{(\{\epsilon_a\})}\big)'(\lambda_a)\, \phi^{(1)}(\lambda_a) \bigg\} \\ & \bigg\{ 1 \;+\; t\, N^{\alpha} \sum_{a<b}^{N} \Big[\sum_{p=1}^{2} \beta\pi\omega_p \operatorname{cotanh}\big[\pi\omega_p N^{\alpha}(\lambda_a-\lambda_b)\big] \Big] \Big[\phi^{(1)}(\lambda_a) - \phi^{(1)}(\lambda_b) \Big] \bigg\} \cdot p_N^{(\{\epsilon_a\})}(\boldsymbol{\lambda}) \;+\; O(t^2)\;. \end{aligned} \tag{3.62}$$

Identifying the terms linear in t leads to:

$$-\Big\langle \phi^{(1)} \partial_1 \big[V_{(\{\epsilon_a\})}\big] \Big\rangle_{L_N^{(\lambda)}}^{(\{\epsilon_a\})} \;+\; \frac{1}{2} \Big\langle \mathcal{D}_N[\phi^{(1)}] \Big\rangle_{L_N^{(\lambda)} \otimes L_N^{(\lambda)}}^{(\{\epsilon_a\})} \;+\; \frac{(1-\beta)}{N^{1+\alpha}} \big\langle \partial_1 \phi^{(1)} \big\rangle_{L_N^{(\lambda)}}^{(\{\epsilon_a\})} \;=\; 0\;. \tag{3.63}$$

The superscript $(\{\epsilon_a\})$ is there to emphasise that the averages should be taken in respect to the probability measure associated with the ϵ-deformed density (3.59). We then centralise the empirical measures in respect to $\mu_{\mathrm{eq}}^{(N)}$. By using the integral equation satisfied by the density of the equilibrium measure $V'(\xi) = \mathcal{S}_N[\rho_{\mathrm{eq}}^{(N)}](\xi)$ for $\xi \in [a_N\,; b_N]$, we obtain:

$$
\begin{aligned}
- \Big\langle \mathcal{U}_N[\phi^{(1)}] \Big\rangle_{\mathcal{L}_N^{(\lambda)}}^{(\{\epsilon_a\})} - \sum_{p=2}^{n+1} \epsilon_a \Big(\langle \phi^{(1)}\, \partial_1 \phi^{(p)} \rangle_{\mu_{\mathrm{eq}}^{(N)}} + \langle \phi^{(1)}\, \partial_1 \phi^{(p)} \rangle_{\mathcal{L}_N^{(\lambda)}}^{(\{\epsilon_a\})} \Big) \\
+ \frac{1}{2} \Big\langle \mathcal{D}_N[\phi^{(1)}] \Big\rangle_{\mathcal{L}_N^{(\lambda)} \otimes \mathcal{L}_N^{(\lambda)}}^{(\{\epsilon_a\})} + \frac{(1-\beta)}{N^{1+\alpha}} \Big(\langle \partial_1 \phi^{(1)} \rangle_{\mu_{\mathrm{eq}}^{(N)}} + \langle \partial_1 \phi^{(1)} \rangle_{\mathcal{L}_N^{(\lambda)}}^{(\{\epsilon_a\})} \Big) = 0\,. \qquad (3.64)
\end{aligned}
$$

Sending ϵ_a's to zero in this equation leads to the desired form of the Schwinger-Dyson equation at level 1. In order to get the Schwinger-Dyson equation at level n, we should compute the ϵ_a derivatives of (3.64) evaluated at $\epsilon_a \equiv 0$. However, first, it is convenient to multiply the above equation by $Z_N(\{\epsilon_a\})/Z_N[V]$ so as to avoid differentiating the $\{\epsilon_a\}$-dependent partition function entering in the definition of the density $p_N^{(\{\epsilon_a\})}(\lambda)$. Doing so, however, produces additional averages in front of the averages solely involving the non-stochastic measures μ_{eq}:

$$
\begin{aligned}
- \Big\langle \mathcal{U}_N[\phi^{(1)}](\xi_1) \prod_{a=2}^{n} \phi^{(a)}(\xi_a) \Big\rangle_{\bigotimes^{n+1} \mathcal{L}_N^{(\lambda)}} + \frac{1}{N^{2+\alpha}} \sum_{p=2}^{n+1} \Big\langle \phi^{(1)}(\xi_1)\, \partial_1 \phi^{(p)}(\xi_1) \prod_{\substack{a=2 \\ \neq p}}^{n+1} \phi^{(a)}(\xi_a^{(p)}) \Big\rangle_{\bigotimes^{n} \mathcal{L}_N^{(\lambda)}} \\
+ \frac{1}{2} \Big\langle \mathcal{D}_N[\phi^{(1)}](\xi_1, \xi_2) \prod_{a=2}^{n+1} \phi^{(a)}(\xi_{a+1}) \Big\rangle_{\bigotimes^{n+2} \mathcal{L}_N^{(\lambda)}} + \frac{(1-\beta)}{N^{1+\alpha}} \Big\langle \partial_1 \phi^{(1)}(\xi_1) \Big\rangle_{\mu_{\mathrm{eq}}^{(N)}} \cdot \Big\langle \prod_{a=2}^{n+1} \phi^{(a)}(\xi_{a-1}) \Big\rangle_{\bigotimes^{n} \mathcal{L}_N^{(\lambda)}} \\
+ \frac{1}{N^{2+\alpha}} \sum_{p=2}^{n+1} \Big\langle \phi^{(1)}(\xi_1) \partial_1 \phi^{(p)}(\xi_1) \Big\rangle_{\mu_{\mathrm{eq}}^{(N)}} \Big\langle \prod_{\substack{a=2 \\ \neq p}}^{n+1} \phi^{(a)}(\xi_{a-1}^{(p)}) \Big\rangle_{\bigotimes^{n-1} \mathcal{L}_N^{(\lambda)}} + \frac{(1-\beta)}{N^{1+\alpha}} \Big\langle \partial_1 \phi^{(1)}(\xi_1) \prod_{a=2}^{n+1} \phi^{(a)}(\xi_a) \Big\rangle_{\bigotimes^{n+1} \mathcal{L}_N^{(\lambda)}} = 0\,.
\end{aligned}
\qquad (3.65)
$$

To any $\boldsymbol{\xi} \in \mathbb{R}^{n-1}$, we associated the vector $\boldsymbol{\xi}^{(p)} \in \mathbb{R}^n$ by $\boldsymbol{\xi}^{(p)} = (\xi_1, \dots, \xi_{p-1}, \xi_1, \xi_p, \dots, \xi_{n-1})$, whose components arise in products of the type $\prod_{\substack{a=2 \\ \neq p}}^{n+1} \phi^{(a)}(\xi_a^{(p)})$. The representation

$$
\mathcal{U}_N[\phi](\xi) = \phi(\xi) V'(\xi) + \int_{a_N}^{b_N} \Big\{ \sum_{p=1}^{2} \beta \pi \omega_p \operatorname{cotanh}\left[\pi \omega_p N^{\alpha} (\xi - \eta) \right] \Big\} (\phi(\eta) - \phi(\xi))\, \rho_{\mathrm{eq}}^{(N)}(\eta)\, \mathrm{d}\eta \qquad (3.66)
$$

readily shows that the operators $\mathcal{U}_N$ and $\mathcal{D}_N$ are both continuous as operators $W_1^{\infty}(K) \to W_0^{\infty}(K)$ for any compact $K \subseteq \mathbb{R}$. This continuity along with the finiteness of the measure $\mathbb{P}_N$ is then enough to conclude, by density of $C_c^{\infty}(\mathbb{R}) \otimes \cdots \otimes C_c^{\infty}(\mathbb{R})$ in $C_c^{\infty}(\mathbb{R}^n)$, that equation (3.58) holds for all functions $\phi_n \in C_c^{\infty}(\mathbb{R}^n)$. Eventually, the assumption of compact support can be dropped. Indeed, given any $\phi_n \in C_c^{\infty}(\mathbb{R}^n)$, the Schwinger-Dyson equation at level 1 can be recast as

$$
\begin{aligned}
\mathbb{M}_{N;\kappa'}^{(n)} \Big[\int \mathcal{K}_{\kappa'}[\mathcal{U}_{N;1}[\phi_n]] \otimes_{a=1}^{n} \mathrm{d}\mathcal{L}_N^{(\lambda)} \Big] = \frac{1}{N^{2+\alpha}} \sum_{p=2}^{n} \mathbb{M}_{N;\kappa'}^{(n-1)} \Big[\int \Xi^{(p)} \Big[\mathcal{K}_{\kappa'}[\partial_p \phi_n]\, \mathrm{d}L_N^{(\lambda)} \otimes (\otimes_{a=2}^{n-1} \mathrm{d}\mathcal{L}_N^{(\lambda)}) \Big] \\
+ \frac{1}{2} \mathbb{M}_{N;\kappa'}^{(n+1)} \Big[\int \mathcal{K}_{\kappa'}[\mathcal{D}_{N;1}[\phi_n]] \otimes_{a=2}^{n+1} \mathrm{d}\mathcal{L}_N^{(\lambda)} \Big] + \frac{(1-\beta)}{N^{1+\alpha}} \mathbb{M}_{N;\kappa'}^{(n)} \Big[\int \mathcal{K}_{\kappa'}[\partial_1 \phi_n]\, \mathrm{d}L_N^{(\lambda)} \otimes (\otimes_{a=2}^{n} \mathrm{d}\mathcal{L}_N^{(\lambda)}) \Big]\,. \qquad (3.67)
\end{aligned}
$$

with the measures $\mathbb{M}_{N;\kappa}^{(n)}$ introduced in Definition 3.8. It is readily seen due to the sub-exponentiality hypothesis (2.14) that given $0 < \kappa < \kappa'$ and ϕ_n such that $\mathcal{K}_\kappa[\phi_n] \in W_1^\infty(\mathbb{R}^n)$, we have:

$$||\mathcal{K}_{\kappa'}[\mathcal{U}_{N;1}[\phi_n]]||_{W_0^\infty(\mathbb{R}^n)} \;\le\; C||\mathcal{K}_\kappa[\phi_n]]||_{W_0^\infty(\mathbb{R}^n)} \tag{3.68}$$

and likewise for $\mathcal{D}_{N;1}$. Thus, since $\mathcal{K}_\kappa[\phi_n] \in W_1^\infty(\mathbb{R}^n)$ can be approached in $W_1^\infty(\mathbb{R}^n)$ norm by functions $\mathcal{K}_\kappa[\psi_n]$ with $\psi_n \in C_c^\infty(\mathbb{R}^n)$, it remains to invoke the finiteness of the measures $\mathbb{M}_{N;\kappa'}^{(n)}$ so as to get (3.58) in full generality. ■

It follows from the form taken by the Schwinger-Dyson equations that, if we want to solve these equations perturbatively we should, in the very first place, construct the inverse to the operator $\mathcal{U}_N$. This should be done is such a way that one can control explicitly or at least in a manageable way, its dependence on N and its possible singularities. Indeed, the building blocks of $\mathcal{U}_N^{-1}$ exhibit, for instance, square root like singularities at the endpoints of the support $[a_N\,; b_N]$ of the equilibrium measure. In § 8.1, we shall construct a regular representation for $\mathcal{U}_N^{-1}$. By regularity, we mean that the various square root singularities present in its building blocks eventually cancel out, hence showing that $\mathcal{U}_N^{-1}[H]$ is smooth as long as H is. Then, in § 8.2, we shall provide explicit, N-dependent, bounds on the $W_\ell^\infty(\mathbb{R})$ norms of $\mathcal{U}_N^{-1}[H]$. These will play a crucial role in the large-N analysis of the Schwinger-Dyson equations.

3.3 Asymptotic analysis of the Schwinger-Dyson equations

The asymptotic analysis of the Schwinger-Dyson equation builds heavily on a family of N-weighted norms that we introduce below.

Definition 3.14 *For any $\phi \in W_n^\infty(\mathbb{R}^p)$, the N-weighted L^∞ norm of order ℓ is defined by*

$$\mathcal{N}_N^{(\ell)}[\phi] \;=\; \sum_{k=0}^{\ell} \frac{||\phi||_{W_k^\infty(\mathbb{R}^p)}}{N^{k\alpha}}\;. \tag{3.69}$$

This notation does not specify the number of variables of ϕ since this is usually clear from the context.

The weighted norm satisfies the obvious bound:

$$\mathcal{N}_N^{(\ell)}[\phi] \le \ell \cdot ||\phi||_{W_\ell^\infty(\mathbb{R}^p)}\;, \tag{3.70}$$

and, respectively, the operators of differentiation and "repetition of a variable" $\Xi^{(p)}$ are bounded as :

$$\mathcal{N}_N^{(\ell)}[\partial_p\phi] \le N^\alpha\, \mathcal{N}_N^{(\ell+1)}[\phi]\;, \qquad\qquad \mathcal{N}_N^{(\ell)}[\Xi^{(p)}[\phi]] \le \mathcal{N}_N^{(\ell)}[\phi]\;. \tag{3.71}$$

Also, it is important to introduce a specific function that allows one to control the dependence on the potential in the various bounds that issue from the Schwinger-Dyson equations.

Definition 3.15 *The order ℓ estimate of the potential V is defined as*

$$\mathfrak{n}_\ell[V] \;=\; \frac{\max\Big\{\prod\limits_{a=1}^{\ell} ||\mathcal{K}_\kappa[V']||_{W_{k_a}^\infty(\mathbb{R}^n)} \;:\; \sum\limits_{a=1}^{\ell} k_a = 2\ell+1\Big\}}{\Big\{\min\Big(1\,,\, \inf_{[a\,;b]}|V''(\xi)|\,,\, |V'(b+\epsilon)-V'(b)|\,,\, |V'(a-\epsilon)-V'(a)|\Big)\Big\}^{\ell+1}}\;, \tag{3.72}$$

where $\epsilon > 0$ is small enough and fixed once for all, while $\kappa > 0$. We also remind that $\mathcal{K}_\kappa$ is the exponential regularisation of Definition 3.8.

Since κ only plays a minor role due to the sub-exponentiality hypothesis (2.14) in the estimates provided by $\mathfrak{n}_{\ell}[V]$, we chose to keep its dependence implicit. Note also that the constants $\mathfrak{n}_{\ell}[V]$ satisfy

$$\mathfrak{n}_{\ell}[V] \cdot \mathfrak{n}_{\ell'}[V] \;\leq\; \mathfrak{n}_{\ell+\ell'+1}[V]\;. \tag{3.73}$$

Lemma 3.16 *Let $\kappa > 0$. There exist constants $C_{n;\ell}, \widetilde{C}_{n;\ell} > 0$ such that, for any ϕ satisfying*

- $\mathcal{K}_{\kappa/\ell}[\phi] \in W^{\infty}_{2\ell+1}(\mathbb{R}^n)$
- $\xi \mapsto \phi(\xi, \xi_2, \dots, \xi_n) \in \mathfrak{X}_s([a_N\,; b_N])$, $0 < s < 1/2$, *that is to say*[10]

$$\int\limits_{\mathbb{R}+\mathrm{i}\epsilon} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \chi_{11}(\mu) \int\limits_{a_N}^{b_N} \phi(\xi, \xi_2, \dots, \xi_n)\, \mathrm{e}^{\mathrm{i}\mu N^{\alpha}(\xi - b_N)} \mathrm{d}\xi \;=\; 0 \qquad \text{almost everywhere in } (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1} \tag{3.74}$$

we have the bounds:

$$\mathcal{N}_N^{(\ell)}\Big[\mathcal{K}_{\kappa}[\mathcal{U}_{N;1}^{-1}[\phi]]\Big] \;\leq\; C_{n;\ell} \cdot \mathfrak{n}_{\ell}[V] \cdot N^{\alpha} \cdot (\ln N)^{2\ell+1} \cdot \mathcal{N}_N^{(2\ell+1)}[\mathcal{K}_{\kappa}[\phi]]\;, \tag{3.75}$$

$$\mathcal{N}_N^{(\ell)}\Big[\mathcal{K}_{\kappa}[\mathcal{D}_{N;1}[\phi]]\Big] \;\leq\; \widetilde{C}_{n;\ell} \cdot (\ln N)^2 \cdot \mathcal{N}_N^{(\ell+1)}[\mathcal{K}_{\kappa}[\phi]]\;. \tag{3.76}$$

Note that the above lemma implies, in particular, a bound on the weighted norm of $\mathcal{D}_{N;1} \circ \mathcal{U}_{N;1}^{-1}$:

$$\mathcal{N}_N^{(\ell)}\Big[\mathcal{K}_{\kappa}[\mathcal{D}_{N;1} \circ \mathcal{U}_{N;1}^{-1}[\phi]]\Big] \;\leq\; C'_{n,\ell} \cdot \mathfrak{n}_{\ell+1}[V] \cdot N^{\alpha} \cdot (\ln N)^{2\ell+5} \cdot \mathcal{N}_N^{(2\ell+3)}[\mathcal{K}_{\kappa}[\phi]]\;, \tag{3.77}$$

Proof — We first focus on the norm of $\mathcal{K}_{\kappa}[\mathcal{D}_{N;1}[\phi]]$. In order to obtain (3.76), we bound

$$O_{\boldsymbol{k}_{n+1}}(\boldsymbol{\xi}_{n+1}) \;=\; \prod_{a=1}^{n+1} \partial_{\xi_a}^{k_a} \mathcal{K}_{\kappa}[\mathcal{D}_{N;1}[\phi]](\xi_1, \dots, \xi_{n+1}) \qquad \text{with} \quad \sum_{a=1}^{n+1} k_a \;\leq\; \ell \quad k_a \in \mathbb{N} \tag{3.78}$$

by different means in the two cases of interest, *viz.* $N^{\alpha}|\xi_1 - \xi_2| \geq (\ln N)^2$ and $N^{\alpha}|\xi_1 - \xi_2| < (\ln N)^2$.

We first treat the case $N^{\alpha}|\xi_1 - \xi_2| \geq (\ln N)^2$. Observe that for $|N^{\alpha}\xi| \;\geq\; (\ln N)^2$, we have:

$$\forall \ell \geq 0, \qquad \Big|\partial_{\xi}^{\ell}\{S(N^{\alpha}\xi)\}\Big| \leq \delta_{\ell,0}\, c'_0 \;+\; (1 - \delta_{\ell,0})\, c'_{\ell}\, N^{\ell\alpha} \mathrm{e}^{-c'' \ln^2 N} \;\leq\; c_{\ell}(\ln N)^2 \tag{3.79}$$

for some constants c_{ℓ}, where S is defined in (2.42) and $\delta_{\ell,0}$ being the Kronecker symbol. Therefore:

$$\begin{aligned} \Big|O_{\boldsymbol{k}_{n+1}}(\boldsymbol{\xi}_{n+1})\Big| \;&\leq\; \sum_{\substack{p_a + \ell_a = k_a \\ a=1,2}} \prod_{a=1}^{2} \binom{k_a}{p_a} \cdot \Big|\partial_{\xi_1}^{p_1} \partial_{\xi_2}^{p_2} [\phi_{\{k_a\}}(\xi_1, \xi_3, \dots, \xi_{n+1}) - \phi_{\{k_a\}}(\xi_2, \xi_3, \dots, \xi_{n+1})]\Big| \cdot c_{\ell_1+\ell_2} \cdot (\ln N)^2 \\ &\leq\; C \cdot N^{[\max(k_1,k_2)]\alpha} \cdot (\ln N)^2 \sum_{s=0}^{\max(k_1,k_2)} N^{-s\alpha} \max_{\eta \in \{\xi_1, \xi_2\}} \Big|\partial_1^s \phi_{\{k_a\}}(\eta, \xi_3, \dots, \xi_{n+1})\Big| \\ &\leq\; C \cdot N^{\ell\alpha} \cdot (\ln N)^2 \cdot \mathcal{N}_N^{(\ell)}[\mathcal{K}_{\kappa}[\phi]]\;, \end{aligned} \tag{3.80}$$

[10]It is straightforward to check by carrying out contour deformations that, for functions ψ decaying sufficiently fast at infinity in respect to its first variable, the condition (3.74) is equivalent to belonging to $\mathfrak{X}_s(\mathbb{R})$.

where, in the intermediate calculations, we have used:

$$\phi_{\{k_a\}}(\xi_1,\xi_2,\ldots,\xi_n) \;=\; \prod_{a=3}^{n+1} \partial_{\xi_a}^{k_a}\Big\{\mathcal{K}_\kappa[\phi](\xi_1,\xi_2,\ldots,\xi_n)\Big\}\,. \tag{3.81}$$

We now turn to the case when $N^\alpha|\xi_1-\xi_2| < (\ln N)^2$. Observe that for any $\ell\in\mathbb{N}$ and $|N^\alpha\xi| \le (\ln N)^2$, the function $\widetilde{S}$, with $\widetilde{S}(x) = xS(x)$, satisfies

$$\forall \ell \ge 0, \qquad \Big|\partial_\xi^\ell\{\widetilde{S}(N^\alpha\xi)\}\Big| \;\le\; \delta_{\ell,0}\Big|N^\alpha\xi[S(N^\alpha\xi)-\frac{2\beta}{N^\alpha\xi}]+2\beta\Big| \;+\; (1-\delta_{\ell,0})\,N^{\ell\alpha}||\widetilde{S}||_{W^\infty_\ell(\mathbb{R})} \;\le\; c_\ell N^{\alpha\ell}(\ln N)^2 \tag{3.82}$$

for some constants c_ℓ. Starting from the integral representation

$$O_{\boldsymbol{k}_{n+1}}(\boldsymbol{x}_{n+1}) \;=\; \int_0^1 \frac{\mathrm{d}t}{N^\alpha}\partial_{\xi_1}^{k_1}\partial_{\xi_2}^{k_2}\Big\{\partial_1\phi_{\{k_a\}}(\xi_1+t(\xi_2-\xi_1),\xi_3,\ldots,\xi_{n+1})\cdot\widetilde{S}(N^\alpha(\xi_1-\xi_2))\Big\}\,, \tag{3.83}$$

we obtain:

$$\begin{aligned}
\Big|O_{\boldsymbol{k}_{n+1}}(\boldsymbol{x}_{n+1})\Big| \;&\le\; \sum_{\substack{p_a+\ell_a=k_a\\ a=1,2}} \frac{\binom{k_1}{p_1}\binom{k_2}{p_2}c_{\ell_1+\ell_2}}{N^{\alpha(1-\ell_1-\ell_2)}}\int_0^1(1-t)^{p_1}t^{p_2}(\partial_1^{p_1+p_2+1}\phi_{\{k_a\}})(\xi_1+t(\xi_2-\xi_1),\xi_3,\ldots,\xi_{n+1})\cdot(\ln N)^2\cdot\mathrm{d}t\\
&\le\; CN^{(k_1+k_2)\alpha}(\ln N)^2\sum_{s=1}^{k_1+k_2+1}N^{-s\alpha}\max_{\eta\in[\xi_1\,;\xi_2]}\Big|(\partial_1^s\phi_{\{k_a\}})(\eta,\xi_3,\ldots,\xi_{n+1})\Big|\\
&\le\; CN^{\ell\alpha}(\ln N)^2\cdot\mathcal{N}_N^{(\ell+1)}[\mathcal{K}_\kappa[\phi]]\,.
\end{aligned} \tag{3.84}$$

Putting together (3.80) and (3.84) and taking the supremum over $\{k_a\}$ such that $\sum_a k_a \le \ell$, we deduce the desired bound (3.76) for the weighted norm of $\mathcal{D}_N$.

The bounds for the weighted norm of $\mathcal{K}_\kappa[\mathcal{U}_{N;1}^{-1}[\phi]]$ are obtained quite straightforwardly by using the $W^\infty_\ell(\mathbb{R})$ bounds on $\mathcal{K}_\kappa[\psi]$, given that $\mathcal{K}_\kappa[\psi]\in W^\infty_{2\ell+1}(\mathbb{R})$, as derived in Proposition 8.2. ■

With the bounds on the action of the operators $\mathcal{U}_{N;1}^{-1}$ and $\mathcal{D}_{N;1}$, we can improve the *a priori* bounds on the centred expectation values of the correlators through a bootstrap procedure.

Proposition 3.17 *Let $\alpha < 1/4$ and pick $\kappa > 0$. There exist an increasing sequence of integers $(m_n)_n$, positive constants $(C_n)_n$, such that, for any $n \ge 1$ and $\phi \in \mathfrak{X}_s([a_N\,;b_N])$ in the sense of* (3.74) *and satisfying $\mathcal{K}_\kappa[\phi]\in W^\infty_{m_n}(\mathbb{R}^n)$, cf.* (3.28)*, we have:*

$$\Big|\langle\phi\rangle_{\bigotimes_1^n\mathcal{L}_N^{(\lambda)}}\Big| \;\le\; C_n\cdot\mathfrak{n}_{m_n}[V]\cdot\mathcal{N}_N^{(m_n)}[\mathcal{K}_\kappa[\phi]]\,N^{(\alpha-1)n}\quad. \tag{3.85}$$

The whole dependence of the upper bound on the potential V is contained in the constant $\mathfrak{n}_{\ell_n}[V]$, and we can take:

$$m_n = \ell_n^{(q_n)},\qquad q_n = 1+\Big\lfloor\frac{n}{1-4\alpha}\Big\rfloor,\qquad \ell_n^{(q)} = 2^q(n+q)+3(2^q-1)\,. \tag{3.86}$$

Proof — The proof utilises a bootstrap-based improvement of the *a priori* bounds given in Corollary 3.10. Namely, assume the existence of sequences $\eta_N\to 0$, $\varkappa_N\in[0\,;1]$, and constants $C_n>0$ independent of N, and integers ℓ_n increasing with n, such that, for any ϕ such that $\mathcal{K}_\kappa[\phi]\in W^\infty_{\ell_n}(\mathbb{R}^n)$:

$$\Big|\langle\phi\rangle_{\bigotimes_1^n\mathcal{L}_N^{(\lambda)}}\Big| \;\le\; C_n\cdot\mathfrak{n}_{\ell_n}[V]\cdot\mathcal{N}_N^{(\ell_n)}[\mathcal{K}_\kappa[\phi]]\cdot\Big(\eta_N^n\cdot\varkappa_N \;+\; N^{n(\alpha-1)}\Big)\,. \tag{3.87}$$

We will establish that there exists a new constants $C'_n > 0$ and integers $\ell'_n = 2\ell_{n+1} + 3$ such that, for $\mathcal{K}_\kappa[\phi] \in W^{\infty}_{\ell'_n}(\mathbb{R}^n)$:

$$\Big|\langle\phi\rangle_{\bigotimes_1^n \mathcal{L}_N^{(\lambda)}}\Big| \;\leq\; C_n \cdot \mathfrak{n}_{\ell'_n}[V] \cdot \mathcal{N}_N^{(\ell'_n)}[\mathcal{K}_\kappa[\phi]] \cdot \Big(\eta_N^n \cdot \varkappa'_N \;+\; N^{n(\alpha-1)}\Big)\,, \tag{3.88}$$

where

$$\varkappa'_N \;=\; \varkappa_N\,(\ln N)^{\ell'_n+2}\,\max\,(N^{\alpha}\eta_N\,;\; N^{\alpha-2}\eta_N^{-2}\,;\; N^{\alpha-1}\eta_N^{-1})\,. \tag{3.89}$$

Before justifying (3.89), let us examine its consequences. The bootstrap approach can be settled if

$$\varkappa'_N = N^{-\gamma_\varkappa}\varkappa_N \tag{3.90}$$

Assuming momentarily that $\eta_N = N^{-\gamma}$, when $0 < \alpha < 1$, the range of α and γ ensuring (3.90) is:

$$\alpha < \gamma < 1-\alpha \qquad \text{what implies}\;\; \alpha < 1/2\,. \tag{3.91}$$

The rate $\gamma_\varkappa$ at which $\varkappa'_N/\varkappa_N$ goes to zero increases when γ runs from α to $1/2$, is maximal and equal to $1/2-\alpha$ when $\gamma = 1/2$, and then decreases when γ increases between $1/2$ and $1-\alpha$.

The *a priori* estimate proved in Corollary 3.10 gives:

$$\Big|\langle\phi\rangle_{\bigotimes_1^n \mathcal{L}_N^{(\lambda)}}\Big| \;\leq\; C'_n \cdot \|\mathcal{K}_\kappa[\phi]\|^{\frac{1}{2}}_{W_n^\infty(\mathbb{R}^n)} \cdot \|\mathcal{K}_\kappa[\phi]\|^{\frac{1}{2}}_{W_0^\infty(\mathbb{R}^n)} \cdot N^{(\alpha-1)n/2} \;\leq\; C'_n \cdot \mathcal{N}_N^{(n)}[\mathcal{K}_\kappa[\phi]]\,N^{(\alpha-1/2)n}\,. \tag{3.92}$$

Therefore, the assumption (3.88) is satisfied with $\eta_N = N^{-\gamma}$ for $\gamma = 1/2-\alpha$, and the order $\ell_n = n$ for the weighted norm. The bootstrap condition (3.91) then implies $\alpha < 1/4$, and in this case, we find:

$$\varkappa'_N \leq \varkappa_N\,(\ln N)^{\ell'_n}\,N^{-\frac{(1-4\alpha)}{2}}\,. \tag{3.93}$$

Now, we can iterate the bootstrap until the first term in (3.88) becomes less or equal than the second term $N^{(\alpha-1)n}$. This is obtained in a number of steps q_n determined by the equation $N^{-(1/2-\alpha)n}N^{-(1-4\alpha)q_n/2} \ll N^{(\alpha-1)n}$, therefore:

$$q_n = 1 + \Big\lfloor \frac{n}{1-4\alpha} \Big\rfloor\,. \tag{3.94}$$

The order of the weighted norm appearing in the bound of the n point correlations at step q of the recursion satisfies $\ell_n^{(q)} = 2\ell_{n+1}^{(q-1)} + 3$, with initial condition $\ell_n^{(0)} = n$. The solution is

$$\ell_n^{(q)} = 2^q(n+q) + 3(2^q-1)\,. \tag{3.95}$$

Therefore, we get at the end of the recursion:

$$\Big|\langle\phi\rangle_{\bigotimes_1^n \mathcal{L}_N^{(\lambda)}}\Big| \leq C_n \cdot N^{(\alpha-1)n} \cdot \mathcal{N}_N^{(m_n)}[\mathcal{K}_\kappa[\phi]], \qquad m_n = \ell_n^{(q_n)}\,. \tag{3.96}$$

We shall now justify the claim (3.89). Starting from (3.87), we bound $\langle\phi\rangle_{\bigotimes_1^n \mathcal{L}_N^{(\lambda)}}$ given by the Schwinger-Dyson equations of Proposition 3.13, using the norms of the operators $\mathcal{U}_{N;1}$ and $\mathcal{D}_N$ obtained in Lemma 3.16. We stress that it is indeed licit to apply the bound (3.75) for $\mathcal{U}_N^{-1}$ for, if ϕ satisfies the condition (3.74), then so do the functions $\partial_p\phi$ with $p = 2, \ldots, n$. Respecting the order of appearance of terms in (3.58), we get[11]:

$$\begin{aligned}
\Big|\langle\phi\rangle_{\bigotimes_1^n \mathcal{L}_N^{(\lambda)}}\Big| \;\leq\; & C\mathfrak{n}^2_{\ell_{n-1}}[V]\frac{N^{2\alpha}}{N^{2+\alpha}}(\ln N)^{2\ell_{n-1}+1}\mathcal{N}_N^{(2\ell_{n-1}+2)}[\mathcal{K}_\kappa[\phi]]\cdot\Big(\eta_N^{n-1}\cdot\varkappa_N \;+\; N^{(n-1)(\alpha-1)}\Big) \\
& + C\mathfrak{n}_{\ell_{n+1}}[V]\mathfrak{n}_{\ell_{n+1}+1}[V]N^{\alpha}(\ln N)^{2\ell_{n+1}+5}\cdot\mathcal{N}_N^{(2\ell_{n+1}+3)}[\mathcal{K}_\kappa[\phi]]\cdot\Big(\eta_N^{n+1}\cdot\varkappa_N \;+\; N^{(n+1)(\alpha-1)}\Big) \\
& + C\,\mathfrak{n}_{\ell_{n-1}}[V]\mathfrak{n}_{\ell_{n-1}+1}[V]\frac{N^{2\alpha}}{N^{1+\alpha}}(\ln N)^{2\ell_{n-1}+3}\cdot\mathcal{N}_N^{(2\ell_{n-1}+3)}[\mathcal{K}_\kappa[\phi]]\cdot\Big(\eta_N^{n-1}\cdot\varkappa_N \;+\; N^{(n-1)(\alpha-1)}\Big) \\
& + C(\mathfrak{n}_{\ell_{n-2}}[V])^2\frac{N^{2\alpha}}{N^{2+\alpha}}(\ln N)^{2\ell_{n-2}+2}\cdot\mathcal{N}_N^{(2\ell_{n-2}+1)}[\mathcal{K}_\kappa[\phi]]\cdot\Big(\eta_N^{n-2}\cdot\varkappa_N \;+\; N^{(n-2)(\alpha-1)}\Big) \\
& + C\,\mathfrak{n}_{\ell_n}[V]\mathfrak{n}_{\ell_n+1}[V]\frac{N^{2\alpha}}{N^{1+\alpha}}(\ln N)^{2\ell_{n-1}+3}\cdot\mathcal{N}_N^{(2\ell_n+3)}[\mathcal{K}_\kappa[\phi]]\cdot\Big(\eta_N^{n}\cdot\varkappa_N \;+\; N^{n(\alpha-1)}\Big)\,, \qquad (3.97)
\end{aligned}$$

for some constant $C > 0$ depending on n and κ only. Note that terms integrated against the probability measure $\mu_{\mathrm{eq}}^{(N)}$ have been bounded by means of sup norms. The maximal powers of N are exactly as in (3.89) – since we assume $\eta_N \to 0$, the powers arising in the first line are negligible compared to those in the fourth line. We can then use (3.73) to bound the products of $\mathfrak{n}_\ell[V]$'s in terms of $\mathfrak{n}_{\ell'_n}[V]$ for a choice:

$$\ell'_n \geq \max\left(2\ell_{n-1}+2, 2\ell_{n+1}+3, 2\ell_{n-1}+3, 2\ell_{n-2}+2, 2\ell_n+3\right)\,. \qquad (3.98)$$

Since $(\ell_n)_n$ is increasing, we can take $\ell'_n = 2\ell_{n+1}+3$, and we indeed find (3.88) for N large enough. Note that, the new sequence $(\ell'_n)_n$ is, again, increasing. Then, the maximal power of $\ln N$ occurs in the second line, and is $(\ln N)^{2\ell_{n+1}+5} = (\ln N)^{\ell'_n+2}$. So, we have fully justified (3.88). ■

The improved estimates on the multi-point correlators are almost all that is needed for obtaining the large N asymptotic expansion of general one-point functions up to $\mathrm{o}(N^{-(2+\alpha)})$ corrections. Prior to deriving such results, we still need to introduce an operator $\widetilde{\mathcal{X}}_N$ mapping any function $W_p^\infty(O)$, O a bounded open subset in $\mathbb{R}^n$, onto a function belonging to $\mathfrak{X}_s([a_N\,;b_N])$ in the sense of (3.74).

Definition 3.18 *Let $\mathcal{X}_N$ be the linear form on $W_1^\infty([a_N\,;b_N])$:*

$$\mathcal{X}_N[\phi] \;=\; \frac{\mathrm{i}N^\alpha}{\chi_{11;+}(0)}\int\limits_{\mathbb{R}+\mathrm{i}\epsilon}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\chi_{11}(\mu)\int\limits_{a_N}^{b_N}\mathrm{e}^{\mathrm{i}\mu N^\alpha(\xi-b_N)}\,\phi(\xi)\,\mathrm{d}\xi\;\;. \qquad (3.99)$$

Then, we denote by $\widetilde{\mathcal{X}}_N$ the operator

$$\widetilde{\mathcal{X}}_N[\phi](\xi) \;=\; \phi(\xi)\;-\;\mathcal{X}_N[\phi] \qquad (3.100)$$

and also define:

$$\widetilde{\mathcal{U}}_N^{-1} = \mathcal{U}_N^{-1}\circ\widetilde{\mathcal{X}}_N, \qquad \widetilde{\mathcal{W}}_N = \mathcal{W}_N\circ\widetilde{\mathcal{X}}_N\,. \qquad (3.101)$$

It follows readily from the identity

$$\int\limits_{\mathbb{R}+\mathrm{i}\epsilon}\chi_{11}(\mu)\cdot\frac{1-\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}}{\mu}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \;=\; \chi_{11;+}(0) \qquad \text{with} \qquad \overline{x}_N \;=\; N^\alpha(b_N-a_N)\,, \qquad (3.102)$$

[11]The third and fifth line are absent in the case $\beta = 1$, and it gives a larger range of $\alpha > 0$ for which η_N can be chosen so that the bootstrap works. But, eventually, this does not lead to a stronger bound because we can only initialize the bootstrap with the concentration bound (3.10) i.e. $\eta_N = N^{-(1/2-\alpha)}$.

that $\widetilde{X}_N[\phi] \in \mathfrak{X}_s([a_N\,;b_N])$ in the sense of (3.74). The proof of (3.102) follows from the use of the boundary conditions $\mathrm{e}^{-\mathrm{i}\lambda N^{\alpha}(b_N-a_N)}\chi_{11;+}(\lambda) = \chi_{11;-}(\lambda)$, $\lambda \in \mathbb{R}$ the fact that $\chi_{11} \in \mathcal{O}(\mathbb{C}\setminus\mathbb{R})$ and that $\chi_{11}(\lambda) = \mathrm{O}(|\lambda|^{-1/2})$ at infinity.

Likewise, by using the bounds (7.23) obtained in Corollary 7.3 it is readily seen that

$$\mathcal{N}_N^{(p)}\Big[\mathcal{K}_{\kappa}[\widetilde{X}_N[\phi]]\Big] \;\leq\; C\cdot\mathcal{N}_N^{(p)}[\mathcal{K}_{\kappa}[\phi]]\;. \tag{3.103}$$

Proposition 3.19 *Given any $\kappa>0$, and any ϕ satisfying $\mathcal{K}_{\kappa}[\phi]\in W_{\ell}^{\infty}(\mathbb{R})$, we have:*

$$\begin{aligned}\langle\phi\rangle_{\mathcal{L}_N^{(\lambda)}} \;=\; & \frac{(1-\beta)}{N^{1+\alpha}}\cdot\Big\langle\partial_1\widetilde{\mathcal{U}}_N^{-1}[\phi]\Big\rangle_{\mu_{\mathrm{eq}}^{(N)}} \;+\; \frac{1}{2N^{2+\alpha}}\Big\langle\Xi^{(2)}\Big[\partial_2\widetilde{\mathcal{U}}_{N;1}^{-1}\Big[\mathcal{D}_N[\widetilde{\mathcal{U}}_N^{-1}[\phi]]\Big]\Big]\Big\rangle_{\mu_{\mathrm{eq}}^{(N)}} \\ & +\frac{(1-\beta)^2}{2N^{2(1+\alpha)}}\Big\langle\partial_1\partial_2\widetilde{\mathcal{U}}_{N;1}^{-1}\widetilde{\mathcal{U}}_{N;2}^{-1}\Big[\mathcal{D}_N[\widetilde{\mathcal{U}}_N^{-1}[\phi]]\Big]\Big\rangle_{\bigotimes^2\mu_{\mathrm{eq}}^{(N)}} \\ & +\frac{(1-\beta)^2}{N^{2(1+\alpha)}}\Big\langle\partial_1\widetilde{\mathcal{U}}_N^{-1}[\partial_1\widetilde{\mathcal{U}}_N^{-1}[\phi]]\Big\rangle_{\mu_{\mathrm{eq}}^{(N)}} \;+\; \frac{\delta_N[\phi,V]}{N^{2+\alpha}}\;.\end{aligned} \tag{3.104}$$

The remainder $\delta_N[\phi,V]$ is bounded as:

$$\Big|\delta_N[\phi,V]\Big| \;\leq\; C\cdot\mathfrak{n}_{\ell}[V]\cdot\mathcal{N}_N^{(\ell')}[\mathcal{K}_{\kappa}[\phi]]\cdot N^{6\alpha-1}\,(\ln N)^{\ell''} \tag{3.105}$$

for a constant $C>0$ that does not dependent on ϕ nor on the potential V, and the integers:

$$\ell = \max(3m_3+5, 8m_2+18), \qquad \ell' = \max(4m_3+9, 14m_2+37), \qquad \ell'' = \max(14m_2+17, 6m_3+16)$$

given in terms of the sequence $(m_n)_n$ introduced in (3.86).

Proof — The strategy is to exploit the Schwinger-Dyson equation and get rid of expectation values of functions integrated against the random measure $L_N^{(\lambda)}$. This can be done by replacing them approximately by integration against a deterministic measure of a transformed function, up to corrections that we can estimate.

Let ϕ be a sufficiently regular function of one variable. Since the signed measure $\mathcal{L}_N^{(\lambda)}$ has zero mass, it follows that $\langle\phi\rangle_{\mathcal{L}_N^{(\lambda)}} = \langle\widetilde{X}_N[\phi]\rangle_{\mathcal{L}_N^{(\lambda)}}$. We can use the Schwinger Dyson equation at level 1 (3.56) for the function $\widetilde{X}_N[\phi]$, and apply the sharp bounds of Proposition 3.17 to derive:

$$\Big|\langle\phi\rangle_{\mathcal{L}_N^{(\lambda)}} - \frac{1-\beta}{N^{1+\alpha}}\Big\langle\partial_1\mathcal{U}_N^{-1}[\widetilde{X}_N[\phi]]\Big\rangle_{\mu_{\mathrm{eq}}^{(N)}}\Big| \;\leq\; C\cdot\mathfrak{n}_{2m_2+2}[V]\cdot\mathcal{N}_N^{(2m_2+3)}[\mathcal{K}_{\kappa}[\phi]]\cdot N^{3\alpha-2}(\ln N)^{2m_2+5}\;. \tag{3.106}$$

Above, we have stressed explicitly the composition of the operator $\mathcal{U}_N^{-1}$ with $\widetilde{X}_N$. This bound ensures that

$$\Big|\frac{1-\beta}{N^{1+\alpha}}\langle\partial_1\widetilde{\mathcal{U}}_N^{-1}[\phi]\rangle_{\mathcal{L}_N^{(\lambda)}} - \frac{(1-\beta)^2}{N^{2(1+\alpha)}}\Big\langle\partial_1\widetilde{\mathcal{U}}_N^{-1}[\partial_1\widetilde{\mathcal{U}}_N^{-1}[\phi]]\Big\rangle_{\mu_{\mathrm{eq}}^{(N)}}\Big| \;\leq\; C'\cdot\mathfrak{n}_{4m_2+7}[V]\cdot\mathcal{N}_N^{(4m_2+9)}[\mathcal{K}_{\kappa}[\phi]]\cdot N^{4\alpha-3}(\ln N)^{6m_2+14}\;. \tag{3.107}$$

where we remind that $\widetilde{\mathcal{U}}_N^{-1} = \mathcal{U}_N^{-1}\circ\widetilde{X}_N$. Equation (3.107) can be used for a substitution of the term proportional to $(1-\beta)$ in the Schwinger-Dyson equation at level 1 (3.56), and we get:

$$\begin{aligned}\Big|\langle\phi\rangle_{\mathcal{L}_N^{(\lambda)}} - \frac{1-\beta}{N^{1+\alpha}}\langle\partial_1\widetilde{\mathcal{U}}_N^{-1}[\phi]\rangle_{\mu_{\mathrm{eq}}^{(N)}} - \frac{(1-\beta)^2}{N^{2(1+\alpha)}}\Big\langle\partial_1\widetilde{\mathcal{U}}_N^{-1}[\partial_1\widetilde{\mathcal{U}}_N^{-1}[\phi]]\Big\rangle_{\mu_{\mathrm{eq}}^{(N)}} \\ - \frac{1}{2}\langle\mathcal{D}_N\circ\widetilde{\mathcal{U}}_N^{-1}[\phi]\rangle_{\bigotimes^2\mathcal{L}_N^{(\lambda)}}\Big| \;\leq\; C'\cdot\mathfrak{n}_{4m_2+7}[V]\cdot\mathcal{N}_N^{(4m_2+9)}[\mathcal{K}_{\kappa}[\phi]]\cdot N^{4\alpha-3}(\ln N)^{6m_2+14}\;.\end{aligned} \tag{3.108}$$

In order to gain a better control on the term involving $\mathcal{D}_N$ – which is a two-point correlator – we need to study the Schwinger-Dyson equation at level $n = 2$ (3.58). Given a sufficiently regular function ψ_2 in two variables, using the sharp bounds of Proposition 3.17, we find:

$$\Big|\langle\psi_2\rangle_{\otimes^2\mathcal{L}_N^{(\lambda)}} \;-\; \frac{1}{N^{2+\alpha}}\Big\langle\Xi^{(2)}\big[\partial_2\widetilde{\mathcal{U}}_{N;1}^{-1}[\psi_2]\big]\Big\rangle_{\mu_{\mathrm{eq}}^{(N)}} \;-\; \frac{1-\beta}{N^{1+\alpha}}\Big\langle\big(\partial_1\widetilde{\mathcal{U}}_{N;1}^{-1}[\psi_2]\big)\Big\rangle_{\mu_{\mathrm{eq}}^{(N)}\otimes\mathcal{L}_N^{(\lambda)}}\Big| \\ \leq\; C\cdot\mathfrak{n}_{2m_3+2}[V]\cdot\mathcal{N}_N^{(2m_3+3)}[\mathcal{K}_\kappa[\psi_2]]\cdot N^{4\alpha-3}(\ln N)^{2m_3+5}\;. \tag{3.109}$$

We apply this estimate to the particular choice:

$$\psi_2(\xi_1,\xi_2) \;=\; \mathcal{D}_N[\widetilde{\mathcal{U}}_N^{-1}[\phi]](\xi_1,\xi_2)\,. \tag{3.110}$$

Thanks to the bound (3.77) on the norm of $\mathcal{D}_N\circ\mathcal{U}_N^{-1}$ and the sub-multiplicativity (3.73) of the $\mathfrak{n}_\ell[V]$'s, we deduce:

$$\Big|\langle\psi_2\rangle_{\otimes^2\mathcal{L}_N^{(\lambda)}} \;-\; \frac{1}{N^{2+\alpha}}\Big\langle\Xi^{(2)}\big[\partial_2\widetilde{\mathcal{U}}_{N;1}^{-1}[\psi_2]\big]\Big\rangle_{\mu_{\mathrm{eq}}^{(N)}} \;-\; \frac{1-\beta}{N^{1+\alpha}}\Big\langle\big(\partial_1\widetilde{\mathcal{U}}_{N;1}^{-1}[\psi_2]\big)\Big\rangle_{\mu_{\mathrm{eq}}^{(N)}\otimes\mathcal{L}_N^{(\lambda)}}\Big| \\ \leq\; C\cdot\mathfrak{n}_{4m_3+7}[V]\cdot\mathcal{N}_N^{(4m_3+9)}[\mathcal{K}_\kappa[\phi]]\cdot N^{5\alpha-3}\,(\ln N)^{6m_3+16}\,. \tag{3.111}$$

This can be used for a substitution of $\langle\psi_2\rangle = \langle\mathcal{D}_N\circ\mathcal{U}_N^{-1}\rangle$ in the left-hand side of (3.108). Before performing this substitution, we still need to control the term in (3.111) which is proportional to $(1-\beta)$. This is a one-point correlator for the function:

$$\psi_1(\xi) \;=\; \frac{1-\beta}{N^{1+\alpha}}\int\partial_\eta\widetilde{\mathcal{U}}_{N;1}^{-1}[\psi_2(*,\xi)](\eta)\,\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(\eta)\,. \tag{3.112}$$

Applying the one-point estimate (3.106) to the function ψ_1, along with the bounds (3.75)-(3.76) for the norms of $\mathcal{U}_N^{-1}$ and $\mathcal{D}_N$, we find:

$$\Big|\langle\psi_1\rangle_{\mathcal{L}_N^{(\lambda)}} \;-\; \frac{1-\beta}{N^{1+\alpha}}\,\Big\langle\partial_1\widetilde{\mathcal{U}}_N^{-1}[\psi_1]\Big\rangle_{\mu_{\mathrm{eq}}^{(N)}}\Big| \;\leq\; C\cdot\mathfrak{n}_{8m_2+18}[V]\cdot\mathcal{N}_N^{(8m_2+21)}[\mathcal{K}_\kappa[\phi]]\cdot N^{5\alpha-3}(\ln N)^{14m_2+37}\,. \tag{3.113}$$

This leads to:

$$\Big|\langle\psi_2\rangle_{\otimes^2\mathcal{L}_N^{(\lambda)}} \;-\; \frac{1}{N^{2+\alpha}}\Big\langle\Xi^{(2)}\circ\partial_2\widetilde{\mathcal{U}}_{N;1}^{-1}[\psi_2]\Big\rangle_{\mu_{\mathrm{eq}}^{(N)}} \;-\; \frac{(1-\beta)^2}{N^{2(1+\alpha)}}\Big\langle\partial_1\widetilde{\mathcal{U}}_{N;1}^{-1}\partial_2[\widetilde{\mathcal{U}}_{N;2}^{-1}[\psi_2]]\Big\rangle_{\mu_{\mathrm{eq}}^{(N)}\otimes\mu_{\mathrm{eq}}^{(N)}}\Big| \\ \leq\; C\cdot\mathfrak{n}_{8m_2+18}[V]\cdot\mathcal{N}_N^{(8m_3+21)}[\mathcal{K}_\kappa[\phi]]\cdot N^{5\alpha-3}(\ln N)^{14m_2+37}\;. \tag{3.114}$$

The result follows by substituting this inequality in (3.108). ■

3.4 The large-N asymptotic expansion of $\ln Z_N[V]$ up to $\mathrm{o}(1)$

We can use the large-N analysis of the Schwinger-Dyson equations to establish the existence of an asymptotic expansion up to $\mathrm{o}(1)$ of $\ln Z_N[V]$. The coefficients in this asymptotic expansion are single and double integrals whose integrand depends on N. We will work out the large-N asymptotic expansion of these coefficients in Sections 7-9. Prior to writing down this large-N asymptotic expansion, we need to introduce several single and double integrals that will enter in the description of the result. We also remind the notation $\widetilde{\mathcal{W}}_N = \mathcal{W}_N\circ\widetilde{\mathcal{X}}_N$ where $\mathcal{W}_N$ is the inverse of $\mathcal{S}_N$ (*cf.* (2.42)), studied in Section 5.4. Given H, G sufficiently regular on $[a_N\,;b_N]$, we define the one-dimensional integrals:

$$\mathfrak{I}_{\mathrm{s}}[H,G] \;=\; \int\limits_{a_N}^{b_N} H(\xi)\cdot\mathcal{W}_N[G](\xi)\cdot\mathrm{d}\xi\;, \qquad \mathfrak{I}_{\mathrm{s};\beta}^{(1)}[H,G] \;=\; \int\limits_{a_N}^{b_N}\mathcal{W}_N[G'](\xi)\,\partial_\xi\bigg\{\frac{\widetilde{\mathcal{W}}_N[H](\xi)}{\mathcal{W}_N[G'](\xi)}\bigg\}\,\mathrm{d}\xi \tag{3.115}$$

and

$$\mathfrak{I}^{(2)}_{\mathrm{s};\beta}[H,G] \;=\; \int_{a_N}^{b_N} \mathcal{W}_N[G'](\xi)\,\partial_\xi\bigg\{\frac{\widetilde{\mathcal{W}}_N\Big[\partial_1\Big(\frac{\widetilde{\mathcal{W}}_N[H]}{\mathcal{W}_N[G']}\Big)\Big](\xi)}{\mathcal{W}_N[G'](\xi)}\bigg\}\,\mathrm{d}\xi\;. \tag{3.116}$$

We also define the two-dimensional integrals:

$$\mathfrak{I}_{\mathrm{d}}[H,G] \;=\; \int_{a_N}^{b_N} \widetilde{\mathcal{W}}_N\bigg[\partial_\xi\Big\{S\left(N^\alpha(\xi-*)\right)\Big(\frac{\widetilde{\mathcal{W}}_N[H](\xi)}{\mathcal{W}_N[G'](\xi)} \;-\; \frac{\widetilde{\mathcal{W}}_N[H](*)}{\mathcal{W}_N[G'](*)}\Big)\Big\}\bigg](\xi)\,\mathrm{d}\xi \tag{3.117}$$

and

$$\mathfrak{I}_{\mathrm{d};\beta}[H,G] \;=\; \frac{1}{2}\int_{a_N}^{b_N} \mathrm{d}\xi\mathrm{d}\eta\,\mathcal{W}_N[G'](\xi)\cdot\mathcal{W}_N[G'](\eta)$$
$$\times\partial_\xi\partial_\eta\Bigg(\frac{1}{\mathcal{W}_N[G'](\xi)\cdot\mathcal{W}_N[G'](\eta)}\widetilde{\mathcal{W}}_{N;1}\circ\widetilde{\mathcal{W}}_{N;2}\bigg[S\left(N^\alpha(*_1-*_2)\right)\cdot\Big\{\frac{\widetilde{\mathcal{W}}_N[H](*_1)}{\mathcal{W}_N[G'](*_1)} \;-\; \frac{\widetilde{\mathcal{W}}_N[H](*_2)}{\mathcal{W}_N[G'](*_2)}\Big\}\bigg](\xi,\eta)\Bigg)\;. \tag{3.118}$$

Above, $*$ refers to the variables on which the operators act, $*_1$, *viz.* $*_2$, to the first, resp. second, running variable on which the product of operators $\mathcal{W}_{N;1}\cdot\mathcal{W}_{N;2}$ acts. The subscript β reminds that the terms concerned are absent in the case $\beta=1$.

Proposition 3.20 *Let $V_{G;N}(\lambda) = g_N\lambda^2 + t_N\lambda$ be the unique Gaussian potential associated with an equilibrium measure supported on $[a_N\,;b_N]$ as given in Lemma D.1 and assume that $0<\alpha<1/6$. Then there exists $\ell\in\mathbb{N}$ such that one has the large-N asymptotic expansion*

$$\ln\Big(\frac{Z_N[V]}{Z_N[V_{G;N}]}\Big) \;=\; -N^{2+\alpha}\int_0^1 \mathfrak{I}_{\mathrm{s}}[\partial_t V_t, V_t']\cdot\mathrm{d}t \;-\; N(1-\beta)\int_0^1 \mathfrak{I}^{(1)}_{\mathrm{s};\beta}[\partial_t V_t, V_t]\cdot\mathrm{d}t \;-\; \frac{1}{2}\int_0^1 \mathfrak{I}_{\mathrm{d}}[\partial_t V_t, V_t]\cdot\mathrm{d}t$$
$$-\;\frac{(1-\beta)^2}{N^\alpha}\int_0^1\Big\{\mathfrak{I}^{(2)}_{\mathrm{s};\beta}[\partial_t V_t, V_t] \;+\; \mathfrak{I}_{\mathrm{d};\beta}[\partial_t V_t, V_t]\Big\}\cdot\mathrm{d}t \;+\; \mathrm{O}\big(N^{6\alpha-1}(\ln N)^{2\ell}\big)\;. \tag{3.119}$$

Proof — The result follows from (2.50). Indeed, the remarks above (2.54) allow to identify the equilibrium measures $\mu^{(N)}_{\mathrm{eq};V_t} = (1-t)\mu^{(N)}_{\mathrm{eq};V_{G;N}} + t\mu^{(N)}_{\mathrm{eq};V}$ for all $t\in[0,1]$. One can then use Proposition 3.19 to expand $\langle\partial_t V_t\rangle^{V_t}_{L_N^{(\lambda)}}$, along with the representation for $\mathcal{U}_N^{-1}$ on the support of the equilibrium measure which reads

$$\widetilde{\mathcal{U}}_N^{-1}[H](\xi) \;=\; \frac{\widetilde{\mathcal{W}}_N[H](\xi)}{\mathcal{W}_N[V'](\xi)}\;. \tag{3.120}$$

Taking these data into account, it solely remains to write down explicitly the one and two-dimensional integrals arising in Proposition 3.19. ■

Note that the factors $\mathfrak{I}^{(2)}_{\mathrm{s};\beta}[\partial_t V_t, V_t]$ and $\mathfrak{I}_{\mathrm{d};\beta}[\partial_t V_t, V_t]$ are preceded by the negative power of $N^{-\alpha}$. Still, it does not mean that these do not contribute to the leading contribution, *i.e.* up to o(1), to the asymptotics of the partition function. Indeed, the presence of derivatives in their associated integrands generates additional powers of N^α.

4 The Riemann–Hilbert approach to the inversion of $\mathcal{S}_N$

In the present section we focus on a class of singular integral equation driven by a one parameter γ-regularisation of the operator $\mathcal{S}_N$. More precisely, we introduce the singular integral operator $\mathcal{S}_{N;\gamma}$

$$\mathcal{S}_{N;\gamma}[\phi](\xi) \;=\; \fint_{a_N}^{b_N} S_\gamma(N^\alpha(\xi-\eta))\phi(\eta)\cdot \mathrm{d}\eta \qquad \text{where} \qquad \left\{ \begin{array}{ccc} S_\gamma(\xi) & = & S(\xi)\cdot \mathbf{1}_{[-\gamma\overline{x}_N\,;\gamma\overline{x}_N]} \\ \overline{x}_N & = & N^\alpha\cdot(b_N-a_N) \end{array} \right. \quad . \tag{4.1}$$

This operator is a regularisation of the operator $\mathcal{S}_N$ in the sense that, formally, $\mathcal{S}_{N;\infty} = \mathcal{S}_N$. This regularisation enables to set a well defined associated Riemann–Hilbert problem, and is such that, once all calculations have been done and the inverse of $\mathcal{S}_{N;\gamma}$ constructed, we can take send $\gamma \to +\infty$ at the level of the obtained answer. It is then not a problem to check that this limiting operator does indeed provides one with the inverse of $\mathcal{S}_N$.

We start this analysis by, first, recasting the singular integral equation into a form where the variables have been re-scaled. Then, we put the problem of inverting the re-scaled operator associated with $\mathcal{S}_{N;\gamma}$ with a vector valued Riemann-Hilbert problem. The resolution of this vector problem demands the resolution of a 2×2 matrix Riemann–Hilbert problem for an auxiliary matrix χ. We construct the solution to this problem, for N-large enough, in § 4.4 and then exhibit some of the overall properties of the solution χ in § 4.5. We shall build on these results so as to invert $\mathcal{S}_{N;\gamma}$ and then $\mathcal{S}_N$ in subsequent sections.

4.1 A re-parametrisation of the problem: a vector Riemann–Hilbert problem

In the handlings that will follow, it will appear more convenient to consider a properly rescaled problem. Namely define

$$\varphi(\xi) \;=\; \phi((\xi+N^\alpha a_N)N^{-\alpha}) \qquad \text{and} \qquad h(\xi) \;=\; \frac{N^\alpha}{2\mathrm{i}\pi\beta}H((\xi+N^\alpha a_N)N^{-\alpha})\;. \tag{4.2}$$

It is then clear that solutions to $\mathcal{S}_{N;\gamma}[\phi](\xi) = H(\xi)$ are in a one-to-one correspondence with those of

$$\mathscr{S}_{N;\gamma}[\varphi](\xi) \;=\; \fint_{0}^{\overline{x}_N} S_\gamma(\xi-\eta)\varphi(\eta)\cdot\frac{\mathrm{d}\eta}{2\mathrm{i}\pi\beta} \;=\; h(\xi)\;\;. \tag{4.3}$$

For any N and $\gamma \geq 0$, the operator $\mathscr{S}_{N;\gamma}$ is continuous as an operator

$$\mathscr{S}_{N;\gamma} \;:\; H_s([0\,;\overline{x}_N]) \quad\longrightarrow\quad H_s([-\gamma\overline{x}_N\,;\gamma\overline{x}_N]) \;\subseteq\; H_s(\mathbb{R})\;. \tag{4.4}$$

Indeed, this continuity follows readily from the boundedness of the Fourier transform $\mathcal{F}[S_\gamma]$ of the operator's integral kernel, *c.f.* Lemma 4.2 to come.

First, we shall start by focusing on spaces with a negative index $s<0$ and going to construct a class of its inverses

$$\mathscr{S}_{N;\gamma}^{-1} \;:\; H_s([-\gamma\overline{x}_N\,;\gamma\overline{x}_N]) \quad\longrightarrow\quad H_s([0\,;\overline{x}_N])\;. \tag{4.5}$$

What we mean here is that, *per se*, the operator is non-invertible in that, as will be inferred from our analysis, for $-k<s<-(k-1)$

$$\dim\ker\mathscr{S}_{N;\gamma} \;=\; k\;. \tag{4.6}$$

In fact, the analysis that will follow, provides one with a thorough characterisation of its kernel. Furthermore, when restricting the operator $\mathscr{S}_{N;\gamma}$ to more regular spaces like $H_s([0;\overline{x}_N])$ with $s > 0$, we get that the image $\mathscr{S}_{N;\gamma}[H_s([0;\overline{x}_N])]$ is a closed, explicitly characterisable subspace of $H_s([-\gamma\overline{x}_N;(\gamma+1)\overline{x}_N])$, and that the operator becomes continuously invertible on it.

In the following, we shall invert the operator $\mathscr{S}_{N;\gamma}$ by means of the resolution of an auxiliary 2×2 Riemann–Hilbert problem and then by implementing a Wiener–Hopf factorisation. The analysis is inspired by the paper of Novokshenov [73] where a correspondence has been built between singular integral equations on a finite segment subordinate to integral kernels depending on the difference on the one hand and Riemann–Hilbert problems on the other one. The large parameter analysis is, however, new.

In fact the very setting of the Riemann–Hilbert problem-based analysis enables one to naturally construct the pseudo-inverse of $\mathscr{S}_{N;\gamma}$ - *i.e.* modulo elements of $\ker[\mathscr{S}_{N;\gamma}]$ – when the operator is understood to act on H_s spaces with *negative* index $s < 0$. The inversion of $\mathscr{S}_{N;\gamma}$ understood as an operator on H_s spaces with *positive* index $s \geq 0$ goes, however, beyond, the "crude" construction issuing from the Riemann–Hilbert problem-based analysis. It is, in particular, based on an explicit characterisation, through linear constraints, of the image space $\mathscr{S}_{N;\gamma}[H_s([0;\overline{x}_N]])]$, $s \geq 0$. For $0 < s < 1/2$, which is the case of interest for us, we show that $\mathscr{S}_{N;\gamma}[H_s([0;\overline{x}_N])]$ coincides with $\mathfrak{X}_s([-\gamma\overline{x}_N;(\gamma+1)\overline{x}_N])$.

Lemma 4.1 *Let $h \in H_s([0;\overline{x}_N])$, $s < 0$. For any solution $\varphi \in H_s([0;\overline{x}_N])$ of* (4.3)*, there exists a two-dimensional vector function $\Phi \in \mathcal{O}(\mathbb{C}\setminus\mathbb{R})$ such that $\varphi = \mathcal{F}^{-1}[(\Phi_1)_+]$ and Φ is a solution to the boundary value problem:*

- $(\Phi_a)_\pm \in \mathcal{F}[H_s(\mathbb{R}^\pm)]$ *for $a \in \{1,2\}$, and there exists $C > 0$ such that:*

$$\forall \mu > 0, \qquad \forall a \in \{1,2\}, \qquad \int_{\mathbb{R}} \left|\Phi_a(\lambda \pm \mathrm{i}\mu)\right|^2 \cdot (1+|\lambda|+|\mu|)^{2s} \cdot \mathrm{d}\lambda \;<\; C\,. \tag{4.7}$$

- *We have the jump equation for $\Phi_+(\lambda) = G_\chi(\lambda)\cdot\Phi_-(\lambda) + \boldsymbol{H}(\lambda)$ for $\lambda \in \mathbb{R}$, with:*

$$G_\chi(\lambda) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\lambda\overline{x}_N} & 0 \\ \frac{1}{2\mathrm{i}\pi\beta}\cdot\mathcal{F}[S_\gamma](\lambda) & -\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N} \end{pmatrix} \qquad \textit{and} \qquad \boldsymbol{H}(\lambda) = \begin{pmatrix} 0 \\ -\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N}\mathcal{F}[h_{\mathrm{e}}](\lambda) \end{pmatrix}. \tag{4.8}$$

Conversely, for any solution $\Phi \in \mathcal{O}(\mathbb{C}\setminus\mathbb{R})$ of the above boundary value problem, $\varphi = \mathcal{F}^{-1}[\,(\Phi_1)_+]$ is a solution of (4.3)*.*

We do remind that $\pm$ denotes the upper/lower boundary values on $\mathbb{R}$ with the latter being oriented from $-\infty$ to $+\infty$; h_{e} denotes any extension of h to $H_s(\mathbb{R})$; $\mathcal{F}[S_\gamma](\lambda)$ refers to the Fourier transform of the principal value distribution induced by S_γ:

$$\mathcal{F}[S_\gamma](\lambda) = \fint_{-\gamma\overline{x}_N}^{\gamma\overline{x}_N} S(\xi)\,\mathrm{e}^{\mathrm{i}\lambda\xi}\mathrm{d}\xi\,. \tag{4.9}$$

Proof — Assume that one is given a solution φ in $H_s([0;\overline{x}_N])$ to (4.3). Then, let ψ_L, ψ_R be two functions such that

$$\mathrm{supp}(\psi_R) = [\overline{x}_N;+\infty[\qquad , \qquad \mathrm{supp}(\psi_L) =]-\infty;0] \tag{4.10}$$

and

$$\fint_0^{\overline{x}_N} S_\gamma(\xi-\eta)\varphi(\eta)\cdot\frac{\mathrm{d}\eta}{2\mathrm{i}\pi\beta} \;-\; h_{\mathrm{e}}(\xi) \;=\; \psi_L(\xi) \;+\; \psi_R(\xi)\,. \tag{4.11}$$

Then, by going to the Fourier space, we get:

$$\frac{1}{2\mathrm{i}\pi\beta}\cdot\mathcal{F}[S_\gamma](\lambda)\cdot\mathcal{F}[\varphi](\lambda)\;-\;\mathcal{F}[h_{\mathrm{e}}](\lambda)\;=\;\mathcal{F}[\psi_1](\lambda)\;+\;\mathcal{F}[\psi_2](\lambda)\,. \tag{4.12}$$

By Lemma 4.2 that will be proved below, $\mathcal{F}[S_\gamma]\in L^\infty(\mathbb{R})$. Hence $\psi_R\in H_s(\mathbb{R}^+)$ whereas $\psi_L\in H_s(\mathbb{R}^-)$. Then, we introduce the vectors

$$\boldsymbol{F}_\uparrow(\lambda)\;=\;\begin{pmatrix}\mathcal{F}[\varphi](\lambda)\\ \mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N}\mathcal{F}[\psi_R](\lambda)\end{pmatrix}\qquad\text{and}\qquad\boldsymbol{F}_\downarrow(\lambda)\;=\;\begin{pmatrix}\mathcal{F}[\varphi_{\overline{x}_N}](\lambda)\\ \mathcal{F}[\psi_L](\lambda)\end{pmatrix} \tag{4.13}$$

where we agree upon $\varphi_{\overline{x}_N}(\xi)=\varphi(\xi+\overline{x}_N)$. Since $[\boldsymbol{F}_\uparrow]_a\in\mathcal{F}[H_s(\mathbb{R}^+)]$, resp. $[\boldsymbol{F}_\downarrow]_a\in\mathcal{F}[H_s(\mathbb{R}^-)]$, it is readily seen that

$$\widetilde{\boldsymbol{F}}_{\uparrow;a}(\lambda)\;=\;(1-\mathrm{i}\lambda)^s\cdot[\boldsymbol{F}_\uparrow]_a(\lambda)\qquad\text{resp.}\qquad\widetilde{\boldsymbol{F}}_{\downarrow;a}(\lambda)\;=\;(1+\mathrm{i}\lambda)^s\cdot[\boldsymbol{F}_\downarrow]_a(\lambda) \tag{4.14}$$

defines a holomorphic function on $\mathbb{H}^+$, resp. $\mathbb{H}^-$, with $L^2(\mathbb{R})$ +, resp. −, boundary values on $\mathbb{R}$. The Paley-Wiener Theorem A.4 then shows the existence of $C>0$ such that:

$$\forall\mu>0,\quad\forall a\in\{1,2\},\qquad\int_{\mathbb{R}}\big|[\boldsymbol{F}_{\uparrow/\downarrow}]_a(\lambda\pm\mathrm{i}\mu)\big|^2\cdot(1+|\lambda|+|\mu|)^{2s}\cdot\mathrm{d}\lambda\;<\;C\,. \tag{4.15}$$

In other words the function:

$$\Phi\;=\;\boldsymbol{F}_\uparrow\cdot\mathbf{1}_{\mathbb{H}_+}\;+\;\boldsymbol{F}_\downarrow\cdot\mathbf{1}_{\mathbb{H}_-} \tag{4.16}$$

solves the vector valued Riemann–Hilbert problem.

Reciprocally, suppose that one is given a solution Φ to the vector-valued Riemann–Hilbert problem in question. Then, set $\varphi=\mathcal{F}^{-1}[(\Phi_1)_+]$. We clearly have $\varphi\in H_s(\mathbb{R}^+)$, but we now show that the support of φ is actually included in $[0,\overline{x}_N]$. Let $(\cdot,\cdot)$ be the canonical scalar product on $L^2(\mathbb{R},\mathbb{C})$. If ρ_R is a $\mathscr{C}^\infty$ function with compact support included in $]\overline{x}_N,+\infty[$, we have:

$$(\rho_R,\varphi)=(\mathcal{F}[\rho_R]\,,\,\mathcal{F}[\varphi])=(e^{-\mathrm{i}\overline{x}_N*}\mathcal{F}[\rho_R](1-\mathrm{i}*)^{-s}\,,\,(1+\mathrm{i}*)^s(\Phi_1)_-)\,, \tag{4.17}$$

where $*$ denotes the running variable. But this is zero since $(1+\mathrm{i}*)^s(\Phi_1)_-\in\mathcal{F}[L^2(\mathbb{R}^-)]$, whereas, by the Paley-Wiener Theorem A.4, $\mathrm{e}^{-\mathrm{i}\overline{x}_N*}\mathcal{F}[\rho_R](1-\mathrm{i}*)^{-s}\in\mathcal{F}[L^2(\mathbb{R}^-)]$. Finally, the fact that $\varphi\in H_s([0\,;\overline{x}_N])$ satisfies (4.3) follows from taking the Fourier transform of the second line of the jump equation (4.8) for Φ. ■

For further handlings, it is useful to characterise the distributional Fourier transform $\mathcal{F}[S_\gamma]$ slightly better.

Lemma 4.2 *The distributional Fourier transform $\mathcal{F}[S_\gamma](\lambda)$ defined by* (4.9) *admits the representation*

$$\frac{\mathcal{F}[S_\gamma](\lambda)}{2\mathrm{i}\pi\beta}\;=\;R(\lambda)\;+\;\big(\mathrm{e}^{\mathrm{i}\lambda\gamma\overline{x}_N}\;+\;\mathrm{e}^{-\mathrm{i}\lambda\gamma\overline{x}_N}\big)\frac{\kappa_N}{\lambda}\;+\;r_N(\lambda)\qquad\textit{where}\qquad\kappa_N=-\sum_{p=1}^{2}\frac{\omega_p}{2}\operatorname{cotanh}[\pi\omega_p\gamma\overline{x}_N] \tag{4.18}$$

$$R(\lambda)\;=\;\frac{\sinh\Big[\dfrac{\lambda(\omega_1+\omega_2)}{2\omega_1\omega_2}\Big]}{2\sinh\Big[\dfrac{\lambda}{2\omega_1}\Big]\sinh\Big[\dfrac{\lambda}{2\omega_2}\Big]}\,, \tag{4.19}$$

and

$$r_N(\lambda) \;=\; \sum_{p=1}^{2} \frac{(\pi\omega_p)^2}{\mathrm{i}\lambda(1-\mathrm{e}^{-\lambda/\omega_p})} \int\limits_{0}^{\mathrm{i}/\omega_p} \Big\{ \frac{\mathrm{e}^{-\mathrm{i}\lambda\gamma\overline{x}_N}}{\sinh^2[\pi\omega_p(\xi-\gamma\overline{x}_N)]} - \frac{\mathrm{e}^{\mathrm{i}\lambda\gamma\overline{x}_N}}{\sinh^2[\pi\omega_p(\xi+\gamma\overline{x}_N)]} \Big\} \cdot \frac{\mathrm{e}^{\mathrm{i}\lambda\xi}\,\mathrm{d}\xi}{2\mathrm{i}\pi} \;. \tag{4.20}$$

Besides, for $\mathrm{Im}\,\lambda = \epsilon > 0$ *small enough, there exists* $C_\epsilon > 0$ *independent of* N *such that, uniformly in* $\mathrm{Re}\,\lambda \in \mathbb{R}$*:*

$$|r_N(\lambda)| \;\leq\; C_\epsilon\, |\lambda|^{-2} \cdot \exp\{ -\gamma\overline{x}_N(2\pi \min[\omega_1,\omega_2] \;-\; \epsilon)\} \;. \tag{4.21}$$

Proof — One has that

$$\begin{aligned} \frac{\mathcal{F}[S_\gamma](\lambda)}{2\mathrm{i}\pi\beta} \quad &= \quad \frac{1}{2}\sum_{p=1}^{2} \lim_{t\to 0^+} \sum_{\epsilon\in\{\pm 1\}} \int\limits_{-\gamma\overline{x}_N}^{\overline{x}_N} \pi\omega_p \; \mathrm{cotanh}[\pi\omega_p(\xi+\mathrm{i}\epsilon t)] \cdot \frac{\mathrm{e}^{\mathrm{i}\lambda\xi}\,\mathrm{d}\xi}{2\mathrm{i}\pi} \\ &= \quad \frac{1}{2} \sum_{\substack{p\in\{1,2\} \\ \epsilon\in\{\pm 1\}}} \frac{\pi\omega_p}{1-e^{-\lambda/\omega_p}} \lim_{t\to 0^+} \int\limits_{\Gamma_p} \mathrm{cotanh}\,[\pi\omega_p(\xi+\mathrm{i}\epsilon t)] \cdot \frac{\mathrm{e}^{\mathrm{i}\lambda\xi}\,\mathrm{d}\xi}{2\mathrm{i}\pi} \;, \end{aligned} \tag{4.22}$$

where $\Gamma_p \;=\; [-\gamma\overline{x}_N\,;\gamma\overline{x}_N] \cup [\gamma\overline{x}_N + \mathrm{i}/\omega_p\,; -\gamma\overline{x}_N + \mathrm{i}/\omega_p]$, where the second interval is endowed with an opposite orientation. It then remains to add the counter-term:

$$\begin{aligned} r_N(\lambda) = \sum_{p=1}^{2} \frac{\pi\omega_p}{1-\mathrm{e}^{-\lambda/\omega_p}} \int\limits_{0}^{\frac{\mathrm{i}}{\omega_p}} \Big\{ & \mathrm{e}^{-\mathrm{i}\lambda\gamma\overline{x}_N}\Big(\mathrm{cotanh}[\pi\omega_p\gamma\overline{x}_N] + \mathrm{cotanh}[\pi\omega_p(\xi-\gamma\overline{x}_N)]\Big) \\ & + \; \mathrm{e}^{\mathrm{i}\lambda\gamma\overline{x}_N}\Big(\mathrm{cotanh}[\pi\omega_p\gamma\overline{x}_N] - \mathrm{cotanh}[\pi\omega_p(\xi+\gamma\overline{x}_N)]\Big)\Big\} \cdot \frac{\mathrm{e}^{\mathrm{i}\lambda\xi}\,\mathrm{d}\xi}{2\mathrm{i}\pi} \;. \end{aligned} \tag{4.23}$$

to form a closed contour $\widetilde{\Gamma}_p$. Upon integrating by parts, we find the expression (4.20) for $r_N(\lambda)$. Then, we pick up the residues surrounded by $\widetilde{\Gamma}_p$, and we also write aside the term behaving as $O(1/\lambda)$ when $\lambda \to \infty$. This leads to the appearence of κ_N in (4.18). The bounds on the line $|\mathrm{Im}\,\lambda| = \epsilon > 0$, with ϵ small enough are then obtained through straightforward majorations. ■

The resolution of the vector Riemann–Hilbert problem for Φ can be done with the help of a matrix Wiener-Hopf factorization. In order to apply this method, we first need to obtain a $\pm$-factorization of the matrix G_χ given by (4.8). This leads to an 2×2 matrix Riemann–Hilbert problem that we formulate and solve, for N sufficiently large, in the next subsections.

4.2 A scalar Riemann–Hilbert problem

In order to state the main result regarding to the auxiliary 2×2 matrix Riemann–Hilbert problem, we first need to introduce some objects. To start with, we introduce a factorization of $R(\lambda)$ that separates contributions from zeroes and poles between the lower and upper half-planes $\lambda \in \mathbb{H}^\pm$. In other words, we consider the solution υ to the following scalar Riemann–Hilbert problem, depending on $\epsilon > 0$ small enough and given once for all:

- $\upsilon \in \mathcal{O}(\mathbb{C} \setminus \{\mathbb{R}+\mathrm{i}\epsilon\})$ and has continuous $\pm$-boundary values on $\mathbb{R}+\mathrm{i}\epsilon$;
- $\upsilon(\lambda) = \begin{cases} (-\mathrm{i}\lambda)^{\frac{1}{2}} \cdot (1 \;+\; \mathrm{O}(\lambda^{-1})) & \text{if } \mathrm{Im}\,\lambda > \epsilon \\ -\mathrm{i}(\mathrm{i}\lambda)^{\frac{1}{2}} \cdot (1 \;+\; \mathrm{O}(\lambda^{-1})) & \text{if } \mathrm{Im}\,\lambda < \epsilon \end{cases}$ when $\lambda \to \infty$ non-tangentially to $\mathbb{R}+\mathrm{i}\epsilon$;

- $\upsilon_+(\lambda)\cdot R(\lambda) \;=\; \upsilon_-(\lambda)$ for $\lambda\in\mathbb{R}+\mathrm{i}\epsilon$.

This problem admits a unique solution given by

$$\upsilon(\lambda) \;=\; \begin{cases} R_{\uparrow}^{-1}(\lambda) & \text{if } \operatorname{Im}\lambda > \epsilon \\ R_{\downarrow}(\lambda) & \text{if } \operatorname{Im}\lambda < \epsilon \end{cases} \tag{4.24}$$

where

$$R_{\uparrow}(\lambda) = \frac{\mathrm{i}}{\lambda}\cdot\sqrt{\omega_1+\omega_2}\cdot\Big(\frac{\omega_2}{\omega_1+\omega_2}\Big)^{\frac{\mathrm{i}\lambda}{2\pi\omega_1}}\cdot\Big(\frac{\omega_1}{\omega_1+\omega_2}\Big)^{\frac{\mathrm{i}\lambda}{2\pi\omega_2}}\cdot\frac{\prod\limits_{p=1}^{2}\Gamma\Big(1-\dfrac{\mathrm{i}\lambda}{2\pi\omega_p}\Big)}{\Gamma\Big(1-\dfrac{\mathrm{i}\lambda(\omega_1+\omega_2)}{2\pi\omega_1\omega_2}\Big)} \tag{4.25}$$

and

$$R_{\downarrow}(\lambda) = \frac{\lambda}{2\pi\sqrt{\omega_1+\omega_2}}\cdot\Big(\frac{\omega_2}{\omega_1+\omega_2}\Big)^{-\frac{\mathrm{i}\lambda}{2\pi\omega_1}}\cdot\Big(\frac{\omega_1}{\omega_1+\omega_2}\Big)^{-\frac{\mathrm{i}\lambda}{2\pi\omega_2}}\cdot\frac{\prod\limits_{p=1}^{2}\Gamma\Big(\dfrac{\mathrm{i}\lambda}{2\pi\omega_p}\Big)}{\Gamma\Big(\dfrac{\mathrm{i}\lambda(\omega_1+\omega_2)}{2\pi\omega_1\omega_2}\Big)}\,. \tag{4.26}$$

Note that

$$R_{\downarrow}(0) \;=\; -\mathrm{i}\sqrt{\omega_1+\omega_2} \qquad\text{and}\qquad \Big(\lambda R_{\uparrow}(\lambda)\Big)_{|\lambda=0} \;=\; \mathrm{i}\sqrt{\omega_1+\omega_2}\,. \tag{4.27}$$

Also, $R_{\uparrow}$ and $R_{\downarrow}$ satisfy to the relations

$$R_{\uparrow}(-\lambda) \;=\; \lambda^{-1}\cdot R_{\downarrow}(\lambda) \qquad\text{and}\qquad \Big(R_{\uparrow}(\lambda^*)\Big)^* \;=\; \lambda^{-1}\cdot R_{\downarrow}(\lambda)\,. \tag{4.28}$$

Furthermore, $R_{\uparrow/\downarrow}$ exhibit the asymptotic behaviour

$$R_{\uparrow}(\lambda) \;=\; (-\mathrm{i}\lambda)^{-\frac{1}{2}}\cdot\Big(1\;+\;\mathrm{O}(\lambda^{-1})\Big) \qquad\text{for}\quad \lambda \underset{\lambda\in\mathbb{H}^+}{\longrightarrow}\infty \tag{4.29}$$

$$R_{\downarrow}(\lambda) \;=\; -\mathrm{i}(\mathrm{i}\lambda)^{\frac{1}{2}}\cdot\Big(1\;+\;\mathrm{O}(\lambda^{-1})\Big) \qquad\text{for}\quad \lambda \underset{\lambda\in\mathbb{H}^-}{\longrightarrow}\infty \tag{4.30}$$

as it should be. The notation $\uparrow$ and $\downarrow$ indicates the direction in the complex plane where $R_{\uparrow/\downarrow}$ have no pole nor zeroes.

Preliminary definitions

We need a few other definition before describing the solution to the factorisation problem for G_χ. Let:

$$\mathcal{R}_{\uparrow}(\lambda) \;=\; \begin{pmatrix} 0 & -1 \\ 1 & -R(\lambda)\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N} \end{pmatrix} \qquad\text{and}\qquad \mathcal{R}_{\downarrow}(\lambda) \;=\; \begin{pmatrix} -1 & R(\lambda)\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N} \\ 0 & 1 \end{pmatrix}, \tag{4.31}$$

as well as their "asymptotic" versions:

$$\mathcal{R}_{\uparrow}^{(\infty)} \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad\text{and}\qquad \mathcal{R}_{\downarrow}^{(\infty)} \;=\; \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.32}$$

We also need to introduce

$$M_{\uparrow}(\lambda) \;=\; \begin{pmatrix} 1 & 0 \\ -\dfrac{1-R^2(\lambda)}{\upsilon^2(\lambda)\cdot R(\lambda)}\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N} & 1 \end{pmatrix} \qquad \text{and} \qquad M_{\downarrow}(\lambda) \;=\; \begin{pmatrix} 1 & \upsilon^2(\lambda)\cdot\dfrac{1-R^2(\lambda)}{R(\lambda)}\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N} \\ 0 & 1 \end{pmatrix} , \tag{4.33}$$

where υ is given by (4.24), and:

$$P_R(\lambda) \;=\; I_2 \;+\; \frac{\theta_R}{\lambda}\Pi^{-1}(0)\sigma^{-}\Pi(0) \qquad \text{and} \qquad \left\{ \begin{array}{ccc} P_{L;\uparrow}(\lambda) & = & I_2 \;+\; \kappa_N\,\lambda^{-1}\,\mathrm{e}^{\mathrm{i}(\gamma-1)\lambda\overline{x}_N}\cdot\sigma^{-} \\ P_{L;\downarrow}(\lambda) & = & I_2 \;+\; \kappa_N\,\lambda^{-1}\,\mathrm{e}^{-\mathrm{i}(\gamma-1)\lambda\overline{x}_N}\cdot\sigma^{-} \end{array} \right. , \tag{4.34}$$

in which $\Pi(0)$ is a constant matrix that will coincide later with the value at 0 of the matrix function Π, *cf.* (4.49).

$$\theta_R \;=\; \frac{1}{\upsilon^2(0)}\,\frac{\kappa_N}{1+\kappa_N/(\omega_1+\omega_2)}\;. \tag{4.35}$$

4.3 The auxiliary 2×2 matrix Riemann–Hilbert problem for χ: formulation and main result

The factorisation problem for the jump matrix G_χ corresponds to solving the 2×2 matrix Riemann–Hilbert problem given below. This problem is solvable for N large enough.

Proposition 4.3 *There exists N_0 such that, for any $N \geq N_0$, the given below 2×2 Riemann–Hilbert problem has a unique solution. This solution is as given in Fig. 2*

- *the 2×2 matrix function $\chi \in \mathcal{O}(\mathbb{C}\setminus\mathbb{R})$ has continuous $\pm$-boundary values on $\mathbb{R}$;*
- $\chi(\lambda) = \left\{ \begin{array}{ll} P_{L;\uparrow}(\lambda)\cdot\begin{pmatrix} -\mathrm{sgn}(\mathrm{Re}\,\lambda)\cdot\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N} & 1 \\ -1 & 0 \end{pmatrix}\cdot(-\mathrm{i}\lambda)^{-\frac{\sigma_3}{2}}\cdot\Big(I_2 \;+\; \dfrac{\chi_1}{\lambda} \;+\; \mathrm{O}(\lambda^{-2})\Big) & \lambda\in\mathbb{H}^+ \\ P_{L;\downarrow}(\lambda)\cdot\begin{pmatrix} -1 & \mathrm{sgn}(\mathrm{Re}\,\lambda)\cdot\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N} \\ 0 & 1 \end{pmatrix}\cdot(\mathrm{i}\lambda)^{-\frac{\sigma_3}{2}}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\sigma_3}\cdot\Big(I_2 \;+\; \dfrac{\chi_1}{\lambda} \;+\; \mathrm{O}(\lambda^{-2})\Big) & \lambda\in\mathbb{H}^- \end{array} \right.$

  for some constant matrix χ_1, when $\lambda\to\infty$ non-tangentially to $\mathbb{R}$;
- $\chi_+(\lambda) \;=\; G_\chi(\lambda)\cdot\chi_-(\lambda)$ *for* $\lambda\in\mathbb{R}$.

Furthermore, the unique solution to the above Riemann–Hilbert problem satisfies $\det\chi(\lambda) = \mathrm{sgn}(\mathrm{Im}(\lambda))$ *for any* $\lambda\in\mathbb{C}\setminus\mathbb{R}$.

The existence of a solution χ will be established in § 4.4, by a set of transformations:

$$\chi \rightsquigarrow \Psi \rightsquigarrow \Pi \tag{4.36}$$

which maps the initial RHP for χ, to a RHP for Π whose jump matrices are uniformly close to the identity when N is large, and thus solvable by perturbative arguments [7]. The structure of χ in terms of the solution Π is summarized in Figure 2. The uniqueness of χ follows from standard arguments, see *e.g.* [30], that we now reproduce.

Proof — (of uniqueness)
Define, for $\lambda\in\mathbb{C}\setminus\mathbb{R}$,

$$d(\lambda) \;=\; \det[\chi(\lambda)]\mathbf{1}_{\mathbb{H}^+}(\lambda) \;-\; \det[\chi(\lambda)]\mathbf{1}_{\mathbb{H}^-}(\lambda)\;. \tag{4.37}$$

Since χ has continuous $\pm$-boundary on $\mathbb{R}$, it follows that $d \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R})$ has continuous $\pm$ boundary values on $\mathbb{R}$ as well. Furthermore these satisfy $d_+(\lambda) = d_-(\lambda)$. Finally, d admits the asymptotic behaviour $d(\lambda) = 1 + \mathrm{O}(\lambda^{-1})$. d can thus be continued to an entire function that is bounded at infinity. Hence, by Liouville theorem, $d \equiv 1$. Let χ_1, χ_2 be two solutions to the Riemann–Hilbert problem for χ. Since χ_2 can be analytically inverted, it follows that $\widetilde{\chi} = \chi_2^{-1} \cdot \chi_1$ solves the Riemann–Hilbert problem:

- $\widetilde{\chi} \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R})$ and has continuous $\pm$-boundary values on $\mathbb{R}$;
- $\widetilde{\chi}(\lambda) = I_2 + \mathrm{O}(\lambda^{-1}) \quad$ when $\lambda \to \infty$ non-tangentially to $\mathbb{R}$;
- $\widetilde{\chi}_+(\lambda) = \widetilde{\chi}_-(\lambda) \quad \text{for} \quad \lambda \in \mathbb{R}$.

Thus, by analytic continuation through $\mathbb{R}$ and Liouville theorem $\widetilde{\chi} = I_2$, hence ensuring the uniqueness of solutions. ■

4.4 Transformation $\chi \rightsquigarrow \Psi \rightsquigarrow \Pi$ and solvability of the Riemann–Hilbert problem

We construct a piecewise analytic matrix Ψ out of the matrix χ according to Figure 1. It is readily checked that the Riemann–Hilbert problem for χ is equivalent to the following Riemann–Hilbert problem for Ψ:

- $\Psi \in \mathcal{O}(\mathbb{C}^* \setminus \Sigma_\Psi)$ and has continuous boundary values on Σ_Ψ ;
- The matrix $\begin{pmatrix} -1 & 0 \\ -\kappa_N\, \lambda^{-1} & 1 + \kappa_N/(\omega_1 + \omega_2) \end{pmatrix} \cdot [\upsilon(0)]^{-\sigma_3} \cdot \Psi(\lambda)$ has a limit when $\lambda \to 0$;
- $\Psi(\lambda) = I_2 + \mathrm{O}(\lambda^{-1})$ when $\lambda \to \infty$ non-tangentially to Σ_Ψ ;
- $\Psi_+(\lambda) = G_\Psi(\lambda) \cdot \Psi_-(\lambda) \quad \text{for} \quad \lambda \in \Sigma_\Psi$;

where the jump matrix G_Ψ takes the form:

$$\text{for } \lambda \in \Gamma_\uparrow \qquad G_\Psi(\lambda) = I_2 + \frac{\mathrm{e}^{\mathrm{i}\lambda \overline{x}_N}}{\upsilon^2(\lambda) R(\lambda)} \cdot \sigma^- \,, \tag{4.38}$$

$$\text{for } \lambda \in \Gamma_\downarrow \qquad G_\Psi(\lambda) = I_2 + \frac{\upsilon^2(\lambda)\, \mathrm{e}^{-\mathrm{i}\lambda \overline{x}_N}}{R(\lambda)} \cdot \sigma^+ \,, \tag{4.39}$$

and for $\lambda \in \mathbb{R} + \mathrm{i}\epsilon$

$$G_\Psi(\lambda) = I_2 + \frac{r_N(\lambda)}{R(\lambda)} \cdot \begin{pmatrix} 1 & -\upsilon_+(\lambda)\upsilon_-(\lambda)\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N} \\ \dfrac{\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}}{\upsilon_+(\lambda)\upsilon_-(\lambda)} & -1 \end{pmatrix} . \tag{4.40}$$

The motivation underlying the construction of Ψ is that its jump matrix G_Ψ not only satisfies $G_\Psi - I_2 \in \mathcal{M}_2\big((L^2 \cap L^\infty)(\Sigma_\Psi)\big)$, but is, in fact, exponentially in N close to the identity

$$\|G_\Psi - I_2\|_{\mathcal{M}_2(L^2(\Sigma_\Psi))} + \|G_\Psi - I_2\|_{\mathcal{M}_2(L^\infty(\Sigma_\Psi))} = \mathrm{O}(\mathrm{e}^{-\varkappa_\epsilon N^\alpha}) \,, \tag{4.41}$$

with

$$\varkappa_\epsilon = (b_N - a_N) \cdot \min\Big\{ \inf_{\lambda \in \Gamma_\uparrow \cup \Gamma_\downarrow} |\mathrm{Im}\,\lambda| \,;\, 2\gamma\big(\pi \min[\omega_1, \omega_2] - \epsilon\big) \Big\} \,. \tag{4.42}$$

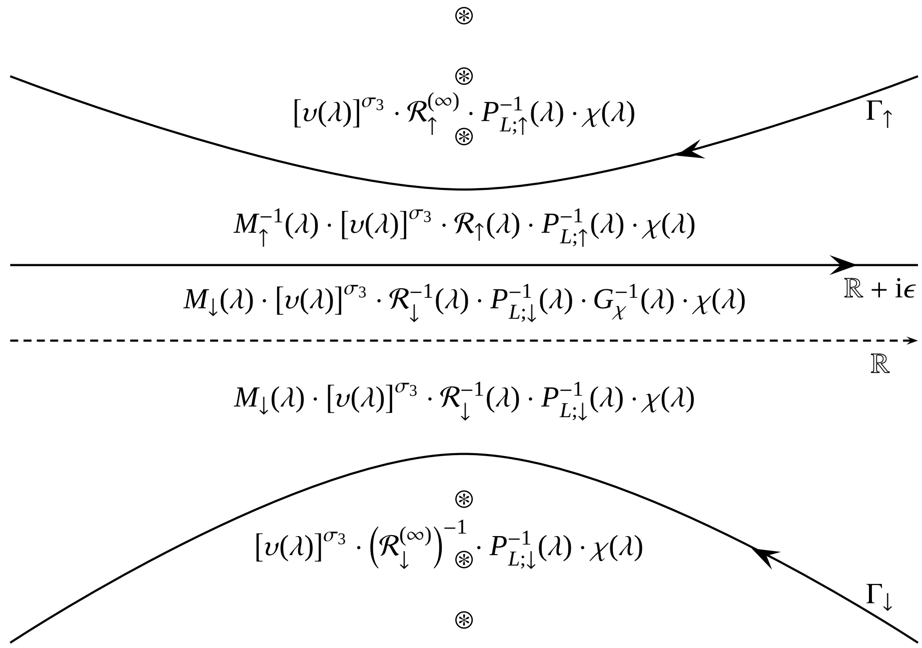

Figure 1: $\Sigma_\Psi = \Gamma_\uparrow \cup \Gamma_\downarrow \cup \{\mathbb{R} + \mathrm{i}\epsilon\}$ is the contour appearing in the Riemann–Hilbert problem for Ψ. $\Gamma_{\uparrow/\downarrow}$ separates all the poles of $R^{-1}(\lambda)$ from $\mathbb{R}$ (they are indicated by $\circledast$), and is such that $\mathrm{dist}(\Gamma_{\uparrow/\downarrow}, \mathbb{R}) > \delta$ for some $\delta > 0$ but sufficiently small.

Note that we have a freedom of choice of the curves $\Gamma_{\uparrow/\downarrow}$, provided that these avoid (resp. from below/above) all the poles of $R^{-1}(\lambda)$ in $\mathbb{H}^{+/-}$. As a consequence, we have the natural bound:

$$\inf_{\lambda \in \Gamma_\uparrow \cup \Gamma_\downarrow} |\mathrm{Im}\,\lambda| \;\leq\; \frac{2\pi\omega_1\omega_2}{\omega_1 + \omega_2} \;. \tag{4.43}$$

These bounds are enough so as to solve the Riemann–Hilbert problem for Ψ. Indeed, introduce the singular integral operator on the space $\mathcal{M}_2(L^2(\Sigma_\Psi))$ of 2×2 matrix-valued $L^2(\Sigma_\Psi)$ functions by

$$C_{\Sigma_\Psi}^{(-)}[\Pi](\lambda) \;=\; \lim_{\substack{z \to \lambda \\ z \in -\text{side of } \Sigma_\Psi}} \int_{\Sigma_\Psi} \frac{(G_\Psi - I_2)(t) \cdot \Pi(t)}{t - z} \cdot \frac{\mathrm{d}t}{2\mathrm{i}\pi} \;. \tag{4.44}$$

Since $G_\Psi - I_2 \in \mathcal{M}_2\Big((L^\infty \cap L^2)(\Sigma_\Psi)\Big)$ and Σ_Ψ is a Lipschitz curve, it follows from Theorem A.3 that $C_{\Sigma_\Psi}^{(-)}$ is a continuous endomorphism on $\mathcal{M}_2(L^2(\Sigma_\Psi))$ that furthermore satisfies:

$$\left|\left|\left| C_{\Sigma_\Psi}^{(-)} \right|\right|\right|_{\mathcal{M}_2(L^2(\Sigma_\Psi))} \;\leq\; C \mathrm{e}^{-\varkappa_\epsilon N^\alpha} \;. \tag{4.45}$$

Hence, since

$$G_\Psi - I_2 \in \mathcal{M}_2\Big(L^2(\Sigma_\Psi)\Big) \quad \text{and} \quad C_{\Sigma_\Psi}^{(-)}[I_2] \in \mathcal{M}_2\Big(L^2(\Sigma_\Psi)\Big) \tag{4.46}$$

provided that N is large enough, it follows that the singular integral equation

$$\Big(I_2 - C_{\Sigma_\Psi}^{(-)}\Big)[\Pi_-] \;=\; I_2 \tag{4.47}$$

admits a unique solution Π_- such that $\Pi_- - I_2 \in \mathcal{M}_2(L^2(\Sigma_\Psi))$. The bound (4.41) also implies that:

$$||\Pi_- - I_2||_{\mathcal{M}_2(L^2(\Sigma_\Psi))} \leq 1 \tag{4.48}$$

for N large enough. It is then a standard fact [7] in the theory of Riemann–Hilbert problems that the matrix

$$\Pi(\lambda) \;=\; I_2 \;+\; \int_{\Sigma_\Psi} \frac{(G_\Psi - I_2)(t)\cdot\Pi_-(t)}{t-\lambda}\cdot\frac{\mathrm{d}t}{2\mathrm{i}\pi} \tag{4.49}$$

is the unique solution to the Riemann–Hilbert problem:

- $\Pi \in \mathcal{O}(\mathbb{C}\setminus\Sigma_\Psi)$ and has continuous $\pm$ boundary values on Σ_Ψ ;
- $\Pi(\lambda) = I_2 \;+\; \mathrm{O}(\lambda^{-1})$ when $\lambda\to\infty$ non-tangentially to Σ_Ψ ;
- $\Pi_+(\lambda) \;=\; G_\Psi(\lambda)\cdot\Pi_-(\lambda)$ for $\lambda\in\Sigma_\Psi$.

We claim that for any open neighbourhood U of Σ_Ψ such that $\mathrm{dist}(\Sigma_\Psi,\partial U) > \delta > 0$, there exists a constant $C > 0$ such that:

$$\forall\lambda\in U, \qquad \max_{a,b\in\{1,2\}} \big[\Pi(\lambda) - I_2\big]_{ab} \leq \frac{C\,e^{-\varkappa_\epsilon N^\alpha}}{1+|\lambda|}\ . \tag{4.50}$$

Indeed, we can write:

$$\max_{a,b\in\{1,2\}} \big[\Pi(\lambda) - I_2\big]_{ab} \leq \max_{a,b\in\{1,2\}} \bigg|\int_{\Sigma_\Psi} \frac{(G_\Psi - I_2)_{ab}(t)}{t-\lambda}\cdot\frac{\mathrm{d}t}{2\mathrm{i}\pi}\bigg|$$
$$+ \sum_{a,b\in\{1,2\}} ||\Pi_- - I_2||_{\mathcal{M}_2(L^2(\Sigma_\Psi))}\cdot\bigg(\int_{\Sigma_\Psi} \frac{\big|(G_\Psi - I_2)_{ab}(t)\big|^2}{|t-\lambda|^2}\cdot\frac{|\mathrm{d}t|}{(2\pi)^2}\bigg)^{1/2}\ . \tag{4.51}$$

The second term is readily bounded with (4.48) and the fact (4.41) that G_Ψ is exponentially close to the identity matrix. For the first term, we study the asymptotic behaviour of $G_\Psi - I_2$ with help of § 4.2:

$$\text{if } t\in\Gamma_\downarrow\cup\Gamma_\uparrow, \qquad |(G_\Psi - I_2)_{ab}(t)| \;\leq\; C\,e^{-|\mathrm{Re}\,t|}\cdot e^{-\varkappa_\epsilon N^\alpha}\ , \tag{4.52}$$

$$\text{if } t\in\mathbb{R}+\mathrm{i}\epsilon, \qquad |(G_\Psi - I_2)_{ab}(t)| \;\leq\; C\,|t|^{-1}\cdot e^{-\varkappa_\epsilon N^\alpha}\ . \tag{4.53}$$

For the contribution on $\mathbb{R}+\mathrm{i}\epsilon$, we split $[G_\Psi - I_2](t) = C_\Psi\cdot t^{-1} + O(t^{-2})$. We compute directly the contour integral of the term in t^{-1}, and find the bound bound $\max_{a,b}\big|[C_\Psi]_{ab}\cdot\lambda^{-1}\big|$ if $\mathrm{Im}\,\lambda > \epsilon$, and 0 otherwise. Hence, it is bounded by $c_1/(1+|\lambda|)$ for some constant $c_1 > 0$. The contribution of the remainder $O(|t|^{-2})$ to the contour integral can be bounded thanks to the lower bound $\mathrm{dist}(\Sigma_\Psi,\lambda) \geq c_2/(1+|\lambda|)$ for some constant $c_2 > 0$. Collecting all these bounds justifies (4.50).

The Riemann–Hilbert problem for Ψ and Π have the same jump matrix G_Ψ, but Ψ must have a zero with prescribed leading coefficient at $\lambda = 0$, while Π has a finite value $\Pi(0)$. We then see that the formula:

$$\Psi(\lambda) \;=\; \Pi(\lambda)\cdot P_R(\lambda) \tag{4.54}$$

with:

$$P_R(\lambda) \;=\; I_2 \;+\; \frac{\theta_R}{\lambda}\cdot\Pi^{-1}(0)\sigma^-\Pi(0)\ , \qquad \text{and} \quad \theta_R \;=\; \frac{1}{\upsilon^2(0)}\,\frac{\kappa_N}{1+\kappa_N/(\omega_1+\omega_2)} \tag{4.55}$$

yields the unique solution to the Riemann–Hilbert problem for Ψ. Tracking back the transformations $\Pi \rightsquigarrow \Psi \rightsquigarrow \chi$, gives the construction of the solution χ of the Riemann–Hilbert problem of Proposition 4.3, summarized in Figure 2. This concludes the proof of Proposition 4.3. ■

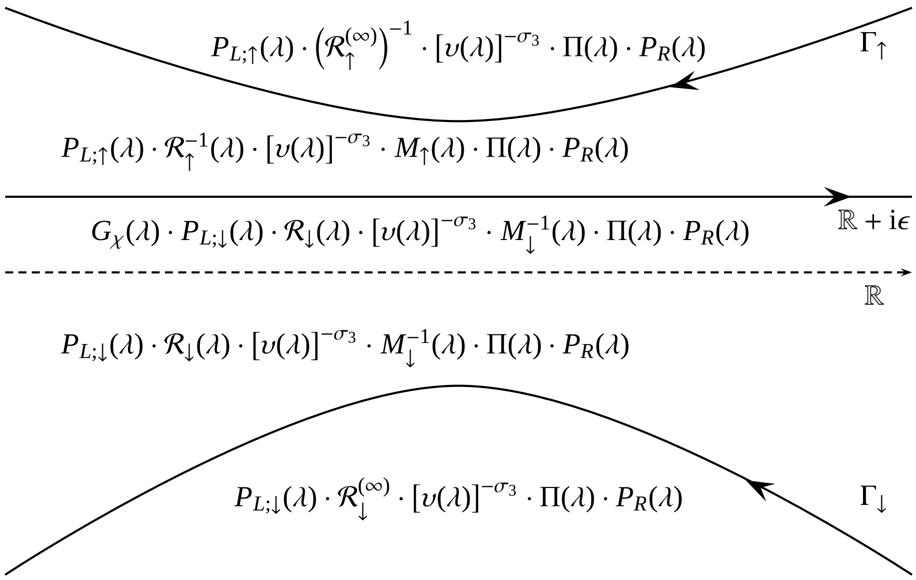

Figure 2: Piecewise definition of the matrix χ. The curves $\Gamma_{\uparrow/\downarrow}$ separate all poles of $\lambda \mapsto R^{-1}(\lambda)$ from $\mathbb{R}$ and are such that $\mathrm{dist}(\Gamma_{\uparrow/\downarrow}, \mathbb{R}) > \delta > \epsilon > 0$ for a sufficiently small δ. The matrix Π appearing here is defined through (4.49).

4.5 Properties of the solution χ

Lemma 4.4 *The solution χ to the Riemann–Hilbert problem given in Proposition 4.3 admits the following symmetries*

$$\chi(-\lambda) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \chi(\lambda) \cdot \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \big(\chi(\lambda^*)\big)^* \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \chi(-\lambda) \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{4.56}$$

where * refers to the component-wise complex conjugation.

Proof — Since $G_{\chi}(-\lambda) \;=\; \mathrm{e}^{\frac{\mathrm{i}\pi\sigma_3}{2}} G_{\chi}^{-1}(\lambda) \mathrm{e}^{-\frac{\mathrm{i}\pi\sigma_3}{2}}$, the matrix:

$$\Xi(\lambda) \;=\; \chi^{-1}(\lambda) \cdot \mathrm{e}^{-\frac{\mathrm{i}\pi\sigma_3}{2}} \cdot \chi(-\lambda) \tag{4.57}$$

is continuous across $\mathbb{R}$ and thus is an entire function. The asymptotic behaviour of $\Xi(\lambda)$ when $\lambda \to \infty$ is deduced from the growth conditions prescribed by the Riemann–Hilbert problem (*cf.* Proposition 4.3):

$$\Xi(\lambda) \;=\; \mathrm{i}\lambda \cdot \sigma^+ \; - \mathrm{i}(\chi_1 \cdot \sigma^+ \; + \; \sigma^+ \cdot \chi_1) \; + \; \mathrm{O}(\lambda^{-1}) \, . \tag{4.58}$$

Since $\Xi(\lambda)$ is entire, by Liouville theorem this asymptotic expression is exact, namely

$$\Xi(\lambda) \;=\; \mathrm{i}\lambda \cdot \sigma^+ \; - \mathrm{i}(\chi_1 \cdot \sigma^+ \; + \; \sigma^+ \cdot \chi_1) \, . \tag{4.59}$$

Observe that

$$\chi_1 \cdot \sigma^+ \; + \; \sigma^+ \cdot \chi_1 \; = \; \begin{pmatrix} [\chi_1]_{21} & \mathrm{tr}[\chi_1] \\ 0 & [\chi_1]_{21} \end{pmatrix} . \tag{4.60}$$

By expanding the relation $\det[\chi(\lambda)] = 1$ for $\lambda \in \mathbb{H}^+$ at large λ, we find that the matrix χ_1 is actually traceless. Finally, the jump condition at $\lambda = 0$ takes the form

$$\chi_-(0) \; = \; \sigma_3 \cdot \chi_+(0) . \tag{4.61}$$

Using this relation and the expression for Ξ given in (4.59), we get:

$$-\mathrm{i}\chi_+(0) \; = \; -\mathrm{i}\chi_+(0) \cdot \begin{pmatrix} [\chi_1]_{21} & 0 \\ 0 & [\chi_1]_{21} \end{pmatrix} \qquad \textit{i.e.} \qquad [\chi_1]_{21} = 1 \tag{4.62}$$

since $\chi_+(0)$ is invertible. This proves the first relation in (4.56). In order to establish the second one, we consider:

$$\widetilde{\Xi}(\lambda) \; = \; \chi^{-1}(-\lambda) \cdot \mathrm{e}^{\frac{\mathrm{i}\pi\sigma_3}{2}} \cdot (\chi(\lambda^*))^* . \tag{4.63}$$

With the relation $(G_\chi(\lambda^*))^* \; = \; G_\chi^{-1}(\lambda)$ and the complex conjugate of the asymptotic behaviour for χ, one shows that $\widetilde{\Xi}$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$, continuous across $\mathbb{R}$ and hence entire. Furthermore, since it admits the asymptotic behaviour

$$\widetilde{\Xi}(\lambda) \; = \; \mathrm{e}^{-\frac{\mathrm{i}\pi\sigma_3}{2}} \cdot \Big(I_2 \; + \; \mathrm{O}(\lambda^{-1}) \Big) , \tag{4.64}$$

by Liouville's theorem, $\widetilde{\Xi}(\lambda) = e^{-\frac{\mathrm{i}\pi\sigma_3}{2}}$. ■

Lemma 4.5 *The matrix χ admits the large-λ, $\lambda \in \mathbb{H}^+$ asymptotic expansion*

$$\chi(\lambda) \; \simeq \; (-\mathrm{i}\lambda)^{1/2} \cdot \sigma^+ \; + \; \sum_{k \geq 0} \frac{K(\lambda) \cdot \chi_k - \mathrm{i}\sigma^+ \cdot \chi_{k+1}}{(-\mathrm{i}\lambda)^{1/2} \lambda^k} , \tag{4.65}$$

where $(\chi_k)_k$ is a sequence of constant, 2×2 matrices, with $\chi_{-1} = 0$ and $\chi_0 = I_2$, and:

$$K(\lambda) \; = \; \begin{pmatrix} -\mathrm{sgn}(\mathrm{Re}\,\lambda)\,\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N} & 0 \\ -\frac{\kappa_N}{\lambda} \cdot \mathrm{sgn}(\mathrm{Re}\,\lambda)\,\mathrm{e}^{\mathrm{i}\lambda\gamma\overline{x}_N} \; - \; 1 & -\mathrm{i}\kappa_N \cdot \mathrm{sgn}(\mathrm{Re}\,\lambda)\,\mathrm{e}^{\mathrm{i}\lambda(\gamma-1)\overline{x}_N} \end{pmatrix} . \tag{4.66}$$

In particular, we have:

$$[\chi]_{11}(\lambda) \;\; \simeq \;\; \frac{1}{(-\mathrm{i}\lambda)^{1/2}} \sum_{k \geq 0} \frac{1}{\lambda^k} \Big[-\mathrm{sgn}(\mathrm{Re}\,\lambda)\,\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N} [\chi_k]_{11} \; - \; \mathrm{i}\,[\chi_{k+1}]_{21} \Big] , \tag{4.67}$$

$$\lambda^{-1} \cdot [\chi]_{12}(\lambda) \;\; \simeq \;\; \frac{1}{(-\mathrm{i}\lambda)^{1/2}} \sum_{k \geq 0} \frac{1}{\lambda^k} \Big[-\mathrm{sgn}(\mathrm{Re}\,\lambda)\,\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N} [\chi_{k-1}]_{12} \; - \; \mathrm{i}\,[\chi_k]_{22} \Big] . \tag{4.68}$$

Note that one should understand the matrix χ_{-1} occurring in (4.68) *as $\chi_{-1} := 0$. We also remind that $[\chi_1]_{21} = 1$.*

Proof — It is enough to establish that Π admits, for any ℓ, the large-λ asymptotic expansion of the form:

$$\Pi(\lambda) \; = \; \sum_{\ell=0}^{k} \lambda^{-\ell}\, \Pi_\ell \; + \; \Delta_{[k]}\Pi(\lambda) \qquad \text{with} \quad \Delta_{[k]}\Pi(\lambda) \; = \; \mathrm{O}\Big(\frac{1}{\lambda^{k+1-\delta}}\Big) \; \text{for any } \delta > 0 \quad \text{and} \qquad \Pi_0 = I_2 . \tag{4.69}$$

Indeed, once this asymptotic expansion is established for Π, the results for χ follow from matrix multiplications prescribed on the top of Figure 2.

Equation (4.50) shows that the expansion (4.69) holds for $k = 0$ uniformly away from Σ_{Ψ}. This is actually valid everywhere, for the jump matrix $G_{\Psi}(\lambda)$ is analytic in a neighbourhood of Σ_{Ψ} and asymptotically close to I_2 at large λ in an open neighbourhood of Σ_{Ψ}, *c.f.* (4.52)-(4.53).

Now assume that the expansion holds up to some order k. Consider the integral representation (4.49) for Π. We recall that $(\Pi_- - I_2) \in L^2(\Sigma_{\Psi})$ and $G_{\Psi} - I_2$ decays exponentially fast along $\Gamma_{\uparrow} \cup \Gamma_{\downarrow}$. Thus, standard manipulations give an asymptotic expansion of the form:

$$\int_{\Gamma_{\uparrow}\cup\Gamma_{\downarrow}} \frac{(G_{\Psi} - I_2)(t) \cdot \Pi_-(t)}{t - \lambda} \cdot \frac{\mathrm{d}t}{2\mathrm{i}\pi} \simeq \sum_{\ell \geq 1} T_{\ell}\, \lambda^{-\ell} . \tag{4.70}$$

It thus remains to focus on the integral on $\mathbb{R} + \mathrm{i}\epsilon$. We can first move the contour to $\mathbb{R} + \mathrm{i}\epsilon'$ for some $0 < \epsilon' < \epsilon$, and insert the assumed asymptotic expansion at order k:

$$\int_{\mathbb{R}+\mathrm{i}\epsilon} \frac{(G_{\Psi} - I_2)(t) \cdot \Pi_-(t)}{t - \lambda} \cdot \frac{\mathrm{d}t}{2\mathrm{i}\pi} = \sum_{\ell=0}^{k} \int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{(G_{\Psi} - I_2)(t) \cdot \Pi_{\ell}}{t^{\ell}(t - \lambda)} \cdot \frac{\mathrm{d}t}{2\mathrm{i}\pi} + \int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{(G_{\Psi} - I_2)(t) \cdot \Delta_{[k]}\Pi(t)}{t - \lambda} \cdot \frac{\mathrm{d}t}{2\mathrm{i}\pi} . \tag{4.71}$$

It follows from (4.20) that we can decompose $r_N(\lambda) = r_N^{(+)}(\lambda)\mathrm{e}^{\mathrm{i}\lambda\gamma\overline{x}_N} + r_N^{(-)}(\lambda)\mathrm{e}^{-\mathrm{i}\lambda\gamma\overline{x}_N}$, with $r_N^{(\pm)}(\lambda)$ bounded in λ away from its poles. This induces a decomposition $G_{\Psi} - I_2 = (G_{\Psi} - I_2)^{(+)} + (G_{\Psi} - I_2)^{(-)}$ on $\mathbb{R} + \mathrm{i}\epsilon'$. Inspecting the expression (4.40), we can convince oneself that there exist curves $\mathscr{C}^{\pm}_{G_{\Psi}} \subseteq \mathbb{H}^{\pm}$ going to ∞ when $\Re(t) \to \pm\infty$, $t \in \mathscr{C}^{\pm}_{G_{\Psi}}$ and such that:

- $t \mapsto \dfrac{(G_{\Psi} - I_2)^{(\pm)}(t) \cdot \Pi_{\ell}}{t^{\ell} \cdot (t - \lambda)}$ has no pole between $\mathbb{R} + \mathrm{i}\epsilon'$ and $\mathscr{C}^{\pm}_{G_{\Psi}}$,
- $(G_{\Psi} - I_2)^{(\pm)}(t)$ decays exponentially fast in t when $t \to \infty$ along $\mathscr{C}^{\pm}_{G_{\Psi}}$.

Therefore, we obtain:

$$\int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{(G_{\Psi} - I_2)(t) \cdot \Pi_{\ell}}{t^{\ell}(t - \lambda)} \cdot \frac{\mathrm{d}t}{2\mathrm{i}\pi} = \int_{\mathscr{C}^{+}_{G_{\Psi}}} \frac{(G_{\Psi} - I_2)^{(+)}(t) \cdot \Pi_{\ell}}{t^{\ell}(t - \lambda)} \cdot \frac{\mathrm{d}t}{2\mathrm{i}\pi} + \int_{\mathscr{C}^{-}_{G_{\Psi}}} \frac{(G_{\Psi} - I_2)^{(-)}(t) \cdot \Pi_{\ell}}{t^{\ell}(t - \lambda)} \cdot \frac{\mathrm{d}t}{2\mathrm{i}\pi} \tag{4.72}$$

and the properties of this decomposition ensure the existence of an all order asymptotic expansion in λ^{-1} when $\lambda \to \infty$. It thus remains to focus on the last term present in (4.71). For $\delta > 0$ but small, we write:

$$\int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{(G_{\Psi} - I_2)(t) \cdot \Delta_{[k]}\Pi(t)}{t - \lambda} \cdot \frac{\mathrm{d}t}{2\mathrm{i}\pi} = -\sum_{\ell=0}^{k} \frac{1}{\lambda^{\ell+1}} \int_{\mathbb{R}+\mathrm{i}\epsilon'} t^{\ell}(G_{\Psi} - I_2)(t) \cdot \Delta_{[k]}\Pi(t) \cdot \frac{\mathrm{d}t}{2\mathrm{i}\pi} + \frac{\Delta_{[k]}T(\lambda)}{\lambda^{k+1}|\lambda|^{1-2\delta}} . \tag{4.73}$$

The decay at ∞ of $\Delta_{[k]}\Pi$ and $(G_{\Psi} - I_2)$ guarantees the existence of an asymptotic expansion of the first term in the right-hand side, this up to a $\mathrm{O}(\lambda^{-k-2})$ remainder. Finally, we have:

$$|\Delta_{[k]}T(\lambda)| = \Big| \int_{\mathbb{R}+\mathrm{i}\epsilon'} t^{k+1} \frac{(G_{\Psi} - I_2)(t) \cdot \Delta_{[k]}\Pi(t)}{|\lambda|^{2\delta-1} \cdot (t - \lambda)} \cdot \frac{\mathrm{d}t}{2\mathrm{i}\pi} \Big| \leq C \int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{|\lambda|^{1-2\delta}\, \mathrm{d}t}{|t|^{1-\delta}\, |t - \lambda|} , \tag{4.74}$$

where we used the assumed bound (4.69) for $\Delta_{[k]}\Pi(t)$ and the $O(1/t)$ decay (4.53) for $G_\Psi - I_2$. The growth of the right-hand side at large λ is then estimated by cutting the integral into pieces:

$$\frac{|\lambda|^{1-2\delta}}{|t|^{1-\delta}\,|t-\lambda|} \;\le\; \begin{cases} \widetilde{C}\,|\lambda|^{-2\delta}\,|t|^{-(1-\delta)} & \text{if } \;|\mathrm{Re}\, t| \le |\lambda|/2 \\ \widetilde{C}\,|\lambda|^{-\delta}\,|t-\lambda|^{-1} & \text{if } \;|\lambda|/2 \le |\mathrm{Re}\, t| \le 3\,|\lambda|/2 \\ \widetilde{C}\,|t|^{-(1+\delta)} & \text{if } \;|\mathrm{Re}\, t| \ge 3\,|\lambda|/2 \end{cases} \tag{4.75}$$

for some $\widetilde{C} > 0$ independent of λ and t. The integral over t of the right-hand side on each of piece is finite, and collecting all the pieces, we get $\Delta_{[k]}T(\lambda) = \mathrm{o}(1)$ when $\lambda \to \infty$. ■

5 The inverse of the master operator

5.1 Solving $\mathscr{S}_{N;\gamma}[\varphi] = h$ for $h \in H_s([0\,;\overline{x}_N])$, $-1 < s < 0$

With the 2×2 matrix χ in hand, we can come back to the inversion of the integral operator $\mathscr{S}_{N;\gamma}$ according to Lemma 4.1.

Proposition 5.1 *Assume* $-1 < s < 0$, *and* $h \in H_s([0\,;\overline{x}_N])$. *Any solution to* $\mathscr{S}_{N;\gamma}[\varphi](\xi) = h(\xi)$ *is of the form* $\varphi = \widetilde{\mathscr{W}}_{\vartheta;z_0}[h_{\mathrm{e}}]$ *where*

$$\widetilde{\mathscr{W}}_{\vartheta;z_0}[h_{\mathrm{e}}] \;=\; \mathcal{F}^{-1}\Big[(*-z_0)\chi_{11;+} + C_+\big[f_{1;z_0}\big] \;+\; \chi_{12;+}\cdot C_+\big[f_{2;z_0}\big] \;+\; \vartheta\cdot\chi_{11;+}\Big]\,. \tag{5.1}$$

Above, $\vartheta \in \mathbb{C}$ *and* $z_0 \in \mathbb{C}\setminus\mathbb{R}$ *are arbitrary constants. We remind that* χ_+ *is the upper boundary value of* χ *on* $\mathbb{R}$, C *is the Cauchy transform* (1.24), $C_\pm$ *its* $\pm$ *boundary values and* h_{e} *is any extension of* h *to* $H_s(\mathbb{R})$.

$$\begin{pmatrix} f_{1;z_0}(\lambda) \\ f_{2;z_0}(\lambda) \end{pmatrix} \;=\; \mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N}\mathcal{F}[h_{\mathrm{e}}](\lambda)\cdot\begin{pmatrix} (\lambda-z_0)^{-1}\chi_{12;+}(\lambda) \\ -\chi_{11;+}(\lambda) \end{pmatrix}\,. \tag{5.2}$$

The transform $\widetilde{\mathscr{W}}_{\vartheta;z_0}$ *is continuous on* $H_s(\mathbb{R})$, $-1 < s < 0$*:*

$$\|\widetilde{\mathscr{W}}_{0;z_0}[h_{\mathrm{e}}]\|_{H_s(\mathbb{R})} \;\le\; C_N\,\|h_{\mathrm{e}}\|_{H_s(\mathbb{R})}\,, \tag{5.3}$$

the continuity constant C_N *being however dependent, a priori, on* N. *Finally, when* $h \in C^1([0\,;\overline{x}_N])$ *the transform can be recast as*

$$\widetilde{\mathscr{W}}_{\vartheta;z_0}[h](\xi) \;=\; \int\limits_{\mathbb{R}+2\mathrm{i}\epsilon'}\frac{\mathrm{d}\lambda}{2\pi}\int\limits_{\mathbb{R}+\mathrm{i}\epsilon'}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\,\frac{\mathrm{e}^{-\mathrm{i}\xi\lambda-\mathrm{i}\overline{x}_N\mu}}{\mu-\lambda}\Big\{\frac{\lambda-z_0}{\mu-z_0}\chi_{11}(\lambda)\chi_{12}(\mu)-\chi_{11}(\mu)\chi_{12}(\lambda)\Big\}\cdot\int\limits_0^{\overline{x}_N}\mathrm{e}^{\mathrm{i}\eta\mu}h(\eta)\cdot\mathrm{d}\eta \\ +\;\vartheta\int\limits_{\mathbb{R}+\mathrm{i}\epsilon'}\mathrm{e}^{-\mathrm{i}\lambda\xi}\chi_{11}(\lambda)\cdot\frac{\mathrm{d}\lambda}{2\pi}\,. \tag{5.4}$$

where $\epsilon' > \epsilon$ *is arbitrary but small enough and such that* $\mathrm{Im}\, z_0 > \epsilon'$ *in the case when* $z_0 \in \mathbb{H}^+$.

We stress that the integrals, as written in (5.4), are to be understood in the Riemann sense in that they only converge as oscillatory integrals.

Proof — The proof is based on a Wiener-Hopf factorisation. For the moment, we only assume that $s < 0$. Let Φ be any solution to the vector Riemann–Hilbert problem for Φ outlined in Lemma 4.1. Then, define a piecewise holomorphic function Υ by

$$\Upsilon(\lambda) \; = \; \begin{cases} \chi^{-1}(\lambda)\Phi(\lambda) \; - \; \widehat{\boldsymbol{H}}(\lambda) & \lambda \in \mathbb{H}^+ \\ \chi^{-1}(\lambda)\Phi(\lambda) \; - \; \widehat{\boldsymbol{H}}(\lambda) & \lambda \in \mathbb{H}^- \end{cases} \tag{5.5}$$

where, for some $z_0 \in \mathbb{C} \setminus \mathbb{R}$

$$\widehat{\boldsymbol{H}}(\lambda) \; = \; \begin{pmatrix} (\lambda - z_0)^{\iota_s} \int\limits_{\mathbb{R}} \dfrac{g_{1;\iota_s}(t)\,\mathrm{d}t}{2\mathrm{i}\pi(t-\lambda)} \\ (\lambda - z_0)^{\iota_s - 1} \int\limits_{\mathbb{R}} \dfrac{g_{1;\iota_s - 1}(t)\,\mathrm{d}t}{2\mathrm{i}\pi(t-\lambda)} \end{pmatrix} \qquad \text{with} \qquad \begin{pmatrix} g_1(\lambda) \\ g_2(\lambda) \end{pmatrix} = \chi_+^{-1}(\lambda) \cdot \boldsymbol{H}(\lambda) \; . \tag{5.6}$$

Above, taking into account that $s < 0$, we have set

$$g_{a;\iota_s}(t) \; = \; (t - z_0)^{-\iota_s} g_a(t) \quad \text{with} \;\; \iota_s = k \quad \text{for} \; -k < s < -(k-1) \; . \tag{5.7}$$

It follows from the asymptotic behaviour for $\chi_+(\lambda)$ at large λ that $g_1 \in \mathcal{F}[H_{s-1/2}]$ and $g_2 \in \mathcal{F}[H_{s+1/2}]$. Recall that Theorem A.1 ensures that the $\pm$ boundary values $C_\pm$ of the Cauchy transform on $\mathbb{R}$ are continuous operators on $H_\tau(\mathbb{R})$ for any $|\tau| < 1/2$. Thus, $C_\pm[g_{1;\iota_s}] \in H_{s+k-1/2}(\mathbb{R})$ as well as $C_\pm[g_{2;\iota_s - 1}] \in H_{s+k-1/2}(\mathbb{R})$, which implies:

$$\widehat{\boldsymbol{H}}_{a;\pm} \in \mathcal{F}[H_{s_a}(\mathbb{R})] \qquad \text{with} \;\; s_1 \, = \, s - 1/2 \;\; \text{and} \;\; s_2 \, = \, s + 1/2 \; . \tag{5.8}$$

Equation (4.7) ensures that, uniformly in $\mu > 0$,

$$\forall a \in \{1,2\}, \qquad \int\limits_{\mathbb{R}} \big|\Upsilon_a(\lambda \pm \mathrm{i}\mu)\big|^2 \, (1 + |\lambda| + |\mu|)^{2s_a} \, \mathrm{d}\lambda < C \;\; . \tag{5.9}$$

The discontinuity equation satisfied by Φ along with $\widehat{\boldsymbol{H}}_{a;+} - \widehat{\boldsymbol{H}}_{a;-} \; = \; g_a$ guarantee that $\Upsilon_a \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R})$ admits $\mathcal{F}[H_{s_a}(\mathbb{R})]$ $\pm$ boundary values that are equal. Then, straightforward manipulations show that, in fact, Υ is entire. Furthermore, for any $\ell \in \mathbb{N}$ such that $s_a + \ell > -1/2$ and for any $\mu > |\mathrm{Im}\, z|$, we have:

$$\partial_z^\ell \Upsilon_a(z) \; = \; \sum_{\epsilon = \pm} \epsilon \int\limits_{\mathbb{R}} \frac{\ell!\, \Upsilon_a(\lambda + \mathrm{i}\epsilon\mu)}{(\lambda + \mathrm{i}\epsilon\mu - z)^{\ell+1}} \, \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \; . \tag{5.10}$$

Thus

$$\big|\partial_z^\ell \Upsilon_a(z)\big| \; \le \; \frac{1}{\pi} \max_{\epsilon = \pm} \bigg(\int_{\mathbb{R}} \frac{(1 + |\lambda| + |\mu|)^{-2s_a}}{|\lambda + \mathrm{i}\epsilon\mu - z|^{2(\ell+1)}} \, \mathrm{d}\lambda \bigg)^{1/2} \bigg(\int_{\mathbb{R}} |\Upsilon_a(\lambda + \mathrm{i}\epsilon\mu)|^2 \, (1 + |\lambda| + |\mu|)^{2s_a} \, \mathrm{d}\lambda \bigg)^{1/2} \tag{5.11}$$

where the last integral factor is bounded. So far, the parameter μ was arbitrary. We stress that the constant C in (5.9) is uniform in μ. Thus taking $\mu = 2|z|$ and assuming that $|z| > 1/2$, we find:

$$\big|\partial_z^\ell \Upsilon_a(z)\big| \; \le \; C' \, |z|^{-(s_a + \ell + 1/2)} \cdot \bigg(\int_{\mathbb{R}} \frac{(|\lambda| + 2)^{-2s_a}}{\big[(\lambda - 1)^2 + 1\big]^{\ell+1}} \, \mathrm{d}\lambda \bigg)^{1/2} \; . \tag{5.12}$$

In particular, reminding the values of s_a in (5.8), we find that $\partial_z^{k-1}\Upsilon_2(z)$ and $\partial_z^k\Upsilon_1(z)$ are entire and bounded, so they must be constant. These constants are zero due to (5.12). Hence, there exist polynomials $P_1 \in \mathbb{C}_{k-1}[X]$ and $P_2 \in \mathbb{C}_{k-2}[X]$ such that

$$\Upsilon(z) \;=\; \begin{pmatrix} P_1(z) \\ P_2(z) \end{pmatrix} . \tag{5.13}$$

Reciprocally, it is readily seen that the piecewise analytic vector

$$\Phi(\lambda) \;=\; \chi(\lambda)\cdot\widehat{\boldsymbol{H}}(\lambda) \;+\; \chi(\lambda)\cdot\begin{pmatrix} P_1(z) \\ P_2(z) \end{pmatrix} \qquad \text{with} \quad P_a \in \mathbb{C}_{k-a}[X] \quad \text{for} \quad -k < s < -(k-1) \tag{5.14}$$

provides solutions to the Riemann–Hilbert problem for Φ.

From now on, we focus on the case $k = 1$, i.e. $h \in H_s([0\,;\overline{x}_N])$ for $-1 < s < 0$. Then, it follows from Lemma 4.1 that any solution to $\mathscr{S}_{N;\gamma}[\varphi] \;=\; h$ takes the form $\varphi = \widetilde{\mathscr{W}}_{\vartheta;z_0}[h_{\mathfrak{e}}]$, with:

$$\mathcal{F}\big[\widetilde{\mathscr{W}}_{\vartheta;z_0}[h_{\mathfrak{e}}]\big](\lambda) \;=\; \Phi_{1;+}(\lambda) \;=\; \chi_{11;+}(\lambda)\cdot(\lambda-z_0)C_+[f_{1;z_0}](\lambda) \;+\; \chi_{12;+}(\lambda)\cdot C_+[f_{2;z_0}](\lambda) \;+\; \vartheta\cdot\chi_{11;+}(\lambda) \tag{5.15}$$

with $f_{a;z_0}$'s given by (5.2).

It is then readily inferred from the asymptotic expansion for χ at $\lambda \to \infty$ given in Lemma 4.5, and from the jump conditions satisfied by χ, that indeed $\Phi_{1;+} \in \mathcal{F}[H_s([0\,;\overline{x}_N])]$. Also the continuity on $\mathcal{F}[H_\tau(\mathbb{R})]$ with $|\tau| < 1/2$ of the $\pm$ boundary values $C_\pm$ of the Cauchy transform, *cf.* Theorem A.1, ensures that

$$||\Phi_{1;+}||_{\mathcal{F}[H_s(\mathbb{R})]} \;\leq\; C\,||h_{\mathfrak{e}}||_{H_s(\mathbb{R})} \;, \tag{5.16}$$

which in turn implies the bound (5.3).

It solely remains to prove the regularised expression (5.4). Given $h \in C^1([0\,;\overline{x}_N])$ it is clear that $h \in H_s([0\,;\overline{x}_N])$ for any $s < 1/2$. We chose the specific extension $h_{\mathfrak{e}} = h$. Then, it follows from the previous discussion that $\widetilde{\mathscr{W}}_{\vartheta;z_0}[h] \in H_s([0\,;\overline{x}_N])$. The integral in the right-hand side of (5.4), considered in the Riemann sense, defines a continuous function on $[0\,;\overline{x}_N]$, that we denote momentarily $\widetilde{\mathscr{V}}_{\vartheta;z_0}[h]$. Now, for any $f \in C^\infty([0\,;\overline{x}_N])$, starting with the expression (5.1) for $\widetilde{\mathscr{W}}_{\vartheta;z_0}[h]$, we have:

$$\big(f, \widetilde{\mathscr{W}}_{\vartheta;z_0}[h]\big) \;=\; \big(\mathcal{F}[f], \Phi_{1;+}\big) \;=\; \int\limits_{\mathbb{R}} \mathcal{F}[f^*](-\lambda)\cdot\Phi_{1;+}(\lambda)\,\mathrm{d}\lambda = \int\limits_{\mathbb{R}+2\mathrm{i}\epsilon'} \mathcal{F}[f^*](-\lambda)\cdot\Phi_1(\lambda)\,\mathrm{d}\lambda \tag{5.17}$$

$$= \int\limits_{\mathbb{R}} \Big(\mathcal{F}[\mathrm{e}^{2\epsilon'\bullet}f](\lambda)\Big)^*\cdot\mathcal{F}[\mathrm{e}^{-2\epsilon'\bullet}\widetilde{\mathscr{W}}_{\vartheta;z_0}[h]](\lambda)\mathrm{d}\lambda \;=\; \big(f, \widetilde{\mathscr{V}}_{\vartheta;z_0}[h]\big) \;. \tag{5.18}$$

in $\bullet$ represents the running variable in respect to which the Fourier transform. There, we have equality $\widetilde{\mathscr{W}}_{\vartheta;z_0}[h] = \widetilde{\mathscr{V}}_{\vartheta;z_0}[h]$ for $h \in C^1 \cap H_s([0\,;\overline{x}_N])$. ■

A *priori*, the solutions $\widetilde{\mathscr{W}}_{\vartheta;z_0}[h_{\mathfrak{e}}]$ given in (5.4) has two free parameters ϑ and z_0. This "double" freedom is, however, illusory.

Lemma 5.2 *Given* $z_0, z_0' \in \mathbb{C}\setminus\mathbb{R}$ *and* $\vartheta \in \mathbb{C}$, *there exists* $\vartheta' \in \mathbb{C}$ *such that* $\widetilde{\mathscr{W}}_{\vartheta;z_0} = \widetilde{\mathscr{W}}_{\vartheta';z_0'}$.

Proof — By carrying out the decomposition $\lambda - z_0 = \lambda - \mu \;+\; \mu - z_0$ in the first term present in the integrand of (5.4), we get that $\widetilde{\mathscr{W}}_{\vartheta;z_0} = \widetilde{\mathscr{W}}_{\vartheta(z_0);\infty}$

$$\vartheta(z_0) \;=\; \vartheta \;-\; \int\limits_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\chi_{12}(\mu)\cdot\mathcal{F}[h_{\mathfrak{e}}](\mu)\cdot\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}}{\mu - z_0}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \;, \tag{5.19}$$

and ∞ means that one should send $z_0 \to \infty$ under the integral sign of (5.4). ■

Hence, with the above lemma in mind, we retrieve that the kernel of $\mathscr{S}_{N;\gamma}$ is one dimensional when considered as an operator on $H_s([0\,;\overline{x}_N])$, with $-1 < s < 0$. The above lemma of course implies that we can choose z_0 arbitrarily in (5.4). It is most suitable to consider the specific form of solutions obtained by taking $z_0 \to 0$ with $\operatorname{Im} z_0 < 0$. For $h \in C^1([0\,;\overline{x}_N])$, this yields a family of solution parametrized by $\vartheta \in \mathbb{C}$:

$$\widetilde{\mathscr{W}_\vartheta}[h](\xi) \;=\; \int\limits_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\pi} \int\limits_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \frac{\mathrm{e}^{-\mathrm{i}\lambda\xi-\mathrm{i}\mu\overline{x}_N}}{\mu-\lambda} \bigg\{ \frac{\lambda}{\mu}\cdot\chi_{11}(\lambda)\chi_{12}(\mu) - \chi_{11}(\mu)\chi_{12}(\lambda) \bigg\} \mathcal{F}[h](\mu) + \vartheta \int\limits_{\mathbb{R}+\mathrm{i}\epsilon'} \chi_{11}(\lambda)\,\mathrm{e}^{-\mathrm{i}\lambda\xi}\cdot\frac{\mathrm{d}\lambda}{2\pi} \;. \tag{5.20}$$

It is possible to find real-valued solutions to $\mathscr{S}_{N;\gamma}[\varphi] = h$ by taking h purely imaginary:

Lemma 5.3 *Let $\vartheta \in \mathrm{i}\mathbb{R}$ and let $h \in C^1([0\,;\overline{x}_N])$ satisfy $h^* = -h$. Then, $(\widetilde{\mathscr{W}_\vartheta}[h_{\mathrm{e}}])^* = \widetilde{\mathscr{W}_\vartheta}[h_{\mathrm{e}}]$.*

Proof — From Lemma 4.4, we have $-\chi_{11}(-\lambda) = (\chi_{11}(\lambda^*))^*$ and $\chi_{12}(-\lambda) = (\chi_{12}(\lambda^*))^*$. Hence, under the assumptions of the present lemma

$$\begin{aligned}(\widetilde{\mathscr{W}_\vartheta}[h](\xi))^* \;=\; & \int\limits_{\mathbb{R}} \frac{\mathrm{d}\lambda}{2\pi} \int\limits_{\mathbb{R}} \frac{\mathrm{d}\mu}{-2\mathrm{i}\pi} \frac{\mathrm{e}^{\mathrm{i}\lambda\xi+\mathrm{i}\mu\overline{x}_N}\mathrm{e}^{2\epsilon'\xi+\epsilon'\overline{x}_N}}{\mu-\lambda+\mathrm{i}\epsilon'} \bigg\{ -\frac{\lambda-2\mathrm{i}\epsilon'}{\mu-\mathrm{i}\epsilon'}\cdot\chi_{11}(-\lambda+2\mathrm{i}\epsilon')\chi_{12}(-\mu+\mathrm{i}\epsilon') \\ & + \chi_{11}(-\mu+\mathrm{i}\epsilon')\chi_{12}(-\lambda+2\mathrm{i}\epsilon') \bigg\} \underbrace{\mathcal{F}[h^*]}_{-\mathcal{F}[h]}(-\mu+\mathrm{i}\epsilon') \;-\; \underbrace{\vartheta^*}_{-\vartheta} \int\limits_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda\xi}\mathrm{e}^{2\epsilon'\xi}\chi_{11}(-\lambda+\mathrm{i}\epsilon')\,\frac{\mathrm{d}\lambda}{2\pi} \;. \end{aligned} \tag{5.21}$$

The change of variables $(\lambda,\mu) \mapsto (-\lambda,-\mu)$ in the first integral and $\lambda \mapsto -\lambda$ in the second integral entails the claim. ■

5.2 Local behaviour of the solution $\widetilde{\mathscr{W}_\vartheta}[h]$ at the boundaries

In the present subsection, we shall establish the local behaviour of $\widetilde{\mathscr{W}_\vartheta}[h](\xi)$ at the boundaries of the segment $[0\,;\overline{x}_N]$, *viz.* when $\xi \to 0$ or $\xi \to \overline{x}_N$, this in the case where $h \in C^1([0\,;\overline{x}_N])$. We shall demonstrate that there exist constants $C_0, C_{\overline{x}_N}$ affine in ϑ and depending on h, such that $\widetilde{\mathscr{W}_\vartheta}[h]$ exhibits the local behaviour

$$\widetilde{\mathscr{W}_\vartheta}[h](\xi) \;=\; \frac{C_0}{\sqrt{\xi}} + \mathrm{O}(1) \quad \text{for} \quad \xi \to 0^+ \qquad \text{and} \qquad \widetilde{\mathscr{W}_\vartheta}[h](\xi) \;=\; \frac{C_{\overline{x}_N}}{\sqrt{\overline{x}_N-\xi}} + \mathrm{O}(1) \quad \text{for} \quad \xi \to (\overline{x}_N)^- \;. \tag{5.22}$$

Let us recall that our motivation for studying $\widetilde{\mathscr{W}_\vartheta}$ takes its origin in the need to construct the density of equilibrium measure $\rho_{\mathrm{eq}}^{(N)}$ which solves $\mathcal{S}_N[\rho_{\mathrm{eq}}^{(N)}] = V'$ as well as to invert the master operator $\mathcal{U}_N$ arising in the Schwinger-Dyson equations described in § 3.2. The density has a square root behaviour at the edges what translates itself into a square root behaviour at $\xi = 0$ and $\xi = \overline{x}_N$ in the rescaled variables. Having this in mind, we would like to enforce $C_0 = C_{\overline{x}_N} = 0$. For this purpose, we can exploit the freedom of choosing ϑ. This is however not enough and, as it will be shown in the present section, in order to have a milder behaviour of $\widetilde{\mathscr{W}_\vartheta}[h]$ at the edges, one also needs to impose a linear constraint on h. In fact, we shall see later on that the latter solely translates the fact that $h \in \mathscr{S}_{N;\gamma}[H_s(\mathbb{R})]$ with $0 < s < 1/2$.

This informal discussion only serves as a guideline and motivation for the results of this subsection, in particular:

Proposition 5.4 *Let*

$$\mathscr{I}_{12}[h] \;=\; \int\limits_{\mathbb{R}+\mathrm{i}\epsilon} \frac{\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}}{\mu}\chi_{12}(\mu)\cdot\mathcal{F}[h](\mu)\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\;. \tag{5.23}$$

Then, for any $h \in C^1([0\,;\overline{x}_N])$ such that

$$\mathscr{I}_{11}[h] \;:=\; \int\limits_{\mathbb{R}+\mathrm{i}\epsilon} \mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}\chi_{11}(\mu)\cdot\mathcal{F}[h](\mu)\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \;=\; 0 \tag{5.24}$$

we have $\widetilde{\mathscr{W}}_{\mathscr{I}_{12}[h]}[h] \in (L^1\cap L^\infty)([0\,;\overline{x}_N])$.

Prior to proving the above lemma, we shall first establish a lemma characterising the local behaviour at 0 and $\overline{x}_N$ of functions belonging to the kernel of $\mathscr{S}_{N;\gamma}$.

Lemma 5.5 *The function*

$$\psi(\xi) = \int\limits_{\mathbb{R}+2\mathrm{i}\epsilon'} \mathrm{e}^{-\mathrm{i}\lambda\xi}\chi_{11}(\lambda)\,\frac{\mathrm{d}\lambda}{2\pi} \qquad \text{satisfies} \qquad \mathscr{S}_{N;\gamma}[\psi](\xi) = 0 \quad \xi\in]0\,;\overline{x}_N[\tag{5.25}$$

and admits the asymptotic behaviour

$$\psi(\xi) \;=\; \frac{1}{\mathrm{i}\sqrt{\pi\xi}} + \mathrm{O}(1) \quad \text{when} \quad \xi\to 0^+ \qquad \text{and} \qquad \psi(\xi) \;=\; \frac{1}{\mathrm{i}\sqrt{\pi(\overline{x}_N-\xi)}} + \mathrm{O}(1) \quad \text{when} \quad \xi\to(\overline{x}_N)^-\;. \tag{5.26}$$

Proof — One has, for $\xi\in]0\,;\overline{x}_N[$ and in the distributional sense,

$$\begin{aligned}
\mathscr{S}_{N;\gamma}[\psi](\xi) \;&=\; \int\limits_{\mathbb{R}}\frac{\mathrm{d}\mu}{2\pi}\int\limits_{\mathbb{R}}\frac{\mathrm{d}\lambda}{2\pi}\,\frac{\mathrm{e}^{-\mathrm{i}\mu\xi}\,\mathcal{F}[S_\gamma](\mu)}{2\mathrm{i}\pi\beta}\cdot\chi_{11;+}(\lambda)\cdot\frac{\mathrm{e}^{\mathrm{i}(\mu-\lambda)\overline{x}_N}-1}{\mathrm{i}(\lambda-\mu)} \\
&=\; \int\limits_{\mathbb{R}}\frac{\mathrm{d}\mu}{2\pi}\int\limits_{\mathbb{R}-\mathrm{i}\epsilon}\frac{\mathrm{d}\lambda}{2\pi}\,\frac{\mathrm{e}^{-\mathrm{i}\mu\xi}\,\mathcal{F}[S_\gamma](\mu)}{2\mathrm{i}\pi\beta}\cdot\frac{\chi_{11}(\lambda)\mathrm{e}^{\mathrm{i}\mu\overline{x}_N}-\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}\chi_{11}(\lambda)}{\mathrm{i}(\lambda-\mu)} \\
&=\; \int\limits_{\mathbb{R}}\frac{\mathrm{d}\mu}{2\pi}\,\frac{\mathrm{e}^{-\mathrm{i}\mu\xi}\,\mathcal{F}[S_\gamma](\mu)}{2\mathrm{i}\pi\beta}\bigg\{-\chi_{11;+}(\mu) \;+\; \int\limits_{\mathbb{R}-\mathrm{i}\epsilon}\frac{\mathrm{d}\lambda}{2\pi}\,\frac{\chi_{11}(\lambda)\,\mathrm{e}^{\mathrm{i}\mu\overline{x}_N}}{\mathrm{i}(\lambda-\mu)}\bigg\} \\
&=\; -\int\limits_{\mathbb{R}}\frac{\mathrm{d}\mu}{2\pi}\Big\{\chi_{21;-}(\mu)\mathrm{e}^{-\mathrm{i}\mu\xi} \;+\; \mathrm{e}^{\mathrm{i}\mu(\overline{x}_N-\xi)}\chi_{21;+}(\mu)\Big\}\;.
\end{aligned} \tag{5.27}$$

Note that, in the intermediate steps, we have used that $\chi_{11;+}(\lambda) = \mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}\chi_{11;-}(\lambda)$, and deformed the integral over λ to the lower half-plane. Further, we have also used that

$$\chi_{21;-}(\lambda) \;+\; \mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}\chi_{21;+}(\lambda) \;=\; \frac{\mathcal{F}[S_\gamma](\lambda)}{2\mathrm{i}\pi\beta}\chi_{11;+}(\lambda)\;. \tag{5.28}$$

Observe that, when $0<\xi<\overline{x}_N$, the function $\mu\mapsto\chi_{21;-}(\mu)\mathrm{e}^{-\mathrm{i}\mu\xi}$ (respectively, $\mu\mapsto\chi_{21;+}(\mu)\mathrm{e}^{\mathrm{i}\mu(\overline{x}_N-\xi)}$) admits an analytic continuation to the lower (resp. upper) half-plane that is Riemann-integrable on $\mathbb{R}-\mathrm{i}\tau$ (resp. $\mathbb{R}+\mathrm{i}\tau$), this for any $\tau>0$, and that decays exponentially fast when $\tau\to+\infty$. As a consequence,

$$\forall\xi\in]0\,;\overline{x}_N[, \qquad \int\limits_{\mathbb{R}}\frac{\mathrm{d}\mu}{2\pi}\Big(\mathrm{e}^{-\mathrm{i}\mu\xi}\chi_{21;-}(\mu) \;+\; \mathrm{e}^{\mathrm{i}\mu(\overline{x}_N-\xi)}\chi_{21;+}(\mu)\Big) \;=\; 0\;, \tag{5.29}$$

which is equivalent to $\mathscr{S}_{N;\gamma}[\psi](\xi) = 0$.

From the large-λ expansion of $\chi(\lambda)$ given in Lemma 4.5, we have for $\lambda \in \mathbb{R} + 2\mathrm{i}\epsilon'$,

$$W(\lambda) \;\equiv\; \chi_{11}(\lambda) \;+\; \frac{\mathrm{sgn}(\mathrm{Re}\,\lambda)\,\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N} + \mathrm{i}}{(-\mathrm{i}\lambda)^{1/2}} \;=\; \mathrm{O}\Big(|\lambda|^{-3/2}\Big)\,. \tag{5.30}$$

Hence,

$$\psi(\xi) \;=\; \int\limits_{\mathbb{R}+2\mathrm{i}\epsilon'} W(\lambda)\,\mathrm{e}^{-\mathrm{i}\lambda\xi}\cdot\frac{\mathrm{d}\lambda}{2\pi} \;-\; \int\limits_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{sgn}(\mathrm{Re}\,\lambda)\,e^{\mathrm{i}\lambda(\overline{x}_N-\xi)}}{(-\mathrm{i}\lambda)^{1/2}}\cdot\frac{\mathrm{d}\lambda}{2\pi} \;+\; \int\limits_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{e}^{-\mathrm{i}\lambda\xi}}{(-\mathrm{i}\lambda)^{1/2}}\cdot\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\,. \tag{5.31}$$

By dominated convergence, the first term is $\mathrm{O}(1)$ in the limit $\xi \to 0^+$. The second term is also a $\mathrm{O}(1)$. This is most easily seen by deforming the contour of integration into a loop in $\mathbb{H}_+$ around $\mathrm{i}\mathbb{R}^+ + 2\mathrm{i}\epsilon'$, hence making the integral strongly convergent, and then applying dominated convergence. Finally, the third term (5.31) can be explicitly computed by deforming the integration contour to $-\mathrm{i}\mathbb{R}^+$:

$$\int\limits_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{e}^{-\mathrm{i}\lambda\xi}}{(-\mathrm{i}\lambda)^{1/2}}\cdot\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \;=\; \frac{-1}{\sqrt{\xi}}\int\limits_0^{+\infty}\bigg\{\frac{1}{(-\mathrm{e}^{\mathrm{i}0^+}t)^{1/2}} \;-\; \frac{1}{(-\mathrm{e}^{-\mathrm{i}0^+}t)^{1/2}}\bigg\}\frac{\mathrm{e}^{-t}\,\mathrm{d}t}{2\pi} \;=\; \frac{\Gamma(1/2)}{\mathrm{i}\pi\sqrt{\xi}} = \frac{1}{\mathrm{i}\sqrt{\pi\xi}}\,. \tag{5.32}$$

Similar arguments ensure that the first and last term in (5.31) are a $\mathrm{O}(1)$ in the $\xi \to (\overline{x}_N)^-$ limit. The middle term can be estimated as

$$\int\limits_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{sgn}(\mathrm{Re}\,\lambda)}{(-\mathrm{i}\lambda)^{\frac{1}{2}}}\mathrm{e}^{\mathrm{i}\lambda(\overline{x}_N-\xi)}\cdot\frac{\mathrm{d}\lambda}{2\pi} \;=\; \frac{\mathrm{e}^{-2(\overline{x}_N-\xi)\epsilon'}}{2\pi\sqrt{\overline{x}_N-\xi}}\int\limits_{\mathbb{R}} \frac{\mathrm{sgn}(\lambda)\,\mathrm{e}^{\mathrm{i}\lambda}\,\mathrm{d}\lambda}{\big(-\mathrm{i}\lambda + 2\epsilon'(\overline{x}_N-\xi)\big)^{1/2}}$$

$$=\; \mathrm{i}\frac{\mathrm{e}^{-2(\overline{x}_N-\xi)\epsilon'}}{\pi\sqrt{\overline{x}_N-\xi}}\int\limits_0^{+\infty}\frac{\mathrm{e}^{-t}\,\mathrm{d}t}{\big(t+2\epsilon'(\overline{x}_N-\xi)\big)^{1/2}} \;=\; \frac{\mathrm{i}}{\sqrt{\pi(\overline{x}_N-\xi)}} \;+\; \mathrm{O}\big(\sqrt{\overline{x}_N-\xi}\,\big)\,. \tag{5.33}$$

Putting together all of the terms entails the claim. ■

Before carrying on with the proof of Proposition 5.4 we still need to prove a technical lemma relative to the large-λ behaviour of certain building blocks of $\widetilde{\mathscr{W}_\vartheta}[h]$.

Lemma 5.6 *Let $h \in C^{p+1}([0\,;\overline{x}_N])$. Then, the integrals*

$$\mathscr{J}_{1a}[h](\lambda) \;=\; \int\limits_{\mathbb{R}\,+\,\mathrm{i}\epsilon'} \frac{\chi_{1a}(\mu)\cdot\mathcal{F}[h](\mu)\cdot\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}}{\mu^{\delta_{2a}}(\mu-\lambda)}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \qquad \textit{with} \quad \delta_{2a} = \begin{cases} 1 & \text{if } a=2 \\ 0 & \text{if } a=1 \end{cases} \tag{5.34}$$

admit the $|\lambda| \to \infty$, $\mathrm{Im}\,\lambda > 2\epsilon' > 0$, asymptotic behaviour:

$$\mathscr{J}_{1a}[h](\lambda) \;=\; -\lambda^{-1}\mathscr{I}_{1a}[h] \;+\; \sum_{k=1}^{p}\frac{w^{(1/2)}_{k;a}(\lambda)}{(-\mathrm{i}\lambda)^{1/2}\lambda^k} \;+\; \sum_{k=1}^{p}\frac{w^{(1)}_{k;a}}{\lambda^{k+1}} \;+\; \mathrm{O}(\lambda^{-(p+3/2)}) \tag{5.35}$$

where

$$w^{(1/2)}_{k;a}(\lambda) \;=\; \sum_{\ell=0}^{k-1}\mathrm{i}^{k-\ell}h^{(k-\ell-1)}(\overline{x}_N)\Big\{\mathrm{sgn}(\mathrm{Re}\,\lambda)\,\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}\,[\chi_{\ell-\delta_{2a}}]_{1a} \;+\; \mathrm{i}\,[\chi_{\ell+1-\delta_{2a}}]_{2a}\Big\}\,, \tag{5.36}$$

and $w^{(1)}_{k;a}$ are constants whose explicit expression is given in the core of the proof .

Proof — The regularity of h implies the following decomposition for its Fourier transform:

$$\mathcal{F}[h](\mu) \;=\; -\sum_{k=0}^{p} \frac{h^{(k)}(\overline{x}_N)\,\mathrm{e}^{\mathrm{i}\mu\overline{x}_N} \;-\; h^{(k)}(0)}{(-\mathrm{i}\mu)^{k+1}} \;+\; \frac{(-1)^{p+1}}{(\mathrm{i}\mu)^{p+1}} \int_{0}^{\overline{x}_N} h^{(p+1)}(t)\,\mathrm{e}^{\mathrm{i}\xi\mu}\,\mathrm{d}\xi\;. \tag{5.37}$$

It gives directly access to the large-μ expansion:

$$\mu^{-\delta_{2a}}\chi_{1a}(\mu)\cdot\mathcal{F}[h](\mu) \;=\; \sum_{k=1}^{p} \frac{T_a^{(k)}(\mu)}{(-\mathrm{i}\mu)^{1/2}\mu^k} \;+\; R_{1a}^{(p)}(\mu)\;. \tag{5.38}$$

The remainder is $R_{1a}^{(p)}(\mu) \;=\; \mathrm{O}(\mu^{-p-3/2})$ when μ is large, whereas $T_a^{(k)}(\mu)$ remains bounded as long as $\mathrm{Im}\,\mu$ is bounded. Explicitly, these functions read:

$$T_a^{(k)}(\mu) \;=\; \sum_{\ell=0}^{k-1} \mathrm{i}^{k-\ell}\Big(h^{(k-1-\ell)}(0) - \mathrm{e}^{\mathrm{i}\mu\overline{x}_N}h^{(k-1-\ell)}(\overline{x}_N)\Big)\Big\{ -\mathrm{sgn}(\mathrm{Re}\,\mu)\,\mathrm{e}^{\mathrm{i}\mu\overline{x}_N}\,[\chi_{\ell-\delta_{2a}}]_{a1} \;-\; \mathrm{i}\,[\chi_{\ell+1-\delta_{2a}}]_{a2}\Big\} \tag{5.39}$$

where χ_m are the matrices appearing in the asymptotic expansion of χ, see (4.65). The integral of interest can be recast as

$$\begin{aligned}\mathscr{J}_{1a}[h](\lambda) \;&=\; \sum_{k=1}^{p}\int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{T_a^{(k)}(\mu)\,\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}}{(-\mathrm{i}\mu)^{1/2}\mu^k(\mu-\lambda)}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} - \sum_{\ell=0}^{p}\frac{1}{\lambda^{\ell+1}}\int_{\mathbb{R}+\mathrm{i}\epsilon'} \mu^{\ell}R_{1a}^{(p)}(\mu)\,\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \\ &\quad+ \int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\mu^{p+1}R_{1a}^{(p)}(\mu)}{\lambda^{p+1}(\mu-\lambda)}\,\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\;.\end{aligned} \tag{5.40}$$

In virtue of the bound on $R_{1a}^{(p)}$, the last term is a $\mathrm{O}(\lambda^{-p-\frac{3}{2}})$. In order to obtain the asymptotic expansion of the first term we study the model integral

$$J_k(\lambda) \;=\; \int_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{(c_1\mathrm{sgn}(\mathrm{Re}\,\mu) - c_2\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N})(\kappa_1\mathrm{e}^{\mathrm{i}\mu\overline{x}_N} - \kappa_2)}{(-\mathrm{i}\mu)^{1/2}\mu^k(\mu-\lambda)}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\;, \tag{5.41}$$

where $\mathrm{Im}\,\lambda > \epsilon'$ while c_1, c_2 and κ_1, κ_2 are free parameters. By deforming appropriately the contours, we get that:

$$J_k(\lambda) \;=\; \kappa_1\frac{c_1\mathrm{sgn}(\mathrm{Re}\,\lambda)\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N} - c_2}{(-\mathrm{i}\lambda)^{1/2}\lambda^k} \;-\; c_1\kappa_2\oint_{-\Gamma([0\,;\mathrm{i}\epsilon'])} \frac{\mathrm{sgn}(\mathrm{Re}\,\mu)}{(-\mathrm{i}\mu)^{1/2}\mu^k(\mu-\lambda)}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \tag{5.42}$$

$$+\, c_1\kappa_1(-\mathrm{i})^k\int_{\epsilon'}^{+\infty}\frac{t^{-k-1/2}\mathrm{e}^{-t\overline{x}_N}}{\mathrm{i}t-\lambda}\cdot\frac{\mathrm{d}t}{\pi} \;+\; c_2\kappa_2\oint_{-\Gamma(\mathrm{i}\mathbb{R}^-)}\frac{\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}\mu^{-k}}{(-\mathrm{i}\mu)^{1/2}(\mu-\lambda)}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \tag{5.43}$$

$$=\; \kappa_1\,\frac{c_1\mathrm{sgn}(\mathrm{Re}\,\lambda)\,\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N} - c_2}{(-\mathrm{i}\lambda)^{1/2}\lambda^k} \;-\; \sum_{q=0}^{p}\lambda^{-(q+1)}\,L_k^{(q)} \;+\; \lambda^{-(p+2)}\Delta_{[p]}\,M_k(\lambda)\;. \tag{5.44}$$

The constant $L_k^{(q)}$ occurring above is expressed in terms of integrals

$$L_k^{(q)} = -c_1\kappa_2\oint_{-\Gamma([0\,;\mathrm{i}\epsilon'])}\frac{\mathrm{sgn}(\mathrm{Re}\,\mu)\cdot\mu^{q-k}}{(-\mathrm{i}\mu)^{1/2}}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \;+\; c_1\kappa_1(-\mathrm{i})^{k-q}\int_{\epsilon'}^{+\infty}t^{q-k-1/2}\cdot\mathrm{e}^{-\overline{x}_N t}\cdot\frac{\mathrm{d}t}{\pi} + c_2\kappa_2\oint_{-\Gamma(\mathrm{i}\mathbb{R}^-)}\frac{\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}\cdot\mu^{q-k}}{(-\mathrm{i}\mu)^{1/2}}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\;.$$

(5.45)

and the remainder function reads:

$$\Delta_{[p]}M_k(\lambda) \;=\; c_1\kappa_2 \oint\limits_{-\Gamma([0\,;\mathrm{i}\epsilon'])} \frac{\lambda\cdot \mathrm{sgn}(\mathrm{Re}\,\mu)\cdot \mu^{p+1-k}}{(\mu-\lambda)\cdot(-\mathrm{i}\mu)^{1/2}}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \;-\; c_2\kappa_2 \oint\limits_{\Gamma(\mathrm{i}\mathbb{R}^-)} \frac{\lambda\cdot \mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}\cdot\mu^{p+1-k}}{(\,-\mathrm{i}\mu)^{1/2}(\mu-\lambda)}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}$$
$$+\; c_1\kappa_1(-\mathrm{i})^{k-p}\int\limits_{\epsilon'}^{+\infty}\frac{\lambda\cdot t^{p-k+1/2}\cdot\mathrm{e}^{-t\overline{x}_N}}{(t+\mathrm{i}\lambda)}\cdot\frac{\mathrm{d}t}{\pi}\;. \quad (5.46)$$

If we define:

$$\widetilde{w}^{(1)}_{k;a} \;=\; -\sum_{k=1}^{p}\sum_{\ell=0}^{k-1}\left\{L^{(q)}_k \;\middle|\; \begin{array}{ll} c_1\to-[\chi_{\ell-\delta_{2a}}]_{1a} & \kappa_1\to-\mathrm{i}^{k-\ell}h^{(k-\ell-1)}(\overline{x}_N) \\ c_2\to\mathrm{i}\,[\chi_{\ell+1-\delta_{2a}}]_{2a} & \kappa_2\to\mathrm{i}^{k-\ell}h^{(k-\ell-1)}(0)\end{array}\right\}. \quad (5.47)$$

we obtain:

$$\sum_{k=1}^{p}\int\limits_{\mathbb{R}+\mathrm{i}\epsilon'}\frac{T^{(k)}_a(\mu)\,\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}}{(-\mathrm{i}\mu)^{1/2}\mu^k(\mu-\lambda)}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \;=\; \sum_{k=1}^{p}\frac{w^{(1/2)}_{k;a}(\lambda)}{(-\mathrm{i}\lambda)^{1/2}\lambda^k} \;+\; \sum_{q=0}^{p}\frac{\widetilde{w}^{(1)}_{k;a}}{\lambda^{q+1}} \;+\; \mathrm{O}(\lambda^{-(p+2)})\;. \quad (5.48)$$

Furthermore, the above relation and equations (5.34) and (5.40), ensure that

$$\int\limits_{\mathbb{R}+\mathrm{i}\epsilon'} R^{(p)}_{1a}(\mu)\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \;=\; \mathscr{I}_{1a}[h] \;+\; \widetilde{w}^{(1)}_{0;a}\;. \quad (5.49)$$

Hence, putting all the terms together, we arrive to the expansion (5.35) with the constants $w^{(1)}_{k;a}$ given by

$$w^{(1)}_{k;a} \;=\; \widetilde{w}^{(1)}_{k;a} \;-\; \int\limits_{\mathbb{R}+i\epsilon'}\mu^k R^{(p)}_{1a}(\mu)\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \qquad k\geq 1\;. \quad (5.50)$$

■

Proof — (*of Proposition* 5.4). Given $h\in C^1([0\,;\overline{x}_N])$ and for $\xi\in]0\,;\overline{x}_N[$, we can represent $\widetilde{\mathscr{W}_0}$ as an integral taken in the Riemann sense[12]

$$\widetilde{\mathscr{W}_0}[h](\xi) \;=\; \int\limits_{\mathbb{R}+2\mathrm{i}\epsilon'}\mathrm{e}^{-\mathrm{i}\lambda\xi}\Big[\lambda\cdot\chi_{11}(\lambda)\mathscr{I}_{12}[h](\lambda) \;-\; \chi_{12}(\lambda)\mathscr{I}_{11}[h](\lambda)\Big]\frac{\mathrm{d}\lambda}{2\pi}\;, \quad (5.51)$$

where we remind that $\mathscr{I}_{1a}[h](\lambda)$ have been defined in (5.34). Using the asymptotic expansions of Lemma 4.5 for χ and those of Lemma 5.6 for $\mathscr{I}_{1a}[h]$, we can decompose:

$$\lambda\cdot\chi_{11}(\lambda)\mathscr{I}_{12}[h](\lambda)-\chi_{12}(\lambda)\mathscr{I}_{11}[h](\lambda) \;=\; \mathscr{I}_{12}[h]\cdot\frac{\mathrm{sgn}(\mathrm{Re}\,\lambda)\,e^{\mathrm{i}\lambda\overline{x}_N}+\mathrm{i}}{(-\mathrm{i}\lambda)^{1/2}} - \frac{\mathrm{i}\mathscr{I}_{11}[h]}{(-\mathrm{i}\lambda)^{1/2}} \quad (5.52)$$
$$+\;\frac{w^{(1/2)}_{1;2}(\lambda)\{\mathrm{sgn}(\mathrm{Re}\,\lambda)\,e^{\mathrm{i}\lambda\overline{x}_N}+\mathrm{i}\}-\mathrm{i}w^{(1/2)}_{1;1}(\lambda)}{\mathrm{i}\lambda} + \mathrm{O}(\lambda^{-3/2})\;.$$

[12]The fact that the integral (5.51) is well-defined in the Riemann sense will follow from the analysis carried out in this proof.

As a matter of fact, the coefficient of $1/(-\mathrm{i}\lambda)$ in this formula vanishes, as can be seen from the expressions (5.36) for $w_{1;a}^{(1/2)}$. Besides, integrating the $O(\lambda^{-3/2})$ in (5.51) yields a contribution remaining finite at $\xi = 0$ and $\xi = \overline{x}_N$, that we denote $\widetilde{\mathscr{W}}_0^{\mathrm{c}}[h] \in C^0([0\,;\overline{x}_N])$. Eventually, the effect of the first line of (5.52) once inserted in (5.51) is already described in (5.32)-(5.33). All in all, we find:

$$\widetilde{\mathscr{W}_0}[h](\xi) = \mathscr{I}_{12}[h]\bigg\{\frac{\mathrm{i}}{\sqrt{\pi\xi}} + \frac{\mathrm{i}}{\sqrt{\pi(\overline{x}_N-\xi)}} + \mathrm{O}(\sqrt{\overline{x}_N-\xi})\bigg\} - \frac{\mathrm{i}\mathscr{I}_{11}[h]}{\sqrt{\pi\xi}} + \widetilde{\mathscr{W}}_0^{\mathrm{c}}[h](\xi)\;. \tag{5.53}$$

Since we have $\widetilde{\mathscr{W}_\vartheta}[h](\xi) = \widetilde{\mathscr{W}_0}[h](\xi) + \vartheta\psi(\xi)$ in terms of the function ψ of Lemma 5.5, we deduce that:

$$\widetilde{\mathscr{W}}_{\mathscr{I}_{12}[h]}[h](\xi) \;=\; -\frac{\mathrm{i}\mathscr{I}_{11}[h]}{\sqrt{\pi\xi}} + O(\sqrt{\overline{x}_N-\xi}) + \widetilde{\mathscr{W}}_0^{\mathrm{c}}[h](\xi) \tag{5.54}$$

and this function is continuous on $[0\,;\overline{x}_N]$ if and only if $\mathscr{I}_{11}[h] = 0$. ■

5.3 A well-behaved inverse operator of $\mathcal{S}_{N;\gamma}$

Since, *in fine*, we are solely interested in solutions belonging to $(L^1\cap L^\infty)([0\,;\overline{x}_N])$ we shall henceforth only focus on $\widetilde{\mathscr{W}}_{\mathscr{I}_{12}[h]}[h]$ and denote this specific solution as $\mathscr{W}_{N;\gamma}[h]$. Furthermore, we shall restrict our reasoning to a class of functions such that $\mathscr{I}_{11}[h] = 0$. We now establish:

Proposition 5.7 *Let* $0 < s < 1/2$. *The subspace*

$$\mathscr{X}_s([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N]) \;=\; \Big\{h \in H_s([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N]) \;:\; \mathscr{I}_{11}[h] \;=\; 0\Big\} \tag{5.55}$$

is closed in $H_s([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N])$, *and the operator:*

$$\mathscr{S}_{N;\gamma} \;:\; H_s([0\,;\overline{x}_N]) \longrightarrow \mathscr{S}_{N;\gamma}[H_s([0\,;\overline{x}_N])] \;=\; \mathscr{X}_s([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N]) \tag{5.56}$$

is continuously invertible. Its inverse is the operator

$$\mathscr{W}_{N;\gamma} \;:\; \mathscr{X}_s([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N]) \longrightarrow H_s([0\,;\overline{x}_N])\;. \tag{5.57}$$

On functions $h \in C^1([0\,;\overline{x}_N])$, *it is defined as:*

$$\mathscr{W}_{N;\gamma}[h](\xi) \;=\; \int\limits_{\mathbb{R}+2\mathrm{i}\epsilon'}\frac{\mathrm{d}\lambda}{2\pi}\int\limits_{\mathbb{R}+\mathrm{i}\epsilon'}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\,\frac{\mathrm{e}^{-\mathrm{i}\lambda\xi-\mathrm{i}\mu\overline{x}_N}}{\mu-\lambda}\Big\{\chi_{11}(\lambda)\chi_{12}(\mu) - \frac{\mu}{\lambda}\cdot\chi_{11}(\mu)\chi_{12}(\lambda)\Big\}\mathcal{F}[h](\mu)\;. \tag{5.58}$$

For $h \in C^1([0\,;\overline{x}_N])$, $\mathscr{W}_{N;\gamma}[h](\xi)$ *is a continuous function on* $[0\,;\overline{x}_N]$, *which vanishes at least like a square root at* 0 *and* $\overline{x}_N$. *The operator* $\mathscr{W}_{N;\gamma}$ *extends continuously to* $H_s([0\,;\overline{x}_N])$, $0 < s < 1/2$ *although the constant of continuity of* $\mathscr{W}_{N;\gamma}$ *depends, a priori, on* N.

Comparing (5.58) with the double integral defining $\mathscr{W}_{N;\theta}$ in (5.20), one observes that λ/μ in front of $\chi_{11}(\lambda)$ is absent and that there is an additional pre-factor μ/λ in front of $\chi_{12}(\lambda)$.

Proof — *Continuity of* $\mathscr{W}_{N;\gamma}$.

Take $h \in C^1([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N])$. We first establish that $\mathscr{W}_{N;\gamma}[h]$, as defined by (5.58), extends as a continuous operator from $H_s([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N])$ to $H_s(\mathbb{R})$. We observe that:

$$\mathcal{F}\big[\mathrm{e}^{-2\epsilon' *}\mathscr{W}_{N;\gamma}[h]\big](\lambda) \;=\; \chi_{11}(\lambda+2\mathrm{i}\epsilon')\cdot C[\widehat{\chi}_{12}\mathcal{F}[h_{\epsilon'}]](\lambda+\mathrm{i}\epsilon') - \frac{\chi_{12}(\lambda+2\mathrm{i}\epsilon')}{\lambda+2\mathrm{i}\epsilon'}\cdot C[\widehat{\chi}_{11}\mathcal{F}[h_{\epsilon'}]](\lambda+\mathrm{i}\epsilon') \tag{5.59}$$

with $h_{\epsilon'}(\xi) \;=\; \mathrm{e}^{-\epsilon'\xi}\,h(\xi)$,

$$\widehat{\chi}_{11}(\mu) \;=\; (\mu+\mathrm{i}\epsilon')\,\chi_{11}(\mu+\mathrm{i}\epsilon')\,\mathrm{e}^{-\mathrm{i}(\mu+\mathrm{i}\epsilon')\overline{x}_N} \qquad \text{and} \qquad \widehat{\chi}_{12}(\mu) \;=\; \chi_{12}(\mu+\mathrm{i}\epsilon')\,\mathrm{e}^{-\mathrm{i}(\mu+\mathrm{i}\epsilon')\overline{x}_N}\;. \tag{5.60}$$

It thus follows from the growth at infinity of χ_{11} and χ_{12} and the continuity on $H_\tau(\mathbb{R})$, $|\tau|\leq 1/2$, of the transforms C_ϵ, where $C_\epsilon[f](\lambda)=C[f](\lambda+\mathrm{i}\epsilon')$, *cf.* Proposition A.2, that

$$\begin{aligned}
||\mathscr{W}_{N;\gamma}[h]||_{H_s(\mathbb{R})} \quad &\leq \quad C\Big\{\big\|C_{\epsilon'}[\widehat{\chi}_{12}\cdot\mathcal{F}[h_{\epsilon'}]]\big\|_{\mathcal{F}[H_{s-1/2}(\mathbb{R})]} + \big\|C_{\epsilon'}[\widehat{\chi}_{11}\cdot\mathcal{F}[h_{\epsilon'}]]\big\|_{\mathcal{F}[H_{s-1/2}(\mathbb{R})]}\Big\} &(5.61)\\
&\leq \quad C'\Big\{\big\|\widehat{\chi}_{12}\cdot\mathcal{F}[h_{\epsilon'}]\big\|_{\mathcal{F}[H_{s-1/2}(\mathbb{R})]} + \big\|\widehat{\chi}_{11}\cdot\mathcal{F}[h_{\epsilon'}]\big\|_{\mathcal{F}[H_{s-1/2}(\mathbb{R})]}\Big\} &(5.62)\\
&\leq \quad C''\,||h_{\epsilon'}||_{H_s(\mathbb{R})} \;\leq\; C'''\,||h||_{H_s([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N])}\;. &(5.63)
\end{aligned}$$

Proof — The space $\mathscr{X}_s([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N])$.

Given $h\in H_s([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N])$, we have:

$$\big|\mathscr{I}_{11}[h]\big| \;\leq\; \Big(\int_{\mathbb{R}}(1+|\mu|)^{-2s}|\chi_{11}(\mu)|^2\,\mathrm{d}\mu\Big)^{1/2}\cdot||h||_{H_s([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N])}\;. \tag{5.64}$$

As a consequence, $\mathscr{I}_{11}$ is a continuous linear form on $H_s([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N])$. In particular, its kernel is closed, what ensures that $\mathscr{X}_s([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N])$ is a closed subspace of $H_s([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N])$. We now establish that:

$$\mathscr{S}_{N;\gamma}[H_s([0\,;\overline{x}_N])]\subseteq\mathscr{X}_s([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N])\;. \tag{5.65}$$

Let $\varphi\in C^1([0\,;\overline{x}_N])$ and define $h=\mathscr{S}_{N;\gamma}[\varphi]$. Then, using the jump condition (5.28):

$$\mathscr{I}_{11}[h]=\int_0^{\overline{x}_N}\mathrm{d}\eta\,\varphi(\eta)\int_{\mathbb{R}}\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}\chi_{11;+}(\mu)\,\frac{\mathcal{F}[S_\gamma](\mu)}{2\mathrm{i}\pi\beta}\,\mathrm{e}^{\mathrm{i}\mu\eta}=\int_0^{\overline{x}_N}\mathrm{d}\eta\,\varphi(\eta)\int_{\mathbb{R}}\Big(\chi_{21;-}(\mu)\mathrm{e}^{\mathrm{i}\mu(\eta-\overline{x}_N)}+\chi_{21;+}(\mu)\mathrm{e}^{\mathrm{i}\mu\eta}\Big)\,\mathrm{d}\mu \tag{5.66}$$

and this quantity vanishes according to (5.29). The equality can then be extended to the whole of $H_s([0\,;\overline{x}_N])$, $0<s<1/2$ since $\mathscr{I}_{11}$ and $\mathscr{S}_{N;\gamma}$ are continuous on this space and $C^1([0\,;\overline{x}_N])$ is dense in $H_s([0\,;\overline{x}_N])$.

Proof — Relation to the inverse.

By definition, for any $h \in (H_s \cap C^1)([-\gamma\overline{x}_N\,;(1+\gamma)\overline{x}_N])$, we have:

$$
\begin{aligned}
\widetilde{\mathscr{W}}_{\mathscr{I}_{12}[h]}[h](\xi) \;=\; & \int\limits_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\pi} \int\limits_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \frac{\mathrm{e}^{-\mathrm{i}\lambda\xi-\mathrm{i}\mu\overline{x}_N}}{\mu-\lambda} \Big\{ \frac{\lambda}{\mu}\cdot\chi_{11}(\lambda)\chi_{12}(\mu) - \chi_{11}(\mu)\chi_{12}(\lambda) \Big\} \cdot \mathcal{F}[h](\mu) \\
& + \Big(\int\limits_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{-\mathrm{i}\lambda\xi} \chi_{11}(\lambda) \Big) \cdot \Big(\int\limits_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \frac{\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}}{\mu} \chi_{12}(\mu)\cdot\mathcal{F}[h](\mu) \Big) \\
= & \int\limits_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\pi} \int\limits_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \frac{\mathrm{e}^{-\mathrm{i}\lambda\xi-\mathrm{i}\mu\overline{x}_N}}{\mu-\lambda} \Big\{ \chi_{11}(\lambda)\chi_{12}(\mu) - \chi_{11}(\mu)\chi_{12}(\lambda) \Big\} \cdot \mathcal{F}[h](\mu) \\
& - \Big(\int\limits_{\mathbb{R}+2\mathrm{i}\epsilon'} \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{-\mathrm{i}\lambda\xi} \frac{\chi_{12}(\lambda)}{\lambda} \Big) \cdot \underbrace{\Big(\int\limits_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \mathrm{e}^{-\mathrm{i}\mu\overline{x}_N} \chi_{11}(\mu)\cdot\mathcal{F}[h](\mu) \Big)}_{=0} \;=\; \mathscr{W}_{N;\gamma}[h](\xi)\,. \qquad (5.67)
\end{aligned}
$$

In the last line, we used the freedom to add a term proportional to $\mathscr{I}_{11}[h]=0$, so that the combination retrieves the announced expression (5.58). The continuity of the linear functional $\mathscr{I}_{12}$ on $H_s([0\,;\overline{x}_N])$ is proven analogously to (5.64), hence ensuring the continuity of the operator $\widetilde{\mathscr{W}}_{\mathscr{I}_{12}[h]}$. Since both operators $\mathscr{W}_{N;\gamma}$ and $\widetilde{\mathscr{W}}_{\mathscr{I}_{12}[h]}$ are continuous on $H_s([0\,;\overline{x}_N])$ and coincide on C^1 functions which form a dense subspace, they coincide on the whole $H_s([0\,;\overline{x}_N])$. From there we deduce two facts:

- we indeed have $\mathscr{S}_{N;\gamma}[\mathscr{W}_{N;\gamma}[h]] \;=\; h$, as a consequence of $\mathscr{S}_{N;\gamma}[\widetilde{\mathscr{W}}_{\mathscr{I}_{12}[h]}[h]] \;=\; h$. This shows that the reverse inclusion to (5.65) holds as well.
- The function $\mathscr{W}_{N;\gamma}[h]$ is supported on $[0\,;\overline{x}_N]$ (and thus belongs to $H_s([0\,;\overline{x}_N])$) since Lemma 4.1 ensures that $\widetilde{\mathscr{W}}_{\mathscr{I}_{12}[h]}[h]$ is supported on $[0\,;\overline{x}_N]$ this for any $h \in H_s([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N]) \subseteq H_\tau([-\gamma\overline{x}_N\,;(\gamma+1)\overline{x}_N])$ with $0<s<1/2$ and $-1<\tau<0$.

Proof — Local behaviour for $C^1([0\,;\overline{x}_N])$ functions.
It follows from a slight improvement of the local estimates carried out in the proof of Proposition 5.4 that, given $h \in C^1([0\,;\overline{x}_N])$, we have:

$$
\mathscr{W}_{N;\gamma}[h](\xi) \;=\; C_L^{(0)} + C_L^{(1/2)}\sqrt{\xi} + \mathrm{O}(\xi) \qquad \mathscr{W}_{N;\gamma}[h](\xi) \;=\; C_R^{(0)} + C_R^{(1/2)}\sqrt{\overline{x}_N-\xi} + \mathrm{O}(\overline{x}_N-\xi)\,,
$$

form some constants $C_{L/R}^{(a)}$ with $a \in \{0, 1/2\}$. It thus remains to check that $C_L^{(0)} = C_R^{(0)} = 0$. It follows also from the proof of Proposition 5.4 that $\mathscr{W}_{N;\gamma}[h]$ is, in fact, continuous on $\mathbb{R}$. Since $\operatorname{supp}[\mathscr{W}_{N;\gamma}[h]] = [0\,;\overline{x}_N]$, the function has to vanish at 0 and $\overline{x}_N$ so as to ensure its continuity. Thence, $C_L^{(0)} = C_R^{(0)} = 0$. ■

5.4 $\mathcal{W}_N$: the inverse operator of $\mathcal{S}_N$

In order to construct the inverse to $\mathscr{S}_N$, we should take the limit $\gamma \to +\infty$ in the previous formulae. It so happens that this limit is already well-defined at the level of the solution to the Riemann–Hilbert problem for χ as defined through Figure 2. More precisely, from now on, let χ be as defined in Figure 3 where the matrix Π is as defined through (4.47)-(4.49) with the exception that one should send $\gamma \to +\infty$ in the jump matrices for Ψ (4.39)-(4.40). Note that, in this limit, $G_\Psi = I_2$ on $\mathbb{R}+\mathrm{i}\epsilon$, *viz.* Ψ is continuous across $\mathbb{R}+\mathrm{i}\epsilon$. Then, we can come back to the inversion of the initial operator $\mathcal{S}_N$ in unrescaled variables – compare (4.1), (4.2) and (4.3).

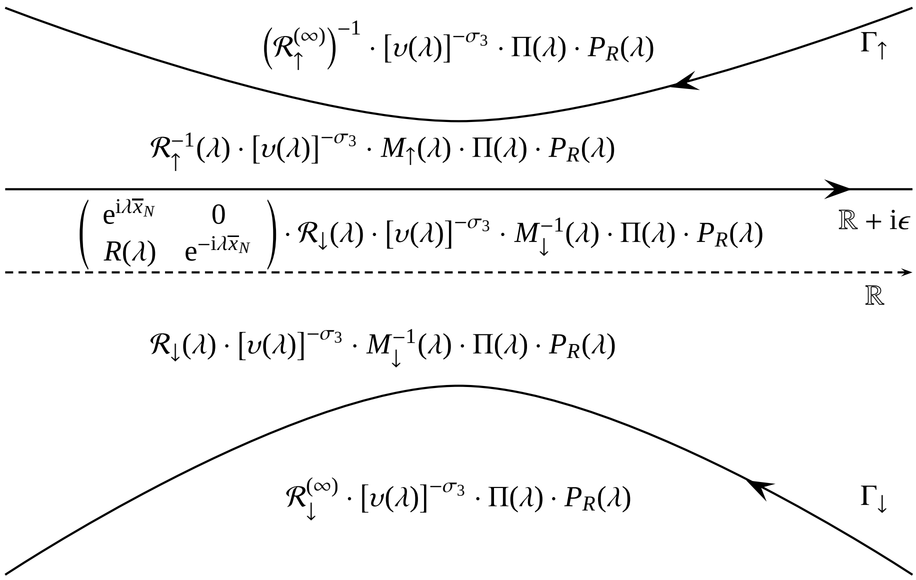

Figure 3: Piecewise definition of the matrix χ (at $\gamma \to +\infty$). The curves $\Gamma_{\uparrow/\downarrow}$ separate all poles of $\lambda \mapsto \lambda R(\lambda)$ from $\mathbb{R}$ and are such that $\mathrm{dist}(\Gamma_{\uparrow/\downarrow}, \mathbb{R}) > \delta$ for some $\delta > 0$ but sufficiently small.

Proposition 5.8 *Let $0 < s < 1/2$. The operator $\mathcal{S}_N \; : \; H_s([a_N\,; b_N]) \longrightarrow H_s(\mathbb{R})$ is continuous and invertible on its image:*

$$\mathfrak{X}_s(\mathbb{R}) = \left\{ H \in H_s(\mathbb{R}) \quad : \quad \int\limits_{\mathbb{R}+i\epsilon} \chi_{11}(\mu) \mathcal{F}[H](N^{\alpha}\mu) e^{-iN^{\alpha}\mu b_N} \cdot \frac{d\mu}{2i\pi} \; = \; 0 \right\} . \tag{5.68}$$

The inverse is then given by the operator $\mathcal{W}_N \; : \; \mathfrak{X}_s(\mathbb{R}) \longrightarrow H_s([a_N\,; b_N])$ defined in (2.44)*:*

$$\mathcal{W}_N[H](\xi) \;=\; \frac{N^{2\alpha}}{2\pi\beta} \int\limits_{\mathbb{R}+2i\epsilon} \frac{d\lambda}{2i\pi} \int\limits_{\mathbb{R}+i\epsilon} \frac{d\mu}{2i\pi} \, \frac{e^{-iN^{\alpha}\lambda(\xi - a_N)}}{\mu - \lambda} \left\{ \chi_{11}(\lambda)\chi_{12}(\mu) - \frac{\mu}{\lambda} \cdot \chi_{11}(\mu)\chi_{12}(\lambda) \right\} e^{-iN^{\alpha}\mu b_N} \mathcal{F}[H](N^{\alpha}\mu) \tag{5.69}$$

with χ being understood as defined in Figure 3.

Proof — Starting from the expression for the inverse operator to $\mathscr{S}_{N;\gamma}$ and carrying out the change of variables, one obtains an operator $\mathcal{W}_{N;\gamma}$ which corresponds to the inverse of the operator $\mathcal{S}_{N;\gamma}$. Then, in this expression we replace χ at finite γ by the solution χ at $\gamma \to +\infty$, as it is given in Figure 3. This corresponds to the operator $\mathcal{W}_N$, as defined in (2.44). One can then verify explicitly on the integral representation for $\mathcal{W}_N$ by using certain elements of the Riemann–Hilbert problem satisfied by χ that the equation $\mathcal{S}_N[\mathcal{W}_N[H]] = H$ does hold on $[a_N\,; b_N]$. All the other conclusions of the theorem can be proved similarly to Proposition 5.7. ■

We describe a symmetry of the integral transform $\mathcal{W}_N$ that will appear handy in the remaining of the text.

Lemma 5.9 *The operator $\mathcal{W}_N$ has the reflection symmetry:*

$$\mathcal{W}_N[H](a_N + b_N - \xi) \;=\; -\mathcal{W}_N[H^\wedge](\xi) \tag{5.70}$$

where we agree upon $H^\wedge(\xi) = H(a_N + b_N - \xi)$.

Proof — It follows from the jump conditions satisfies by χ and from Lemma 4.4 that, for $\lambda \in \mathbb{R}$,

$$\chi_{11;+}(-\lambda) \;=\; \mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N} \cdot \chi_{11;+}(\lambda) \quad \text{and} \quad \chi_{12;+}(-\lambda) \;=\; \mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N} \cdot \Big(\chi_{12;+}(\lambda) - \lambda\chi_{11;+}(\lambda)\Big)\,. \tag{5.71}$$

Upon squeezing the contours of integration in the integral representation for $\mathcal{W}_N$ to $\mathbb{R}$ we get, in particular, the + boundary values of χ_{1a}. It is then enough to implement the change of variables $(\lambda,\mu,\eta) \mapsto (-\lambda,-\mu,b_N + a_N - \eta)$ and observe that, all in all, the unwanted terms cancel out. ■

In the case of a constant argument (which clearly does *not* belong to $\mathfrak{X}_s(\mathbb{R})$) the expression for $\mathcal{W}_N$ simplifies:

Lemma 5.10 *The function $\mathcal{W}_N[1](\xi)$ admits the one-fold integral representation*

$$\mathcal{W}_N[1](\xi) \;=\; -\frac{N^\alpha\chi_{12;+}(0)}{2\mathrm{i}\pi\beta}\int\limits_{\mathbb{R}+\mathrm{i}\epsilon'} \frac{\chi_{11}(\lambda)}{\lambda}\mathrm{e}^{-\mathrm{i}N^\alpha\lambda(\xi-a_N)} \cdot \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\,. \tag{5.72}$$

Proof — Starting from the representation (2.44) we get, for any $\xi \in]a_N\,;b_N[$,

$$\mathcal{W}_N[1](\xi) \;=\; \frac{N^\alpha}{2\pi\mathrm{i}\beta}\int\limits_{\mathbb{R}+2\mathrm{i}\epsilon}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\int\limits_{\mathbb{R}+\mathrm{i}\epsilon}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\frac{\mathrm{e}^{-\mathrm{i}N^\alpha(\xi-a_N)\lambda}}{\mu-\lambda}\bigg\{\frac{1}{\mu}\cdot\chi_{11}(\lambda)\chi_{12}(\mu) - \frac{1}{\lambda}\cdot\chi_{11}(\mu)\chi_{12}(\lambda)\bigg\}\cdot\Big(1 \,-\, \mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}\Big)\,. \tag{5.73}$$

One should then treat the terms involving the function 1 and $\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}$ arising in the right-hand side differently. The part involving 1 is zero as can be seen by deforming the μ-integral up to $+\mathrm{i}\infty$. In what concerns the part involving $\mathrm{e}^{-\mathrm{i}\mu\overline{x}_N}$, we deform the μ-integral up to $-\mathrm{i}\infty$ by using the jump conditions $\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N}\chi_{1a;+}(\lambda) \;=\; \chi_{1a;-}(\lambda)$. Solely the pole at $\mu = 0$ contributes, hence leading to (5.72). ■

6 Local behaviour of $\mathcal{W}_N[H](\xi)$ in ξ, uniformly in N

In this section we derive a local (in ξ), uniform (in N), behaviour of the inverse $\mathcal{W}_N[H](\xi)$. This will allow an effective simplification, in the large-N limit, of the various integrals involving $\mathcal{W}_N[H]$ arising from the Schwinger-Dyson equations of Proposition 3.13. Furthermore, these local asymptotics will provide a base for estimating the W_p^∞ norms of the inverse of the master operator $\mathcal{U}_N$, *cf.* (3.54). In fact, such estimates demand to have a control on the leading and sub-leading contributions issuing from $\mathcal{W}_N$ in respect to W_p^∞ norms. We shall demonstrate in § 6.1 that the operator $\mathcal{W}_N$ can be decomposed as

$$\mathcal{W}_N \;=\; \mathcal{W}_R \,+\, \mathcal{W}_{\mathrm{bk}} \,+\, \mathcal{W}_L \,+\, \mathcal{W}_{\exp}\,. \tag{6.1}$$

The operator $\mathcal{W}_{\exp}$ represents an exponentially small remainder in W_p^∞ norm, while the three other operators contribute to the leading order asymptotics when $N \to \infty$. Their expression is constructed solely out of the leading asymptotics in N of the solution χ to the Riemann–Hilbert problem given in Proposition 4.3.

In § 6.2 we shall build on this decomposition so as to show that there arise two regimes for the large-N asymptotic behaviour of $\mathcal{W}_N[H]$ namely when

- ξ is in the "bulk" of $[a_N\,;b_N]$, *i.e.* uniformly in N away from the endpoints a_N and b_N.
- ξ is close to the boundaries, *viz.* in the vicinity of the endpoints a_N (resp b_N).

In addition to providing the associated asymptotic expansions, we shall also establish certain properties of the remainders which will turn out to be crucial for our further purposes.

6.1 An appropriate decomposition of $\mathcal{W}_N$

We remind that any function $H \in C^k([a_N\,;b_N])$ admits a continuation into a function $C^k_{\mathrm{c}}(]a_N-\eta\,;b_N+\eta[)$ for some $\eta>0$. We denote any such extension by H_{e}, as it was already specified in the notation and basic definition section. In the present subsection we establish a decomposition that is adapted for deriving the local and uniform in N asymptotic expansion for $\mathcal{W}_N$.

In this section and the next ones, we will use extensively the following notations:

Definition 6.1 *To a variable ξ on the real line, we associate $x_R = N^{\alpha}(b_N-\xi)$ and $x_L = N^{\alpha}(\xi-a_N)$ the corresponding rescaled and centred around the right (resp. left) boundary variables. Similarly, for a variable η, we denote y_R and y_L its rescaled and centred variable.*

Definition 6.2 *If H is a function of a variable ξ, we denote $H^{\wedge}(\xi) = H(a_N+b_N-\xi)$ its reflection around the centre of $[a_N\,;b_N]$ (as already met in Lemma 5.9). This exchanges the role of the left and right boundaries. If H is a function of many variables, by $H^{\wedge}$ we mean that all variables are simultaneously reflected. If O is an operator, we define the reflected operator by:*

$$O^{\wedge}[H] = (O[H^{\wedge}])^{\wedge} \tag{6.2}$$

Definition 6.3 *Let $\mathscr{C}^{(+)}_{\mathrm{reg}}$ (resp. $\mathscr{C}^{(-)}_{\mathrm{reg}}$) be a contour such that:*

- *it passes between $\mathbb{R}$ and $\Gamma_{\uparrow}$ (resp. $\Gamma_{\downarrow}$).*
- *it comes from infinity in the direction of angle $\mathrm{e}^{\pm 3\mathrm{i}\pi/4}$ and goes to infinity in the direction of angle $\mathrm{e}^{\pm\mathrm{i}\pi/4}$.*

These contours are depicted in Figure 4, and we denote $\varsigma/2 = \mathrm{dist}(\mathscr{C}^{(+)}_{\mathrm{reg}},\mathbb{R}) > 0$. We also introduce an odd function J by setting, for $x>0$:

$$J(x) \;=\; \int\limits_{\mathscr{C}^{(+)}_{\mathrm{reg}}} \frac{\mathrm{e}^{\mathrm{i}\lambda x}}{R(\lambda)}\,\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\;. \tag{6.3}$$

Proposition 6.4 *Given any function $H\in C^k([a_N\,;b_N])$ with $k\geq 1$ belonging $\mathfrak{X}_s(\mathbb{R})$ (the image of $\mathcal{S}_N$, see (5.68)), the function $\mathcal{W}_N[H]$ is $C^{k-1}(]a_N\,;b_N[)$ and admits the representation*

$$\mathcal{W}_N[H](\xi) \;=\; \mathcal{W}_R[H_{\mathrm{e}}](x_R,\xi) \;+\; \mathcal{W}_{\mathrm{bk}}[H_{\mathrm{e}}](\xi) \;+\; \mathcal{W}_L[H_{\mathrm{e}}](x_L,\xi) \;+\; \mathcal{W}_{\exp}[H](\xi) \tag{6.4}$$

with:

$$\mathcal{W}_{\mathrm{bk}}[H_{\mathrm{e}}](\xi) \;=\; \frac{N^{\alpha}}{2\pi\beta}\int\limits_{\mathbb{R}}\big[H_{\mathrm{e}}(\xi+N^{-\alpha}y)-H_{\mathrm{e}}(\xi)\big]\,J(y)\cdot\mathrm{d}y\,, \tag{6.5}$$

$$\begin{aligned}\mathcal{W}_R[H_{\mathrm{e}}](x,\xi) \;=\;& -\frac{N^{\alpha}}{2\pi\beta}\int\limits_{x}^{+\infty}\big[H_{\mathrm{e}}(\xi+N^{-\alpha}y)-H_{\mathrm{e}}(\xi)\big]\,J(y)\cdot\mathrm{d}y\,,\\ & -\frac{N^{2\alpha}}{2\pi\beta}\int\limits_{\mathscr{C}^{(+)}_{\mathrm{reg}}}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\int\limits_{\mathscr{C}^{(-)}_{\mathrm{reg}}}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\,\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{(\mu-\lambda)R_{\downarrow}(\lambda)R_{\uparrow}(\mu)}\bigg\{\int\limits_{a_N}^{b_N}H_{\mathrm{e}}(\eta)\mathrm{e}^{\mathrm{i}\mu N^{\alpha}(\eta-b_N)}\mathrm{d}\eta\;-\;\frac{H_{\mathrm{e}}(\xi)}{\mathrm{i}\mu N^{\alpha}}\bigg\}\end{aligned} \tag{6.6}$$

$$\mathcal{W}_L[H_{\mathrm{e}}](x,\xi) \;=\; -\mathcal{W}_R[H^{\wedge}_{\mathrm{e}}](x,a_N+b_N-\xi)\;. \tag{6.7}$$

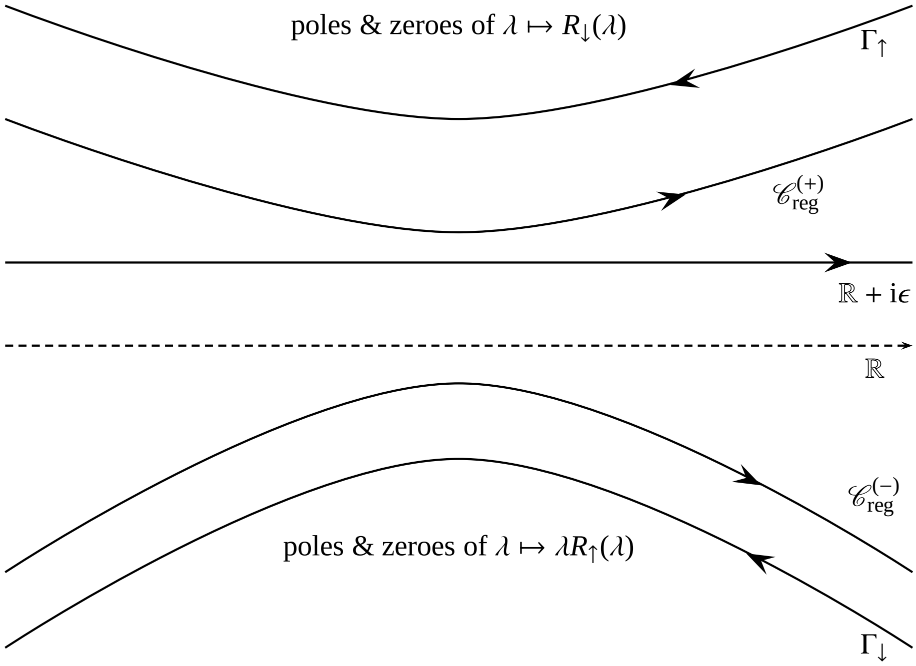

Figure 4: The curves $\mathscr{C}^{(\pm)}_{\mathrm{reg}}$.

The remainder operator $\mathcal{W}_{\exp}[H_{\mathfrak{e}}]$ *reads:*

$$\mathcal{W}_{\exp} \;=\; \mathcal{W}_N^{(++)} \;-\; (\mathcal{W}_N^{(++)})^{\wedge} \;+\; \mathcal{W}_{\mathrm{res}} \;-\; (\mathcal{W}_{\mathrm{res}})^{\wedge} + \Delta\mathcal{W}_N^{(+-)} \;-\; (\Delta\mathcal{W}_N^{(+-)})^{\wedge}\;, \tag{6.8}$$

where the operators $\mathcal{W}_N^{(++)}$ *and* $\Delta\mathcal{W}_N^{(+-)}$ *are given by*

$$\begin{aligned}
\mathcal{W}_N^{(++)}[H](\xi) \;&=\; \frac{N^{2\alpha}}{2\pi\beta}\int_{\mathscr{C}^{(+)}_{\mathrm{reg}}}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\int_{\mathscr{C}^{(+)}_{\mathrm{reg}}}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\int_{a_N}^{b_N}\mathrm{d}\eta\,\frac{\mathrm{e}^{-\mathrm{i}N^{\alpha}\lambda(\xi-b_N)+\mathrm{i}N^{\alpha}\mu(\eta-a_N)}}{(\mu-\lambda)R_{\downarrow}(\lambda)R_{\downarrow}(\mu)}\Big\{\Psi_{11}(\lambda)\Psi_{12}(\mu)-\frac{\mu}{\lambda}\cdot\Psi_{11}(\mu)\Psi_{12}(\lambda)\Big\}H(\eta)\;,\\
\Delta\mathcal{W}_N^{(+-)}[H](\xi) \;&=\; \frac{N^{2\alpha}}{2\pi\beta}\int_{\mathscr{C}^{(+)}_{\mathrm{reg}}}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\int_{\mathscr{C}^{(-)}_{\mathrm{reg}}}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\int_{a_N}^{b_N}\mathrm{d}\eta\frac{\mathrm{e}^{-\mathrm{i}N^{\alpha}\lambda(\xi-b_N)+\mathrm{i}N^{\alpha}\mu(\eta-b_N)}}{(\mu-\lambda)R_{\downarrow}(\lambda)R_{\uparrow}(\mu)}\Big\{1+\frac{\mu}{\lambda}\cdot\Psi_{21}(\mu)\Psi_{12}(\lambda)-\Psi_{11}(\lambda)\Psi_{22}(\mu)\Big\}H(\eta)
\end{aligned} \tag{6.9}$$

while $\mathcal{W}_{\mathrm{res}}$ *is the one-form:*

$$\mathcal{W}_{\mathrm{res}}[H] \;=\; \frac{N^{2\alpha}}{2\pi\beta}\,\frac{\Pi_{12}(0)\theta_R}{R_{\downarrow}(0)}\int_{\mathscr{C}^{(+)}_{\mathrm{reg}}}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\,\frac{\mu\Psi_{11}(\mu)}{R_{\downarrow}(\mu)}\int_{a_N}^{b_N}\mathrm{d}\eta\,H(\eta)\,\mathrm{e}^{\mathrm{i}N^{\alpha}\mu(\eta-a_N)}\;. \tag{6.10}$$

The piecewise holomorphic matrix $\Psi(\mu)$ *corresponds to the solution to the Riemann–Hilbert problem for* Ψ *described in Section 4.4 in which we have taken the limit* $\gamma \to +\infty$.

In the expressions above, we have used an extension H_e of H whenever it was necessary to integrate H over the whole real line, but we can keep H when only the integrals over $[a_N\,;b_N]$ are involved – e.g. in (6.9). The decomposition given in Proposition 6.4 splits $\mathcal{W}_N$ into a sum of four operators. The operator $\mathcal{W}_{\mathrm{bk}}$ takes into account the purely bulk-type contribution of the inverse, namely those which do not feel the presence of the boundaries a_N, b_N of the support of the equilibrium measure. This operator does not single out a specific point but rather takes values which are of the same order of magnitude throughout the whole of the interval $[a_N\,;b_N]$. In their turn the operators $\mathcal{W}_{R/L}$ represent the contributions of the right/left boundaries of the support of the equilibrium measure. These operators localise, with exponential precision, on their respective left or right boundary. Namely, they decay exponentially fast in $x_{R/L}$ when $x_{R/L} \to +\infty$. This fact is a consequence of the exponential decay at $\pm\infty$ of $J(x)$ in what concerns the first integral in (6.6) and an immediate bound of the second one which follows from $\inf\{\mathrm{Im}\,\lambda,\ \lambda \in \mathscr{C}^{(+)}_{\mathrm{reg}}\} > 0$.

Proof — We remind that since we are considering the $\gamma \to +\infty$ limit, the matrix Ψ has no jump across $\mathbb{R} + \mathrm{i}\epsilon$. A straightforward calculation based on the identity:

$$\chi(\lambda) \;=\; \begin{pmatrix} -R_{\downarrow}^{-1}(\lambda)\,\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N} & R_{\uparrow}^{-1}(\lambda) \\ -R_{\uparrow}(\lambda) & 0 \end{pmatrix} \cdot \Psi(\lambda) \qquad \text{valid for } \lambda \text{ between } \mathbb{R} \text{ and } \Gamma_{\uparrow} \tag{6.11}$$

shows that, for such λ's and μ's,

$$\frac{N^{2\alpha}}{2\pi\beta} \cdot \mathrm{e}^{-\mathrm{i}N^{\alpha}\lambda(\xi - a_N)} \Big\{ \chi_{11}(\lambda)\chi_{12}(\mu) \;-\; \frac{\mu}{\lambda} \cdot \chi_{11}(\mu)\chi_{12}(\lambda) \Big\}\, \mathrm{e}^{\mathrm{i}N^{\alpha}\mu(\eta - b_N)} \;=\; \sum_{\epsilon_1,\epsilon_2 \in \{\pm\}} K_{\epsilon_1,\epsilon_2}(\lambda,\mu \mid \xi,\eta)\,. \tag{6.12}$$

The above decomposition contains four kernels

$$K_{--}(\lambda,\mu \mid \xi,\eta) \;=\; \frac{N^{2\alpha}}{2\pi\beta}\, \frac{\mathrm{e}^{-\mathrm{i}N^{\alpha}\lambda(\xi - a_N) + \mathrm{i}N^{\alpha}\mu(\eta - b_N)}}{R_{\uparrow}(\lambda)R_{\uparrow}(\mu)} \Big\{ \Psi_{21}(\lambda)\Psi_{22}(\mu) \;-\; \frac{\mu}{\lambda}\Psi_{21}(\mu)\Psi_{22}(\lambda) \Big\}\,, \tag{6.13}$$

$$K_{++}(\lambda,\mu \mid \xi,\eta) \;=\; \frac{N^{2\alpha}}{2\pi\beta}\, \frac{\mathrm{e}^{-\mathrm{i}N^{\alpha}\lambda(\xi - b_N) + \mathrm{i}N^{\alpha}\mu(\eta - a_N)}}{R_{\downarrow}(\lambda)R_{\downarrow}(\mu)} \Big\{ \Psi_{11}(\lambda)\Psi_{12}(\mu) \;-\; \frac{\mu}{\lambda}\Psi_{11}(\mu)\Psi_{12}(\lambda) \Big\}\,, \tag{6.14}$$

$$K_{+-}(\lambda,\mu \mid \xi,\eta) \;=\; -\frac{N^{2\alpha}}{2\pi\beta}\, \frac{\mathrm{e}^{-\mathrm{i}N^{\alpha}\lambda(\xi - b_N) + \mathrm{i}N^{\alpha}\mu(\eta - b_N)}}{R_{\downarrow}(\lambda)R_{\uparrow}(\mu)} \Big\{ \Psi_{11}(\lambda)\Psi_{22}(\mu) \;-\; \frac{\mu}{\lambda}\Psi_{21}(\mu)\Psi_{12}(\lambda) \Big\}\,, \tag{6.15}$$

$$K_{-+}(\lambda,\mu \mid \xi,\eta) \;=\; -\frac{N^{2\alpha}}{2\pi\beta}\, \frac{\mathrm{e}^{-\mathrm{i}N^{\alpha}\lambda(\xi - a_N) + \mathrm{i}N^{\alpha}\mu(\eta - a_N)}}{R_{\uparrow}(\lambda)R_{\downarrow}(\mu)} \Big\{ \Psi_{21}(\lambda)\Psi_{12}(\mu) \;-\; \frac{\mu}{\lambda}\Psi_{11}(\mu)\Psi_{22}(\lambda) \Big\}\,. \tag{6.16}$$

The labeling of the kernels $K_{\epsilon_1,\epsilon_2}(\lambda,\mu \mid \xi,\eta)$ by the subscripts ϵ_1, ϵ_2 refers to the half-planes $\mathbb{H}^{\epsilon_1} \times \mathbb{H}^{\epsilon_2}$ in which they are exponentially small when $N \to \infty$, provided that the variables $\xi, \eta \in [a_N\,;b_N]$ are uniformly away from the boundaries a_N or b_N.

One should note that the above kernels $K_{\epsilon_1,\epsilon_2}$ have a simple pole at $\lambda = 0$. In particular,

$$\mathrm{Res}\Big(K_{-+}(\lambda,\mu \mid \xi,\eta)\,\mathrm{d}\lambda, \lambda = 0 \Big) \;=\; \frac{\mu\Psi_{11}(\mu)}{R_{\downarrow}(\mu)} \cdot \mathrm{e}^{\mathrm{i}N^{\alpha}\mu(\eta - a_N)}\, \frac{N^{2\alpha} \cdot \theta_R \cdot \Pi_{12}(0)}{2\pi\beta \cdot (\lambda R_{\uparrow}(\lambda))_{|\lambda=0}}\,, \tag{6.17}$$

$$\mathrm{Res}\Big(K_{--}(\lambda,\mu \mid \xi,\eta)\,\mathrm{d}\lambda, \lambda = 0 \Big) \;=\; -\frac{\mu\Psi_{21}(\mu)}{R_{\uparrow}(\mu)} \cdot \mathrm{e}^{\mathrm{i}N^{\alpha}\mu(\eta - b_N)}\, \frac{N^{2\alpha} \cdot \theta_R \cdot \Pi_{12}(0)}{2\pi\beta \cdot (\lambda R_{\uparrow}(\lambda))_{|\lambda=0}}\,. \tag{6.18}$$

Furthermore, the kernels are related. Indeed, according to the definition of Ψ in terms of χ in Figure 1, we have for λ between $\Gamma_{\downarrow}$ and $\mathbb{R}$:

$$\chi(\lambda) \;=\; \begin{pmatrix} -R_{\downarrow}^{-1}(\lambda) & R_{\uparrow}^{-1}(\lambda)\,\mathrm{e}^{-\mathrm{i}\lambda\overline{x}_N} \\ 0 & R_{\downarrow}(\lambda) \end{pmatrix} \cdot \Psi(\lambda) \tag{6.19}$$

and by invoking the reflection relation for χ obtained in Lemma 4.4, we can show that:

$$\Psi(-\lambda) \;=\; \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix} \cdot \Psi(\lambda) \cdot \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} . \tag{6.20}$$

The above equation ensures that

$$K_{-+}(\,-\lambda, -\mu \mid a_N + b_N - \xi, a_N + b_N - \eta) \;=\; K_{+-}(\lambda, \mu \mid \xi, \eta)\,, \tag{6.21}$$

$$K_{--}(\,-\lambda, -\mu \mid a_N + b_N - \xi, a_N + b_N - \eta) \;=\; K_{++}(\lambda, \mu \mid \xi, \eta)\,. \tag{6.22}$$

The decomposition (6.12) of the integral kernel allows one recasting the operator $\mathcal{W}_N$ as:

$$\mathcal{W}_N[H](\xi) \;=\; \sum_{\epsilon_1,\epsilon_2\in\{\pm1\}} \widetilde{\mathcal{W}}_N^{(\epsilon_1\epsilon_2)}[H](\xi) \tag{6.23}$$

where

$$\widetilde{\mathcal{W}}_N^{(\epsilon_1\epsilon_2)}[H](\xi) \;=\; \int_{\mathbb{R}+2\mathrm{i}\epsilon} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \int_{\mathbb{R}+\mathrm{i}\epsilon} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \int_{a_N}^{b_N} \mathrm{d}\eta\, \frac{K_{\epsilon_1\epsilon_2}(\lambda,\mu \mid \xi,\eta)}{\mu-\lambda} H(\eta)\,. \tag{6.24}$$

The next step consists in deforming the contours arising in the definition of $\widetilde{\mathcal{W}}_N^{(\epsilon_1\epsilon_2)}[H]$. We shall discuss these handlings on the example of $\widetilde{\mathcal{W}}_N^{(-+)}[H]$. In this case, one should deform the λ-integration to $\mathbb{R}-2\mathrm{i}\epsilon$. In doing so, we pick the residues at the poles at $\lambda = 0$ and $\lambda = \mu$ leading to

$$\widetilde{\mathcal{W}}_N^{(-+)}[H](\xi) \;=\; \mathcal{W}_{\mathrm{res}}[H] \;+\; \int_{\mathbb{R}} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \int_{a_N}^{b_N} \mathrm{d}\eta\, K_{-+}(\lambda,\lambda \mid \xi,\eta) H(\eta) \;+\; \int_{\mathbb{R}-2\mathrm{i}\epsilon} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \int_{\mathbb{R}-\mathrm{i}\epsilon} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \int_{a_N}^{b_N} \mathrm{d}\eta\, \frac{K_{-+}(\lambda,\mu \mid \xi,\eta)}{\mu-\lambda} H(\eta)\,.$$

It remains to implement the change of variables $(\lambda,\mu)\mapsto(-\lambda,-\mu)$ in the last integral and observe that

$$K_{-+}(\lambda,\lambda \mid \xi,\eta) \;=\; \frac{N^{2\alpha}}{2\pi\beta} \frac{\mathrm{e}^{\mathrm{i}N^{\alpha}\mu(\eta-\xi)}}{R(\lambda)} \quad \text{since} \quad \det\Psi(\lambda) \;=\; 1\,, \tag{6.25}$$

so as to obtain

$$\widetilde{\mathcal{W}}_N^{(-+)}[H](\xi) \;=\; \mathcal{W}_{\mathrm{res}}[H] \;+\; \mathcal{W}_{\mathrm{bk}}^{(0)}[H](\xi) \;-\; \widetilde{\mathcal{W}}_N^{(+-)}[H^{\wedge}](a_N + b_N - \xi)\,, \tag{6.26}$$

with $\mathcal{W}_{\mathrm{res}}$ being given by (6.10) and

$$\mathcal{W}_{\mathrm{bk}}^{(0)}[H](\xi) \;=\; \frac{N^{2\alpha}}{2\pi\beta} \fint_{a_N}^{b_N} J(N^{\alpha}(\eta-\xi))\, H(\eta)\, \mathrm{d}\eta \tag{6.27}$$

with the function J given in (6.3). A similar reasoning applied to the case of $\widetilde{\mathcal{W}}_N^{(--)}[H](\xi)$ yields

$$\widetilde{\mathcal{W}}_N^{(--)}[H](\xi) \;=\; -\widetilde{\mathcal{W}}_N^{(++)}[H^{\wedge}](a_N + b_N - \xi) \;-\; \mathcal{W}_{\mathrm{res}}[H^{\wedge}]\,. \tag{6.28}$$

Hence, eventually, upon deforming the contours to $\mathscr{C}_{\mathrm{reg}}^{(+)}$ or $\mathscr{C}_{\mathrm{reg}}^{(-)}$ in the $\mathcal{W}_N^{(\epsilon,\epsilon')}$ operators,

$$\begin{aligned}\mathcal{W}_N[H](\xi) \;=\; & \mathcal{W}_N^{(++)}[H](\xi) \;-\; \mathcal{W}_N^{(++)}[H^{\wedge}](a_N + b_N - \xi) \;+\; \mathcal{W}_N^{(+-)}[H](\xi) \\ & -\; \mathcal{W}_N^{(+-)}[H^{\wedge}](a_N + b_N - \xi) \;+\; \mathcal{W}_{\mathrm{res}}[H] \;-\; \mathcal{W}_{\mathrm{res}}[H^{\wedge}] \;+\; \mathcal{W}_{\mathrm{bk}}^{(0)}[H](\xi)\,.\end{aligned} \tag{6.29}$$

The operator $\mathcal{W}_N^{(++)}$ appearing above has been defined in (6.9) whereas

$$\mathcal{W}_N^{(+-)}[H](\xi) \;=\; \int\limits_{\mathscr{C}_{\mathrm{reg}}^{(+)}} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \int\limits_{\mathscr{C}_{\mathrm{reg}}^{(-)}} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \int\limits_{a_N}^{b_N} \mathrm{d}\eta \, \frac{K_{+-}(\lambda,\mu \mid \xi,\eta)}{\mu-\lambda} \, H(\eta) \; . \tag{6.30}$$

At this stage, it remains to observe that

$$\mathcal{W}_N^{(+-)}[H](\xi) \;=\; \mathcal{W}_R^{(0)}[H](x_R) \;+\; \Delta\mathcal{W}_N^{(+-)}[H](\xi) \; , \tag{6.31}$$

where $\Delta\mathcal{W}_N^{(+-)}$ is as defined in (6.9), while

$$\mathcal{W}_R^{(0)}[H](x) \;=\; -\frac{N^{2\alpha}}{2\pi\beta} \int\limits_{\mathscr{C}_{\mathrm{reg}}^{(+)}} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \int\limits_{\mathscr{C}_{\mathrm{reg}}^{(-)}} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \int\limits_{a_N}^{b_N} \mathrm{d}\eta \, \frac{H(\eta)\, \mathrm{e}^{\mathrm{i}\lambda x + \mathrm{i}N^{\alpha}\mu(\eta-b_N)}}{(\mu-\lambda)\, R_{\downarrow}(\lambda) R_{\uparrow}(\mu)} \; . \tag{6.32}$$

As a consequence, we obtain the decomposition:

$$\mathcal{W}_N[H](\xi) \;=\; \mathcal{W}_L^{(0)}[H](x_L) \;+\; \mathcal{W}_{\mathrm{bk}}^{(0)}[H](\xi) \;+\; \mathcal{W}_R^{(0)}[H](x_R) \;+\; \mathcal{W}_{\mathrm{exp}}[H](\xi) \; , \tag{6.33}$$

where we have set $\mathcal{W}_L^{(0)}[H](x) \;=\; -\mathcal{W}_R^{(0)}[H^{\wedge}](x)$. In order to obtain the representation (6.4) it is enough to incorporate certain terms present in $\mathcal{W}_{\mathrm{bk}}^{(0)}[H](\xi)$ into the R and L-type operators. Namely, we can recast $\mathcal{W}_{\mathrm{bk}}^{(0)}[H](\xi)$ as

$$\begin{aligned} \mathcal{W}_{\mathrm{bk}}^{(0)}[H](\xi) &= \frac{N^{2\alpha}}{2\pi\beta} \int\limits_{a_N}^{b_N} J(N^{\alpha}(\eta-\xi))\, [H(\eta)-H(\xi)]\, \mathrm{d}\eta \;-\; N^{\alpha} H(\xi)[\varrho_0(x_R) \;-\; \varrho_0(x_L)] \\ &= \mathcal{W}_{\mathrm{bk}}[H_{\mathrm{e}}](\xi) \;-\; N^{\alpha}[\varrho_0(x_R) \;-\; \varrho_0(x_L)]\, H_{\mathrm{e}}(\xi) \;-\; \frac{N^{\alpha}}{2\pi\beta} \Bigg\{ \int\limits_{x_R}^{+\infty} + \int\limits_{-\infty}^{-x_L} \Bigg\} [H_{\mathrm{e}}(\xi + N^{-\alpha}y) - H_{\mathrm{e}}(\xi)]\, J(y)\, \mathrm{d}y \; . \end{aligned}$$

There, we have introduced

$$\varrho_0(x) \;=\; \frac{-1}{2\mathrm{i}\pi\beta} \int\limits_{\mathscr{C}_{\mathrm{reg}}^{(+)}} \frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\lambda R(\lambda)} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \qquad \textit{i.e.} \qquad \varrho_0'(x) = -\frac{J(x)}{2\pi\beta} \; . \tag{6.34}$$

The representation (6.4) for $\mathcal{W}_N[H]$ follows by redistributing the terms. This decomposition also ensures that $\mathcal{W}_N[H] \in C^{k-1}(]a_N\,; b_N[)$. Indeed, this regularity follows from the exponential decay of the integrands in Fourier space when $\xi \in]a_N\,; b_N[$ and derivation under the integral theorems.

■

Note that the integral defining ϱ_0 in (6.34) can be evaluated explicitly leading to:

$$\varrho_0(x) \;=\; \frac{1}{2\pi^2\beta} \cdot \ln \left| \frac{1 - \mathrm{e}^{-\frac{2\pi|x|\omega_1\omega_2}{\omega_1+\omega_2}}}{1 + \mathrm{e}^{-\frac{2\pi|x|\omega_1\omega_2}{\omega_1+\omega_2} + \mathrm{i}\pi\frac{\omega_2-\omega_1}{\omega_1+\omega_2}}} \right| \; . \tag{6.35}$$

In particular, it exhibits a logarithmic singularity at the origin meaning that $J(x)$ has a $1/x$ behaviour around 0.

6.2 Local approximants for $\mathcal{W}_N$

In this subsection, we obtain uniform – in the running variable – asymptotic expansions for the operators $\mathcal{W}_{\rm bk}$, $\mathcal{W}_R$ and $\mathcal{W}_{\exp}$. In particular, we shall establish that if ξ is uniformly away from b_N (resp. a_N), $\mathcal{W}_R$ (resp. $\mathcal{W}_L$) will only generate exponentially small (in N) corrections. Finally, this exponentially small bound will hold uniformly after a finite number of ξ-differentiations. Prior to discussing these matters we need to introduce two families of auxiliary functions on $\mathbb{R}^+$ and constants that come into play during the description of these behaviours.

Definition 6.5 *For any integer $\ell \geq 0$:*

$$\varpi_\ell(x) \;=\; \frac{1}{2\pi\beta}\int_x^{+\infty} y^\ell\, J(y)\,\mathrm{d}y\;, \tag{6.36}$$

$$\varrho_\ell(x) \;=\; \frac{\mathrm{i}^{\ell+1}}{2\pi\beta}\int_{\mathscr{C}^{(+)}_{\rm reg}}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\int_{\mathbb{R}-\mathrm{i}\epsilon'}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\,\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\mu^{\ell+1}R_\uparrow(\mu)(\mu-\lambda)R_\downarrow(\lambda)}\;, \tag{6.37}$$

$$u_\ell \;=\; \frac{\mathrm{i}^\ell}{2\mathrm{i}\pi\beta\,\ell!}\,\frac{\partial^\ell}{\partial\lambda^\ell}\Big(\frac{1}{R(\lambda)}\Big)_{|\lambda=0}\;. \tag{6.38}$$

Note that $u_{2p}=0$ since R is an odd function – given in (4.19).

For $\ell = 0$, this definition of ϱ_0 coincide with (6.34), whose explicit expression is (6.35). Indeed, we remember from § 4.2 that $\uparrow$ means that we can move the contour of integration over μ up to $+\mathrm{i}\infty$ without hitting a pole of $R_\uparrow^{-1}(\mu)$. According to (4.27), $\mu\,R_\uparrow(\mu)$ has a non-zero limit when $\mu\to 0$, so we just pick up the residue at $\mu=\lambda$, which leads to the expression (6.34). For $\ell \geq 1$, the function ϖ_ℓ is continuous at $x = 0$. Furthermore, for any $\ell \geq 0$, $\varrho_\ell(x)$ and $\varpi_p(x)$ decay exponentially fast in x when $x \to +\infty$. Indeed, it is readily seen on the basis of their explicit integral representations that there exists $C_\ell > 0$ such that:

$$|\varrho_\ell(x)| + |\varpi_\ell(x)| \;\leq\; C_\ell\,\mathrm{e}^{-C'_\ell x} \qquad \text{for } \ell\geq 1\;. \tag{6.39}$$

Proposition 6.6 *Let $k \geq 0$ be an integer, $H \in C^{2k+1}([a_N\,;b_N])$, and define:*

$$\mathcal{W}_{R;k}[H](x,\xi) \;=\; \frac{H(\xi)-H(b_N)}{\xi-b_N}\cdot x\varrho_0(x) \;-\; \sum_{\ell=1}^{k}\frac{H^{(\ell)}(\xi)}{N^{(\ell-1)\alpha}\cdot\ell!}\cdot\varpi_\ell(x) \;+\; \sum_{\ell=1}^{k}\frac{H^{(\ell)}(b_N)}{N^{(\ell-1)\alpha}}\cdot\varrho_\ell(x)\;, \tag{6.40}$$

$$\mathcal{W}_{{\rm bk};k}[H](\xi) \;=\; \sum_{\ell=1}^{k}\frac{H^{(\ell)}(\xi)}{N^{\alpha(\ell-1)}}\cdot u_\ell\;. \tag{6.41}$$

These operators provide the asymptotic expansions, uniform for $\xi\in[a_N\,;b_N]$:

$$\mathcal{W}_R[H_{\mathfrak{e}}](x_R,\xi) \;=\; \mathcal{W}_{R;k}[H](x_R,\xi) \;+\; \Delta_{[k]}\mathcal{W}_R[H_{\mathfrak{e}}](x_R,\xi)\;, \tag{6.42}$$

$$\mathcal{W}_{\rm bk}[H_{\mathfrak{e}}](\xi) \;=\; \mathcal{W}_{{\rm bk};k}[H](\xi) \;+\; \Delta_{[k]}\mathcal{W}_{\rm bk}[H_{\mathfrak{e}}](\xi)\;. \tag{6.43}$$

The remainder in (6.42) *takes the form:*

$$\Delta_{[k]}\mathcal{W}_R[H_{\mathfrak{e}}](x,\xi) \;=\; \mathcal{R}^{(0)}_{R;[k]}[H_{\mathfrak{e}}](x,\xi) \;+\; \sum_{\ell=0}^{k} x^{\ell+1/2}\,\mathcal{R}^{(1/2)}_{R;[k];\ell}[H_{\mathfrak{e}}](x)\;, \tag{6.44}$$

with $\mathcal{R}^{(0)}_{R;[k]}[H_{\mathrm{e}}] \in W^{(\infty)}_{k}(\mathbb{R}^{+} \times [a_N\,;b_N])$ *and* $\mathcal{R}^{(1/2)}_{R;[k];\ell}[H_{\mathrm{e}}] \in W^{(\infty)}_{k}(\mathbb{R}^{+})$*, and the more precise bound:*

$$\forall m \in [\![\, 0\,;\, k\,]\!], \qquad \max_{\substack{p\in[\![\, 0\,;\, m\,]\!] \\ \ell\in[\![\, 0\,;\, k\,]\!]}} \Big\{ \Big|\partial_{\xi}^{p}\mathcal{R}^{(0)}_{R;[k]}[H_{\mathrm{e}}](x_R,\xi)\Big| \;+\; \Big|\partial_{\xi}^{p}\mathcal{R}^{(1/2)}_{R;[k];\ell}[H_{\mathrm{e}}](x_R)\Big| \Big\} \;\leq\; \frac{C\mathrm{e}^{-C' x_R}}{N^{(k-m)\alpha}}\, ||H^{(k+1)}_{\mathrm{e}}||_{W^{\infty}_{m}(\mathbb{R})} \tag{6.45}$$

for some $C, C' > 0$ *independent of* N *and* H*. The remainder in* (6.43) *is bounded by:*

$$\Big|\Big|\Delta_{[k]}\mathcal{W}_{\mathrm{bk}}[H_{\mathrm{e}}]\Big|\Big|_{W^{\infty}_{m}([a_N\,;b_N])} \;\leq\; C\, N^{-k\alpha}\, ||H^{(k+1)}_{\mathrm{e}}||_{W^{\infty}_{m}(\mathbb{R})}\ . \tag{6.46}$$

Proposition 6.7 *Let* $k \geq 0$ *be an integer, and* $H \in C^{2k+1}([a_N\,;b_N])$*. The operator* $\mathcal{W}_{\mathrm{exp}}$ *takes the form:*

$$\begin{aligned}\mathcal{W}_{\mathrm{exp}}[H](\xi) \;&=\; \mathcal{R}^{(0)}_{\mathrm{exp};R}[H](x_R,\xi) \;+\; \sum_{\ell=0}^{k} x_R^{\ell+1/2}\,\mathcal{R}^{(1/2)}_{\mathrm{exp};R;\ell}[H](x_R) \\ &\quad +\mathcal{R}^{(0)}_{\mathrm{exp};L}[H](x_L,\xi) \;+\; \sum_{\ell=0}^{k} x_L^{\ell+1/2}\,\mathcal{R}^{(1/2)}_{\mathrm{exp};L;\ell}[H](x_L)\end{aligned} \tag{6.47}$$

with $\mathcal{R}^{(0)}_{\mathrm{exp};R/L}[H_{\mathrm{e}}] \in W^{(\infty)}_{k}(\mathbb{R}^{+} \times [a_N\,;b_N])$ *and* $\mathcal{R}^{(1/2)}_{\mathrm{exp};R/L;\ell}[H_{\mathrm{e}}] \in W^{(\infty)}_{k}(\mathbb{R}^{+})$*, and the more precise bound:*

$$\forall m \in [\![\, 0\,;\, k\,]\!], \qquad \max_{\substack{p\in[\![\, 0\,;\, m\,]\!] \\ \ell\in[\![\, 0\,;\, k\,]\!]}} \Big\{ \Big|\partial_{\xi}^{p}\mathcal{R}^{(0)}_{\mathrm{exp};R/L}[H_{\mathrm{e}}](x_{R/L},\xi)\Big| + \Big|\partial_{\xi}^{p}\mathcal{R}^{(1/2)}_{\mathrm{exp};R/L;\ell}[H_{\mathrm{e}}](x_{R/L})\Big| \Big\} \;\leq\; C N^{m\alpha}\mathrm{e}^{-C' N^{\alpha}}\, ||H^{(k+1)}_{\mathrm{e}}||_{W^{\infty}_{m}(\mathbb{R})} \tag{6.48}$$

for some $C, C' > 0$ *independent of* N *and* H*.*

The idea for obtaining the above form of the asymptotic expansions is to represent H in terms of its Taylor-integral expansion of order k. We can then compute explicitly the contributions issuing from the polynomial part of the Taylor series expansion for H and obtain sharp bounds on the remainder by exploiting the structure of the integral remainder in the Taylor integral series. In particular, the analysis of this integral remainder allows uniform bounds for the remainder as given in (6.45), (6.46) and (6.48). The reason for such handlings instead of more direct bounds issues from the fact that the integrals we manipulate are only weakly convergent. One thus has first to build on the analytic structure of the integrand so as to obtain the desired bounds and expressions and, in particular, carry out some contour deformations. Clearly, such handlings cannot be done anymore upon inserting the absolute value under the integral sign, as then the integrand is no more analytic.

Proof — We carry out the analysis, individually, for each operator.

The operator $\mathcal{W}_{\mathrm{bk}}$

The Taylor integral expansion of H up to order k yields the representation

$$\mathcal{W}_{\mathrm{bk}}[H_{\mathrm{e}}](\xi) \;=\; \sum_{p=1}^{k} \frac{1}{2\pi\beta\, N^{(p-1)\alpha}}\,\frac{H^{(p)}(\xi)}{p!} \int_{\mathbb{R}} y^{p}\, J(y)\,\mathrm{d}y \;+\; \Delta_{[k]}\mathcal{W}_{\mathrm{bk}}[H_{\mathrm{e}}](\xi)\ , \tag{6.49}$$

where

$$\Delta_{[k]}\mathcal{W}_{\mathrm{bk}}[H_{\mathrm{e}}](\xi) \;=\; \frac{1}{2\pi\beta N^{k\alpha}} \int_{0}^{1}\mathrm{d}t\,\frac{(1-t)^{k}}{k!} \int_{\mathbb{R}}\mathrm{d}y\, y^{k+1}\, J(y)\, H^{(k+1)}_{\mathrm{e}}(\xi + N^{-\alpha}ty)\ . \tag{6.50}$$

In the first terms of (6.49) we identify:

$$\int_{\mathbb{R}} y^{\ell} J(y)\mathrm{d}y \;=\; \mathrm{i}^{\ell-1}\frac{\partial^{\ell}}{\partial\lambda^{\ell}}\Big(\frac{1}{R(\lambda)}\Big)_{|\lambda=0} = 2\pi\beta\,\ell!\,u_{\ell}\;, \tag{6.51}$$

and we remind that this is zero when ℓ is even. Finally, we get that the remainder is a C^k function of ξ, and:

$$\forall m\in[\![\,0\,;\,k\,]\!],\qquad ||\Delta_{[k]}\mathcal{W}_{\mathrm{bk}}[H_{\mathrm{e}}]||_{W^{\infty}_{m}([a_N\,;b_N])}\;\leq\;\frac{||H_{\mathrm{e}}^{(k+1)}||_{W^{\infty}_{m}(\mathbb{R})}}{N^{k\alpha}}\int_{\mathbb{R}}|y|^{k+1}|J(y)|\,\frac{\mathrm{d}y}{2\pi\beta}\;. \tag{6.52}$$

Since J decays exponentially at ∞ (see (6.34) and (6.35)), the last integral gives a finite, k-dependent constant.

The operator $\mathcal{W}_R$

The contribution arising in the first line of (6.6) can be treated analogously to $\mathcal{W}_{\mathrm{bk}}$, what leads to

$$-\frac{N^{\alpha}}{2\pi\beta}\int_{x}^{+\infty} J(y)\,[H_{\mathrm{e}}(\xi+N^{-\alpha}y)-H_{\mathrm{e}}(\xi)]\,\mathrm{d}y\;=\;-\sum_{\ell=1}^{k}\frac{H_{\mathrm{e}}^{(\ell)}(\xi)}{N^{(\ell-1)\alpha}\,\ell!}\,\varpi_{\ell}(x)\;+\;\Delta_{[k]}\mathcal{W}_{R}^{(1)}[H_{\mathrm{e}}](x,\xi) \tag{6.53}$$

with

$$\Delta_{[k]}\mathcal{W}_{R}^{(1)}[H_{\mathrm{e}}](x,\xi)\;=\;\frac{-1}{2\pi\beta\,N^{k\alpha}}\int_{x}^{+\infty}\mathrm{d}y\,y^{k+1}J(y)\int_{0}^{1}\mathrm{d}t\,\frac{(1-t)^{k}}{k!}\,H_{\mathrm{e}}^{(k+1)}(\xi+N^{-\alpha}\,ty)\;. \tag{6.54}$$

Since J decays exponentially at infinity, we clearly have:

$$\max_{p\in[\![\,0\,;\,m\,]\!]}\Big|\partial_{\xi}^{p}\cdot\Delta_{[k]}\mathcal{W}_{R}^{(1)}[H_{\mathrm{e}}](x_R,\xi)\Big|\;\leq\;C\,\mathrm{e}^{-C'x_R}\cdot\frac{||H_{\mathrm{e}}^{(k+1)}||_{W^{\infty}_{m}(\mathbb{R})}}{N^{(k-m)\alpha}} \tag{6.55}$$

for some constants C, C' independent of H and N. We remind that the ξ-derivative can act on both variables ξ *and* $x_R = N^{\alpha}(b_N-\xi)$.

We now focus on the contributions issuing from the second line of (6.6). For this purpose, observe that the Taylor-integral series representation for H yields the following representation for the Fourier transform of H:

$$\int_{a_N}^{b_N} H(\eta)\mathrm{e}^{\mathrm{i}\mu N^{\alpha}(\eta-b_N)}\,\mathrm{d}\eta\;=\;\mathcal{F}_{1;k}[H](\mu)\;+\;\mathcal{F}_{2;k}[H_{\mathrm{e}}](\mu)\;+\;\mathcal{F}_{3}[H_{\mathrm{e}}](\mu)\;, \tag{6.56}$$

where we complete the integral over $[a_N\,;b_N]$ to $]-\infty\,;b_N]$ in the first term, while the two last terms come from subtracting the right and left contributions:

$$\mathcal{F}_{1;k}[H](\mu)\;=\;-\sum_{p=0}^{k}\bigg(\frac{\mathrm{i}}{N^{\alpha}\mu}\bigg)^{p+1}\cdot H^{(p)}(b_N)\qquad,\quad\mathcal{F}_{3}[H_{\mathrm{e}}](\mu)\;=\;\int_{-\infty}^{a_N}H_{\mathrm{e}}(\eta)\,\mathrm{e}^{\mathrm{i}\mu N^{\alpha}(\eta-b_N)}\,\mathrm{d}\eta \tag{6.57}$$

and

$$\mathcal{F}_{2;k}[H_{\mathrm{e}}](\mu)\;=\;\int_{-\infty}^{b_N}\mathrm{d}\eta\int_{0}^{1}\mathrm{d}t\,\frac{(1-t)^{k}}{k!}\mathrm{e}^{\mathrm{i}\mu N^{\alpha}(\eta-b_N)}(\eta-b_N)^{k+1}\,H_{\mathrm{e}}^{(k+1)}(b_N+t(\eta-b_N))\;. \tag{6.58}$$

Thus,

$$
-\frac{N^{2\alpha}}{2\pi\beta}\int\limits_{\mathscr{C}^{(+)}_{\mathrm{reg}}}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\int\limits_{\mathscr{C}^{(-)}_{\mathrm{reg}}}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{(\mu-\lambda)R_{\downarrow}(\lambda)R_{\uparrow}(\mu)}\int\limits_{a_N}^{b_N}H(\eta)\mathrm{e}^{-\mathrm{i}\mu y_R}\mathrm{d}\eta
$$

$$
= \sum_{\ell=0}^{k}\frac{H^{(\ell)}(b_N)}{N^{(\ell-1)\alpha}}\,\varrho_{\ell}(x) \; + \; \mathcal{L}_{\Lambda_0}\big[\mathcal{F}_{2;k}[H_{\mathfrak{e}}] \, + \, \mathcal{F}_3[H_{\mathfrak{e}}]\big](x)\;. \quad (6.59)
$$

$\mathcal{L}_{\Lambda_0}$ is an operator with integral kernel – see later equation (6.66):

$$
\Lambda_0(\lambda,\mu) \; = \; \frac{-1}{R_{\uparrow}(\mu)R_{\downarrow}(\lambda)} \quad (6.60)
$$

which satisfies the assumptions of Lemma 6.8 appearing below. Thence, Lemma 6.8 entails the decomposition:

$$
\mathcal{L}_{\Lambda_0}[\mathcal{F}_{2;k}[H_{\mathfrak{e}}] + \mathcal{F}_3[H_{\mathfrak{e}}]](x) \; = \; \sum_{\ell=0}^{k}\Big\{x^{\ell+1/2}\mathrm{e}^{-\varsigma x}\mathcal{L}_{\Lambda_0;\ell}[\mathcal{F}_{2;k}[H_{\mathfrak{e}}] + \mathcal{F}_3[H_{\mathfrak{e}}]](x)\Big\}
$$

$$
+ \; (\Delta_{[k]}\mathcal{L}_{\Lambda_0})[\mathcal{F}_{2;k}[H_{\mathfrak{e}}] + \mathcal{F}_3[H_{\mathfrak{e}}]](x) \quad (6.61)
$$

in which both $\mathcal{L}_{\Lambda_0;\ell}[\mathcal{F}_{2;k}[H_{\mathfrak{e}}] + \mathcal{F}_3[H_{\mathfrak{e}}]](x)$ and $(\Delta_{[k]}\mathcal{L}_{\Lambda_0})[\mathcal{F}_{2;k}[H_{\mathfrak{e}}] + \mathcal{F}_3[H_{\mathfrak{e}}]](x)$ belong to $W^{\infty}_{k}(\mathbb{R}^{+})$ and are as given in (6.68)-(6.69)

By using the bounds:

$$
\Big|\mathcal{F}_{2;k}[H_{\mathfrak{e}}](\mu)\Big| \; \leq \; \frac{c_k\|H^{(k+1)}_{\mathfrak{e}}\|_{L^{\infty}(\mathbb{R})}}{(N^{\alpha}|\mu|)^{k+2}} \qquad \text{since} \quad \frac{1}{|\mathrm{Im}\,\mu|} \; \leq \; \frac{c'}{|\mu|} \;\; \text{for} \;\; \mu\in\mathscr{C}^{(-)}_{\mathrm{reg}}\;, \quad (6.62)
$$

and

$$
\Big|\mathcal{F}_3[H_{\mathfrak{e}}](\mu)\Big| \; \leq \; c\frac{\|H_{\mathfrak{e}}\|_{L^{\infty}(\mathbb{R})}}{|\mu|N^{\alpha}}\cdot\mathrm{e}^{-\overline{x}_N|\mathrm{Im}\,\mu|}\;. \quad (6.63)
$$

we get that there exists N-independent constants C, C' such that

$$
\max_{\substack{p\in[\![\,0\,;\,m\,]\!]\\ \ell\in[\![\,0\,;\,k\,]\!]}}\Big|\partial_{\xi}^{p}\cdot\Big\{\Big(\mathrm{e}^{-\varsigma x_R}\mathcal{L}_{\Lambda_0;\ell} \, + \, \Delta_{[k]}\mathcal{L}_{\Lambda_0}\Big)\Big[\mathcal{F}_{2;k}[H_{\mathfrak{e}}] \, + \, \mathcal{F}_3[H_{\mathfrak{e}}]\Big](x_R)\Big\}\Big| \; \leq \; C\,\mathrm{e}^{-C'x_R}\,\frac{\|H^{(k+1)}_{\mathfrak{e}}\|_{L^{\infty}(\mathbb{R})}}{N^{(k-m)\alpha}}\;. \quad (6.64)
$$

We have relied on:

Lemma 6.8 *Let $\Lambda(\lambda,\mu)$ be a holomorphic function of λ and μ belonging to the region of the complex plane delimited by $\mathscr{C}^{(+)}_{\mathrm{reg}}$ and $\mathscr{C}^{(-)}_{\mathrm{reg}}$ and such that it admits an asymptotic expansion*

$$
\Lambda(\lambda,\mu) \; = \; \sum_{\ell=0}^{k}\frac{\Lambda_{\ell}(\mu)}{\big[\mathrm{i}(\lambda-\mathrm{i}\varsigma)\big]^{\ell+1/2}} \; + \; \Delta_{[k]}\Lambda(\lambda,\mu) \quad \text{with} \quad \left\{\begin{array}{lcl} |\Lambda_{\ell}(\mu)| & = & \mathrm{O}(|\mu|^{1/2}) \\ |\Delta_{[k]}\Lambda(\lambda,\mu)| & = & \mathrm{O}(|\lambda|^{-(k+3/2)}\cdot|\mu|^{1/2}) \end{array}\right. . \quad (6.65)
$$

Then, the integral operator on $\mu\cdot L^{\infty}(\mathscr{C}^{(-)}_{\mathrm{reg}}) \; \equiv \; \Big\{f \; : \; \mu\mapsto\mu f(\mu)\in L^{\infty}(\mathscr{C}^{(-)}_{\mathrm{reg}})\Big\}$

$$
\mathcal{L}_{\Lambda}[f](x) \; = \; \frac{N^{2\alpha}}{2\pi\beta}\int\limits_{\mathscr{C}^{(+)}_{\mathrm{reg}}}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\int\limits_{\mathscr{C}^{(-)}_{\mathrm{reg}}}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\frac{\Lambda(\lambda,\mu)}{\mu-\lambda}\mathrm{e}^{\mathrm{i}\lambda x}f(\mu) \quad (6.66)
$$

can be recast as

$$\mathcal{L}_{\Lambda}[f](x) \;=\; \sum_{\ell=0}^{k} x^{\ell+1/2}\mathrm{e}^{-\varsigma x}\mathcal{L}_{\Lambda;\ell}[f](x) \;+\; \Delta_{[k]}\mathcal{L}_{\Lambda}[f](x)\;, \tag{6.67}$$

where the operators

$$\mathcal{L}_{\Lambda;k}[f](x) \;=\; \frac{N^{2\alpha}}{2\pi\beta}\int\limits_{\Gamma(\mathrm{i}\mathbb{R}^+)}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\int\limits_{\mathscr{C}_{\mathrm{reg}}^{(-)}}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\frac{\Lambda_k(\mu)\mathrm{e}^{\mathrm{i}\lambda}f(\mu)}{[x(\mu-\mathrm{i}\varsigma)-\lambda](\mathrm{i}\lambda)^{\ell+\frac{1}{2}}}\;, \tag{6.68}$$

$$\Delta_{[k]}\mathcal{L}_{\Lambda}[f](x) \;=\; \frac{N^{2\alpha}}{2\pi\beta}\int\limits_{\mathscr{C}_{\mathrm{reg}}^{(+)}}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\int\limits_{\mathscr{C}_{\mathrm{reg}}^{(-)}}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\frac{\Delta_{[k]}\Lambda(\lambda,\mu)}{\mu-\lambda}\mathrm{e}^{\mathrm{i}\lambda x}f(\mu)\;, \tag{6.69}$$

are continuous as operators $\mu\cdot L^{\infty}(\mathscr{C}_{\mathrm{reg}}^{(-)})\to W_k^{\infty}(\mathbb{R}^+)$. *Note that, above,* $\Gamma(\mathrm{i}\mathbb{R}^+)$ *corresponds to a small counter-clockwise loop around* $\mathrm{i}\mathbb{R}^+$.

Proof — It is enough to insert the large-μ expansion of Λ and then, in the part subordinate to the inverse power-law expansion, deform the λ-integrals to $\Gamma(\mathrm{i}\mathbb{R}+\mathrm{i}\varsigma)$, translate by $+\mathrm{i}\varsigma$ and, finally, rescale by x. The statements about continuity are evident. ■

The operator $\mathcal{W}_{\exp}$ (Proposition 6.7)

The analysis relative to the structure of $\mathcal{W}_{\exp}[H_{\mathfrak{e}}]$ follows basically the same steps as above so we shall not detail them here again. The main point, though, is the presence of an exponential prefactor $\mathrm{e}^{-cN^{\alpha}}$ which issues from the bound (4.50) on $\Pi-I_2$. ■

6.3 Large N asymptotics of the approximants of $\mathcal{W}_N$

The results of Propositions 6.6 and 6.7 induce the representation

$$\mathcal{W}_N[H](\xi) \;=\; \mathcal{W}_{R;k}[H](x_R,\xi) \;+\; \mathcal{W}_{\mathrm{bk};k}[H](\xi) \;-\; \mathcal{W}_{R;k}[H](x_L,a_N+b_N-\xi) \;+\; \Delta_{[k]}\mathcal{W}_N[H_{\mathfrak{e}}](\xi)\;, \tag{6.70}$$

with all remainders at order k are collected in the last term. In this subsection, we shall derive asymptotic expansion (in N) of the approximants $\mathcal{W}_{\mathrm{bk};k}$ and $\mathcal{W}_{R;k}$ in the case when their unrescaled variable ξ scales towards b_N as $\xi=b_N-N^{-\alpha}x$ with x being independent of N. We, however, first need to establish properties of certain auxiliary functions that appear in this analysis.

Definition 6.9 *Let $\ell\geq 0$ be an integer. As a supplement to Definition 6.5, we introduce, for any integer $\ell\geq 0$:*

$$\mathfrak{b}_{\ell}(x) \;=\; \varrho_{\ell+1}(x) - \frac{(-x)^{\ell+1}}{(\ell+1)!}\varrho_0(x) \;-\; \sum_{\substack{s+p=\ell\\ s,p\geq 0}}\frac{(-x)^p\varpi_{s+1}(x)}{p!(s+1)!} \qquad \textit{and} \qquad \mathfrak{u}_{\ell}(x) \;=\; \sum_{\substack{s+p=\ell\\ s,p\geq 0}}\frac{(-x)^p u_{s+1}}{p!} \tag{6.71}$$

and:

$$\mathfrak{a}_0(x) \;=\; \mathfrak{b}_0(x)+\mathfrak{u}_0(x)\;, \qquad \mathfrak{a}_{\ell}(x) \;=\; \frac{\mathfrak{b}_{\ell}(x)+\mathfrak{u}_{\ell}(x)}{\mathfrak{a}_0(x)} \quad \text{for } \ell\geq 1\;. \tag{6.72}$$

It will be important for the estimates of § 8.2 to remark that $x^{-1/2}\mathfrak{a}_0(x)$ is a smooth and positive function:

Lemma 6.10 *Let $\ell, n, m \geq 0$ be three integers such that $n > m$. There exist polynomials $p_{\ell;m,n}$ of degree at most $n+\ell$ and functions $f_{\ell;m,n} \in W^{\infty}_{n-m}(\mathbb{R}^+)$ such that, for any $x > 0$:*

$$\mathfrak{a}_0(x) \;=\; \sqrt{x}\, p_{0;m,n}(x)\mathrm{e}^{-\varsigma x} \;+\; x^m\, f_{0;m,n}(x) \qquad \textit{and} \qquad \mathfrak{a}_0(x)\cdot\mathfrak{a}_\ell(x) \;=\; \sqrt{x}\, p_{\ell;m,n}(x)\mathrm{e}^{-\varsigma x} \;+\; x^m\, f_{\ell;m,n}(x)\,. \tag{6.73}$$

The function $\mathfrak{a}_0(x)$ is positive for $x > 0$ and satisfies

$$\mathfrak{a}_0(x) \underset{x\to 0}{=} \frac{1}{\pi\beta}\sqrt{\frac{x}{\pi(\omega_1+\omega_2)}} \;+\; \mathrm{O}(x) \tag{6.74}$$

Finally, one has, in the $x \to +\infty$ regime,

$$\mathfrak{a}_0(x) \;=\; u_1 \;+\; \mathrm{O}(\mathrm{e}^{-\varsigma x}) \qquad , \qquad \mathfrak{a}_0(x)\cdot\mathfrak{a}_\ell(x) \;=\; \mathfrak{u}_\ell(x) \;+\; \mathrm{O}(\mathrm{e}^{-\varsigma x}) \tag{6.75}$$

and the bound on the remainder is stable with respect to finite-order differentiations.

Proof — By using the integral representation (6.3) for the function J, we can readily recast $\varpi_\ell(x)$, for $x > 0$ as:

$$\varpi_\ell(x) \;=\; \frac{\mathrm{i}^{\ell+1}}{2\pi\beta}\int_{\mathscr{C}^{(+)}_{\mathrm{reg}}} \frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\lambda}\,\frac{\partial^\ell}{\partial\lambda^\ell}\Big(\frac{1}{R(\lambda)}\Big)\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\,. \tag{6.76}$$

The μ-integral arising in the definition (6.37) of ϱ_ℓ can be computed by moving the contour of integration over μ up to $+\mathrm{i}\infty$, and picking the residues at $\mu = \lambda$ and $\mu = 0$:

$$\varrho_\ell(x) \;=\; \frac{\mathrm{i}^{\ell+1}}{2\pi\beta}\int_{\mathscr{C}^{(+)}_{\mathrm{reg}}} \frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\lambda^{\ell+1}R(\lambda)}\,\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \;+\; \tau_\ell(x) \quad \text{with} \quad \tau_\ell(x) \;=\; \frac{\mathrm{i}^{\ell+1}}{2\pi\beta}\int_{\mathscr{C}^{(+)}_{\mathrm{reg}}} \frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\ell!\,R_\downarrow(\lambda)}\cdot\frac{\partial^\ell}{\partial\mu^\ell}\Big(\frac{1}{(\mu-\lambda)R_\uparrow(\mu)}\Big)_{|\mu=0}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\,.$$

The first term can be related to the functions ϱ_0 and ϖ_s of Definition 6.5 by an ℓ-fold integration by parts based on the identities:

$$\frac{1}{\lambda^{\ell+1}} \;=\; \frac{\partial^\ell}{\partial\lambda^\ell}\Big\{\frac{(-1)^\ell}{\lambda\,\ell!}\Big\} \qquad \text{and} \qquad \frac{\partial^\ell}{\partial\lambda^\ell}\Big\{\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{R(\lambda)}\Big\} \;=\; \sum_{\substack{s+p=\ell\\ s,p\geq 0}} \frac{\ell!}{s!\,p!}(\mathrm{i}x)^p\mathrm{e}^{\mathrm{i}\lambda x}\cdot\frac{\partial^s}{\partial\lambda^s}\Big\{\frac{1}{R(\lambda)}\Big\}\,. \tag{6.77}$$

Namely, we obtain – writing the identity for $\ell+1$ instead of ℓ – that:

$$\varrho_{\ell+1}(x) \;-\; \tau_{\ell+1}(x) \;=\; \frac{(-x)^{\ell+1}}{(\ell+1)!}\,\varrho_0(x) \;+\; \sum_{\substack{s+p=\ell\\ s,p\geq 0}} \frac{(-x)^p\varpi_{s+1}(x)}{p!(s+1)!}\,. \tag{6.78}$$

According to Definition 6.9, we can thus identify $\tau_{\ell+1}(x) = \mathfrak{b}_\ell(x)$ – in this proof, we will nevertheless keep the notation τ_ℓ. Hence, it remains to focus on $\tau_\ell(x)$. Computing the ℓ^{th}-order μ-derivative appearing in its integrand and then repeating the same integration by parts trick, we obtain that:

$$\tau_\ell(x) \;=\; -\frac{\mathrm{i}^{\ell+1}}{2\pi\beta}\sum_{s+r+p=\ell}\frac{(\mathrm{i}x)^r}{s!\,p!\,r!}\frac{\partial^s}{\partial\mu^s}\Big(\frac{1}{R_\uparrow(\mu)}\Big)_{|\mu=0}\int_{\mathscr{C}^{(+)}_{\mathrm{reg}}}\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\lambda}\cdot\frac{\partial^p}{\partial\lambda^p}\Big\{\frac{1}{R_\downarrow(\lambda)}\Big\}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\,. \tag{6.79}$$

In the second integral, let us move a bit the contour $\mathscr{C}^{(+)}_{\mathrm{reg}}$ to a contour $\mathscr{C}^{(+)}_{\mathrm{reg},0}$ which passes below 0 while keeping the same asymptotic directions as $\mathscr{C}^{(+)}_{\mathrm{reg}}$. Doing so, we pick up the residue at $\lambda = 0$:

$$\int\limits_{\mathscr{C}^{(+)}_{\mathrm{reg}}} \frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\lambda} \cdot \frac{\partial^p}{\partial \lambda^p}\Big\{\frac{1}{R_{\downarrow}(\lambda)}\Big\} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} = -\frac{\partial^p}{\partial \lambda^p}\Big\{\frac{1}{R_{\downarrow}(\lambda)}\Big\}_{|\lambda=0} + \int\limits_{\mathscr{C}^{(+)}_{\mathrm{reg},0}} \frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\lambda} \cdot \frac{\partial^p}{\partial \lambda^p}\Big\{\frac{1}{R_{\downarrow}(\lambda)}\Big\} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \tag{6.80}$$

We observe that there exist constants $c_{p;q}$ such that:

$$\frac{1}{\lambda} \cdot \frac{\partial^p}{\partial \lambda^p}\Big\{\frac{1}{R_{\downarrow}(\lambda)}\Big\} \;=\; \sum_{q=p+1}^{n} \frac{c_{p;q}}{[\mathrm{i}(\lambda - \mathrm{i}\varsigma)]^{q+1/2}} \;+\; \Delta^{(p)}_{[n]}[R^{-1}_{\downarrow}](\lambda)\;, \tag{6.81}$$

This decomposition ensures that $\Delta^{(p)}_{[n]}[R^{-1}_{\downarrow}](\lambda)$ is holomorphic in $\mathbb{H}^{-}$, has a simple pole at $\lambda = 0$ and satisfies $\Delta^{(p)}_{[n]}[R^{-1}_{\downarrow}](\lambda) = \mathrm{O}\big(\lambda^{-(n+3/2)}\big)$.

Since $\varsigma/2$ is the distance between $\mathscr{C}^{(+)}_{\mathrm{reg}}$ and $\mathbb{R}$, we can choose this contour – for a fixed ς – such that the branch cut of the denominators in (6.81) is located on a vertical half-line above $\mathscr{C}^{(+)}_{\mathrm{reg},0}$. This implies that the remainder in (6.81) is holomorphic below $\mathscr{C}^{(+)}_{\mathrm{reg},0}$. So, in the second integral of (6.80), we obtain with the first sum contributions involving:

$$\int\limits_{\mathscr{C}^{(+)}_{\mathrm{reg}}} \frac{e^{\mathrm{i}\lambda x}}{[\mathrm{i}(\lambda - \mathrm{i}\varsigma)]^{q+1/2}} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} = \frac{e^{-\varsigma x}\, x^{q-1/2}}{\mathrm{i}\Gamma(q+1/2)} \tag{6.82}$$

in which (after the change of variable $t = -\mathrm{i}x(\lambda - \mathrm{i}\varsigma)$) we have recognised the Hankel contour integral representation of $\{\Gamma(q+1/2)\}^{-1}$. In its turn, the contribution of the remainder in (6.81) can be written:

$$\int\limits_{\mathscr{C}^{(+)}_{\mathrm{reg},0}} \mathrm{e}^{\mathrm{i}\lambda x} \Delta^{(p)}_{[n]}[R^{-1}_{\downarrow}](\lambda)\, \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \;=\; \int\limits_{\mathscr{C}^{(+)}_{\mathrm{reg}}} \bigg(\mathrm{e}^{\mathrm{i}\lambda x} - \sum_{r=0}^{m-1} \frac{(\mathrm{i}\lambda)^r x^r}{r!}\bigg) \cdot \Delta^{(p)}_{[n]}[R^{-1}_{\downarrow}](\lambda) \cdot \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} + \underbrace{\sum_{r=0}^{m-1} \int\limits_{\mathscr{C}^{(+)}_{\mathrm{reg},0}} \frac{(\mathrm{i}\lambda)^r x^r}{r!} \cdot \Delta^{(p)}_{[n]}[R^{-1}_{\downarrow}](\lambda) \cdot \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}}_{=0}\;. \tag{6.83}$$

Note that the last sum vanishes since we can deform the contour of integration to $-\mathrm{i}\infty$ provided $m \leq n$. Also, we could deform $\mathscr{C}^{(+)}_{\mathrm{reg},0}$ back to $\mathscr{C}^{(+)}_{\mathrm{reg}}$ in the first term since the integrand has no pole at $\lambda = 0$. All-in-all, we get

$$\begin{aligned}\int\limits_{\mathscr{C}^{(+)}_{\mathrm{reg}}} \frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\lambda} \frac{\partial^p}{\partial \lambda^p}\Big\{\frac{1}{R_{\downarrow}(\lambda)}\Big\} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \;&=\; -\frac{\partial^p}{\partial \lambda^p}\Big\{\frac{1}{R_{\downarrow}(\lambda)}\Big\}_{|\lambda=0} + \sum_{q=p+1}^{n} \frac{c_{p;q}\, e^{-\varsigma x}\, x^{q-1/2}}{\mathrm{i}\Gamma(q+1/2)} \\ &\quad + \int\limits_{\mathscr{C}^{(+)}_{\mathrm{reg}}} \Delta^{(p)}_{[n]}[R^{-1}_{\downarrow}](\lambda)\bigg(\mathrm{e}^{\mathrm{i}\lambda x} \;-\; \sum_{r=0}^{m-1} \frac{(\mathrm{i}x)^r \lambda^r}{r!}\bigg) \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\;. \end{aligned} \tag{6.84}$$

With the bound

$$\bigg|\mathrm{e}^{\mathrm{i}\lambda x} \;-\; \sum_{r=0}^{m-1} \frac{(\mathrm{i}x)^r \lambda^r}{r!}\bigg| \leq x^m |\lambda|^m \tag{6.85}$$

and theorems of derivation under the integral, we can conclude that the last term in (6.84) is at least $n-m$ times differentiable and that it has, at least, an m-fold zero at $x = 0$. With the decomposition (6.84), we can come back to τ_ℓ given by (6.79). The second term in (6.84) – which contain derivatives of $1/R_\downarrow$ – can be recombined with its prefactor – containing derivatives of $1/R_\uparrow$ – by using the Leibniz rule backwards for the representation of the derivative at 0 of $1/R = 1/(R_\uparrow R_\downarrow)$. Subsequently, we find there exist a polynomial $p_{\ell;m,n}$ of degree at most $n+\ell$ and a function $f_{\ell;m,n} \in W^\infty_{n-m}(\mathbb{R}^+)$ such that

$$\tau_{\ell+1}(x) \;=\; \sqrt{x}\, p_{\ell;m,n}(x)\mathrm{e}^{-\varsigma x} \;+\; x^m\, f_{\ell;m,n}(x) \;-\; \frac{\mathrm{i}^\ell}{2\pi\beta} \sum_{s+p=\ell} \frac{(\mathrm{i}x)^p}{(s+1)!\,p!}\, \frac{\partial^{s+1}}{\partial \lambda^{s+1}} \Big\{ \frac{1}{R(\lambda)} \Big\}_{|\lambda=0} \,. \tag{6.86}$$

The claim then follows upon adding up all of the terms. Finally, the estimates at $x \to +\infty$ of $\mathfrak{a}_\ell$ follow readily from the exponential decay at $x \to +\infty$ of the functions ϱ and ϖ.

To compute the behaviour at $x \to 0$, we remind that:

$$\mathfrak{a}_0(x) \;=\; \mathfrak{b}_0(x) + \mathfrak{u}_1 \;=\; \tau_1(x) + \mathfrak{u}_1 \,. \tag{6.87}$$

We already know from (6.86) that $\mathfrak{a}_0(0) = 0$, and we just have to look in (6.79)-(6.84) for the coefficient of $\sqrt{x}$ in the case $\ell = 1$. For this purpose, it is enough to write (6.79) with $n = 1$. Then, the square-root behaviour occur for $p = r = 0$ and $s = 1$ in the sum, and gives:

$$\mathfrak{a}_0(x) = \frac{c_{0;1}\, x^{1/2}\, \mathrm{e}^{-\varsigma x}}{2\mathrm{i}\pi\beta \cdot \Gamma(3/2)}\, \partial_\mu R_\uparrow^{-1}(\mu)|_{\mu=0} \;+\; \mathrm{O}(x) \,. \tag{6.88}$$

The coefficient $c_{0;1}$ is given by the large λ asymptotics in (6.81), coming from that of $R_\downarrow(\lambda)$ given by (4.30):

$$c_{0;1} = -1 \,. \tag{6.89}$$

On the other hand, we know from (4.27) that:

$$\partial_\mu R_\uparrow^{-1}(\mu)|_{\mu=0} = \frac{1}{\mathrm{i}\,\sqrt{\omega_1+\omega_2}} \tag{6.90}$$

Therefore:

$$\mathfrak{a}_0(x) = \frac{1}{\pi\beta}\,\sqrt{\frac{x}{\pi(\omega_1+\omega_2)}} \;+\; \mathrm{O}(x) \,. \tag{6.91}$$

We finally turn to proving that $\mathfrak{a}_0 > 0$ on $\mathbb{R}^+$. It follows from the previous calculations that

$$\mathfrak{a}_0(x) \;=\; \frac{1}{2\pi\beta}\Big(\frac{1}{\mu \cdot R_\uparrow(\mu)}\Big)_{|\mu=0} \int_{\mathscr{C}^{(+)}_{\mathrm{reg}}} \frac{\mathrm{e}^{\mathrm{i}\lambda x}-1}{\lambda R_\downarrow(\lambda)}\, \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \,. \tag{6.92}$$

The integral can be computed by deforming the contour up to $+\mathrm{i}\infty$ and, in doing so, we pick up the residues of the poles located at

$$\lambda = \frac{2\mathrm{i}\pi n\, \omega_1\omega_2}{\omega_1+\omega_2}\,, \qquad n \geq 1 \,. \tag{6.93}$$

All-in-all this yields

$$\mathfrak{a}_0(x) \;=\; \sum_{n\geq 1} \mathfrak{a}_{0;n}\Big(1 \;-\; \mathrm{e}^{-\frac{2\pi\omega_1\omega_2}{\omega_1+\omega_2}nx}\Big) \qquad \text{with} \qquad \mathfrak{a}_{0;n} \;=\; \frac{(\omega_1+\omega_2)\cdot(-1)^{n-1}\cdot\kappa^{-n\kappa}\cdot(1-\kappa)^{-n(1-\kappa)}}{2\pi\beta\omega_1\omega_2\cdot n^2\cdot n!\cdot\Gamma(-\kappa n)\cdot\Gamma(-(1-\kappa)n)} \tag{6.94}$$

and $\kappa = \omega_2/(\omega_1+\omega_2) < 1$. By using the Euler reflection formula, we can recast $\mathfrak{a}_{0;n}$ into a manifestly strictly positive form

$$\mathfrak{a}_{0;n} \;=\; \frac{(\omega_1+\omega_2)}{2\pi\beta\omega_1\omega_2}\cdot\Big(\frac{\sin[\pi\kappa n]}{\pi}\Big)^2\cdot\frac{\Gamma(1+\kappa n)\cdot\Gamma(1+(1-\kappa)n)}{n^2\cdot n!\cdot\kappa^{n\kappa}\cdot(1-\kappa)^{n(1-\kappa)}}\;. \tag{6.95}$$

The asymptotics of $\mathfrak{a}_{0;n}$ then takes the form

$$\mathfrak{a}_{0;n} \underset{n\to+\infty}{\sim} \frac{(\omega_1+\omega_2)}{2\beta\omega_1\omega_2}\cdot\sqrt{\frac{2\kappa(1-\kappa)}{\pi n^3}}\cdot\Big(\frac{\sin[\pi\kappa n]}{\pi}\Big)^2\;. \tag{6.96}$$

Thus the series (6.94) defining $\mathfrak{a}_0(x)$ converges uniformly for $x\in\mathbb{R}^+$. Since the series only contains positive summands, $\mathfrak{a}_0(x)$ is positive for $x>0$. ■

The main reason for investigating the properties of the functions $\mathfrak{a}_\ell(x)$ lies in the fact that they describe the large-N asymptotics of the function $\mathcal{W}_{R;k}[H](x, b_N - N^{-\alpha}\,x) + \mathcal{W}_{\mathrm{bk};k}[H](b_N - N^{-\alpha}\,x)$. In particular, $\mathfrak{a}_0(x)$ arises as the first term and plays a particularly important role in the analysis that will follow. Let us remind Definition 3.14 for the weighted norm:

$$\mathcal{N}_N^{(\ell)}[H] = \sum_{k=0}^{\ell}\frac{||H||_{W_k^\infty(\mathbb{R})}}{N^{k\alpha}}\;. \tag{6.97}$$

Lemma 6.11 *Let $k\geq 0$ be an integer, $H\in C^{2k+1}([a_N\,;b_N])$. Define the functions:*

$$\mathcal{W}^{(\mathrm{as})}_{R;k}[H](x) \;=\; H'(b_N)\,\mathfrak{b}_0(x) \;+\; \sum_{\ell=1}^{k-1}\frac{H^{(\ell+1)}(b_N)\,\mathfrak{b}_\ell(x)}{N^{\ell\alpha}}\;, \tag{6.98}$$

$$\mathcal{W}^{(\mathrm{as})}_{\mathrm{bk};k}[H](x) \;=\; H'(b_N)\,u_1 + \sum_{\ell=1}^{k-1}\frac{H^{(\ell+1)}(b_N)\,\mathfrak{u}_\ell(x)}{N^{\ell\alpha}}\;. \tag{6.99}$$

The approximants at order k, $\mathcal{W}_{R;k}[H](x, b_N - N^{-\alpha}\,x)$ and $\mathcal{W}_{\mathrm{bk};k}[H](b_N - N^{-\alpha}\,x)$, admit the large-$N$ asymptotic expansions:

$$\mathcal{W}_{R;k}[H](x, b_N - N^{-\alpha}x) \;=\; \mathcal{W}^{(\mathrm{as})}_{R;k}[H](x) \;+\; \Delta_{[k]}\mathcal{W}^{(\mathrm{as})}_{R}[H](x)\;, \tag{6.100}$$

$$\mathcal{W}_{\mathrm{bk};k}[H](b_N - N^{-\alpha}x) \;=\; \mathcal{W}^{(\mathrm{as})}_{\mathrm{bk};k}[H](x) \;+\; \Delta_{[k]}\mathcal{W}^{(\mathrm{as})}_{\mathrm{bk}}[H](x)\;. \tag{6.101}$$

The remainders have the following structure:

$$\Delta_{[k]}\mathcal{W}^{(\mathrm{as})}_{R}[H](x) \;=\; N^{-k\alpha}\cdot\mathrm{e}^{-\varsigma x}\Big\{(\ln x)\,\mathcal{R}^{(1)}_{\mathrm{as};k}[H](x) \;+\; \mathcal{R}^{(2)}_{\mathrm{as};k}[H](x)\Big\}\;, \tag{6.102}$$

$$\Delta_{[k]}\mathcal{W}^{(\mathrm{as})}_{\mathrm{bk}}[H](x) \;=\; N^{-k\alpha}\cdot\mathcal{R}^{(3)}_{\mathrm{as};k}[H](x)\;, \tag{6.103}$$

where $\mathcal{R}^{(a)}_{\mathrm{as};k}[H]\in W^\infty_\ell(\mathbb{R}^+)$ for $a=1,2,3$. For $a=1$, we have:

$$\big|\mathcal{R}^{(1)}_{\mathrm{as};k}[H](x)\big| = \mathrm{O}(x^{k+1}) \tag{6.104}$$

uniformly in N. Moreover, we have uniform bounds for $x\in[0\,;\epsilon N^\alpha]$, namely for $\ell\in[\![\,0\,;\,k\,]\!]$:

$$\big|\partial_\xi^\ell\mathcal{R}^{(1)}_{\mathrm{as};k}[H](x_R)\big| \;\leq\; C_{k,\ell}\cdot x_R^{k-\ell+1}\cdot N^{\ell\alpha}\cdot\mathcal{N}_N^{(\ell)}[H_{\mathfrak{e}}^{(k+1)}]\;, \tag{6.105}$$

$$a=2,3,\qquad \big|\partial_\xi^\ell\mathcal{R}^{(a)}_{\mathrm{as};k}[H](x_R)\big| \;\leq\; C_{k,\ell}\cdot N^{\ell\alpha}\cdot\mathcal{N}_N^{(\ell)}[H_{\mathfrak{e}}^{(k+1)}]\;, \tag{6.106}$$

where we remind $x_R = N^\alpha(b_N-\xi)$.

Note that we can combine the operators into the asymptotic expansion

$$\mathcal{W}^{(\mathrm{as})}_{R;k}[H](x) \; + \; \mathcal{W}^{(\mathrm{as})}_{\mathrm{bk};k}[H](x) \; = \; H'(b_N)\,\mathfrak{a}_0(x)\bigg\{1 + \sum_{\ell=1}^{k} \frac{H^{(\ell+1)}(b_N)\,\mathfrak{a}_\ell(x)}{H'(b_N)\,N^{\alpha\ell}}\bigg\}\,. \tag{6.107}$$

Proof — The form of the large-N asymptotic expansion follows from straightforward manipulations on the Taylor integral representation for $H^{(\ell)}(\xi)$ around $\xi = b_N$ for $\ell \in [\![\, 0\,;\, k \,]\!]$. The control on the remainder arising in (6.100), (6.101) and (6.107) follows from the explicit integral representation for the remainder in the Taylor-integral series:

$$\Delta_{[k]}\mathcal{W}^{(\mathrm{as})}_{R}[H](x) \;\; = \;\; N^{-k\alpha}\int_0^1 \mathrm{d}t\, H^{(k+1)}(b_N - N^{-\alpha}\,tx)\bigg\{-\frac{(1-t)^k\,(-x)^{k+1}\varrho_0(x)}{k!} - \sum_{\ell=1}^{k}\frac{(1-t)^{k-\ell}(-x)^{1+k-\ell}\,\varpi_\ell(x)}{\ell!(k-\ell)!}\bigg\}\,,$$

$$\Delta_{[k]}\mathcal{W}^{(\mathrm{as})}_{\mathrm{bk}}[H](x) \;\; = \;\; N^{-k\alpha}\sum_{\ell=1}^{k} u_\ell\,\frac{(-x)^{k+1-\ell}}{(k-\ell)!}\int_0^1 \mathrm{d}t\,(1-t)^{k-\ell}\,H^{(k+1)}(b_N - N^{-\alpha}\,tx)\,. \tag{6.108}$$

and we remark that $\varrho_0(x)$ – given by (6.35) – has a logarithmic singularity when $x \to 0$. The details to arrive to (6.105)-(6.106) are left to the reader. ■

Collecting the bounds, we have obtained in sup norms, we find in particular $\mathcal{W}_N[H]$ is bounded when H is C^1:

Corollary 6.12 *There exists $C > 0$ independent of N such that, for any $H \in C^1([a_N\,;b_N])$,*

$$||\mathcal{W}_N[H]||_{W_0^\infty([a_N\,;b_N])} \;\leq\; C\,||H_{\mathrm{e}}||_{W_1^\infty(\mathbb{R})}\,. \tag{6.109}$$

7 Asymptotic analysis of single integrals

7.1 Asymptotic analysis of the constraint functionals $\mathcal{X}_N[H]$

Recall that for any $H \in C^1([a_N\,;b_N])$ the linear form $\mathcal{X}_N[H]$ defined in (3.99):

$$\mathcal{X}_N[H] \;=\; \frac{\mathrm{i}N^\alpha}{\chi_{11;+}(0)}\int_{\mathbb{R}+\mathrm{i}\epsilon'}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\,\chi_{11}(\mu)\int_{a_N}^{b_N} H(\eta)\mathrm{e}^{\mathrm{i}N^\alpha\mu(\eta-b_N)}\,\mathrm{d}\eta \tag{7.1}$$

is related to the constraint $\mathscr{I}_{11}[h]$ defined in (5.24) where H and h are related by the rescaling (4.2):

$$\mathscr{I}_{11}[h] \;=\; -\frac{N^\alpha\,\chi_{11;+}(0)}{2\pi\beta}\,\mathcal{X}_N[H]\,, \qquad h(x) = \frac{N^\alpha}{2\mathrm{i}\pi\beta}\,H(a_N + N^{-\alpha}x)\,. \tag{7.2}$$

In the following, we shall obtain the large-N expansion of the linear form $\mathcal{X}_N[h]$ introduced in (3.99) and defining the hyperplane $\mathfrak{X}_s$ where we inverse operators. We first need to define new constants:

Definition 7.1 *If $p \geq 0$ is an integer, we define:*

$$\daleth_p \;=\; -\frac{R_\downarrow(0)}{2}\int_{\mathbb{R}+\mathrm{i}\epsilon'}\frac{1}{\mu^{p+1}R_\downarrow(\mu)}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \;=\; (-1)^{p+1}\,\frac{R_\downarrow(0)}{2}\int_{\mathbb{R}-\mathrm{i}\epsilon'}\frac{1}{\mu^{p+2}R_\uparrow(\mu)}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\,. \tag{7.3}$$

The equality between the two expressions of $\daleth_p$ follows from the symmetry (4.28).

Lemma 7.2 *Let $k \geq 1$ be an integer, and $H \in C^k([a_N\,; b_N])$. We have an asymptotic expansion:*

$$\mathcal{X}_N[H] = \sum_{p=0}^{k-1} \frac{\mathrm{i}^p \urcorner_p}{N^{\alpha p}} \Big\{ H^{(p)}(a_N) \,+\, (-1)^p H^{(p)}(b_N) \Big\} \;+\; \Delta_{[k]}\mathcal{X}_N[H]\,, \tag{7.4}$$

where:

$$\Big|\Delta_{[k]}\mathcal{X}_N[H]\Big| \;\leq\; C\,N^{-k\alpha}\,||H||_{W_k^{\infty}([a_N\,;b_N])}\;. \tag{7.5}$$

Proof — For λ between $\Gamma_\uparrow$ and $\mathbb{R}$, we decompose χ into:

$$\chi(\lambda) \;=\; \chi_\uparrow^{(\mathrm{as})}(\lambda) \;+\; \chi_\uparrow^{(\mathrm{exp})}(\lambda) \tag{7.6}$$

In terms of the various matrices used § 4.4, the main part is:

$$\chi_\uparrow^{(\mathrm{as})}(\lambda) \;=\; \mathcal{R}_\uparrow^{-1}(\lambda)\cdot[\upsilon(\lambda)]^{-\sigma_3}\cdot M_\uparrow(\lambda)\cdot\left(I_2+\frac{\sigma^-}{\lambda}\right) = \begin{pmatrix} -\dfrac{\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}}{R_\downarrow(\lambda)} \;+\; \dfrac{1}{\lambda R_\uparrow(\lambda)} & \dfrac{1}{R_\uparrow(\lambda)} \\ -\,R_\uparrow(\lambda) & 0 \end{pmatrix} \tag{7.7}$$

and is such that the remainder is exponentially small in N:

$$\chi_\uparrow^{(\mathrm{exp})}(\lambda) \;=\; \chi_\uparrow^{(\mathrm{as})}(\lambda)\cdot[\delta\Pi](\lambda) \qquad \text{with} \qquad [\delta\Pi](\lambda) \;=\; \left(I_2 \,+\, \frac{\sigma^-}{\lambda}\right)^{-1}\cdot\Pi(\lambda)\cdot P_R(\lambda) \;-\; I_2\;. \tag{7.8}$$

Indeed, the large-N behaviour of θ_R inferred from (4.18) and (4.35) as well as the estimate (4.50) on the matrix $\Pi - I_2$ imply that, for ϵ' fixed but small enough, and uniformly in $\lambda \in \mathbb{R} + \mathrm{i}\tau$, $0 < \tau < \epsilon'$:

$$\Big|[\delta\Pi]_{ab}(\lambda)\Big| \;\leq\; \frac{C\,\mathrm{e}^{-\varkappa_{\epsilon'}\,N^{\alpha}}}{1+|\lambda|}\;. \tag{7.9}$$

Furthermore, a direct calculation shows that

$$[\chi_\uparrow^{(\mathrm{exp})}]_{11}(\lambda) \;=\; \left(\frac{1}{\lambda R_\uparrow(\lambda)} - \frac{\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}}{R_\downarrow(\lambda)}\right)[\delta\Pi]_{11}(\lambda) \;+\; \frac{[\delta\Pi]_{21}(\lambda)}{R_\uparrow(\lambda)}\;, \tag{7.10}$$

and taking into account the large-λ behaviour of $R_{\uparrow/\downarrow}$ given in (4.25)- (4.26), we also get a uniform bound for$\lambda \in \mathbb{R} + \mathrm{i}\tau$, $0 < \tau < \epsilon'$:

$$\Big|[\chi_\uparrow^{(\mathrm{exp})}]_{11}(\lambda)\Big| \;\leq\; \frac{C'\,\mathrm{e}^{-\varkappa_{\epsilon'}N^{\alpha}}}{\sqrt{1+|\lambda|}}\;. \tag{7.11}$$

In particular, this estimate (7.11) implies:

$$\frac{1}{\chi_{11;+}(0)} \;=\; -\frac{R_\downarrow(0)}{2} \;+\; \mathrm{O}(e^{-\varkappa_{\epsilon'}N^{\alpha}})\;. \tag{7.12}$$

The decomposition (7.6) in formula (7.1) induces a decomposition:

$$\mathcal{X}_N[H] \;=\; \mathcal{X}_N^{(\mathrm{as})}[H] \;+\; \mathcal{X}_N^{(\mathrm{exp})}[H] \tag{7.13}$$

where

$$\mathcal{X}^{(\mathrm{exp})}[H] \;=\; \frac{\mathrm{i}N^{\alpha}}{\chi_{11;+}(0)} \int\limits_{\widetilde{\mathscr{C}}^{(-)}} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \, [\chi_{\uparrow}^{(\mathrm{exp})}(\mu)]_{11} \int\limits_{a_N}^{b_N} H(\eta) \mathrm{e}^{\mathrm{i}N^{\alpha}\mu(\eta-b_N)} \cdot \mathrm{d}\eta \tag{7.14}$$

and $\widetilde{\mathscr{C}}^{(-)}$ is a contour surrounding 0 from above, going to ∞ in $\mathbb{H}^{-}$ along the rays $t\mathrm{e}^{-\frac{3\mathrm{i}\pi}{4}}$ and $t\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}$ and such that $\max\{\mathrm{Im}(\lambda) \;:\; \lambda \in \widetilde{\mathscr{C}}^{(-)}\} = \epsilon'$. Note that we could have carried out this contour deformation since $\Pi(\lambda)$ is holomorphic in the domain delimited by $\mathbb{R} + \mathrm{i}\epsilon'$ and $\widetilde{\mathscr{C}}^{(-)}$.
Since for $\lambda \in \widetilde{\mathscr{C}}^{(-)}$, we have:

$$\Big| \int\limits_{a_N}^{b_N} H(\eta) \mathrm{e}^{\mathrm{i}N^{\alpha}\lambda(\eta-b_N)} \, \mathrm{d}\eta \Big| \;\leq\; \frac{C\,\mathrm{e}^{\overline{x}_N \epsilon'}}{|\lambda|} \, ||H||_{L^{\infty}([a_N\,;b_N])} \;, \tag{7.15}$$

it is readily seen that

$$\big|\mathcal{X}_N^{(\mathrm{exp})}[H]\big| \;\leq\; C' \cdot N^{\alpha} \mathrm{e}^{-\frac{\varkappa_{\epsilon'}}{2} N^{\alpha}} \, ||H||_{L^{\infty}([a_N\,;b_N])} \;. \tag{7.16}$$

It thus remains to estimate

$$\mathcal{X}_N^{(\mathrm{as})}[H] = \mathcal{X}_R^{(\mathrm{as})}[H] \;+\; \mathcal{X}_R^{(\mathrm{as})}[H^{\wedge}] \tag{7.17}$$

where

$$\mathcal{X}_R^{(\mathrm{as})}[H] \;=\; \frac{\mathrm{i}N^{\alpha}}{\chi_{11;+}(0)} \int\limits_{\mathscr{C}_{\mathrm{reg}}^{(-)}} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \, \frac{1}{\mu R_{\uparrow}(\mu)} \int\limits_{a_N}^{b_N} H(\eta) \mathrm{e}^{\mathrm{i}N^{\alpha}\mu(\eta-b_N)} \, \mathrm{d}\eta \;, \tag{7.18}$$

and the second term arises upon the change of variables $(\mu, \eta) \mapsto (-\mu, a_N + b_N - \eta)$ in the initial expression. The dependence in N is implicit in these new notations. Note that we could deform the contour from $\mathbb{R} + \mathrm{i}\epsilon'$ up to $\mathbb{R} - \mathrm{i}\epsilon'$ or $\mathscr{C}_{\mathrm{reg}}^{(-)}$ since the integrand is holomorphic in the domain swapped in between. Replacing H by its Taylor series with integral remainder at order k, we get:

$$\mathcal{X}_R^{(\mathrm{as})}[H] \;=\; \mathcal{X}_{R;k}^{(\mathrm{as})}[H] \;+\; \Delta_{[k]}\mathcal{X}_R^{(\mathrm{as})}[H] \;. \tag{7.19}$$

The first term is:

$$\mathcal{X}_{R;k}^{(\mathrm{as})}[H] \;=\; \mathrm{i}N^{\alpha} \sum_{p=0}^{k-1} \frac{H^{(p)}(b_N)}{p!\,\chi_{11;+}(0)} \int\limits_{\mathbb{R}-\mathrm{i}\epsilon'} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \frac{1}{\mu R_{\uparrow}(\mu)} \int\limits_{-\infty}^{0} \eta^{p} \mathrm{e}^{\mathrm{i}N^{\alpha}\mu\eta} \, \mathrm{d}\eta \;=\; \frac{-2}{R_{\downarrow}(0)\chi_{11;+}(0)} \sum_{p=0}^{k-1} \frac{(-\mathrm{i})^{p} \, \daleth_p \, H^{(p)}(b_N)}{N^{p\alpha}} \tag{7.20}$$

where we have recognised the constants $\daleth_p$ of Definition 7.1. The remainder is:

$$\begin{aligned} \Delta_{[k]}\mathcal{X}_R^{(\mathrm{as})}[H] \;=\; \frac{1}{\mathrm{i}\chi_{11;+}(0)} \bigg\{ & \sum_{p=0}^{k-1} \frac{H^{(p)}(b_N)}{p!\,N^{p\alpha}} \int\limits_{\mathscr{C}_{\mathrm{reg}}^{(-)}} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \frac{1}{\mu R_{\uparrow}(\mu)} \int\limits_{-\infty}^{-\overline{x}_N} \eta^{p} \mathrm{e}^{\mathrm{i}\mu\eta} \, \mathrm{d}\eta \\ & + \int\limits_{\mathscr{C}_{\mathrm{reg}}^{(-)}} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \frac{N^{\alpha}}{\mu R_{\uparrow}(\mu)} \int\limits_{a_N}^{b_N} \mathrm{d}\eta \, (\eta - b_N)^{k} \int\limits_{0}^{1} \mathrm{d}t \, \frac{(1-t)^{k-1}}{(k-1)!} \mathrm{e}^{\mathrm{i}N^{\alpha}\mu(\eta-b_N)} \, H^{(k)}(b_N + t(\eta - b_N)) \bigg\} \;. \end{aligned} \tag{7.21}$$

$\mathcal{X}^{(\mathrm{as})}_{R;k}[H]$ yields the leading terms of the asymptotic expansion announced in (7.4). Hence, it remains to bound $\Delta_{[k]}\mathcal{X}^{(\mathrm{as})}_{R}[H]$. The first line in (7.21) is exponentially small and bounded by a term proportional to $||H||_{W^{\infty}_{k-1}([a_N\,;b_N])}$. The second line is bounded by

$$N^{\alpha}\cdot|R_{\downarrow}(0)|\cdot||H||_{W^{\infty}_{k}([a_N\,;b_N])}\int\limits_{\mathscr{C}^{(-)}_{\mathrm{reg}}}\frac{|\mathrm{d}\mu|}{2\pi\,k!}\,\frac{1}{\big|\mu R_{\uparrow}(\mu)\big|}\int\limits_{-\infty}^{b_N}\mathrm{d}\eta\,(b_N-\eta)^{k}\,\mathrm{e}^{-N^{\alpha}\left[\mathrm{Im}\,\mu(\eta-b_N)\right]}\leq C\,N^{-k\alpha}\,||H||_{W^{\infty}_{k}([a_N\,;b_N])}\;. \tag{7.22}$$

It thus solely remains to put all the pieces together. ∎

Using these estimates, we obtain the continuity of the linear form $\mathcal{X}_N$ in sup norms:

Corollary 7.3 *There exists $C>0$ independent of N, such that:*

$$\big|\widetilde{\mathcal{X}}_N[H]\big|\;\leq\;C\,||H||_{W^{\infty}_{0}([a_N\,;b_N])}\;\;. \tag{7.23}$$

Proof — We have shown in the proof of Lemma 7.2 a decomposition:

$$\mathcal{X}^{(\mathrm{as})}_{N}[H]\;=\;\mathcal{X}^{(\mathrm{as})}_{R}[H]\;+\;\mathcal{X}^{(\mathrm{as})}_{R}[H^{\wedge}]+\mathcal{X}^{(\mathrm{exp})}_{N}[H]\;. \tag{7.24}$$

$\mathcal{X}^{(\mathrm{as})}_{R}[H]$ is given in (7.18). It has $\chi_{11;+}(0)$ as prefactor, and we have seen in (7.12) that this quantity takes the non-zero value $-2/R_{\downarrow}(0)$ up to exponential small (in N) corrections. So, we have the bound:

$$\big|\mathcal{X}^{(\mathrm{as})}_{R}[H]\big|\;\leq\;\frac{|R_{\downarrow}(0)|}{2}\cdot||H||_{W^{\infty}_{0}([a_N\,;b_N])}\cdot\int\limits_{\mathscr{C}^{(-)}_{\mathrm{reg}}}\frac{1}{|\mu||\mathrm{Im}\,\mu|\,R_{\uparrow}(\mu)|}\,\frac{|\mathrm{d}\mu|}{2\pi} \tag{7.25}$$

where the inverse power of $|\mathrm{Im}\,\mu|$ and the loss of the prefactor N^{α} resulted from integrating the decaying exponential $|e^{\mathrm{i}N^{\alpha}\mu(\eta-b_N)}|$ over $[a_N\,;b_N]$, given that $\mathrm{Im}\,\mu<0$ for $\mu\in\mathscr{C}^{(-)}_{\mathrm{reg}}$. We conclude by combining this estimate with (7.16) which shows that the remainder is exponentially small. ∎

7.2 Asymptotic analysis of simple integrals

In the present subsection, we obtain the large-N asymptotic expansion of one-dimensional integrals involving $\mathcal{W}_N[H]$. This provides the first set of results that were necessary in § 3.4 for a thorough calculation of the large-N expansion of the partition function.

Definition 7.4 *If G and H are two functions on $[a_N\,;b_N]$, we define:*

$$\mathfrak{I}_{\mathrm{s}}[G,H]\;=\;\int\limits_{a_N}^{b_N}G(\xi)\cdot\mathcal{W}_N[H](\xi)\,\mathrm{d}\xi \tag{7.26}$$

where the $\mathcal{W}_N$ is the operator defined in (2.44).

To write the large N-expansion of $\mathfrak{I}_{\mathrm{s}}$, we need to introduce some more constants:

Definition 7.5 *If $s, \ell \geq 0$ are integers, we set:*

$$\daleth_{s,\ell} \;=\; \int\limits_{0}^{+\infty} x^{s}\, \mathfrak{b}_{\ell}(x)\, \mathrm{d}x \tag{7.27}$$

where the function $\mathfrak{b}_{\ell}$ has been introduced in Definition 6.9.

Proposition 7.6 *Let $k \geq 1$ be an integer, $G \in C^{k-1}([a_N\,;b_N])$ and $H \in C^{k+1}([a_N\,;b_N])$. We have the asymptotic expansion:*

$$\begin{aligned}\mathfrak{I}_{\mathrm{s}}[G,H] \;=\;& u_1 \int\limits_{a_N}^{b_N} G(\xi)\cdot H'(\xi)\,\mathrm{d}\xi \;+\; \sum_{p=1}^{k-1} \frac{1}{N^{\alpha p}} \Bigg\{ u_{p+1} \int\limits_{a_N}^{b_N} G(\xi) H^{(p+1)}(\xi)\,\mathrm{d}\xi \\ &+ \sum_{\substack{s+\ell=p-1 \\ s,\ell\geq 0}} \frac{\daleth_{s,\ell}}{s!} \Big[(-1)^{s} H^{(\ell+1)}(b_N)\cdot G^{(s)}(b_N) \;+\; (-1)^{\ell} H^{(\ell+1)}(a_N) G^{(s)}(a_N) \Big] \Bigg\} \;+\; \Delta_{[k]}\mathfrak{I}_{\mathrm{s}}[G,H]\;.\end{aligned} \tag{7.28}$$

where we remind that u's are the constants appearing in Definition 6.5. The remainder is bounded as

$$\Big|\Delta_{[k]}\mathfrak{I}_{\mathrm{s}}[G,H]\Big| \;\leq\; C\,N^{-k\alpha}\, ||G||_{W^{\infty}_{k-1}([a_N\,;b_N])}\, ||H_{\mathfrak{e}}||_{W^{\infty}_{k+1}([a_N\,;b_N])} \tag{7.29}$$

for some constant $C > 0$ independent of N, G and H.

Note that the leading asymptotics of $\mathfrak{I}_{\mathrm{s}}[G,H]$, *i.e.* up to the o(1) remainder, correspond precisely to the contribution obtained by replacing the integral kernel $S\,(N^{\alpha}(\xi-\eta))$ of $\mathcal{S}_N$ by the sign function– which corresponds to the almost sure pointwise limit of $S\,(N^{\alpha}(\xi-\eta))$, see (2.42) – and then inverting the formal limiting operator. The corrections, however, are already more complicated as they stem from the fine behaviour at the boundaries.

Proof — Recall from Propositions 6.4 and 6.6 that $\mathcal{W}_N[H]$ decomposes as

$$\mathcal{W}_N[H](\xi) \;=\; \mathcal{W}_{R;k}[H](x_R,\xi) \;+\; \mathcal{W}_{\mathrm{bk};k}[H](\xi) \;-\; \mathcal{W}_{R;k}[H^{\wedge}](x_L, a_N+b_N-\xi) \;+\; \Delta_{[k]}\mathcal{W}_N[H_{\mathfrak{e}}](\xi) \tag{7.30}$$

where

$$\Big|\Big|\Delta_{[k]}\mathcal{W}_N[H_{\mathfrak{e}}]\Big|\Big|_{L^{\infty}([a_N\,;b_N])} \;\leq\; C\,N^{-k\alpha}\, ||H_{\mathfrak{e}}^{(k+1)}||_{L^{\infty}(\mathbb{R})}\;. \tag{7.31}$$

This leads to the decomposition

$$\mathfrak{I}_{\mathrm{s}}[G,H] \;=\; \mathfrak{I}^{(\mathrm{bk})}_{\mathrm{s};k}[G,H] \;+\; \mathfrak{I}^{(\partial)}_{\mathrm{s};k}[G,H] \;-\; \mathfrak{I}^{(\partial)}_{\mathrm{s};k}[G^{\wedge},H^{\wedge}] \;+\; \Delta_{[k]}\mathfrak{I}_{\mathrm{s}}[G,H_{\mathfrak{e}}] \tag{7.32}$$

where:

$$\begin{aligned}\mathfrak{I}^{(\mathrm{bk})}_{\mathrm{s};k}[G,H] \;&=\; \int\limits_{a_N}^{b_N} G(\xi)\cdot \mathcal{W}_{\mathrm{bk};k}[H](\xi)\,\mathrm{d}\xi\;, \\ \mathfrak{I}^{(\partial)}_{\mathrm{s};k}[G,H] \;&=\; \frac{1}{N^{\alpha}} \int\limits_{0}^{\overline{x}_N} G(b_N - N^{-\alpha}x)\cdot \mathcal{W}_{R;k}[H](x, b_N - N^{-\alpha}x)\,\mathrm{d}x\;, \\ \Delta_{[k]}\mathfrak{I}_{\mathrm{s}}[G,H_{\mathfrak{e}}] \;&=\; \int\limits_{a_N}^{b_N} G(\xi)\cdot \Delta_{[k]}\mathcal{W}_N[H_{\mathfrak{e}}](\xi)\,\mathrm{d}\xi\;.\end{aligned} \tag{7.33}$$

Clearly from the estimate (7.31), there exist a constant $C' > 0$ such that:

$$\left|\Delta_{[k]}\mathfrak{I}_{\mathrm{s}}[G,H_{\mathfrak{e}}]\right| \;\le\; C'\,N^{-k\alpha}\cdot||G||_{L^{\infty}([a_N\,;b_N])}\cdot||H_{\mathfrak{e}}^{(k+1)}||_{L^{\infty}(\mathbb{R})}\;. \tag{7.34}$$

The asymptotic expansion of $\mathfrak{I}_{\mathrm{s};k}^{(\mathrm{bk})}$ follows readily from the expression (6.41) for $\mathcal{W}_{\mathrm{bk};k}[H]$. It produces the first line of (7.28). As a consequence, it remains to focus on $\mathfrak{I}_{\mathrm{s};k}^{(\partial)}$. Recall from Proposition 6.11 the decomposition

$$\mathcal{W}_{R;k}[H](x,b_N-N^{-\alpha}x) \;=\; \mathcal{W}_{R;k}^{(\mathrm{as})}[H](x) \;+\; \Delta_{[k]}\mathcal{W}_{R}^{(\mathrm{as})}[H](x) \tag{7.35}$$

and especially the bounds (6.104)-(6.106) on the remainder, which imply:

$$\left|\Delta_{[k]}\mathcal{W}_{R}^{(\mathrm{as})}[H](x)\right| \;\le\; C\,\mathrm{e}^{-\varsigma x}\,x^{k+1}\,\ln x\cdot N^{-k\alpha}\cdot||H_{\mathfrak{e}}||_{W^{\infty}_{k+1}(\mathbb{R})}\;. \tag{7.36}$$

The contribution of the first term of (7.35) involves the functions $\mathfrak{b}_{\ell}$. it remains to replace G by its Taylor series with integral remainder of appropriate order so as to get

$$\mathfrak{I}_{\mathrm{s};k}^{(\partial)}[G,H] \;=\; \sum_{p=0}^{k-1}\frac{1}{N^{(p+1)\alpha}}\sum_{\substack{s+\ell=p\\ s,\ell\ge 0}}\frac{(-1)^s}{s!}H^{(\ell+1)}(b_N)\cdot G^{(s)}(b_N)\int_0^{\overline{x}_N}x^s\,\mathfrak{b}_{\ell}(x)\,\mathrm{d}x \;+\; \Delta_{[k]}\mathfrak{I}_{\mathrm{s}}^{(\partial)}[G,H] \tag{7.37}$$

where

$$\Delta_{[k]}\mathfrak{I}_{\mathrm{s}}^{(\partial)}[G,H] \;=\; \frac{1}{N^{k\alpha}}\sum_{\ell=0}^{k-1}\frac{H^{(\ell+1)}(b_N)}{(k-\ell-2)!}\int_0^{\overline{x}_N}\mathrm{d}x\,\mathfrak{b}_{\ell}(x)\,(-x)^{k-\ell-1}\int_0^1\mathrm{d}t\,(1-t)^{k-2-\ell}G^{(k-\ell-1)}(b_N-N^{-\alpha}tx) \tag{7.38}$$

$$+\;\frac{1}{N^{\alpha}}\int_0^{\overline{x}_N}G(b_N-N^{-\alpha}x)\cdot\Delta_{[k]}\mathcal{W}_{R}^{(\mathrm{as})}[H](x)\,\mathrm{d}x\;. \tag{7.39}$$

Clearly from (7.36), there exists $C'' > 0$ such that:

$$\left|\Delta_k\mathfrak{I}_{\mathrm{s};k}^{(\partial)}[G,H]\right| \;\le\; C''\,N^{-k\alpha}\,||H_{\mathfrak{e}}||_{W^{\infty}_{k+1}(\mathbb{R})}\cdot||G||_{W^{\infty}_{k-1}([a_N\,;b_N])}\;. \tag{7.40}$$

Moreover, one can extend the integration in (7.37) from $[0\,;\overline{x}_N]$ up to $\mathbb{R}^+$, this for the price of exponentially small corrections in N. Adding up all the pieces leads to (7.28). ■

In the case when $G=1$, *i.e.* to estimate the magnitude of the total integral of $\mathcal{W}_N[H]$, we can obtain slightly better bounds, solely involving the sup norm.

Lemma 7.7 *There exists $C > 0$ independent of N such that, for any $H \in C^1([a_N\,;b_N])$,*

$$\left|\int_{a_N}^{b_N}\mathcal{W}_N[H](\xi)\,\mathrm{d}\xi\right| \;\le\; C\,||H_{\mathfrak{e}}||_{W^{\infty}_0(\mathbb{R})}\;. \tag{7.41}$$

Proof — Recall from Propositions 6.4 the decomposition:

$$\mathcal{W}_N[H](\xi) \;=\; \mathcal{W}_R[H_{\mathfrak{e}}](x_R,\xi) \;+\; \mathcal{W}_{\mathrm{bk}}[H_{\mathfrak{e}}](\xi) \;-\; \mathcal{W}_R[H_{\mathfrak{e}}^{\wedge}](x_L,a_N+b_N-\xi) \;+\; \mathcal{W}_{\mathrm{exp}}[H](\xi)\;. \tag{7.42}$$

We focus on the integral of each of the terms taken individually. We have:

$$\int_{a_N}^{b_N} \mathcal{W}_{\mathrm{bk}}[H_{\mathrm{e}}](\xi)\,\mathrm{d}\xi \;=\; \frac{N^{\alpha}}{2\pi\beta}\int_{\mathbb{R}}\mathrm{d}y\,J(y)\int_{0}^{N^{-\alpha}y}\big[H_{\mathrm{e}}(b_N+t)\;-\;H_{\mathrm{e}}(a_N+t)\big]\,\mathrm{d}t\;, \tag{7.43}$$

thus leading to

$$\bigg|\int_{a_N}^{b_N}\mathcal{W}_{\mathrm{bk}}[H_{\mathrm{e}}](\xi)\,\mathrm{d}\xi\bigg| \;\le\; C\,||H_{\mathrm{e}}||_{W_0^{\infty}(\mathbb{R})}\;. \tag{7.44}$$

Next, we have:

$$\begin{aligned}\int_{a_N}^{b_N}\mathcal{W}_R[H_{\mathrm{e}}](x_R,\xi)\,\mathrm{d}\xi \;=\; & -\frac{N^{\alpha}}{2\pi\beta}\int_{\overline{x}_N}^{+\infty}\mathrm{d}y\,J(y)\int_{a_N}^{b_N}\mathrm{d}\xi\,\big[H_{\mathrm{e}}(\xi+N^{-\alpha}y)-H_{\mathrm{e}}(\xi)\big] \\ & -\frac{N^{\alpha}}{2\pi\beta}\int_{0}^{\overline{x}_N}\mathrm{d}y\,J(y)\int_{0}^{N^{-\alpha}y}\big[H_{\mathrm{e}}(b_N+t)-H_{\mathrm{e}}(b_N-N^{-\alpha}y+t)\big]\,\mathrm{d}t \\ & +\frac{N^{\alpha}}{2\mathrm{i}\pi\beta}\int_{\mathscr{C}_{\mathrm{reg}}^{(+)}}\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\int_{\mathscr{C}_{\mathrm{reg}}^{(-)}}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\frac{1}{(\mu-\lambda)R_{\downarrow}(\mu)R_{\uparrow}(\lambda)}\bigg\{\frac{\mathrm{e}^{\mathrm{i}\lambda\overline{x}_N}-1}{\lambda}\int_{a_N}^{b_N}H_{\mathrm{e}}(\eta)\mathrm{e}^{-\mathrm{i}\mu y_R}\mathrm{d}\eta\;+\;\frac{1}{\mu}\int_{a_N}^{b_N}H_{\mathrm{e}}(\xi)\mathrm{e}^{\mathrm{i}\lambda x_R}\mathrm{d}\xi\bigg\}\;. \end{aligned} \tag{7.45}$$

The exponential decay of J at $+\infty$ ensures that the first two lines of (7.45) are indeed bounded by $C\,||H_{\mathrm{e}}||_{W_0^{\infty}(\mathbb{R})}$ for some N-independent $C>0$. The last line is bounded similarly by using

$$\forall\lambda\in\mathscr{C}_{\mathrm{reg}}^{(\pm)},\qquad \bigg|\int_{a_N}^{b_N}H_{\mathrm{e}}(\xi)\mathrm{e}^{\pm\mathrm{i}\lambda N^{\alpha}(b_N-\xi)}\,\mathrm{d}\xi\bigg|\;\le\;\frac{C'\,||H_{\mathrm{e}}||_{W_0^{\infty}(\mathbb{R})}}{|\lambda|N^{\alpha}}\;. \tag{7.46}$$

It thus solely remains to focus on the exponentially small term $\mathcal{W}_{\mathrm{exp}}[H]$. In fact, we only discuss the operator $\mathcal{W}_N^{(++)}$ as all others can be treated in a similar fashion. Thanks to the bound (4.50) for $\Pi(\lambda)-I_2$ and the expression (4.54) of the matrix Ψ in terms of Π, we have:

$$\Psi(\lambda)\;=\;I_2\;+\;\mathrm{O}\bigg(\frac{\mathrm{e}^{-\varkappa_{\epsilon}N^{\alpha}}}{1+|\lambda|}\bigg) \tag{7.47}$$

which is valid for λ uniformly away from the jump contour Σ_{Ψ} (see Figure 1). Therefore, using the definition (6.9) of $\mathcal{W}_N^{(++)}$:

$$\bigg|\int_{a_N}^{b_N}\mathcal{W}_N^{(++)}[H_{\mathrm{e}}](\xi)\,\mathrm{d}\xi\bigg|\;\le\;C''\,||H_{\mathrm{e}}||_{W_0^{\infty}(\mathbb{R})}\,\mathrm{e}^{-\varkappa_{\epsilon}N^{\alpha}}\int_{\mathscr{C}_{\mathrm{reg}}^{(+)}}\frac{|\mathrm{d}\lambda\mathrm{d}\mu|}{(2\pi)^2}\frac{1}{|\lambda-\mu|\,|R_{\downarrow}(\lambda)R_{\downarrow}(\mu)\,\lambda|}\;. \tag{7.48}$$

Adding up all the intermediate bounds readily leads to the claim. ■

By a slight modification of the method leading to Lemma 7.7, we can likewise control the $L^1([a_N\,;b_N])$ norm of $\mathcal{W}_N$ in terms of the W_1^{∞} norm of (an extension of) H.

Lemma 7.8 *For any $H\in C^1([a_N\,;b_N])$ it holds*

$$||\mathcal{W}_N[H]||_{L^1([a_N\,;b_N])}\;\le\;C\,||H_{\mathrm{e}}||_{W_1^{\infty}(\mathbb{R})}\quad\text{and}\quad||\mathcal{W}_{\mathrm{exp}}[H]||_{L^1([a_N\,;b_N])}\;\le\;C\,\mathrm{e}^{-C'N^{\alpha}}\,||H_{\mathrm{e}}||_{W_1^{\infty}(\mathbb{R})}\;. \tag{7.49}$$

7.3 The support of the equilibrium measure

In the present subsection we build on the previous analysis so as to prove the existence of the endpoints (a_N, b_N) of the support of the equilibrium measure and thus the fact that

$$\rho_{\mathrm{eq}}^{(N)}(\xi) \;=\; \mathbf{1}_{[a_N\,;b_N]}(\xi)\cdot \mathcal{W}_N[V'](\xi)\,\mathrm{d}\xi \;, \tag{7.50}$$

where $\mathcal{W}_N$ is as defined in (2.44).

Lemma 7.9 *There exists a unique sequence* (a_N, b_N) *– defining the support of the Lebesgue-continuous equilibrium measure which corresponds to the unique solution to the minimisation problem* (2.35)-(2.36)*. The sequences* a_N *and* b_N *are bounded in* N*.*

Proof — The existence and uniqueness of the solution to the minimisation problem (2.35)-(2.36) is obtained through a straightforward generalisation of the proof arising in the random matrix case, see *e.g.* [30].

The endpoint of the support of the equilibrium measure should be chosen in such a way that, on the one hand, the density of equilibrium measure admits at most a square root behaviour at the endpoints and, on the other hand, that it indeed defines a probability measure. In other words, the endpoints are to be chosen so that the two constraints are satisfied

$$\mathcal{X}_N[V'] \;=\; 0 \qquad \text{and} \qquad \mathfrak{I}_s[1,V'] = \int\limits_{a_N}^{b_N} \mathcal{W}_N[V'](\xi)\,\mathrm{d}\xi \;=\; 1 \;. \tag{7.51}$$

The asymptotic expansion of $\mathcal{X}_N[V']$ and $\mathfrak{I}_s[1,V']$ is given, respectively, in Lemma 7.2 and Proposition 7.6. However, the control on the remainder obtained there does depend on a_N and b_N. Should a_N or b_N be unbounded in N this could brake the *a priori* control on the remainder. Still, observe that if (a_N, b_N) solve the system of equations (7.51) then $\xi \mapsto \mathcal{W}_N[V'](\xi)$ with $\mathcal{W}_N$ associated with the support $[a_N\,;b_N]$ provides one with a solution to the minimisation problem of $\mathcal{E}_N$ defined in (2.33). By uniqueness of solutions to this minimisation problem, it thus corresponds to the density of equilibrium measure. As a consequence, there exists at most one solution (a_N, b_N) to the system of equations (7.51).

Assume that the sequence a_N and b_N are bounded in N. Then, the leading asymptotic expansion of the two functionals in (7.51) yields

$$\left\{ \begin{array}{lcl} V'(b_N) + V'(a_N) & = & \mathrm{O}(N^{-\alpha}) \\ V'(b_N) - V'(a_N) & = & u_1^{-1} + \mathrm{O}(N^{-\alpha}) \end{array} \right. \qquad \textit{viz.} \qquad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} V'(b_N) - V'(b) \\ V'(a_N) - V'(a) \end{pmatrix} \;=\; \mathrm{O}(N^{-\alpha}) \;. \tag{7.52}$$

Note that the control on the remainder follows from the fact that $|a_N|$ and $|b_N|$ are bounded by an N-independent constant. Also, (a, b) appearing above corresponds to the unique solution to the system

$$V'(b) \,+\, V'(a) \;=\; 0 \quad \text{and} \quad V'(b) \,-\, V'(a) \;=\; u_1^{-1} \;. \tag{7.53}$$

We do stress that the existence and uniqueness of this solution is ensured by the strict convexity of V.

The smoothness of the remainder in (a_N, b_N) away from 0, the control on its magnitude (guaranteed by the boundedness of a_N and b_N) as well as the strict convexity of V and the invertibility of the matrix occurring in (7.52) ensure the existence of solutions (a_N, b_N) by the implicit function theorem, this provided that N is large enough. Hence, since a solution to (7.51) with a_N and b_N bounded in N does exists, by uniqueness of the solutions to (7.51), it is the one that defines the endpoints of the support of the equilibrium measure. ■

Corollary 7.10 *Let the pair* (a,b) *correspond to the unique solution to the system*

$$V'(b) \; + \; V'(a) \; = \; 0 \quad \textit{and} \quad V'(b) \; - \; V'(a) \; = \; u_1^{-1} \; . \tag{7.54}$$

Then the endpoints (a_N, b_N) *of the support of the equilibrium measure admit the asymptotic expansion*

$$a_N \; = \; \sum_{\ell=0}^{k-1} \frac{a_{N;\ell}}{N^{\ell\alpha}} \; + \; \mathrm{O}(N^{-k\alpha}) \qquad \textit{and} \qquad b_N \; = \; \sum_{\ell=0}^{k-1} \frac{b_{N;\ell}}{N^{\ell\alpha}} \; + \; \mathrm{O}(N^{-k\alpha}) \; , \tag{7.55}$$

where $a_{N;0} = a$ *and* $b_{N;0} = b$*.*

Note that the existence and uniqueness of solutions to the system (7.54) follows from the strict convexity of the potential V.

Proof — The invertibility of the matrix occurring in (7.52) as well as the strict convexity of the potential V ensure that a_N and b_N admit the expansion (7.55) for $k = 1$, *viz.* up to $\mathrm{O}(N^{-\alpha})$ corrections. Now suppose that this expansion holds up to $\mathrm{O}(N^{-(k-1)\alpha})$. It follows from Lemma 7.2 and Proposition 7.6 that the asymptotic expansion of $\mathcal{X}_N[V']$ and $\mathfrak{I}_s[1,V']$ up to $\mathrm{O}(N^{-k\alpha})$ can be recast as

$$\begin{pmatrix} \mathcal{X}_N[V'] \cdot \urcorner_0^{-1} \\ \mathfrak{I}_{\mathrm{s}}[1,V'] \cdot u_1^{-1} \end{pmatrix} \; = \; \begin{pmatrix} V'(b_N) + V'(a_N) \; + \; \mathcal{B}_{1;k-1}[V'] \; + \; \urcorner_0^{-1} \cdot \Delta_{[k]}\mathcal{X}_N[V'] \\ V'(b_N) - V'(a_N) \; + \; \mathcal{B}_{2;k-1}[V'] \; + \; u_1^{-1} \cdot \Delta_{[k]}\mathfrak{I}_{\mathrm{s}}[1,V'] \end{pmatrix} \; . \tag{7.56}$$

In this expression, we have $\big|\urcorner_0^{-1} \cdot \Delta_{[k]}\mathcal{X}_N[V']\big| \; + \; \big|u_1^{-1} \cdot \Delta_{[k]}\mathfrak{I}_{\mathrm{s}}[1,V']\big| \; \leq \; CN^{-k\alpha}$ since a_N and b_N are bounded uniformly in N, while

$$\begin{pmatrix} \mathcal{B}_{1;k-1}[V'] \\ \mathcal{B}_{2;k-1}[V'] \end{pmatrix} = \sum_{p=1}^{k-1} \frac{1}{N^{p\alpha}} \begin{pmatrix} \mathrm{i}^p \cdot \urcorner_p \urcorner_0^{-1} \cdot \Big(V^{(p+1)}(a_N) \; + \; (-1)^p V^{(p+1)}(b_N) \Big) \\ (u_{p+1} + \urcorner_{0,p-1}) u_1^{-1} \cdot V^{(p+1)}(b_N) - (u_{p+1} + (-1)^p \urcorner_{0,p-1}) u_1^{-1} \cdot V^{(p+1)}(a_N) \end{pmatrix} . \tag{7.57}$$

We remind that $\urcorner_p$ was introduced in Definition 7.1, u_p in Definition 6.5, and $\urcorner_{0,p}$ in Definition 7.5.

Since both $\mathcal{B}_{1;k-1}[V']$ and $\mathcal{B}_{2;k-1}[V']$ have $N^{-\alpha}$ as a prefactor, by composition of asymptotic expansions, there exist functions $B_{p;\ell}(b_{N;1},\dots,b_{N;\ell-1} \mid a_{N;1},\dots,a_{N;\ell-1})$, indexed by $p \in \{1,2\}$ and $\ell \in [\![1 \, ; \, k-1]\!]$, independent of k, such that

$$\begin{pmatrix} \mathcal{B}_{1;k-1}[V'] \\ \mathcal{B}_{2;k-1}[V'] \end{pmatrix} \; = \; \sum_{\ell=1}^{k-1} \frac{1}{N^{\ell\alpha}} \begin{pmatrix} B_{1;\ell}(b_{N;1},\dots,b_{N;\ell-1} \mid a_{N;1},\dots,a_{N;\ell-1}) \\ B_{2;\ell}(b_{N;1},\dots,b_{N;\ell-1} \mid a_{N;1},\dots,a_{N;\ell-1}) \end{pmatrix} \; + \; \mathrm{O}(N^{-\alpha k}) \; . \tag{7.58}$$

As a consequence, we have the relation:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} V'(b_N) - V'(b) \\ V'(a_N) - V'(a) \end{pmatrix} \; = \; \sum_{\ell=1}^{k-1} \frac{-1}{N^{\ell\alpha}} \begin{pmatrix} B_{1;\ell}(b_{N;1},\dots,b_{N;\ell-1} \mid a_{N;1},\dots,a_{N;\ell-1}) \\ B_{2;\ell}(b_{N;1},\dots,b_{N;\ell-1} \mid a_{N;1},\dots,a_{N;\ell-1}) \end{pmatrix} \; + \; \mathrm{O}(N^{-k\alpha}) \; . \tag{7.59}$$

This implies the existence of an asymptotic expansion of a_N and b_N up to a remainder of the order $\mathrm{O}(N^{-k\alpha})$. ■

8 The operator $\mathcal{U}_N^{-1}$

Let us remind the definition of the operators $\mathcal{U}_N$ and $\mathcal{S}_N$:

$$\mathcal{U}_N[\phi](\xi) \;\; = \;\; \phi(\xi) \cdot \Big\{ V'(\xi) \; - \; \mathcal{S}_N[\rho_{\mathrm{eq}}^{(N)}](\xi) \Big\} + \mathcal{S}_N[\phi \cdot \rho_{\mathrm{eq}}^{(N)}](\xi) \tag{8.1}$$

$$\mathcal{S}_N[\phi](\xi) \;\; = \;\; \fint_{a_N}^{b_N} S\,[N^{\alpha}(\xi - \eta)]\phi(\eta)\,\mathrm{d}\eta \qquad \text{and} \qquad S(\xi) \; = \; \sum_{p=1}^{2} \beta\pi\omega_p \operatorname{cotanh}[\pi\omega_p\xi] \; . \tag{8.2}$$

and the fact that $\mathcal{W}_N$ defined in § 5.4 is the inverse operator to $\mathcal{S}_N$. We also remind that the density $\rho_{\mathrm{eq}}^{(N)}$ of the N-dependent equilibrium measure satisfies the integral equation:

$$\forall \xi \in [a_N\,; b_N], \qquad \mathcal{S}_N[\rho_{\mathrm{eq}}^{(N)}](\xi) = V'(\xi)\,. \tag{8.3}$$

This makes the first term of (8.1) vanish for $\xi \in [a_N\,; b_N]$, but it can be non-zero outside of this segment.

In this section we obtain an integral representation for the inverse of $\mathcal{U}_N$, which shows that $\mathcal{U}_N^{-1}[H]$ is smooth as long as H is. Then, in § 8.2, we shall provide explicit, N-dependent, bounds on the $W_\ell^\infty(\mathbb{R})$ norms of $\mathcal{U}_N^{-1}[H]$. This technical result is crucial in the analysis of the Schwinger-Dyson equation performed in § 3.3.

8.1 An integral representation for $\mathcal{U}_N^{-1}$

Proposition 8.1 *The operator $\mathcal{U}_N$ is invertible on $(\mathfrak{X}_s \cap C^1)(\mathbb{R})$, $0 < s < 1/2$, and its inverse admits the representation*

$$\mathcal{U}_N^{-1}[H](\xi) \;=\; \frac{\mathcal{V}_N[H](\xi)}{\mathcal{V}_N[V'](\xi)}\,, \tag{8.4}$$

where $\mathcal{V}_N \;=\; \mathcal{V}_N^{[1]} \;+\; \mathcal{V}_N^{[2]}$ with

$$\mathcal{V}_N^{[1]}[H](\xi) \;=\; \int\limits_{a_N}^{b_N} \frac{[H(\xi) - H(s)]\,\mathrm{d}s}{(\xi - s)\sqrt{(s-a_N)(b_N-s)}} \qquad \textit{and} \qquad \mathcal{V}_N^{[2]}[H](\xi) \;=\; \int\limits_{a_N}^{b_N} V_N^{[2]}(\xi,\eta)\cdot \mathcal{W}_N[H](\eta)\,\mathrm{d}\eta\,. \tag{8.5}$$

and the integral kernel of the operator $\mathcal{V}_N^{[2]}$ reads:

$$V_N^{[2]}(\xi,\eta) \;=\; \int\limits_{a_N}^{b_N} \frac{S_{\mathrm{reg}}[N^\alpha(s-\eta)] \;-\; S_{\mathrm{reg}}[N^\alpha(\xi-\eta)]}{(\xi - s)\sqrt{(s-a_N)(b_N-s)}}\,\mathrm{d}s \qquad \textit{with} \qquad S_{\mathrm{reg}}(\xi) \;=\; S(\xi) \;-\; \frac{2\beta}{\xi}\,. \tag{8.6}$$

Finally, we have that, for any $\xi \in [a_N\,; b_N]$, $\mathcal{V}_N[V'](\xi) \neq 0$.

Note that the above representation is not completely fit for obtaining a fine bound of the $W_\ell^\infty(\mathbb{R})$ norm of $\mathcal{U}_N^{-1}[H]$ in the large-N limit. Indeed, we will show in Appendix C that $\mathcal{V}_N[V'](\xi) > c_N > 0$ for N large enough. Unfortunately, the constant $c_N \to 0$ and thus does not provide an optimal bound for the $W_\ell^\infty(\mathbb{R})$ norm. Gaining a more precise control on c_N (*eg.* its dependence on N) is much harder, but a more precise control is one of the ingredients that are necessary for obtaining sharp N-dependent bounds for the $W_\ell^\infty(\mathbb{R})$ norm of $\mathcal{U}_N^{-1}[H]$. We shall obtain such a more explicit control on c_N in the course of the proof of Theorem 8.2.

Proof — Given $H \in (\mathfrak{X}_s \cap C^1_c)(\mathbb{R})$, let ϕ be the unique solution to the equation $\mathcal{S}_N[\phi](\xi) = H(\xi)$ on $[a_N\,; b_N]$. Reminding the definition of $\mathcal{S}_N$ in (2.42), it means that, for $\xi \in]a_N\,; b_N[$:

$$\fint\limits_{a_N}^{b_N} \frac{\phi(\eta)\,\mathrm{d}\eta}{(\xi-\eta)\mathrm{i}\pi} \;=\; U(\xi) \qquad \text{where} \qquad U(\xi) \;=\; \frac{N^\alpha}{2\mathrm{i}\pi\beta}\bigg\{H(\xi) \;-\; \int\limits_{a_N}^{b_N} S_{\mathrm{reg}}[N^\alpha(\xi-\eta)]\phi(\eta)\,\mathrm{d}\eta\bigg\}\,. \tag{8.7}$$

As a consequence, the function

$$F(z) \;=\; \frac{1}{q(z)}\int\limits_{a_N}^{b_N} \frac{\phi(\eta)}{z-\eta}\cdot\frac{\mathrm{d}\eta}{2\mathrm{i}\pi} \qquad \text{with} \qquad q(z) \;=\; \sqrt{(z-a_N)(z-b_N)} \tag{8.8}$$

solves the scalar Riemann–Hilbert problem

- $F \in \mathcal{O}(\mathbb{C} \setminus [a_N ; b_N])$ and admits $\pm$ $L^p([a_N ; b_N])$ boundary values for $p \in]1 ; 2[$;
- $F(z) = \mathrm{O}(z^{-1})$ when $z \to \infty$;
- $F_+(x) - F_-(x) = U(x)/q_+(x)$ for any $x \in]a_N ; b_N[$.

Note that the L^p character of the boundary values follows from the fact that both ϕ and the principal value integral are continuous on $[a_N ; b_N]$. The former follows from Propositions 6.4-6.6 whereas the latter is a consequence of (8.7). By uniqueness of the solution to such a Riemann–Hilbert problem, it follows that

$$F(z) = \int_{a_N}^{b_N} \frac{U(s)}{q_+(s)(s-z)} \frac{\mathrm{d}s}{2\mathrm{i}\pi} \qquad \text{for} \qquad z \in \mathbb{C} \setminus [a_N ; b_N] \, . \tag{8.9}$$

By using that, for $\xi \in]a_N ; b_N[$,

$$-\phi(\xi) = q_+(\xi) \cdot \Big(F_+(\xi) + F_-(\xi) \Big) \qquad \text{and} \qquad \fint_{a_N}^{b_N} \frac{1}{q_+(s) \cdot (s-\xi)} \cdot \frac{\mathrm{d}s}{\mathrm{i}\pi} = 0 \, , \tag{8.10}$$

we obtain that:

$$\phi(\xi) = \frac{\sqrt{N^{2\alpha}(\xi - a_N)(b_N - \xi)}}{2\pi^2 \beta} \mathcal{V}_N[H](\xi) \tag{8.11}$$

with the expression of $\mathcal{V}_N$ given by (8.5). Further, given any $\xi \in \mathbb{R} \setminus [a_N ; b_N]$, we have:

$$\mathcal{S}_N[\phi](\xi) = \int_{a_N}^{b_N} S_{\mathrm{reg}}[N^{\alpha}(\xi - \eta)]\phi(\eta) \, \mathrm{d}\eta + \frac{4\mathrm{i}\pi\beta}{N^{\alpha}} q(\xi) F(\xi) \, . \tag{8.12}$$

It then remains to use that, for such ξ's

$$\int_{a_N}^{b_N} \frac{1}{q_+(s)(s-\xi)} \cdot \frac{\mathrm{d}s}{\mathrm{i}\pi} = \frac{1}{q(\xi)} \tag{8.13}$$

so as to get the representation

$$\mathcal{S}_N[\phi](\xi) = H(\xi) - \frac{q(\xi)}{\pi} \cdot \mathcal{V}_N[H](\xi) \, . \tag{8.14}$$

We can now go back to the original problem. Let ψ be any solution to $\mathcal{U}_N[\psi] = H$. Due to the integral equation satisfied by the density of equilibrium measure on $[a_N ; b_N]$, it follows that, for any $\xi \in [a_N ; b_N]$ such that $\mathcal{W}_N[V'](\xi) \neq 0$,

$$\psi(\xi) = \frac{\mathcal{W}_N[H](\xi)}{\mathcal{W}_N[V'](\xi)} \, . \tag{8.15}$$

and we can conclude thanks to the relation (8.11). For $\xi \in \mathbb{R} \setminus [a_N ; b_N]$, we rather have:

$$\psi(\xi) = \frac{\mathcal{S}_N[\mathcal{W}_N[H]](\xi) - H(\xi)}{\mathcal{S}_N[\mathcal{W}_N[V']](\xi) - V'(\xi)} \tag{8.16}$$

at any point where the denominator does not vanish. It then solely remains to invoke the relation (8.14). Note that

$$\mathcal{V}_N[V'](\xi) \;=\; \frac{2\pi^2\beta\,\rho_{\mathrm{eq}}^{(N)}(\xi)}{\sqrt{N^{2\alpha}(\xi-a_N)(b_N-\xi)}}\ . \tag{8.17}$$

It is shown in proof of Theorem 2.4 given in Appendix C, point (*ii*), that $\rho_{\mathrm{eq}}^{(N)}(\xi) > 0$ for $\xi\ \in]a_N\,;b_N[$ for N large enough and that it vanishes as a square root at the edges. Furthermore, it is also shown in that appendix, Equation (C.8), that $V'(\xi) - \mathcal{S}_N[\mathcal{W}_N[V']](\xi) \neq 0$ on $\mathbb{R}\setminus[a_N\,;b_N]$. Thus the denominator in (8.4) never vanishes and thus holds for any $\xi\in\mathbb{R}$ and any $H\in\mathfrak{X}_s\cap C^1_{\mathrm{c}}(\mathbb{R})$. The result then follows by density of $\mathfrak{X}_s\cap C^1_{\mathrm{c}}(\mathbb{R})$ in $\mathfrak{X}_s\cap C^1(\mathbb{R})$. ■

8.2 Sharp weighted bounds for $\mathcal{U}_N^{-1}$

The aim of the present subsection is to prove one of the most important technical propositions of the paper, namely sharp N-dependent bounds on the $W^{\infty}_{\ell}(\mathbb{R})$ norm of $\mathcal{U}_N^{-1}[H]$. Part of the difficulties of the proof consists in obtaining lower bounds for $\mathcal{W}_N[V']$ in the vicinity of a_N and b_N as well as in gaining a sufficiently precise control on the square root behaviour of $\mathcal{W}_N[H]$ at the edges.

Proposition 8.2 below is the key tool for the large-N analysis of the Schwinger-Dyson equations. We insist that although our result is effective in what concerns our purposes, it is *not* optimal. More optimal results can be obtained in respect to local W^{∞}_{ℓ} norms, *viz.* $W^{\infty}_{\ell}(J)$ with J being specific subintervals of $\mathbb{R}$, or in respect to milder ones such as the $W^{p}_{\ell}(\mathbb{R})$ ones. However, obtaining these results demands more efforts on the one hand and, on the other hand, requires much more technical handlings so as to make the best of them when dealing with the Schwinger-Dyson equations. We therefore chose not to venture further in these technicalities.

Before stating the theorem, we remind the expression for the weighted norm (Definition 3.14):

$$\mathcal{N}_N^{(\ell)}[\phi] \;=\; \sum_{k=0}^{\ell}\frac{||\phi||_{W^{\infty}_{\ell}(\mathbb{R})}}{N^{\ell\alpha}}\ . \tag{8.18}$$

and the *ad hoc* norms on the potential (Definition 3.15):

$$\mathfrak{n}_\ell[V] \;=\; \frac{\max\Big\{\prod_{a=1}^{\ell}||\mathcal{K}_\kappa[V']||_{W^{\infty}_{k_a}(\mathbb{R}^n)}\ :\ \sum_{a=1}^{\ell}k_a=2\ell+1\Big\}}{\Big\{\min\Big(1\,,\ \inf_{[a\,;b]}|V''(\xi)|\,,\ |V'(b+\epsilon)-V'(b)|\,,\ |V'(a-\epsilon)-V'(a)|\Big)\Big\}^{\ell+1}} \tag{8.19}$$

for some $\epsilon>0$ small enough but independent of N. We also remind that $\mathcal{K}_\kappa[H]$ is an exponential regularisation of H, see Definition 3.8.

Proposition 8.2 *Let $\ell\geq 0$ be an integer, and C_V, κ be positive constants. There exist a constant $C_\ell>0$ such that for any H and V satisfying*

- $\mathcal{K}_{\kappa/\ell}[H]\in W^{\infty}_{2\ell+1}(\mathbb{R})$ *and* $\mathcal{K}_{\kappa/\ell}[V]\in W^{\infty}_{2\ell+2}(\mathbb{R})$ *;*
- $||V||_{W^{\infty}_{3}([a-\delta\,;b+\delta])} < C_V$ *for some $\delta>0$ where (a,b) are such that* $(a_N,b_N)\underset{N\to+\infty}{\longrightarrow}(a,b)$ *;*
- $H\in\mathfrak{X}_s([a_N\,;b_N])$ *;*

we have the following bound:

$$\Big|\Big|\mathcal{K}_\kappa[\mathcal{U}_N^{-1}[H]]\Big|\Big|_{W^{\infty}_{\ell}(\mathbb{R})} \;\leq\; C_\ell\cdot\mathfrak{n}_\ell[V]\cdot N^{(\ell+1)\alpha}\cdot(\ln N)^{2\ell+1}\cdot\mathcal{N}_N^{(2\ell+1)}[\mathcal{K}_\kappa[H]]\ . \tag{8.20}$$

Proof — As discussed in the proof of Proposition 8.1, the operator $\mathcal{U}_N^{-1}$ can be recast as

$$\mathcal{U}_N^{-1}[H](\xi) \;=\; \frac{\mathcal{W}_N[H](\xi)}{\mathcal{W}_N[V'](\xi)} \cdot \mathbf{1}_{[a_N\,;b_N]}(\xi) \;+\; \frac{\mathcal{S}_N[\mathcal{W}_N[H]](\xi) \,-\, H(\xi)}{\mathcal{S}_N[\mathcal{W}_N[V']](\xi) \,-\, V'(\xi)} \cdot \mathbf{1}_{[a_N\,;b_N]^c}(\xi)\;. \tag{8.21}$$

Therefore, obtaining sharp bounds on $\mathcal{U}_N^{-1}[H]$ demands to control, with sufficient accuracy, both ratios appearing in the formula above. Observe that the same Proposition 8.1 and, in particular, equations (8.11)-(8.14) ensure that, given $\epsilon > 0$ small enough and H of class C^{k+1}, the functions

$$\xi \mapsto \frac{\mathcal{W}_N[H](\xi)}{q_R(\xi)} \qquad \text{and} \qquad \xi \mapsto \frac{\mathcal{S}_N[\mathcal{W}_N[H]](\xi) - H(\xi)}{q_R(\xi)} \tag{8.22}$$

with:

$$q_R(\xi) \;=\; \sqrt{N^{\alpha}(b_N - \xi)} = x_R^{1/2} \tag{8.23}$$

are respectively $C^k([b_N - \epsilon\,; b_N])$ and $C^k([b_N\,; b_N + \epsilon])$. A similar statement holds at the left boundary. Furthermore, the same proposition readily ensures that both functions are clearly C^{k+1} uniformly away from the boundaries.

The large-N behaviour of both functions in (8.22) is not uniform on $\mathbb{R}$ and depends on whether one is in a vicinity of the endpoints a_N, b_N or not. Therefore, we will split the analysis for ξ in one of the four regions, from right to left on the real axis:

$$\begin{aligned}
\mathbb{J}_N^{(R;\mathrm{out})} &= [b_N + \epsilon(\ln N)^2 \cdot N^{-\alpha}\,; +\infty[&& (8.24)\\
\mathbb{J}_N^{(R;\mathrm{ext})} &= [b_N\,; b_N + \epsilon(\ln N)^2 \cdot N^{-\alpha}] && (8.25)\\
\mathbb{J}_N^{(R;\mathrm{in})} &= [b_N - \epsilon(\ln N)^2 \cdot N^{-\alpha}\,; b_N] && (8.26)\\
\mathbb{J}_N^{(\mathrm{bk})} &= [a_N + \epsilon(\ln N)^2 \cdot N^{-\alpha}\,; b_N - \epsilon(\ln N)^2 \cdot N^{-\alpha}]\;. && (8.27)
\end{aligned}$$

Indeed, the behaviour on the three other regions:

$$\begin{aligned}
\mathbb{J}_N^{(L;\mathrm{in})} &= [a_N\,; a_N + \epsilon(\ln N)^2 \cdot N^{-\alpha}] && (8.28)\\
\mathbb{J}_N^{(L;\mathrm{ext})} &= [a_N - \epsilon(\ln N)^2 \cdot N^{-\alpha}\,; a_N] && (8.29)\\
\mathbb{J}_N^{(L;\mathrm{out})} &=]-\infty\,; a_N - \epsilon(\ln N)^2 \cdot N^{-\alpha}] && (8.30)
\end{aligned}$$

can be deduced by the reflection symmetry

$$\mathcal{W}_N[H](\xi) \;=\; -\mathcal{W}_N[H^{\wedge}](a_N + b_N - \xi) \tag{8.31}$$

from the analysis on the local intervals (8.24)-(8.26).

The proof consists in several steps. First of all, we bound the $W_\ell^{\infty}(\mathbb{J}_N^{(*)})$ norm of the functions in (8.22), this depending on the interval of interest. Also, we obtain *lower* bounds for the same functions with $H \leftrightarrow V'$. Finally, we use the partitioning of $\mathbb{R}$ into the local intervals (8.24)-(8.26) so as to raise the local bounds into global bounds on $\mathcal{U}_N^{-1}[H]$ issuing from those on $\mathcal{W}_N[H] \cdot q_R^{-1}$ and $\{\mathcal{S}_N[\mathcal{W}_N[H]] - H\} \cdot q_R^{-1}$.

Lower and upper bounds on $\mathbb{J}_N^{(R;\text{out})}$

Let us decompose S given in (8.2) into:

$$S(x) = S_\infty(x) + (\Delta S)(x), \qquad \text{with} \quad S_\infty(x) = \beta\pi(\omega_1+\omega_2)\mathrm{sgn}(x) \tag{8.32}$$

We observe that when $\xi \in \mathbb{J}_N^{(R;\text{out})}$ and $\eta \in [a_N\,;b_N]$ one avoids the simple pole in the kernel functions $S[N^\alpha(\xi-\eta)]$ of the integral operator $\mathcal{S}_N$. Besides, the decomposition (8.32) has the property that, for any integer $\ell \geq 0$, there exists constants $c, C_\ell > 0$ independent of N such that:

$$\forall \xi \in \mathbb{J}_N^{(R;\text{out})},\ \ \forall \eta \in [a_N\,;b_N], \qquad \Big|\partial_\xi^\ell (\Delta S)[N^\alpha(\xi-\eta)]\Big| \ \leq\ C_\ell\, N^{\ell\alpha}\, \mathrm{e}^{-c(\ln N)^2}\,. \tag{8.33}$$

We have proved in Lemma 7.7 and 7.8 that

$$\bigg|\int_{a_N}^{b_N} \mathcal{W}_N[H](\xi)\,\mathrm{d}\xi\bigg| \ \leq\ C\,||H_{\mathrm{e}}||_{W_0^\infty(\mathbb{R})}\,, \qquad ||\mathcal{W}_N[H]||_{L^1([a_N\,;b_N])} \ \leq\ C\,||H_{\mathrm{e}}||_{W_1^\infty(\mathbb{R})} \tag{8.34}$$

for some $C > 0$ independent of N. Subsequently:

$$\Big|\Big|\mathcal{S}_N[\mathcal{W}_N[H]]\Big|\Big|_{W_\ell^\infty(\mathbb{J}_N^{(R;\text{out})})} \ \leq\ \delta_{\ell 0}\, C\,||H_{\mathrm{e}}||_{W_0^\infty(\mathbb{R})} \ +\ C_\ell\, N^{\ell\alpha}\, \mathrm{e}^{-c(\ln N)^2}\,||\mathcal{W}_N[H]||_{L^1([a_N\,;b_N])} \tag{8.35}$$

$$\leq\ \delta_{\ell 0}\, C'\,\mathcal{N}_N^{(0)}[\mathcal{K}_\kappa[H]] \ +\ C'_\ell\, N^{(\ell+1)\alpha}\,\mathrm{e}^{-c(\ln N)^2}\,(b_N-a_N)\,\mathcal{N}_N^{(1)}[\mathcal{K}_\kappa[H]]\,. \tag{8.36}$$

We have used: in the first line, the estimates (8.34) ; in the second line, the definition (8.18) of the weighted norm, and we have included exponential regularisations via $\mathcal{K}_\kappa$, whose only effect is to change the value of the constant prefactors. Since $(a_N, b_N) \to (a, b)$ in virtue of Corollary 7.10, we can write for N large enough:

$$||\mathcal{K}_\kappa[\mathcal{S}_N[\mathcal{W}_N[H]] - H]||_{W_\ell^\infty(\mathbb{J}_N^{(R;\text{out})})} \ \leq\ \widetilde{C}_\ell \cdot N^{\ell\alpha}\cdot \mathcal{N}_N^{(\ell)}[\mathcal{K}_\kappa[H]]\,. \tag{8.37}$$

Indeed, a bound from the left-hand side is obtained by adding the W_ℓ^∞ norm of H to (8.36), which is itself bounded by a multiple of $N^{\ell\alpha}\mathcal{N}_N^{(\ell)}[\mathcal{K}_\kappa[H]]$.

Thanks to the decomposition (8.32) using that $\mathrm{sgn}(\xi-\eta) = 1$ for $\xi \in \mathbb{J}_N^{R;(\text{out})}$ and $\eta \in [a_N\,;b_N]$, as well as the exponential estimate (8.33) and the L^1 bound of $\mathcal{W}_N$ from Lemma 7.8, we can also write:

$$\mathcal{S}_N[\mathcal{W}_N[V']](\xi) - V'(\xi) = \underbrace{\pi\beta(\omega_1+\omega_2)}_{=V'(b)}\underbrace{\int_{a_N}^{b_N}\mathcal{W}_N[V'](\xi)\,\mathrm{d}\xi}_{=1} \ -\ V'(\xi) + \mathrm{O}\Big(e^{-c(\ln N)^2}\,||V'||_{W_1^\infty([a_N\,;b_N])}\Big)\,. \tag{8.38}$$

The identification of the first term comes from (3.3). Further, we have for $|\xi - b| \leq \epsilon$ and $\xi \in \mathbb{J}_N^{(R;\text{out})}$:

$$\Big|V'(b) - V'(\xi)| \ \geq\ |\xi - b|\cdot \inf_{\xi\in[b\,;b+\epsilon]}|V''(\xi)| \ \geq\ \frac{\epsilon}{2}\frac{(\ln N)^2}{N^\alpha}V''(b) \geq \frac{\epsilon}{2}\,\frac{V''(b)}{N^\alpha} \tag{8.39}$$

To obtain the last bound we have assumed that ϵ was small enough – but still independent of N – and made use of $|b - b_N| = \mathrm{O}(N^{-\alpha})$ as well as of $||V||_{W_3^\infty([a-\delta\,;b+\delta])} < +\infty$ and N large enough. Finally, it is clear from the strict convexity of V that in the case $|b - \xi| > \epsilon$:

$$|V'(b) - V'(\xi)| \ \geq\ |V'(b+\epsilon) - V'(b)| \geq \frac{\epsilon}{2}\,\frac{V'(b+\epsilon) - V'(b)}{N^\alpha}\ , \tag{8.40}$$

where the last inequality is a trivial one. Therefore, in any case, for N large enough:

$$\left|\mathcal{S}_N[\mathcal{W}_N[V']] - V'(\xi)\right| \geq \frac{\epsilon}{4N^{\alpha}} \min\left\{ \inf_{\xi\in[a\,;b]} V''(\xi) \, , \, |V'(b+\epsilon) - V'(b)| \right\} . \tag{8.41}$$

The combination of the numerator upper bound (8.37) applied to $H = V'$ (using that the weighted norm is dominated by the W^{∞} norm) and the denominator lower bound (8.41) implies that, for any $\kappa > 0$ such that both sides below are well-defined:

$$\frac{\|\mathcal{K}_{\kappa}[\mathcal{S}_N[\mathcal{W}_N[V']] - V']\|_{W_{\ell}^{\infty}(\mathbb{J}_N^{(R;\mathrm{out})})}}{\left|\mathcal{S}_N[\mathcal{W}_N[V']](\xi) \, - \, V'(\xi)\right|} \;\leq\; \frac{N^{(\ell+1)\alpha} \cdot C_{\ell} \cdot \|V'\|_{W_{\ell}^{\infty}(\mathbb{R})}}{\min\left\{ \inf_{\xi\in[a\,;b]} |V''(\xi)| \, , \, |V'(b+\epsilon) - V'(b)| \right\}} \, . \tag{8.42}$$

Implicitly, we have treated ϵ from (8.41) like a constant.

Lower and upper bounds on $\mathbb{J}_N^{(\mathrm{bk})}$

Consider the decomposition of $\mathcal{W}_N$ from (6.70):

$$\begin{aligned}\mathcal{W}_N[H](\xi) \;=\; \mathcal{W}_{\mathrm{bk};k}[H](\xi) \;+\; \Delta_{[k]}\mathcal{W}_{\mathrm{bk};k}[H_{\mathfrak{e}}](\xi) \;+\; \mathcal{W}_R[H_{\mathfrak{e}}](x_R) \\ -\; \mathcal{W}_R[H^{\wedge}](x_L, b_N + a_N - \xi) \;+\; \mathcal{W}_{\exp}[H_{\mathfrak{e}}](\xi) \, .\end{aligned} \tag{8.43}$$

From the expression of $\mathcal{W}_{\mathrm{bk};k}$ in (6.41), we have the bound:

$$\|\mathcal{W}_{\mathrm{bk};k}[H]\|_{W_{\ell}^{\infty}(\mathbb{J}_N^{(\mathrm{bk})})} \;\leq\; c_{k;\ell} \cdot \max_{s\in[\![\,0\,;\,\ell\,]\!]} \mathcal{N}_N^{(k-1)}[H^{(s+1)}] \tag{8.44}$$

and recollecting the estimates of the other terms from Propositions 6.4 and 6.6, we also find:

$$\left\|\Delta_{[k]}\mathcal{W}_{\mathrm{bk};k}[H_{\mathfrak{e}}] \,+\, \mathcal{W}_R[H_{\mathfrak{e}}] \,-\, (\mathcal{W}_R)^{\wedge}[H_{\mathfrak{e}}] \,+\, \mathcal{W}_{\exp}[H_{\mathfrak{e}}]\right\|_{W_{\ell}^{\infty}(\mathbb{J}_N^{(\mathrm{bk})})} \;\leq\; c_{\ell}\, N^{-k\alpha} \|H_{\mathfrak{e}}^{(k+1)}\|_{W_{\ell}^{\infty}(\mathbb{R})} \, , \tag{8.45}$$

with the reflected operator $\mathcal{W}_R^{\wedge}$ as introduced in Definition 6.2. We do stress that, in the present context, $H_{\mathfrak{e}}$ denotes a compactly supported extension of H from $[a_N\,;b_N]$ to $\mathbb{R}$ that, furthermore, satisfies the same regularity properties as H. All in all, the bounds (8.44)-(8.45) yield

$$\|\mathcal{W}_N[H](\xi)\|_{W_{\ell}^{\infty}(\mathbb{J}_N^{(\mathrm{bk})})} \;\leq\; c'_{k;\ell} \cdot \max_{s\in[\![\,0\,;\,\ell\,]\!]} \left\{\mathcal{N}_N^{(k)}[H^{(s+1)}]\right\} . \tag{8.46}$$

Besides, for $k = 1$ we have from (6.41):

$$\mathcal{W}_{\mathrm{bk};k}[V'](\xi) \;=\; u_1\, V''(\xi) \, . \tag{8.47}$$

The constant u_1 was introduced in Definition 6.5, and according to the expression of $R(\lambda)$ in (4.19), it takes the value:

$$u_1 = \frac{1}{2\pi\beta(\omega_1 + \omega_2)} > 0 \, . \tag{8.48}$$

So, using the bound (8.45) for $k = 1$ and $\ell = 0$ to control the extra terms in $\mathcal{W}_N$ in sup norm, we get

$$\left|\mathcal{W}_N[V'](\xi)\right| \;\geq\; u_1 \inf_{\xi\in[a\,;b]} V''(\xi) \;-\; \frac{C}{N^{\alpha}}\|V_{\mathfrak{e}}\|_{W_3^{\infty}(\mathbb{R})} \;\geq\; \frac{u_1}{2} \cdot \inf_{\xi\in[a\,;b]} \{V''(\xi)\} \tag{8.49}$$

where the last lower bound holds for N large enough. The above lower bound leads to

$$\frac{\|\mathcal{W}_{\mathrm{bk};k}[V']\|_{W_{\ell}^{\infty}(\mathbb{J}_N^{(\mathrm{bk})})}}{\left|\mathcal{W}_N[V'](\xi)\right|} \;\leq\; \frac{C_{\ell} \cdot \|V'\|_{W_{k+\ell+1}^{\infty}(\mathbb{J}_N^{(\mathrm{bk})})}}{\inf\limits_{\xi\in[a\,;b]} V''(\xi)} \, . \tag{8.50}$$

Lower and upper bounds on $\mathbb{J}_N^{(R;\mathrm{in})}$

In virtue of Lemma 6.11 and Proposition 6.6, given $k \in \mathbb{N}^*$, we have the decomposition

$$\mathcal{W}_N[H](\xi) \;=\; (\mathcal{W}_{R;k}^{(\mathrm{as})} \;+\; \mathcal{W}_{\mathrm{bk};k}^{(\mathrm{as})})[H](x_R) \;+\; \Omega_{R;k}[H_{\mathfrak{e}}](x_R, \xi) \tag{8.51}$$

$$\Omega_{R;k}[H_{\mathfrak{e}}](x_R, \xi) \;=\; \Delta_{[k]}\mathcal{W}_R^{(\mathrm{as})}[H](x_R) + \Delta_{[k]}\mathcal{W}_{\mathrm{bk}}^{(\mathrm{as})}[H](x_R) - \mathcal{W}_{R;k}[H^{\wedge}](x_L, b_N + a_N - \xi) + \Delta_{[k]}\mathcal{W}_N[H_{\mathfrak{e}}](\xi) \tag{8.52}$$

where $\Delta_{[k]}\mathcal{W}_N[H_{\mathfrak{e}}]$ has been introduced in (6.70). We remind from (6.107) that:

$$(\mathcal{W}_{R;k}^{(\mathrm{as})} \;+\; \mathcal{W}_{\mathrm{bk};k}^{(\mathrm{as})})[H](x_R) = H'(b_N)\mathfrak{a}_0(x_R) \;+\; \sum_{r=1}^{k} \frac{H^{(r+1)}(b_N)}{N^{r\alpha}}\,(\mathfrak{a}_0 \cdot \mathfrak{a}_r)(x_R) \tag{8.53}$$

For any integers n, ℓ such that $n \geq \ell + 2$, Lemma 6.10 applied to $(\ell, m, n) \hookleftarrow (r, \ell + 1, n)$ tells us:

$$\frac{\mathfrak{a}_0(x)}{\sqrt{x}} \;=\; p_{0;\ell+1,n}(x)\mathrm{e}^{-\varsigma x} \;+\; x^{\ell+1/2} f_{0;\ell+1,n}(x), \qquad \frac{(\mathfrak{a}_0 \cdot \mathfrak{a}_r)(x)}{\sqrt{x}} \;=\; p_{r;\ell+1,n}(x)\mathrm{e}^{-\varsigma x} + x^{\ell+1/2} f_{r;\ell+1,n}(x) \tag{8.54}$$

for some polynomials $p_{k;\ell+1,n}(x)$ of degree at most $n + k$ and functions $f_{k;\ell+1,n} \in W^{\infty}_{n-(\ell+1)}(\mathbb{R}_+)$. We therefore get:

$$\big\| q_R^{-1}(\mathcal{W}_{R;k}^{(\mathrm{as})} + \mathcal{W}_{\mathrm{bk};k}^{(\mathrm{as})})[H] \big\|_{W^{\infty}_{\ell}(\mathbb{J}_N^{(R;\mathrm{in})})} \leq c_{k;\ell} \cdot N^{\ell\alpha} \cdot (\ln N)^{2\ell+1} \cdot \mathcal{N}_N^{(k-1)}[H'_{\mathfrak{e}}]\;. \tag{8.55}$$

In this inequality, one power of N^{α} pops up at each action of the derivative of $x_R = N^{\alpha}(b_N - \xi)$. Furthermore, by putting together the control of the remainders in Proposition 6.6 and Lemma 6.11, we get that:

$$\Omega_{R;k}[H_{\mathfrak{e}}](x_R, \xi) \;=\; \sum_{m=0}^{k} \Big\{ c_{k;m}^{(0)} x_R^m \;+\; c_{k;m}^{(1/2)} x_R^{m+\frac{1}{2}} \Big\} \;+\; f_k(x_R) \tag{8.56}$$

where, for any $0 \leq \ell \leq k$, the function f_k satisfies:

$$\Big| \partial_{\xi}^{\ell}\big(x_R^{-1/2} f_k(x_R)\big) \Big| \;\leq\; C_{k;\ell} \cdot x_R^{k+\frac{1}{2}-\ell} \cdot N^{(\ell-k)\alpha} \cdot \mathcal{N}_N^{(\ell)}[H_{\mathfrak{e}}^{(k+1)}] \cdot \big(\, \ln(x_R)\, \mathrm{e}^{-Cx_R} \;+\; 1 \big)\;. \tag{8.57}$$

Since the functions $(\mathcal{W}_{R;k}^{(\mathrm{as})} + \mathcal{W}_{\mathrm{bk};k}^{(\mathrm{as})})[H] \cdot q_R^{-1}$ and $\mathcal{W}_N[H] \cdot q_R^{-1}$ are smooth on $J_N^{(R;\mathrm{in})}$, so must be $\Omega_{R;k}[H_{\mathfrak{e}}] \cdot q_R^{-1}$. As a consequence, we necessarily have $c_{k;m}^{(0)} = 0$. The properties of the remainders then ensure that, for any $0 \leq \ell \leq k$,

$$\big| c_{k;m}^{(1/2)} \big| \;\leq\; C_{k;m} \cdot N^{-k\alpha} \cdot \big\| H_{\mathfrak{e}}^{(k+1)} \big\|_{W^{\infty}_m(\mathbb{R})}\;. \tag{8.58}$$

Thus, all-in-all, by choosing properly the compactly supported regular extension $H_{\mathfrak{e}}$ of H from $[a_N\,; b_N]$ to $\mathbb{R}$ we get

$$\| q_R^{-1} \cdot \mathcal{W}_N[H] \|_{W^{\infty}_{\ell}(\mathbb{J}_N^{(R;\mathrm{in})})} \;\leq\; C_{\ell} \cdot (\ln N)^{2\ell+1} \cdot N^{(\ell+1)\alpha} \cdot \mathcal{N}_N^{(2\ell+1)}[\mathcal{K}_{\kappa}[H_{\mathfrak{e}}]] \tag{8.59}$$

upon choosing $k = \ell$. This holds for any $\kappa > 0$, the right-hand side being possibly $+\infty$.

In what concerns the lower bounds, observe that

$$x_R^{-1/2} \cdot (\mathcal{W}_{R;1}^{(\mathrm{as})} + \mathcal{W}_{\mathrm{bk};1}^{(\mathrm{as})})[H](x_R) \;=\; \frac{\mathfrak{a}_0(x_R)}{\sqrt{x_R}} V''(b_N) \bigg(1 \;+\; \frac{V^{(3)}(b_N)}{V''(b_N)} \cdot \frac{\mathfrak{a}_1(x_R)}{N^{\alpha}} \bigg) \tag{8.60}$$

as well as

$$\Big| c_{1;0}^{(1/2)} + c_{1;1}^{(1/2)} x_R \,+\, x_R^{-1/2} f_1(x_R) \Big| \;\leq\; C \cdot \Big\{ N^{-\alpha} \cdot (x_R + 1) \| V''_{\mathfrak{e}} \|_{W^{\infty}_1(\mathbb{R})} \;+\; N^{-\alpha}\, x_R^{\frac{3}{2}} \big(\ln x_R\, \mathrm{e}^{-Cx_R} + 1 \big) \| V''_{\mathfrak{e}} \|_{W^{\infty}_0(\mathbb{R})} \Big\}\;. \tag{8.61}$$

These estimates imply, for N large enough:

$$\Big|\frac{\mathcal{W}_N[V'](\xi)}{q_R(\xi)}\Big| \;>\; \frac{\mathfrak{a}_0(x_R)}{\sqrt{x_R}}V''(b_N) \;-\; \frac{(\ln N)^3}{N^{\alpha}}||V_{\mathfrak{e}}||_{W_3^{\infty}(\mathbb{R})} \;. \tag{8.62}$$

The function $x \to \mathfrak{a}_0(x)/\sqrt{x}$ is bounded from below on $\mathbb{R}^+$, *cf.* Lemma 6.10 and $(a_N, b_N) \to (a, b)$ in virtue of Corollary 7.10. As a consequence, for any potential V such that $||V_{\mathfrak{e}}||_{W_3^{\infty}([a\,;b])} < C$, there exists N_0 large enough and $c > 0$ such that

$$\Big|\frac{\mathcal{W}_N[V'](\xi)}{q_R(\xi)}\Big| \;>\; c\inf_{[a\,;b]}\{V''(\xi)\} \;. \tag{8.63}$$

We can deduce from the above bounds that, for any $\xi \in \mathbb{J}_N^{(R;\mathrm{in})}$,

$$\frac{||q_R^{-1}\cdot\mathcal{W}_N[V']||_{W_\ell^{\infty}(\mathbb{J}_N^{(R;\mathrm{in})})}}{q_R^{-1}(\xi)\cdot\mathcal{W}_N[V'](\xi)} \;\leq\; C_\ell\cdot(\ln N)^{2\ell+1}\cdot N^{\ell\alpha}\cdot\frac{||V'||_{W_{2\ell+1}^{\infty}(\mathbb{J}_N^{(R;\mathrm{in})})}}{\inf_{[a\,;b]}\{V''(\xi)\}} \;. \tag{8.64}$$

Lower and upper bounds on $\mathbb{J}_N^{(R;\mathrm{ext})}$

Let us go back to the vector Riemann–Hilbert problem discussed in Lemma 4.1. The representation (4.11) and the fact that the solution Φ to this vector Riemann–Hilbert problem allows one the reconstruction of the functions ψ_1 and ψ_2 arising in (4.11) through (4.16). Using the reconstruction formula (5.14) with $P_1 = P_2 = 0$ and $z_0 = \infty$ and applying the regularisation trick exactly as in (5.67), we get $\xi \in [b_N\,; +\infty[$:

$$\mathcal{S}_N[\mathcal{W}_N[H]](\xi) \;=\; N^{\alpha}\int\limits_{\mathbb{R}+2\mathrm{i}\epsilon}\frac{\mathrm{d}\lambda}{2\pi}\int\limits_{\mathbb{R}+\mathrm{i}\epsilon}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\int\limits_{a_N}^{b_N}\mathrm{d}\eta H(\eta)\frac{\mathrm{e}^{\mathrm{i}\lambda N^{\alpha}(b_N-\xi)-\mathrm{i}\mu N^{\alpha}(b_N-\eta)}}{\mu-\lambda}\cdot\Big\{\chi_{21}(\lambda)\chi_{12}(\mu)-\frac{\mu}{\lambda}\cdot\chi_{11}(\mu)\chi_{22}(\lambda)\Big\} \,. \tag{8.65}$$

The local behaviour of the above integral representation can be studied with the set of tools developed throughout Section 6. We do not reproduce this reasoning again. All-in-all, we obtain:

$$||q_R^{-1}\cdot\mathcal{K}_\kappa[\mathcal{S}_N[\mathcal{W}_N[H]]-H]||_{W_\ell^{\infty}(\mathbb{J}_N^{(R;\mathrm{ext})})} \;\leq\; C_\ell(\ln N)^{2\ell+1}\cdot N^{(\ell+1)\alpha}\cdot\mathcal{N}_N^{(2\ell+1)}[\mathcal{K}_\kappa[H]] \tag{8.66}$$

and, for any $\xi \in \mathbb{J}_N^{(R;\mathrm{ext})}$,

$$\Big|q_R^{-1}(\xi)\cdot\{\mathcal{S}_N[\mathcal{W}_N[V'](\xi)]-V'(\xi)\}\Big| \;>\; c\inf_{[a\,;b]}V''(b_N) \tag{8.67}$$

provided that N is large enough. Likewise, we have the bounds:

$$\frac{||q_R^{-1}\mathcal{K}_\kappa[\mathcal{S}_N[\mathcal{W}_N[V']]-V']||_{W_\ell^{\infty}(\mathbb{J}_N^{(R;\mathrm{ext})})}}{q_R^{-1}(\xi)\cdot\Big\{\mathcal{S}_N[\mathcal{W}_N[V']]-V'\Big\}} \;\leq\; C_\ell\cdot(\ln N)^{2\ell+1}\cdot N^{\ell\alpha}\cdot\frac{||\mathcal{K}_\kappa[V']||_{W_{2\ell+1}^{\infty}(\mathbb{J}_N^{(R;\mathrm{ext})})}}{\inf_{[a\,;b]}\{V''(\xi)\}} \;. \tag{8.68}$$

Synthesis

Let us now write:

$$\begin{aligned}\mathcal{U}_N^{-1}[H](\xi) \;=\; \sum_{A=L,R}\Bigg\{&\frac{\mathcal{S}_N[\mathcal{W}_N[H]](\xi)-H(\xi)}{\mathcal{S}_N[\mathcal{W}_N[V']](\xi)-V'(\xi)}\cdot\mathbf{1}_{\mathbb{J}_N^{(A;\mathrm{out})}}(\xi) \;+\; \frac{\mathcal{W}_N[H](\xi)}{\mathcal{W}_N[V'](\xi)}\cdot\mathbf{1}_{\mathbb{J}_N^{(A;\mathrm{in})}}(\xi) \\ &+\; \frac{q_R^{-1}(\xi)\cdot\{\mathcal{S}_N[\mathcal{W}_N[H]](\xi)-H(\xi)\}}{q_R^{-1}(\xi)\cdot\{\mathcal{S}_N[\mathcal{W}_N[V']](\xi)-V'(\xi)\}}\cdot\mathbf{1}_{\mathbb{J}_N^{(A;\mathrm{ext})}}(\xi)\Bigg\} \;+\; \frac{\mathcal{W}_N[H](\xi)}{\mathcal{W}_N[V'](\xi)}\cdot\mathbf{1}_{\mathbb{J}_N^{(\mathrm{bk})}}(\xi) \;,\end{aligned} \tag{8.69}$$

The piecewise bounds (8.37)-(8.42) on $\mathbb{J}_N^{(R;\mathrm{out})}$, (8.46)-(8.50) on $\mathbb{J}_N^{(\mathrm{bk})}$, (8.59)-(8.64) on $\mathbb{J}_N^{(\mathrm{bk})}$, (8.66)-(8.68) on $\mathbb{J}_N^{(R;\mathrm{in})}$, and those which can be deduced by reflection symmetry on the three other segments defined in (8.28)-(8.30), can now be used together with the Faá di Bruno formula

$$\frac{\mathrm{d}^\ell}{\mathrm{d}\xi^\ell}\Big(\frac{f}{g}\Big)(\xi) \;=\; \sum_{n+m=\ell}\sum_{\sum kn_k=n} \frac{\ell!(\sum_{k=1}^n n_k)!}{m!}\cdot\frac{f^{(m)}(\xi)}{g(\xi)}\cdot\prod_{j=1}^{n}\Big\{\frac{1}{n_j!}\Big(\frac{-g^{(j)}(\xi)}{j!g(\xi)}\Big)^{n_j}\Big\} \tag{8.70}$$

to establish the global bound. Note that, in the intermediate bounds, one should use the obvious property of the exponential regularisation:

$$K_\kappa[f_1\cdots f_p] \;=\; \prod_{a=1}^{p}\mathcal{K}_{\kappa/p}[f_a]\;. \tag{8.71}$$

The details are left to the reader. ■

9 Asymptotic evaluation of the double integral

In this section we study the large-N asymptotic expansion for the double integral in:

Definition 9.1

$$\mathfrak{I}_{\mathrm{d}}[H,V] \;=\; \int_{a_N}^{b_N}\mathcal{W}_N\circ\widetilde{\mathcal{X}}_N\Big[\partial_\xi\{S\left(N^\alpha(\xi-*)\right)\cdot\mathcal{G}_N[\widetilde{\mathcal{X}}_N[H],V](\xi,*)\}\Big](\xi)\,\mathrm{d}\xi\;, \tag{9.1}$$

with

$$\mathcal{G}_N[H,V](\xi,\eta) \;=\; \frac{\mathcal{W}_N[H](\xi)}{\mathcal{W}_N[V'](\xi)} - \frac{\mathcal{W}_N[H](\eta)}{\mathcal{W}_N[V'](\eta)}\;. \tag{9.2}$$

We remind that * indicates the variable on which the operator $\mathcal{W}_N$ acts. The asymptotic analysis of the double integral $\mathfrak{I}_{\mathrm{d};\beta}$ arising in the $\beta\neq 1$ large-N asymptotics (*cf*. (3.118)) can be carried out within the setting of the method developed in this section. However, in order to keep the discussion minimal, we shall not present this calculation here.

In order to carry out the large-N asymptotic analysis of $\mathfrak{I}_{\mathrm{d}}[H,V]$, it is convenient to write down a decomposition for $\mathcal{G}_N[H,V]$ ensuing from the decomposition of $\mathcal{W}_N$ that has been described in Propositions 6.4 and 6.6. We omit the proof since it consists of straightforward algebraic manipulations.

Lemma 9.2 *The function $\mathcal{G}_N[H,V](\xi,\eta)$ can be recast as*

$$\begin{aligned}\mathcal{G}_N[H,V](\xi,\eta) \;=\; & \mathcal{G}_{\mathrm{bk};k}[H,V](\xi,\eta) \;+\; \mathcal{G}_{R;k}^{(\mathrm{as})}[H,V](x_R,y_R;\xi,\eta)\\ & -\;\mathcal{G}_{R;k}^{(\mathrm{as})}[H^\wedge,V^\wedge](x_L,y_L;a_N+b_N-\xi,a_N+b_N-\eta) \;+\; \Delta_{[k]}\mathcal{G}_N[H,V](\xi,\eta)\;.\end{aligned} \tag{9.3}$$

The functions arising in this decomposition read

$$\mathcal{G}_{\mathrm{bk};k}[H,V](\xi,\eta) \;=\; \frac{\mathcal{W}_{\mathrm{bk};k}[H](\xi)}{\mathcal{W}_{\mathrm{bk};k}[V'](\xi)} \;-\; (\xi\leftrightarrow\eta)\;, \tag{9.4}$$

and

$$\mathcal{G}^{(\mathrm{as})}_{R;k}[H,V](x,y;\xi,\eta) \;=\; \bigg\{ \frac{\mathcal{W}^{(\mathrm{as})}_{R;k}[H](x)}{\mathcal{W}_{\mathrm{bk};k}[V'](\xi)} \;-\; \frac{\mathcal{W}^{(\mathrm{as})}_{R;k}[V'](x)}{\mathcal{W}_{\mathrm{bk};k}[V'](\xi)} \cdot \frac{(\mathcal{W}^{(\mathrm{as})}_{\mathrm{bk};k}+\mathcal{W}^{(\mathrm{as})}_{R;k})[H](x)}{(\mathcal{W}^{(\mathrm{as})}_{\mathrm{bk};k}+\mathcal{W}^{(\mathrm{as})}_{R;k})[V'](x)} \bigg\} \;-\; \left(\begin{array}{c} \xi \leftrightarrow \eta \\ x \leftrightarrow y \end{array} \right). \quad (9.5)$$

Finally, the remainder $\Delta_{[k]}\mathcal{G}_N$ *takes the form*

$$\Delta_{[k]}\mathcal{G}_N[H,V](\xi,\eta) \;=\; \frac{1}{\mathcal{W}_{\mathrm{bk};k}[V'](\xi)} \bigg\{ \Delta_{[k]}\mathcal{W}_N[H](\xi) \;-\; \Delta_{[k]}\mathcal{W}_N[V'](\xi) \cdot \frac{\mathcal{W}_N[H](\xi)}{\mathcal{W}_N[V'](\xi)} \bigg\} \;-\; \big(\xi \leftrightarrow \eta\big)$$
$$+\; \Delta_{[k]}\mathcal{G}^{(\mathrm{as})}_N[H,V](x_R,y_R;\xi,\eta) \;-\; \Delta_{[k]}\mathcal{G}^{(\mathrm{as})}_N[H^{\wedge},V^{\wedge}](x_L,y_L;a_N+b_N-\xi,a_N+b_N-\eta)\;. \quad (9.6)$$

The reminder $\Delta_{[k]}\mathcal{W}_N$ *of order k has been introduced in* (6.70)*, while*

$$\Delta_{[k]}\mathcal{G}^{(\mathrm{as})}_N[H,V](x,y;\xi,\eta) \;=\; \frac{1}{\mathcal{W}_{\mathrm{bk};k}[V'](\xi)} \bigg\{ \Delta_{[k]}\mathcal{W}^{(\mathrm{as})}_R[H](x) \;-\; \Delta_{[k]}\mathcal{W}^{(\mathrm{as})}_R[V'](x) \cdot \frac{\mathcal{W}_N[H](\xi)}{\mathcal{W}_N[V'](\xi)}$$
$$-\Big[(\Delta_{[k]}\mathcal{W}_N)_R[H](\xi) \;-\; (\Delta_{[k]}\mathcal{W}_N)_R[V'](\xi) \cdot \frac{\mathcal{W}_N[H](\xi)}{\mathcal{W}_N[V'](\xi)} \Big] \cdot \frac{\mathcal{W}^{(\mathrm{as})}_{R;k}[V'](x)}{(\mathcal{W}^{(\mathrm{as})}_{\mathrm{bk};k}+\mathcal{W}^{(\mathrm{as})}_{R;k})[V'](x)} \bigg\} \;-\; \left(\begin{array}{c} \xi \leftrightarrow \eta \\ x \leftrightarrow y \end{array} \right). \quad (9.7)$$

The local right boundary remainder arising above is defined as

$$(\Delta_{[k]}\mathcal{W}_N)_R \;=\; \mathcal{W}_N \;-\; \mathcal{W}^{(\mathrm{as})}_{R;k} \;-\; \mathcal{W}^{(\mathrm{as})}_{\mathrm{bk};k}\;. \quad (9.8)$$

Note that the two terms $\mathcal{G}^{(\mathrm{as})}_{R;k}$ present in (9.3) correspond to the parts of $\mathcal{G}_N$ that localise at the right and left boundary. The way in which they appear is reminiscent of the symmetry satisfied by $\mathcal{W}_N$:

$$\mathcal{W}_N[H](a_N+b_N-\xi) \;=\; -\mathcal{W}_N[H^{\wedge}](\xi)\;. \quad (9.9)$$

Lemma 9.3 *The double integral* $\mathfrak{I}_{\mathrm{d}}[H,V]$ *can be recast as*

$$\mathfrak{I}_{\mathrm{d}}[H,V] \;=\; \mathfrak{I}_{\mathrm{d;bk}}\Big[\mathcal{G}_{\mathrm{bk};k}[H,V]\Big] \;+\; \mathfrak{I}_{\mathrm{d;bk}}\Big[\mathcal{G}^{(\mathrm{as})}_{R;k}[H,V] \;+\; \mathcal{G}^{(\mathrm{as})}_{R;k}[H^{\wedge},V^{\wedge}]\Big]$$
$$+\; \mathfrak{I}_{\mathrm{d};R}\Big[(\mathcal{G}_{\mathrm{bk};k}+\mathcal{G}^{(\mathrm{as})}_{R;k})[H,V] \;+\; (\mathcal{G}_{\mathrm{bk};k}+\mathcal{G}^{(\mathrm{as})}_{R;k})[H^{\wedge},V^{\wedge}]\Big] \;+\; \Delta_{[k]}\mathfrak{I}_{\mathrm{d}}\Big[\widetilde{\mathcal{X}}_N[H],V\Big]\;. \quad (9.10)$$

The bulk part of the double integral is described by the functional

$$\mathfrak{I}_{\mathrm{d;bk}}[F] \;=\; \frac{-N^{2\alpha}}{4\pi\beta} \int\limits_{[a_N\,;b_N]^2} J(N^{\alpha}(\xi-\eta)) \cdot (\partial_{\xi} \;-\; \partial_{\eta})\{S\,(N^{\alpha}(\xi-\eta))F(\xi,\eta)\}\,\mathrm{d}\xi\mathrm{d}\eta\;. \quad (9.11)$$

The local (right) part of the double integral is represented as

$$\mathfrak{I}_{\mathrm{d};R}[F] \;=\; -\frac{N^{2\alpha}}{2\pi\beta} \int\limits_{\mathscr{C}^{(+)}_{\mathrm{reg}}} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \int\limits_{\mathscr{C}^{(-)}_{\mathrm{reg}}} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \int\limits_{a_N}^{b_N} \frac{\mathrm{d}\xi\,\mathrm{e}^{\mathrm{i}\lambda N^{\alpha}(b_N-\xi)}}{(\mu-\lambda)R_{\downarrow}(\lambda)R_{\uparrow}(\mu)} \int\limits_{a_N}^{b_N} \mathrm{d}\eta\,\mathrm{e}^{-\mathrm{i}\mu N^{\alpha}(b_N-\eta)}\,\partial_{\xi}\{S\,(N^{\alpha}(\xi-\eta))F(\xi,\eta)\}\;. \quad (9.12)$$

Finally, $\Delta_{[k]}\mathfrak{I}_{\mathrm{d}}$ represents the remainder which decomposes as

$$\Delta_{[k]}\mathfrak{I}_{\mathrm{d}}[H,V] \;=\; \sum_{p=1}^{4}\Delta_{[k]}\mathfrak{I}_{\mathrm{d};p}[H,V] \tag{9.13}$$

$$\Delta_{[k]}\mathfrak{I}_{\mathrm{d};1}[H,V] \;=\; \int\limits_{a_N}^{b_N}\mathcal{W}_{\exp}\Big[\partial_\xi\{S\,(N^\alpha(\xi-*))\cdot(\mathcal{G}_N-\Delta_{[k]}\mathcal{G}_N)[H,V](\xi,*)\}\Big](\xi)\,\mathrm{d}\xi \tag{9.14}$$

$$\Delta_{[k]}\mathfrak{I}_{\mathrm{d};2}[H,V] \;=\; \int\limits_{a_N}^{b_N}\mathcal{W}_N\Big[\partial_\xi\{S\,(N^\alpha(\xi-*))\cdot\Delta_{[k]}\mathcal{G}_N[H,V](\xi,*)\}\Big](\xi)\,\mathrm{d}\xi \tag{9.15}$$

$$\Delta_{[k]}\mathfrak{I}_{\mathrm{d};3}[H,V] \;=\; -\int\limits_{a_N}^{b_N}\mathcal{W}_N[1](\xi)\cdot\mathcal{X}_N\Big[\partial_\xi\{S\,(N^\alpha(\xi-*))\cdot\mathcal{G}_N[H,V](\xi,*)\}\Big](\xi)\,\mathrm{d}\xi\,. \tag{9.16}$$

$$\Delta_{[k]}\mathfrak{I}_{\mathrm{d};4}[H,V] \;=\; -\mathfrak{I}_{\mathrm{d};R}\Big[(\mathcal{G}^{(\mathrm{as})}_{R;k}[H,V])^{\wedge} \;+\; (\mathcal{G}^{(\mathrm{as})}_{R;k}[H^{\wedge},V^{\wedge}])^{\wedge}\Big] \tag{9.17}$$

where $\mathcal{W}_{\exp}$ is as defined in (6.33).

Proof — We first invoke the Definition 3.18 of the operator $\widetilde{\mathcal{X}}_N$ so as to recast $\mathfrak{I}_{\mathrm{d}}[H,V]$ as an integral involving solely $\mathcal{W}_N$, and another one containing the action of $\mathcal{X}_N$. Then, in the first integral, we decompose the operator $\mathcal{W}_N$ arising in the "exterior" part of the double integral $\mathfrak{I}_{\mathrm{d}}[H,V]$ as $\mathcal{W}_N = (\mathcal{W}^{(0)}_R + \mathcal{W}^{(0)}_{\mathrm{bk}} + \mathcal{W}^{(0)}_L + \mathcal{W}_{\exp})$, *cf.* (6.33). Then, it remains to observe that

$$\int\limits_{a_N}^{b_N}\mathcal{W}^{(0)}_L\Big[\partial_\xi\{S\,(N^\alpha(\xi-*))\cdot\mathcal{G}_N[H,V](\xi,*)\}\Big](x_L)\,\mathrm{d}\xi \;=\; \int\limits_{a_N}^{b_N}\mathcal{W}^{(0)}_R\Big[\partial_\xi\{S\,(N^\alpha(\xi-*))\cdot\mathcal{G}_N[H^{\wedge},V^{\wedge}](\xi,*)\}\Big](x_R)\,\mathrm{d}\xi \tag{9.18}$$

and that

$$-\mathfrak{I}_{\mathrm{d;bk}}\Big[(\mathcal{G}^{(\mathrm{as})}_{R;k}[H^{\wedge},V^{\wedge}])^{\wedge}\Big] \;=\; \mathfrak{I}_{\mathrm{d;bk}}\Big[\mathcal{G}^{(\mathrm{as})}_{R;k}[H^{\wedge},V^{\wedge}]\Big]\,. \tag{9.19}$$

Putting all these results together, and using that the functions $\mathcal{G}_{\mathrm{bk};k}[H,V]$ and $\mathcal{G}^{(\mathrm{as})}_{R;k}[H,V]$ solely involve derivatives of H which implies:

$$\mathcal{G}_{\mathrm{bk};k}[\widetilde{\mathcal{X}}_N[H],V] \;=\; \mathcal{G}_{\mathrm{bk};k}[H,V] \qquad\text{and}\qquad \mathcal{G}^{(\mathrm{as})}_{R;k}[\widetilde{\mathcal{X}}_N[H],V] \;=\; \mathcal{G}^{(\mathrm{as})}_{R;k}[H,V]\,, \tag{9.20}$$

we obtain the desired decomposition of the double integral. ■

9.1 The asymptotic expansion related to $\mathfrak{I}_{\mathrm{d;bk}}$ and $\mathfrak{I}_{\mathrm{d};R}$

Once again, we need to introduce new constants:

Definition 9.4 *If $\ell \geq 0$ is an integer, we set:*

$$\daleth_{2\ell} \;=\; \int\limits_{\mathbb{R}}\frac{J(u)\,u^{2\ell}}{4\pi\beta\,(2\ell)!}\,[uS'(u)+S(u)]\,\mathrm{d}u \qquad\textit{and}\qquad \daleth_{2\ell+1} \;=\; \int\limits_{\mathbb{R}}\frac{J(u)\,S(u)\,u^{2(\ell+1)}}{4\pi\beta\,(2\ell+1)!}\,\mathrm{d}u \tag{9.21}$$

where the function J comes from Definition 6.3 and S is the kernel of $\mathcal{S}_N$ and appears lately in (8.2).

They are useful in the following:

Lemma 9.5 *Assume $F \in C^{2k+2}([a_N\,;b_N]^2)$ and antisymmetric viz. $F(\xi,\eta) = -F(\eta,\xi)$. We have the asymptotic expansion:*

$$\mathfrak{I}_{\mathrm{d;bk}}[F] \;=\; -N^{\alpha}\cdot \daleth_0 \cdot \mathcal{T}_{\mathrm{even}}[F](0) - \sum_{\ell=1}^{k} \frac{1}{N^{(2\ell-1)\alpha}} \Big\{ \daleth_{2\ell}\cdot \mathcal{T}^{(2\ell)}_{\mathrm{even}}[F](0) \,+\, \daleth_{2\ell-1}\cdot \mathcal{T}^{(2\ell-1)}_{\mathrm{odd}}[F](0) \Big\} \;+\; \mathrm{O}(N^{-2k\alpha}) \quad (9.22)$$

in terms of the integral transforms:

$$\mathcal{T}_{\mathrm{even}}[F](s) \;=\frac{1}{s}\int\limits_{2a_N-|s|}^{2b_N-|s|} F[(v+s)/2,(v-s)/2]\,\mathrm{d}v \qquad \text{and} \qquad \mathcal{T}_{\mathrm{odd}}[F](s) \;=\int\limits_{2a_N-|s|}^{2b_N-|s|} \partial_s\{s^{-1}\,F[(v+s)/2,(v-s)/2]\}\,\mathrm{d}v\,. \quad (9.23)$$

The integral transforms $\mathcal{T}_{\mathrm{even}},\mathcal{T}_{\mathrm{odd}}$ can be slightly simplified in the case of specific examples of the function F. In particular, if F takes the form $F(\xi,\eta) = g(\xi) - g(\eta)$ for some sufficiently regular function g, then we have:

$$\mathcal{T}_{\mathrm{even}}[F](0) \;=\; \int\limits_{2a_N}^{2b_N} g'(v/2)\,\mathrm{d}v \;=\; 2[g(b_N)-g(a_N)]\,. \quad (9.24)$$

Proof — We first implement the change of variables

$$\left\{ \begin{array}{ccc} u & = & N^{\alpha}(\xi-\eta) \\ v & = & \xi+\eta \end{array}\right. \qquad i.e. \qquad \left\{ \begin{array}{ccc} \xi & = & (v+N^{-\alpha}u)/2 \\ \eta & = & (v-N^{-\alpha}u)/2 \end{array}\right. \quad (9.25)$$

in the integral representation for $\mathfrak{I}_{\mathrm{d;bk}}[F]$. This recasts the integral as

$$\begin{aligned} \mathfrak{I}_{\mathrm{d;bk}}[F] \;&=\; -\frac{N^{2\alpha}}{4\pi\beta}\int\limits_{-\overline{x}_N}^{\overline{x}_N} \mathrm{d}u\, J(u) \int\limits_{2a_N+|u|N^{-\alpha}}^{2b_N-|u|N^{-\alpha}} \partial_u\Big\{ S(u)\cdot F\Big[\frac{v+uN^{-\alpha}}{2},\frac{v-uN^{-\alpha}}{2}\Big]\Big\}\,\mathrm{d}v \\ &=\; -\frac{N^{\alpha}}{4\pi\beta}\int\limits_{-\overline{x}_N}^{\overline{x}_N}\Big(J(u)\,[uS'(u)+S(u)]\,\mathcal{T}_{\mathrm{even}}[F](uN^{-\alpha}) \;+\; N^{-\alpha}J(u)\cdot uS(u)\cdot \mathcal{T}_{\mathrm{odd}}[F](uN^{-\alpha})\Big)\mathrm{d}u \quad (9.26) \end{aligned}$$

Both $J(u)\cdot uS(u)$ and $J(u)[uS'(u)+S(u)]$ decay exponentially fast at infinity. Hence, the expansion (9.22) readily follows by using the Taylor expansion with integral remainder for the functions $\mathcal{T}_{\mathrm{even/odd}}[F](uN^{-\alpha})$ around $u=0$, and the parity properties of $\mathcal{T}_{\mathrm{even/odd}}[F]$. ■

Lemma 9.6 *Let $F(x,y;\xi,\eta)$ be such that*

- *$F(x,y;\xi,\eta) = -F(y,x;\eta,\xi)$;*
- *the map $(x,y;\xi,\eta)\mapsto F(x,y;\xi,\eta)$ is $C^3(\mathbb{R}^+\times\mathbb{R}^+\times[a_N\,;b_N]^2)$;*
- *F – and any combination of partial derivatives of total order at most 3 – decays exponentially fast in x,y uniformly in $(\xi,\eta)\in[a_N\,;b_N]$, viz.*

$$\max\Big\{ \big|\partial_x^{p_1}\partial_y^{p_2}\partial_\xi^{p_3}\partial_\eta^{p_4}F(x,y;\xi,\eta)\big| \;:\; \sum_{a=1}^{4} p_a \;\le\; 3\Big\} \;\le\; C\,\mathrm{e}^{-c\min(x,y)}\,. \quad (9.27)$$

- *the following asymptotic expansion holds uniformly in $(x,y) \in [0\,;\epsilon N^{\alpha}]$, for some $\epsilon > 0$ and with a differentiable remainder in the sense of* (9.27).

$$F(x,y;b_N - N^{-\alpha}x, b_N - N^{-a}\,y) \;=\; \sum_{\ell=1}^{k} \frac{f_{\ell}(x,y)}{N^{\ell\alpha}} \;+\; \mathrm{O}\Big(\frac{C_k\,\mathrm{e}^{-c\min(x,y)}}{N^{(k+1)\alpha}}\Big)\,, \tag{9.28}$$

where $f_{\ell} \in C^3(\mathbb{R}^+ \times \mathbb{R}^+)$ for $\ell \in [\![\, 1\,;\,k\,]\!]$ while

$$\max\Big\{\big|\partial_x^p\partial_y^q f_{\ell}(x,y)\big| \;:\; p+q \,\leq\, 3 \quad and \quad \ell \in [\![\, 1\,;\,k\,]\!]\Big\} \;\leq\; C_k\,\mathrm{e}^{-c\min(x,y)}\,. \tag{9.29}$$

Then, denoting $F_N(\xi,\eta) \,=\, F(N^{\alpha}(b_N-\xi), N^{\alpha}(b_N-\eta);\xi,\eta)$, we have an asymptotic expansion:

$$\mathfrak{I}_{\mathrm{d;bk}}[F_N] \;=\; -\sum_{\ell=1}^{k} \frac{1}{N^{(\ell-1)\alpha}} \int_{\mathbb{R}} \frac{\mathrm{d}u\,J(u)}{4\pi\beta} \int_{|u|}^{+\infty} \mathrm{d}v\;\partial_u\Big\{S(u)\cdot f_{\ell}[(v-u)/2,(v+u)/2]\Big\} \;+\; \mathrm{O}\Big(\frac{1}{N^{k\alpha}}\Big)\,, \tag{9.30}$$

Note that, necessarily, f_{ℓ} are antisymmetric functions of (x,y).

Proof — The change of variables

$$\left\{\begin{array}{lcl} u & = & N^{\alpha}(\xi-\eta) \\ v & = & N^{\alpha}(2b_N-\xi-\eta) \end{array}\right. \qquad i.e. \qquad \left\{\begin{array}{lcl} \xi & = & b_N - N^{-\alpha}(v-u)/2 \\ \eta & = & b_N - N^{-\alpha}(v+u)/2 \end{array}\right. \tag{9.31}$$

recasts the integral as

$$\mathfrak{I}_{\mathrm{d;bk}}[F] \;=\; -N^{\alpha}\int_{-\overline{x}_N}^{\overline{x}_N} \frac{\mathrm{d}u\,J(u)}{4\pi\beta} \int_{|u|}^{2\overline{x}_N-|u|} \mathrm{d}v\;\partial_u\Big\{S(u)\cdot F\Big[\frac{v-u}{2},\frac{v+u}{2};b_N-\frac{v-u}{2N^{\alpha}},b_N-\frac{v+u}{2N^{\alpha}}\Big]\Big\}\,. \tag{9.32}$$

At this stage, we can limit all the domains of integration to $|u|,|v| \leq \epsilon N^{\alpha}$, this for the price of exponentially small corrections. Then, we insert the asymptotic expansion (9.28) and extend the domains of integration up to $+\infty$ this, again, for the price of exponentially small corrections, and we get the claim. ■

Very similarly, but under slightly different assumptions on the function F, we have the large-N asymptotic expansion of the right edge double integral.

Lemma 9.7 *Let $F(x,y;\xi,\eta)$ be such that*

- *$F(x,y;\xi,\eta) = -F(y,x;\eta,\xi)$;*
- *the map $(x,y;\xi,\eta) \mapsto F(x,y;\xi,\eta)$ is $C^3(\mathbb{R}^+ \times \mathbb{R}^+ \times [a_N\,;b_N]^2)$;*
- *F decays exponentially fast in x,y this uniformly in $(\xi,\eta) \in [a_N\,;b_N]$ and for any combination of partial derivatives of total order at most 3, viz.:*

$$\max\Big\{\big|\partial_x^{p_1}\partial_y^{p_2}\partial_{\xi}^{p_3}\partial_{\eta}^{p_4}F(x,y;\xi,\eta)\big| \;:\; \sum_{a=1}^{4} p_a \,\leq\, 3\Big\} \;\leq\; C\,\mathrm{e}^{-c\min(x,y)}\,. \tag{9.33}$$

- *the following asymptotic expansion holds uniformly in* $(x,y) \in [0\,; \epsilon N^{\alpha}]$, *for some* $\epsilon > 0$ *and with a differentiable remainder in the sense of* (9.33).

$$F(x,y;b_N - N^{-\alpha}x, b_N - N^{-\alpha}y) \; = \; \sum_{\ell=1}^{k} \frac{f_{\ell}(x,y)}{N^{\ell\alpha}} \; + \; \mathrm{O}\Big(\frac{C_k\,(x^k+y^k+1)}{N^{(k+1)\alpha}}\Big)\,, \tag{9.34}$$

where $f_{\ell} \in C^3(\mathbb{R}^+ \times \mathbb{R}^+)$ *for* $\ell \in [\![\, 1\,;\, k \,]\!]$ *while*

$$\max\Big\{\big|\partial_x^p \partial_y^q f_{\ell}(x,y)\big| \; : \; p+q \;\leq\; 3 \quad and \quad \ell \in [\![\, 1\,;\, k\,]\!]\Big\} \;\leq\; C_k\,(x^k+y^k+1)\,. \tag{9.35}$$

Then, we have the following asymptotic expansions

$$\mathfrak{I}_{\mathrm{d};R}[F_N] \; = \; \sum_{\ell=1}^{k} \frac{1}{N^{(\ell-1)\alpha}} \int\limits_{\mathscr{C}_{\mathrm{reg}}^{(+)}} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \int\limits_{\mathscr{C}_{\mathrm{reg}}^{(-)}} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \frac{(2\pi\beta)^{-1}}{(\lambda-\mu)R_{\downarrow}(\lambda)R_{\uparrow}(\mu)} \int\limits_{0}^{+\infty} \mathrm{e}^{\mathrm{i}\lambda x - \mathrm{i}\mu y}\partial_x\{S\,(x{-}y){\cdot}\, f_{\ell}(x,y)\}\mathrm{d}x\,\mathrm{d}y \; + \; \mathrm{O}\Big(\frac{1}{N^{\alpha k}}\Big)\,. \tag{9.36}$$

The function F_N *occurring above is as defined in the previous Lemma.*

Proof — The change of variables $x = N^{\alpha}(b_N - \xi)$ and $y = N^{\alpha}(b_N - \eta)$ recasts the integral in the form

$$\mathfrak{I}_{\mathrm{d};R}[F] \; = \; N^{\alpha}\int\limits_{\mathscr{C}_{\mathrm{reg}}^{(+)}} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \int\limits_{\mathscr{C}_{\mathrm{reg}}^{(-)}} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \frac{(2\pi\beta)^{-1}}{(\lambda-\mu)R_{\downarrow}(\lambda)R_{\uparrow}(\mu)} \int\limits_{0}^{\overline{x}_N} \mathrm{e}^{\mathrm{i}\lambda x - \mathrm{i}\mu y}\partial_x\{S\,(x-y)\cdot F(x,y;b_N - N^{-\alpha}x, b_N - N^{-\alpha}y)\}\mathrm{d}x\mathrm{d}y\,.$$

We can then conclude exactly as in the proof of Lemma 9.6. ∎

9.2 Estimation of the remainder $\Delta_{[k]}\mathfrak{I}_{\mathrm{d}}[H,V]$.

Lemma 9.8 *Let* $k \geq 1$ *be an integer. Given* $C_V > 0$, *assume* V *strictly convex, smooth enough and* $\|V_{\mathrm{e}}\|_{W_3^{\infty}(\mathbb{R})} < C_V$. *There exists* $C > 0$ *such that, for any* $H \in \mathfrak{X}_s(\mathbb{R})$ *smooth enough, the remainder integral* $\Delta_{[k]}\mathfrak{I}_{\mathrm{d}}[H,V]$ *satisfies:*

$$\Big|\Delta_{[k]}\mathfrak{I}_{\mathrm{d}}[H,V]\Big| \;\leq\; C\,N^{-(k-5)\alpha}\cdot \mathfrak{n}_{k+4}[V_{\mathrm{e}}]\cdot \|H_{\mathrm{e}}\|_{W^{\infty}_{\max\{k,5\}+4}(\mathbb{R})}\,. \tag{9.37}$$

Proof — It follows from Lemma 6.11 and 6.10, as well as $V''(b_N) \neq 0$ by strict convexity, that:

$$x \mapsto \frac{(\mathcal{W}_{\mathrm{bk};k}^{(\mathrm{as})} \; + \; \mathcal{W}_{R;k}^{(\mathrm{as})})[H](x)}{(\mathcal{W}_{\mathrm{bk};k}^{(\mathrm{as})} \; + \; \mathcal{W}_{R;k}^{(\mathrm{as})})[V'](x)} \tag{9.38}$$

is smooth at $x = 0$. As a consequence, the function

$$\begin{aligned}(\xi,\eta) \mapsto \mathcal{G}_{\mathrm{bk};k}&[H,V](\xi,\eta) + \mathcal{G}_{R;k}^{(\mathrm{as})}[H,V](x_R,y_R;\xi,\eta) \\ &= \; \frac{\Delta_{[k]}\mathcal{W}_{\mathrm{bk}}^{(\mathrm{as})}[H](\xi)}{\mathcal{W}_{\mathrm{bk};k}[V'](\xi)} \; + \; \frac{\mathcal{W}_{\mathrm{bk};k}^{(\mathrm{as})}[V'](x_R)}{\mathcal{W}_{\mathrm{bk};k}[V'](\xi)} \cdot \frac{(\mathcal{W}_{\mathrm{bk};k}^{(\mathrm{as})} \; + \; \mathcal{W}_{R;k}^{(\mathrm{as})})[H](x_R)}{(\mathcal{W}_{\mathrm{bk};k}^{(\mathrm{as})} \; + \; \mathcal{W}_{R;k}^{(\mathrm{as})})[V'](x_R)} \; - \; \big(\xi \leftrightarrow \eta\big)\end{aligned} \tag{9.39}$$

is smooth in (ξ,η). Furthermore, it follows from Theorem 8.1 that $(\xi,\eta) \mapsto \mathcal{G}_N[H,V](\xi,\eta)$ is smooth on $[a_N\,; b_N]$ as well. Since $\mathcal{G}_{R;k}^{(\mathrm{as})}[H,V](x_R,y_R;\xi,\eta)$ is smooth in ξ – resp. η – as soon as the latter variable is away from b_N or a_N, it follows that $(\xi,\eta) \mapsto \Delta_{[k]}\mathcal{G}_N[H,V](\xi,\eta)$ is smooth as well.

The remainder $\Delta_{[k]}\mathcal{G}_N$ described in (8.36) involves the remainders $\Delta_{[k]}\mathcal{W}^{(\mathrm{as})}_{R/\mathrm{bk}}$ studied in Lemma 6.11, and $(\Delta_{[k]}\mathcal{W}_N)_R$ defined in (9.8) and for which Proposition 6.6 and Lemma 6.11 also provide estimates. Using the properties of the $\mathfrak{a}$'s obtained in Lemma 6.10 and involved in the asymptotic expansion of the (as) quantities, it shows the existence of constants $c^{(0)}_{\ell;k}, c^{(1/2)}_{\ell;k}$ and of functions $f_{m;k} \in W^{\infty}_m(\mathbb{R}^+)$ bounded uniformly in N and satisfying $f_{m;k}(x) = \mathrm{O}(x^{m+1/2})$ such that

$$\Delta_{[k]}\mathcal{G}_N[H,V](\xi,\eta) \;=\; \frac{1}{N^{k\alpha}} \sum_{\ell=0}^{m} \big(c^{(0)}_{\ell;k}\, x_R^{\ell} \;+\; c^{(1/2)}_{\ell;k}\, x_R^{\ell-1/2}\big) \;+\; \frac{f_{m;k}(x_R)}{N^{k\alpha}} \;-\; \Big(x_R \leftrightarrow y_R\Big)\,, \tag{9.40}$$

for $(x_R, y_R) \in [0\,;\epsilon]^2$. Since $\Delta_{[k]}\mathcal{G}_N[H,V](\xi,\eta)$ is smooth, we necessarily have that $c^{(1/2)}_{\ell;k} = 0$ for $\ell \in [\![\, 0\,;\, m \,]\!]$. The representation (9.40) thus ensures that

$$\max_{0\le \ell+p\le n}\ \max_{\substack{(x_R,y_R)\\ \in[0\,;\epsilon]^2}} \Big|\partial_\xi^{\ell}\partial_\eta^{p}\Delta_{[k]}\mathcal{G}_N[H,V](\xi,\eta)\Big| \;\le\; \frac{C_n}{N^{(k-n)\alpha}} \cdot \mathfrak{n}_{n+k}[V] \cdot ||H_{\mathfrak{e}}||_{W^{\infty}_{n+1+k}(\mathbb{R})}\ . \tag{9.41}$$

Here, the explicit control on the dependence of the bound on V and H issues from the control on the remainders entering in the expression for $\Delta_{[k]}\mathcal{G}_N[H,V]$.
Similar types of bounds can, of course, be obtained for $(x_L, y_L) \in [0\,;\epsilon]^2$. Finally, as soon as a variable, be it ξ or η, is uniformly (in N) away from an immediate neighbourhood of the endpoints a_N and b_N, we can use more crude expressions for the remainders so as to bound derivatives of the remainder $\Delta_{[k]}\mathcal{G}_N[H,V]$. This does not spoil (9.41) and we conclude:

$$\max_{0\le \ell+p\le n}\ \max_{\substack{(\xi,\eta)\\ \in[a_N\,;b_N]^2}} \Big|\partial_\xi^{\ell}\partial_\eta^{p}\Delta_{[k]}\mathcal{G}_N[H,V](\xi,\eta)\Big| \;\le\; \frac{C_n}{N^{(k-n)\alpha}} \cdot \mathfrak{n}_{n+k}[V] \cdot ||H_{\mathfrak{e}}||_{W^{\infty}_{n+1+k}(\mathbb{R})}\ . \tag{9.42}$$

Having at disposal such a control on the remainder $\Delta_{[k]}\mathcal{G}_N[H,V]$, we are in position to bound the double integral of interest. The latter decomposes into a sum of four terms

$$\Delta_{[k]}\mathfrak{I}_{\mathrm{d}}[H,V] \;=\; \sum_{p=1}^{4} \Delta_{[k]}\mathfrak{I}_{\mathrm{d};p}[H,V] \tag{9.43}$$

that have been defined in (9.14)-(9.17).

Bounding $\Delta_{[k]}\mathfrak{I}_{\mathrm{d};1}[H,V]$

Let

$$\tau(\xi,\eta) \;=\; \partial_\xi\Big\{S\,(N^{\alpha}(\xi-\eta))\cdot\mathcal{G}_N[H,V](\xi,\eta)\Big\} \quad \text{and} \quad \Delta_{[k]}\tau(\xi,\eta) \;=\; \partial_\xi\Big\{S\,(N^{\alpha}(\xi-\eta))\cdot\Delta_{[k]}\mathcal{G}_N[H,V](\xi,\eta)\Big\}\,. \tag{9.44}$$

Observe that given $(\xi,\eta) \mapsto f(\xi,\eta)$ sufficiently regular, we have the decomposition:

$$\int\limits_{a_N}^{b_N} \mathcal{W}_{\exp}[\,f(\xi,*)](\xi)\,\mathrm{d}\xi \;=\; \int\limits_{a_N}^{b_N} \mathcal{W}_{\exp}[\,f(a_N,*)](\xi)\,\mathrm{d}\xi \;+\; \int\limits_{a_N}^{b_N}\mathrm{d}\xi \int\limits_{a_N}^{\xi}\mathrm{d}\eta\ \mathcal{W}_{\exp}[\partial_\eta f(\eta,*)](\xi)\ . \tag{9.45}$$

The latter ensures that

$$\Big|\int\limits_{a_N}^{b_N} \mathcal{W}_{\exp}[\,f(\xi,*)](\xi)\,\mathrm{d}\xi\Big| \;\le\; \Big|\Big|\mathcal{W}_{\exp}[\,f(a_N,*)]\Big|\Big|_{L^1([a_N\,;b_N])} \;+\; (b_N - a_N) \sup_{\eta\in[a_N\,;b_N]} \Big|\Big|\mathcal{W}_{\exp}[\partial_\eta f(\eta,*)]\Big|\Big|_{L^1([a_N\,;b_N])}\ .$$

$$(9.46)$$

The two terms can be estimated directly using the L^1 bound (7.49) obtained in Lemma 7.8. For the first one:

$$\Big\|\mathcal{W}_{\exp}[f(a_N,*)]\Big\|_{L^1([a_N\,;b_N])} \;\leq\; C_1 \mathrm{e}^{-C_2 N^{\alpha}}\, \|f_{\mathrm{e}}(a_N,*)\|_{W_1^{\infty}(\mathbb{R})} \leq C_1\, \mathrm{e}^{-C_2 N^{\alpha}}\, \|f\|_{W_1^{\infty}(\mathbb{R}^2)} \tag{9.47}$$

for some $C_1, C_2 > 0$ independent of N and f, and likewise for the second term. But the $W_p^{\infty}(\mathbb{R}^2)$ norm of f_{e} is also bounded by a constant times the $W_p^{\infty}([a_N\,;b_N]^2)$ norm of f, and we can make the constant depends only on the compact support of the extension f_{e}. Therefore:

$$\Big\|\mathcal{W}_{\exp}[f(a_N,*)]\Big\|_{L^1([a_N\,;b_N])} \leq C_1'\, \mathrm{e}^{-C_2' N^{\alpha}} \|f\|_{W_1^{\infty}([a_N\,;b_N]^2)} \tag{9.48}$$

for some $C_1', C_2' > 0$. Taking $f = \tau - \Delta_{[k]}\tau$ to match the definition (9.14) of $\Delta_{[k]}\mathfrak{I}_{\mathrm{d};1}$, this implies:

$$\Big|\Delta_{[k]}\mathfrak{I}_{\mathrm{d};1}[H,V]\Big| \;\leq\; C_1' \mathrm{e}^{-C_2 N^{\alpha}} \Big\{\|\tau\|_{W_2^{\infty}([a_N\,;b_N]^2)} + \|\Delta_{[k]}\tau\|_{W_2^{\infty}([a_N\,;b_N]^2)}\Big\}\,. \tag{9.49}$$

It solely remains to bound the $W_2^{\infty}([a_N\,;b_N]^2)$ norm of τ and $\Delta_{[k]}\tau$. We remind that, for $\xi \in [a_N\,;b_N]$, we have from the definition (9.2) and the expression of $\mathcal{U}_N^{-1}$ given in (8.21):

$$\mathcal{G}_N[H,V](\xi) = \mathcal{U}_N^{-1}[H](\xi) - \mathcal{U}_N^{-1}[H](\eta)\,. \tag{9.50}$$

By invoking the mean value theorem and the estimate of Proposition 8.2 for W_{ℓ}^{∞} norm of $\mathcal{U}_N^{-1}[H]$, we obtain:

$$\begin{aligned}
\|\tau\|_{W_{\ell}^{\infty}([a_N\,;b_N]^2)} \;&\leq\; C\,N^{\alpha}\, \left\|(\xi,\eta)\mapsto \frac{\mathcal{G}_N[H,V](\xi,\eta)}{\xi-\eta}\right\|_{W_{\ell+1}^{\infty}([a_N\,;b_N]^2)} \;\leq\; C'\,N^{\alpha}\, \Big\|\mathcal{U}_N^{-1}[H]\Big\|_{W_{\ell+2}^{\infty}([a_N\,;b_N])} &(9.51)\\
&\leq\; C_{\ell}'\cdot(\ln N)^{2\ell+5}\cdot N^{(\ell+4)\alpha}\cdot \mathfrak{n}_{\ell+2}[V]\cdot \mathcal{N}_N^{(2\ell+5)}[\mathcal{K}_{\kappa}[H]] &(9.52)\\
&\leq\; C_{\ell}''\cdot(\ln N)^{2\ell+5}\cdot N^{(\ell+4)\alpha}\cdot \mathfrak{n}_{\ell+2}[V]\cdot \|H_{\mathrm{e}}\|_{W_{2\ell+5}^{\infty}(\mathbb{R})} &(9.53)
\end{aligned}$$

where the last step comes from domination of the weighted norm by the W^{∞} norm of the same order – and the exponential regularisation can easily be traded for a compactly supported extension up to increasing the constant prefactor. Similarly, in virtue of the bounds (9.42), we get:

$$\|\Delta_{[k]}\tau\|_{W_{\ell}^{\infty}([a_N\,;b_N]^2)} \;\leq\; C'\cdot N^{(\ell+3-k)\alpha}\cdot \mathfrak{n}_{k+\ell+2}[V]\cdot \|H_{\mathrm{e}}\|_{W_{k+\ell+3}^{\infty}(\mathbb{R})}\,. \tag{9.54}$$

Putting these two estimates back in (9.49) with $\ell = 2$, we see that:

$$\Big|\Delta_{[k]}\mathfrak{I}_{\mathrm{d};1}[H,V]\Big| \;\leq\; C_1'\cdot N^{6\alpha}\cdot \mathrm{e}^{-C_2 N^{\alpha}}\,\mathfrak{n}_{k+4}[V]\cdot \|H_{\mathrm{e}}\|_{W_{\max\{k,5\}+4}^{\infty}(\mathbb{R})}\,. \tag{9.55}$$

which is exponentially small when $N \to \infty$.

Bounding $\Delta_{[k]}\mathfrak{I}_{\mathrm{d};2}[H,V]$

$\Delta_{[k]}\mathfrak{I}_{\mathrm{d};2}[H,V]$ has been defined in (9.15) and can be bounded by repeating the previous handlings. Indeed, using (7.49) on the L^1 norm of $\mathcal{W}_N$ and then following the previous steps, one finds:

$$\Big|\Delta_{[k]}\mathfrak{I}_{\mathrm{d};2}[H,V]\Big| \;\leq\; \|\Delta_{[k]}\tau\|_{W_2^{\infty}([a_N\,;b_N]^2)} \tag{9.56}$$

with $\Delta_{[k]}\tau$ defined in (9.44) and bounded in W_{ℓ}^{∞} norms in (9.54). Hence, we find:

$$\Big|\Delta_{[k]}\mathfrak{I}_{\mathrm{d};2}[H,V]\Big| \;\leq\; C_1'\cdot N^{(5-k)\alpha}\cdot \mathfrak{n}_{k+4}[V]\cdot \|H_{\mathrm{e}}\|_{W_{k+5}^{\infty}(\mathbb{R})}\,. \tag{9.57}$$

Bounding $\Delta_{[k]}\mathfrak{I}_{\mathrm{d};3}[H,V]$

This quantity is defined in (9.16), and it follows from the explicit expression for $\mathcal{W}_N[1](\xi)$ given in (5.72) that

$$\Big|\Delta_{[k]}\mathfrak{I}_{\mathrm{d};3}[H,V]\Big| \;\leq\; C\,N^{\alpha}\|\tau\|_{W_0^{\infty}([a_N\,;b_N]^2)}\cdot\Big|\chi_{12;+}(0)\Big|\cdot|b_N-a_N|\cdot\sup_{\xi\in[a_N\,;b_N]}\left|\int\limits_{\mathbb{R}+\mathrm{i}\epsilon'}\frac{\chi_{11}(\lambda)}{\lambda}\mathrm{e}^{-\mathrm{i}N^{\alpha}\lambda(\xi-a_N)}\cdot\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\right| \quad (9.58)$$

where τ is as defined in (9.44). The decomposition (7.6) for χ and its properties show the existence of $C,C'>0$ such that:

$$\forall\lambda\in\mathbb{R}+\mathrm{i}\epsilon',\qquad |\chi_{11}(\lambda)|\;\leq\;C\,|\lambda|^{-1/2},\qquad\text{and}\qquad|\chi_{12}(\lambda)|\;\leq\;C'\,\mathrm{e}^{-N^{\alpha}\varkappa_{\epsilon'}}\;. \quad (9.59)$$

Hence, by invoking the bounds (9.53) satisfied by τ, we get:

$$\Big|\Delta_{[k]}\mathfrak{I}_{\mathrm{d};3}[H,V]\Big|\;\leq\;C''\cdot N^{5\alpha}\cdot(\ln N)^5\cdot\mathrm{e}^{-N^{\alpha}[\varkappa_{\epsilon'}-\epsilon'(b_N-a_N)]}\cdot\mathfrak{n}_2[V]\cdot\|H_{\mathrm{e}}\|_{W_5^{\infty}(\mathbb{R})}\;. \quad (9.60)$$

Since $\varkappa_{\epsilon'}>0$ is bounded away from 0 when $\epsilon'\to0$ according to its definition (4.42), we also have $\varkappa_{\epsilon}-\epsilon' x_N>0$ uniformly in N for some choice of ϵ' small enough but independent of N.

Bounding $\Delta_{[k]}\mathfrak{I}_{\mathrm{d};4}[H,V]$

This quantity is defined in (9.17), and it involves integration of:

$$\tau_L(\xi,\eta)\;=\;\partial_{\xi}\Big\{S\big(N^{\alpha}(\xi-\eta)\big)\cdot\mathcal{G}_{R;k}^{(\mathrm{as})}[H,V](x_L,y_L;b_N+a_N-\xi,b_N+a_N-\eta)\Big\} \quad (9.61)$$

where $\mathcal{G}_{R;k}^{(\mathrm{as})}$ was defined in (9.5). It only involves the operators $\mathcal{W}_{\mathrm{bk};k}^{(\mathrm{as})}$ and $\mathcal{W}_{R;k}^{(\mathrm{as})}$, whose expression is given in Lemma 6.11. Let us fix $\epsilon>0$. Straightforward manipulations show that, for $(\xi,\eta)\in[a_N+\epsilon\,;b_N]^2$, we have:

$$\Big|\tau_L(\xi,\eta)\Big|\;\leq\;CN^{3\alpha}\mathrm{e}^{-C'\min(x_L,y_L)}\cdot\mathfrak{n}_{k+1}[V]\cdot\|H_{\mathrm{e}}\|_{W_k^{\infty}(\mathbb{R})}\;\leq\;\widetilde{C}\cdot N^{3\alpha}\cdot\mathrm{e}^{-\epsilon C'N^{\alpha}}\cdot\mathfrak{n}_{k+1}[V]\cdot\|H_{\mathrm{e}}\|_{W_k^{\infty}(\mathbb{R})} \quad (9.62)$$

which is thus exponentially small in N. Similar steps show that, for

$$(\xi,\eta)\in\Big\{[a_N+\epsilon\,;b_N]\times[a_N\,;a_N+\epsilon]\Big\}\cup\Big\{[a_N\,;a_N+\epsilon]\times[a_N+\epsilon\,;b_N]\Big\}\cup\Big\{[a_N\,;a_N+\epsilon]\times[a_N\,;a_N+\epsilon]\Big\}\;, \quad (9.63)$$

we have:

$$\Big|\tau_L(\xi,\eta)\Big|\;\leq\;C\,N^{3\alpha}\mathfrak{n}_{k+1}[V]\,\|H\|_{W_k^{\infty}(\mathbb{R})}\;. \quad (9.64)$$

Here, the exponential decay in N will come after integration of τ_L as it appears in (9.17). Indeed, given $\operatorname{Im}\lambda>0$ and $\operatorname{Im}\mu<0$ we have:

$$\begin{aligned}\left|\int\limits_{a_N}^{b_N}\mathrm{e}^{\mathrm{i}\lambda x_R}\mathrm{e}^{-\mathrm{i}\mu y_R}\tau_L(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta\right| &\leq C\,N^{3\alpha}\mathrm{e}^{-\epsilon C'N^{\alpha}}\mathfrak{n}_{k+1}[V]\,\|H\|_{W_k^{\infty}(\mathbb{R})}\int\limits_{a_N+\epsilon}^{b_N}\mathrm{e}^{-|\operatorname{Im}\lambda|N^{\alpha}(b_N-\xi)-|\operatorname{Im}\mu|N^{\alpha}(b_N-\eta)}\,\mathrm{d}\eta\,\mathrm{d}\xi\\ &+CN^{3\alpha}\mathfrak{n}_{k+1}[V]\,\|H\|_{W_k^{\infty}(\mathbb{R})}\Bigg\{\int\limits_{a_N+\epsilon}^{b_N}\mathrm{d}\xi\int\limits_{a_N}^{a_N+\epsilon}\mathrm{d}\eta\;+\;\int\limits_{a_N}^{a_N+\epsilon}\mathrm{d}\xi\int\limits_{a_N+\epsilon}^{b_N}\mathrm{d}\eta\;+\;\int\limits_{a_N}^{a_N+\epsilon}\int\limits_{a_N}^{a_N+\epsilon}\mathrm{d}\xi\mathrm{d}\eta\Bigg\}\mathrm{e}^{-|\operatorname{Im}\lambda|N^{\alpha}(b_N-\xi)-|\operatorname{Im}\mu|N^{\alpha}(b_N-\eta)}\\ &\leq\;\mathfrak{n}_{k+1}[V]\,\|H\|_{W_k^{\infty}(\mathbb{R})}\cdot\frac{\widetilde{C}\,N^{3\alpha}\mathrm{e}^{-\widetilde{C}'N^{\alpha}}}{|\lambda\cdot\mu|}\;. \end{aligned} \quad (9.65)$$

Note that, above, we have used that for $\lambda \in \mathscr{C}_{\mathrm{reg}}^{(+)}$ and $\mu \in \mathscr{C}_{\mathrm{reg}}^{(-)}$, we can bound:

$$|\mathrm{Im}\,\lambda|^{-1} \le c_1 |\lambda|^{-1}\,, \qquad |\mathrm{Im}\,\mu|^{-1} \le c_1 |\mu|^{-1} \tag{9.66}$$

for some constant $c_1 > 1$. Hence, all in all, we have:

$$\begin{aligned} \Big|\mathfrak{I}_{\mathrm{d};R}\Big[\big(\mathcal{G}_{R;k}^{(\mathrm{as})}[H,V]\big)^{\wedge}\Big]\Big| \;&\le\; C'' N^{3\alpha} \mathrm{e}^{-\widetilde{C}' N^{\alpha}} \cdot \int_{\mathscr{C}_{\mathrm{reg}}^{(+)}} |\mathrm{d}\lambda| \int_{\mathscr{C}_{\mathrm{reg}}^{(-)}} |\mathrm{d}\mu| \cdot \frac{\mathfrak{n}_{k+1}[V]\,||H||_{W_k^{\infty}(\mathbb{R})}}{|\mu-\lambda| \cdot |\lambda R_{\downarrow}(\lambda) R_{\uparrow}(\mu)\mu|} \\ &\le\; C'''\, N^{-3\alpha}\, \mathrm{e}^{-\widetilde{C}' N^{\alpha}} \cdot \mathfrak{n}_{k+1}[V]\,||H||_{W_k^{\infty}(\mathbb{R})}\,. \end{aligned} \tag{9.67}$$

Then, putting together all of the results for each $\Delta_{[k]}\mathfrak{I}_{\mathrm{d};p}$ for $p \in [\![\, 1\,;\, 4\,]\!]$ entails the global bound (9.37). ■

9.3 Leading asymptotics of the double integral

We need to introduce two new quantities before writing down the asymptotic expansion of the double integral $\mathfrak{I}_{\mathrm{d}}$.

Definition 9.9 *We define the function:*

$$\mathfrak{c}(x) = \frac{\mathfrak{b}_1(x) - \mathfrak{b}_0(x)\mathfrak{a}_1(x)}{u_1} \tag{9.68}$$

and the constant:

$$\begin{aligned} \aleph_0 \;=\; &-\int_{\mathbb{R}} \frac{\mathrm{d}u\, J(u)}{4\pi\beta} \int_{|u|}^{+\infty} \mathrm{d}v\, \partial_u \Big\{ S(u) \cdot \Big(\mathfrak{c}[\frac{v-u}{2}] \;-\; \mathfrak{c}[\frac{v+u}{2}] \Big) \Big\} \\ &+\; \frac{1}{2\pi\beta} \int_{\mathscr{C}_{\mathrm{reg}}^{(+)}} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \int_{\mathscr{C}_{\mathrm{reg}}^{(-)}} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \frac{1}{(\lambda-\mu) R_{\downarrow}(\lambda) R_{\uparrow}(\mu)} \int_0^{+\infty} \mathrm{e}^{\mathrm{i}\lambda x - \mathrm{i}\mu y} \partial_x \Big\{ S(x-y)[\mathfrak{c}(x) - \mathfrak{c}(y)] \;-\; x + y \Big\} \mathrm{d}x\, \mathrm{d}y \end{aligned} \tag{9.69}$$

Proposition 9.10 *We have the large-N behaviour:*

$$\mathfrak{I}_{\mathrm{d}}[H,V] \;=\; -2\daleth_0 \cdot N^{\alpha} \cdot \Big\{ \frac{H'(b_N)}{V''(b_N)} \;-\; \frac{H'(a_N)}{V''(a_N)} \Big\} \;+\; \aleph_0 \cdot \Big\{ \Big(\frac{H'}{V''}\Big)'(b_N) \;+\; \Big(\frac{H'}{V''}\Big)'(a_N) \Big\} \;+\; \Delta\mathfrak{I}_{\mathrm{d}}[H,V] \tag{9.70}$$

and the remainder is bounded as:

$$\Delta\mathfrak{I}_{\mathrm{d}}[H,V] \;\le\; \frac{C}{N^{\alpha}} \cdot \mathfrak{n}_{10}[V_{\mathfrak{e}}] \cdot ||H_{\mathfrak{e}}||_{W_{11}^{\infty}(\mathbb{R})}\,. \tag{9.71}$$

Proof — We first need to introduce two universal sequences of polynomials $\mathcal{P}_{\ell}(\{x_p\}_1^{\ell})$ and $\mathcal{Q}_{\ell}(\{y_p\}_1^{\ell};\{a_p\}_1^{\ell})$. Given formal power series

$$f(z) = 1 \,+\, \sum_{\ell \ge 1} f_{\ell}\, z^{\ell} \qquad \text{and} \qquad g(z) = 1 \,+\, \sum_{\ell \ge 1} g_{\ell}\, z^{\ell} \tag{9.72}$$

they are defined to be the coefficients arising in the formal power series

$$\frac{1}{f(z)} \;=\; 1 + \sum_{\ell \ge 1} \mathcal{P}_{\ell}(\{f_p\}_1^{\ell})\, z^{\ell} \quad \text{and} \quad \frac{g(z)}{f(z)} \;=\; 1 + \sum_{\ell \ge 1} \mathcal{Q}_{\ell}(\{g_p\}_1^{\ell};\{f_p\}_1^{\ell})\, z^{\ell}\,. \tag{9.73}$$

Note that

$$\mathcal{Q}_\ell(\{g_p\}_1^\ell;\{f_p\}_1^\ell) \; = \; \sum_{\substack{r+s=\ell \\ r,s\geq 0}} g_r \cdot \mathcal{P}_s(\{f_p\}_1^s) \,, \tag{9.74}$$

where we agree upon the convention $\mathcal{P}_0 = 1$ and $g_0 = 1$. This notation is convenient to write down the large-N expansion of $\mathcal{G}_{\mathrm{bk};k}$ – defined in (9.4) – ensuing from the large N-expansion of $\mathcal{W}_{\mathrm{bk};k}$ provided by Lemma 6.11. We find, uniformly in $(\xi,\eta)\in[a_N\,;b_N]^2$:

$$\mathcal{G}_{\mathrm{bk};k}[H,V](\xi,\eta) \; = \; \sum_{\ell=0}^{k-1} \frac{\mathfrak{G}_{\mathrm{bk};\ell}[H,V](\xi,\eta)}{N^{\alpha\ell}} \; + \; \mathrm{O}(N^{-k\alpha}) \tag{9.75}$$

where

$$\mathfrak{G}_{\mathrm{bk};\ell}[H,V](\xi,\eta) \; = \; \mathfrak{g}_{\mathrm{bk};\ell}[H,V](\xi) \; - \; \mathfrak{g}_{\mathrm{bk};\ell}[H,V](\eta) \tag{9.76}$$

with

$$\mathfrak{g}_{\mathrm{bk};\ell}[H,V](\xi) \; = \; \frac{H'(\xi)}{V''(\xi)}\cdot \mathcal{Q}_\ell\bigg(\bigg\{ \frac{H^{(\ell+1)}(\xi)}{H'(\xi)}\,\frac{u_{\ell+1}}{u_1}\bigg\}_\ell ; \bigg\{ \frac{V^{(\ell+2)}(\xi)}{V''(\xi)}\,\frac{u_{\ell+1}}{u_1}\bigg\}_\ell\bigg) \,. \tag{9.77}$$

Also, in the case of a localisation of the variables around b_N, we have:

$$\begin{aligned} \mathcal{G}_{\mathrm{bk};k}[H,V](b_N-N^{-\alpha}x, b_N-N^{-a}y) \; = \; & \frac{H'(b_N)}{V''(b_N)} \sum_{\ell=1}^{k} N^{-\ell\alpha}\cdot \mathcal{Q}_\ell\bigg(\bigg\{ \frac{H^{(p+1)}(b_N)}{H'(b_N)}\,\frac{\mathfrak{u}_p(x)}{u_1}\bigg\} ; \bigg\{ \frac{V^{(p+2)}(b_N)}{V''(b_N)}\,\frac{\mathfrak{u}_p(x)}{u_1}\bigg\}\bigg) \\ & - \; (x\leftrightarrow y) + \; \mathrm{O}\bigg(\frac{x^k+y^k+1}{N^{(k+1)\alpha}}\bigg) \,. \end{aligned} \tag{9.78}$$

Finally, we also have the expansion,

$$\mathcal{G}_{R;k}^{(\mathrm{as})}[H,V](x,y;b_N-N^{-\alpha}x,b_N-N^{-\alpha}y) \; = \; \sum_{\ell=1}^{k} \frac{\mathfrak{G}_{R;\ell}[H,V](x,y)}{N^{\alpha\ell}} \; + \; \mathrm{O}\bigg(\frac{\mathrm{e}^{-c\min(x,y)}}{N^{(k+1)\alpha}}\bigg) \tag{9.79}$$

where $\mathfrak{G}_{R;\ell}[H,V](x,y) \; = \; \mathfrak{g}_{R;\ell}[H,V](x)-\mathfrak{g}_{R;\ell}[H,V](y)$ and

$$\begin{aligned} \mathfrak{g}_{R;\ell}[H,V](x) \; = \; & \frac{1}{u_1V''(b_N)} \sum_{\substack{m+s=\ell \\ m,s\geq 0}} \mathcal{P}_m\bigg(\bigg\{ \frac{V^{(q+2)}(b_N)}{V''(b_N)}\,\frac{\mathfrak{u}_q(x)}{u_1}\bigg\}_q\bigg)\cdot \frac{H^{(s+1)}(b_N)}{H'(b_N)}\cdot \mathfrak{b}_s(x) \\ & - \; \frac{H'(b_N)}{u_1[V''(b_N)]^2} \sum_{\substack{m+s+p=\ell \\ m,s,p\geq 0}} \mathcal{P}_m\bigg(\bigg\{ \frac{V^{(q+2)}(b_N)}{V''(b_N)}\,\frac{\mathfrak{u}_q(x)}{u_1}\bigg\}_q\bigg) \\ & \times \mathcal{Q}_p\bigg(\bigg\{ \frac{H^{(q+1)}(b_N)}{H'(b_N)}\,\mathfrak{a}_q(x)\bigg\}_q ; \bigg\{ \frac{V^{(q+2)}(b_N)}{V''(b_N)}\,\mathfrak{a}_q(x)\bigg\}_q\bigg)\cdot \frac{V^{(s+2)}(b_N)}{V''(b_N)}\cdot \mathfrak{b}_s(x) \,. \end{aligned} \tag{9.80}$$

We can now come back to the double integral $\mathfrak{I}_{\mathrm{d}}$. It has been decomposed in Lemma 9.3. If we want a remainder $\Delta_{[k]}\mathfrak{I}_{\mathrm{d}}$ decaying with N, we should take $k=6$ in Lemma 9.8. Then, up to $\mathrm{O}(N^{-\alpha})$, we are thus left with operators $\mathfrak{I}_{\mathrm{d;bk}}$ and $\mathfrak{I}_{\mathrm{d};R}$, and Lemmas 9.5 and 9.7 describe for us their asymptotic expansion knowing the asymptotic expansion of the functions to which they are applied. Here, they are applied to the various functions involving $\mathcal{G}_{\mathrm{bk};k}$ and $\mathcal{G}^{(\mathrm{as})}_{\mathrm{R};k}$ whose expansion has been described in (9.78) and (9.79). As these expression shows, in order to get $\mathfrak{I}_{\mathrm{d}}$ up to $\mathrm{O}(N^{-\alpha})$, one just need the expressions of $\mathfrak{g}_{\mathrm{bk};0}[H,V](\xi)$ from (9.77) and $\mathfrak{g}_{R;1}[H,V](x)$ from (9.80). These only involve the universal polynomials $\mathcal{P}_1$ and $\mathcal{Q}_1$, whose expression follows from their definitions in (9.73):

$$\mathcal{P}_1(\{f_1\}) = -f_1 \qquad \mathcal{Q}_1(\{g_1\};\{f_1\}) = g_1 - f_1 \,. \tag{9.81}$$

Therefore, we get

$$\mathfrak{g}_{\mathrm{bk};1}[H,V](\xi) \;=\; \frac{H'(\xi)}{V''(\xi)} \qquad \text{and} \qquad \mathfrak{g}_{R;1}[H,V](x) \;=\; \frac{\mathfrak{b}_1(x) \,-\, \mathfrak{a}_1(x)\mathfrak{b}_0(x)}{u_1}\cdot\Big(\frac{H'}{V''}\Big)'(b_N)\,. \tag{9.82}$$

and we recognize in the prefactor of the second equation the function $\mathfrak{c}(x)$ of Definition 9.9. Finally, we remind that we take the remainder at order $k = 6$. The claim then follows upon recognising the constant $\aleph_0$ from Definition 9.4 in the computation of the leading term by Lemma 9.7. ■

A Several theorems and properties of use to the analysis

Theorem A.1 (Hunt, Muckenhoupt, Wheeden [57]) *The Hilbert transform, defined as an operator*

$$\mathcal{H} \,:\, L^2(\mathbb{R}, w(x)\mathrm{d}x) \;\to\; L^2(\mathbb{R}, w(x)\mathrm{d}x)$$

is bounded if and only if there exists a constant $C > 0$ such that, for any interval $I \subseteq \mathbb{R}$:

$$\Big\{\frac{1}{|I|}\int_I w(x)\mathrm{d}x\Big\}\cdot\Big\{\frac{1}{|I|}\int_I \frac{\mathrm{d}x}{w(x)}\Big\} \;<\; C \tag{A.1}$$

In particular, the operators "upper/lower boundary values" $C_{\pm} \;:\; \mathcal{F}[H_s(\mathbb{R})] \;\to\; \mathcal{F}[H_s(\mathbb{R})]$ are bounded if and only if $|s| < 1/2$.

A less refined version of this theorem takes the form :

Proposition A.2 *For any $\gamma > 0$, the shifted Cauchy operators $C_\gamma : f \mapsto C_\gamma[f]$ with $C_\gamma[f](\lambda) = C[f](\lambda + \mathrm{i}\gamma)$ are continuous on $\mathcal{F}[H_s(\mathbb{R})]$ with $|s| < 1/2$.*

Theorem A.3 (Calderon [22]) *Let Σ be a non-self intersecting Lipschitz curve in $\mathbb{C}$ and C_Σ the Cauchy transform on $L^2(\Sigma, \mathrm{d}s)$:*

$$\forall f \in L^2(\Sigma, \mathrm{d}s) \qquad C_\Sigma[f](z) \;=\; \int_\Sigma \frac{f(s)}{s-z}\cdot\frac{\mathrm{d}s}{2\mathrm{i}\pi} \in \mathcal{O}(\mathbb{C}\setminus\Sigma)\,. \tag{A.2}$$

For any $f \in L^2(\Sigma, \mathrm{d}s)$, $C_\Sigma[f]$ admits $L^2(\Sigma, \mathrm{d}s)$ $\pm$ boundary values $C_{\Sigma;\pm}[f]$. The operators $C_{\Sigma;\pm}[f]$ are continuous operators on $L^2(\Sigma, \mathrm{d}s)$ which, furthermore, satisfy $C_{\Sigma;+} - C_{\Sigma;-} = \mathrm{id}$.

Theorem A.4 (Paley, Wiener [74]) *Let $u \in L^2(\mathbb{R}^{\pm})$. Then $\mathcal{F}[u]$ is the $L^2(\mathbb{R})$ boundary value on $\mathbb{R}$ of a function $\widehat{u}$ that is holomorphic on $\mathbb{H}^{\pm}$, and there exists a constant $C > 0$ such that:*

$$\forall \mu > 0, \qquad \int_{\mathbb{R}} \big|\widehat{u}(\lambda \pm \mathrm{i}\mu)\big|^2\cdot\mathrm{d}\lambda \;<\; C \tag{A.3}$$

Reciprocally, every holomorphic function on $\widehat{u}$ on $\mathbb{H}^{\pm}$ that satisfies the bounds (A.3) *and admits $L^2(\mathbb{R})$ $\pm$ boundary values $\widehat{u}_{\pm}$ on $\mathbb{R}$, is the Fourier transform of a function $u \in L^2(\mathbb{R}^{\pm})$, viz. $\widehat{u}(z) = \mathcal{F}[u](z)$, $z \in \mathbb{H}^{\pm}$.*

B Proof of Theorem 2.1

We denote by $\mathfrak{p}_N$ the rescaled probability density on $\mathbb{R}^N$ associated with $\mathfrak{z}_N$, namely

$$\mathfrak{p}_N(\boldsymbol{\lambda}) \; = \; \frac{N^{\alpha_q N}}{\mathfrak{z}_N} \prod_{a<b}^{N} \Big\{ \sinh\big[\pi\omega_1 N^{\alpha_q}(\lambda_a - \lambda_b)\big] \sinh\big[\pi\omega_2 N^{\alpha_q}(\lambda_a - \lambda_b)\big] \Big\}^{\beta} \cdot \prod_{a=1}^{N} \mathrm{e}^{-W(N^{\alpha_q}\lambda_a)} \qquad \text{with} \quad \alpha_q \; = \; \frac{1}{q-1} \, .$$

To obtain the above probability density, we have rescaled in the variables in (1.9) as $y_a = N^{\alpha_q}\lambda_a$ with the value of α_q guided by the heuristic arguments that followed the statement of Theorem 2.1. We shall denote by $\mathcal{P}_N$ the probability measure on $\mathcal{M}^1(\mathbb{R})$ induced by $\mathfrak{p}_N$, *viz.* the measurable sets in $\mathcal{M}^1(\mathbb{R})$ are generated by the Borel σ-algebra for the weak topology, and for any open subset in $\mathcal{M}^1(\mathbb{R})$, we have:

$$\mathcal{P}_N[O] \; = \int\limits_{\{L_N^{(\boldsymbol{\lambda})} \in O\}} \mathfrak{p}_N(\boldsymbol{\lambda}) \, \mathrm{d}^N\boldsymbol{\lambda} \, . \tag{B.1}$$

The strategy of the proof consists in proving that $\mathcal{P}_N$ is exponentially tight and then establishing a weak large deviation principle, namely upper and lower bounding $\mathcal{P}_N$ on balls of shrinking radius this for balls relatively to the bounded Lipschitz topology, see *e.g.* [4].

B.1 Exponential tightness

Lemma B.1 *The sequence of measures $\mathcal{P}_N$ is exponentially tight, i.e.:*

$$\limsup_{L\to+\infty} \limsup_{N\to\infty} N^{-(2+\alpha_q)} \, \ln \mathcal{P}_N[K_L^c] = -\infty \, . \tag{B.2}$$

where $K_L = \{\mu \in \mathcal{M}^1(\mathbb{R}) \; : \; \int_{\mathbb{R}} |x|^q \, \mathrm{d}\mu(x) \leq L\}$.

Proof — By the monotone convergence theorem,

$$\int\limits_{\mathbb{R}} |x|^q \; \mathrm{d}\mu(x) = \sup_{M\in\mathbb{N}} \int\limits_{\mathbb{R}} \min(|x|^q \, , M) \, \mathrm{d}\mu(x). \tag{B.3}$$

The left-hand side is lower semi-continuous as a supremum of a continuous family of functionals on $\mathcal{M}_1(\mathbb{R})$. Thus, K_L is closed as a level set of a lower semi-continuous function. For any $\mu \in K_L$, we have by Chebyshev inequality:

$$\mu[[-M\, ; M]^c] \; \leq \; \frac{1}{M^q} \int\limits_{[-M\, ;M]^c} |x|^q \, \mathrm{d}\mu(x) \leq \frac{L}{M^q} \, . \tag{B.4}$$

As a consequence,

$$K_L \subseteq \bigcap_{M\in\mathbb{N}} \Big\{ \mu \in \mathcal{M}^1(\mathbb{R}) \; : \; \mu[[-M\, ; M]^c] \leq \frac{L}{M^q} \Big\} \, . \tag{B.5}$$

The right-hand side is uniformly tight, by construction and is closed as an intersection of level sets of lower semi-continuous functions on $\mathcal{M}^1(\mathbb{R})$. Thence by Prokhorov theorem, it is compact. As K_L is closed, it must be as well compact.

We now estimate $\mathcal{P}_N[K_L^c]$. We start by a rough estimate for the partition function. It follows by Jensen inequality applied to the probability measure of $\mathbb{R}^N$

$$\prod_{a=1}^{N} \frac{e^{-W(\lambda_a)} d\lambda_a}{\int_{\mathbb{R}} e^{-W(\lambda)} d\lambda}, \tag{B.6}$$

that

$$\begin{aligned} \ln\left[\mathfrak{z}_N[W]\right] &\geq N \ln\Big[\int e^{-W(\lambda)} d\lambda\Big] + \int_{\mathbb{R}^N} \sum_{a<b} \beta \ln\left\{\sinh\left[\pi\omega_1(\lambda_a-\lambda_b)\right]\sinh\left[\pi\omega_2(\lambda_a-\lambda_b)\right]\right\} \prod_{a=1}^{N} \frac{e^{-W(\lambda_a)} d\lambda_a}{\int e^{-W(\lambda)} d\lambda} \\ &\geq N \ln\Big[\int e^{-W(\lambda)} d\lambda\Big] + \frac{\beta N(N-1)}{2} \int_{\mathbb{R}^2} \ln\left\{\sinh\left[\pi\omega_1(\lambda_1-\lambda_2)\right]\sinh\left[\pi\omega_2(\lambda_1-\lambda_2)\right]\right\} \cdot \frac{e^{-W(\lambda_1)-W(\lambda_2)} d\lambda_1 d\lambda_2}{\Big(\int e^{-W(\lambda)} d\lambda\Big)^2} \end{aligned} \tag{B.7}$$

As a consequence, $\mathfrak{z}_N \geq e^{-N^2\kappa}$ for some $\kappa \in \mathbb{R}$. It now remains to estimate the integral arising from the integration over K_L^c. Using that $|\sinh(\lambda)| \leq e^{|\lambda|}$ we get:

$$\begin{aligned} \prod_{a<b}^{N} \Big\{\sinh\left[\pi\omega_1 N^{\alpha_q}(\lambda_a-\lambda_b)\right]\sinh\left[\pi\omega_2 N^{\alpha_q}(\lambda_a-\lambda_b)\right]\Big\}^{\beta} &\leq \prod_{a<b}^{N} \exp\Big\{\pi\beta(\omega_1+\omega_2)N^{\alpha_q}|\lambda_a-\lambda_b|\Big\} \\ \leq \prod_{a<b}^{N} \exp\Big\{\pi\beta(\omega_1+\omega_2)N^{\alpha_q}(|\lambda_a|+|\lambda_b|)\Big\} &\leq \prod_{a=1}^{N} \exp\Big\{\pi\beta(\omega_1+\omega_2)N^{\alpha_q+1}|\lambda_a|\Big\}. \end{aligned} \tag{B.8}$$

Hence,

$$\mathcal{P}_N[K_L^c] \leq e^{\kappa N^2} N^{\alpha_q N} \int_{\{L_N^{(\lambda)} \in K_L^c\}} \prod_{a=1}^{N} \exp\Big\{\pi\beta(\omega_1+\omega_2)N^{\alpha_q+1}|\lambda_a| - W(N^{\alpha_q}\lambda_a)\Big\} \cdot d^N\boldsymbol{\lambda} \tag{B.9}$$

Since $|\xi|^{1-q} \underset{|\xi|\to+\infty}{\longrightarrow} 0$ there exists a constant $C \in \mathbb{R}$ such that

$$\forall \xi \in \mathbb{R}, \qquad \pi\beta(\omega_1+\omega_2)|\xi| \leq \frac{c_q\,|\xi|^q}{2} + C. \tag{B.10}$$

Likewise it follows from (2.2) that given any $\epsilon > 0$ there exists $\tau_\epsilon \in \mathbb{R}^+$ such that

$$\forall \xi \in \mathbb{R}, \qquad -c_q(1+\epsilon)\,|\xi|^q - \tau_\epsilon \leq -W(\xi) \leq -c_q(1-\epsilon)\,|\xi|^q + \tau_\epsilon. \tag{B.11}$$

In the following, ϵ will be taken small. Taking into account that $q\alpha_q = \alpha_q + 1$, (B.10) and the upper bound of (B.11) lead to:

$$\begin{aligned}
\mathcal{P}_N[K_L^c] &\leq \mathrm{e}^{\kappa N^2} N^{\alpha_q N} \int\limits_{\{L_N^{(\lambda)} \in K_L^c\}} \prod_{a=1}^{N} \exp\Big\{ N^{\alpha_q+1} C + \frac{c_q}{2} N^{\alpha_q+1} |\lambda_a|^q + \tau_\epsilon - c_q(1-\epsilon) N^{q\alpha_q} |\lambda_a|^q \Big\} \cdot \mathrm{d}^N \boldsymbol{\lambda} \\
&\leq N^{\alpha_q N} \mathrm{e}^{\kappa N^2 + C N^{2+\alpha_q} + \tau_\epsilon N} \int\limits_{\{L_N^{(\lambda)} \in K_L^c\}} \Big(\prod_{a=1}^{N} \mathrm{e}^{-\epsilon c_q N^{\alpha_q+1} |\lambda_a|^q} \Big) \exp\Big\{ - \frac{c_q(1-4\epsilon)}{2} N^{2+\alpha_q} \int\limits_{\mathbb{R}} |x|^q \, \mathrm{d}L_N^{(\lambda)}(x) \Big\} \cdot \mathrm{d}^N \boldsymbol{\lambda} \\
&\leq N^{\alpha_q N} \mathrm{e}^{C' N^{2+\alpha_q} + \tau_\epsilon N - [c_q(1-4\epsilon)/2] L N^{2+\alpha_q}} \Big(\int\limits_{\mathbb{R}} \mathrm{e}^{-\epsilon c_q |\lambda_a|^q} \mathrm{d}\lambda \Big)^N
\end{aligned} \tag{B.12}$$

for some constant $C' > C$ and N large enough. As a consequence,

$$\limsup_{N \to +\infty} N^{-(2+\alpha_q)} \ln \mathcal{P}_N[K_L^c] \leq C - L c_q (1-4\epsilon)/2 \,,$$

and this upper bound goes to $-\infty$ when $L \to +\infty$. ∎

B.2 Lower bound

In the following we focus on the renormalised measure on $\mathcal{M}_1(\mathbb{R})$ defined as $\overline{\mathcal{P}}_N = \mathfrak{z}_N[W] \cdot \mathcal{P}_N$. We will now derive a lower bound for the $\overline{\mathcal{P}}_N$ volume of small Vasershtein balls, in terms of the energy functional $\mathcal{E}_{(\mathrm{ply})}$ of (2.1), namely:

$$\mathcal{E}_{(\mathrm{ply})}[\mu] = \int E(\xi, \eta) \, \mathrm{d}\mu(\xi) \mathrm{d}\mu(\eta), \qquad E(\xi, \eta) = \frac{c_q}{2} (|\xi|^q + |\eta|^q) - \frac{\beta \pi (\omega_1 + \omega_2)}{2} \, |\xi - \eta| \tag{B.13}$$

Lemma B.2 *Let $B_\delta(\mu)$ be the ball in $\mathcal{M}^1(\mathbb{R})$ centred at μ and of radius δ in respect to D_V. Then, for any $\mu \in \mathcal{M}^1(\mathbb{R})$, it holds*

$$\liminf_{\delta \to 0} \liminf_{N \to \infty} N^{-(2+\alpha_q)} \ln \overline{\mathcal{P}}_N[B_\delta(\mu)] \geq -\mathcal{E}_{(\mathrm{ply})}[\mu] \tag{B.14}$$

Proof — Let $\mu \in \mathcal{M}^1(\mathbb{R})$ and $\delta > 0$. If $\int |x|^q \, \mathrm{d}\mu(x) = +\infty$, then $\mathcal{E}_{(\mathrm{ply})}[\mu] = +\infty$ and there is nothing to prove. Thus we may assume from the very beginning that $\int |x|^q \, \mathrm{d}\mu(x) < +\infty$. If $M > 0$ is large enough, we have $\mu([-M\,;M]) \neq 0$, and we can introduce:

$$\mu_M = \frac{\mathbf{1}_{[-M\,;M]} \cdot \mu}{\mu([-M\,;M])} \tag{B.15}$$

which is now a compactly supported measure. We will obtain the lower bound for $\overline{\mathcal{P}}_N[B_\delta(\mu)]$ by restricting to configurations close enough to the classical positions of μ_M, and only at the end, see how the estimate behaves when $M \to \infty$. For any given integer N, we define:

$$\forall a \in [\![1, N]\!], \qquad x_a^{N,M} = \inf \Big\{ x \in \mathbb{R} \;:\; \int\limits_{-\infty}^{x} \mathrm{d}\mu_M \geq \frac{a}{N+1} \Big\}. \tag{B.16}$$

When $N \to \infty$, $L_N^{(\boldsymbol{x}^{N,M})}$ approximates μ_M for the Vasershtein distance, so there exists N_δ such that, for any $N \geq N_\delta$, we have the inclusion:

$$\Omega_\delta := \Big\{ \boldsymbol{\lambda} \in \mathbb{R}^N \;\;:\;\; \forall a \in [\![1,N]\!],\; \big|\lambda_a - x_a^{N,M}\big| < \delta/2 \Big\} \;\subseteq\; \Big\{ \boldsymbol{\lambda} \in \mathbb{R}^N \;\;:\;\; D_V(\mu_M, L_N^{(\boldsymbol{\lambda})}) < \delta \Big\} \,. \tag{B.17}$$

Subsequently:

$$\overline{\mathcal{P}}_N[B_\delta(\mu_M)] \;\;\geq\;\; N^{N\alpha_q} \int_{\Omega_\delta} \prod_{a<b}^{N} \Big\{ \sinh\left[\pi\omega_1 N^{\alpha_q}(\lambda_a - \lambda_b)\right] \sinh\left[\pi\omega_2 N^{\alpha_q}\,|\lambda_a - \lambda_b|\,\right] \Big\}^{\beta} \prod_{a=1}^{N} \mathrm{e}^{-W(N^{\alpha_q}\lambda_a)} \cdot \mathrm{d}^N\boldsymbol{\lambda} \,. \tag{B.18}$$

It follows from the lower bound

$$|\sinh(x)| \geq \frac{\mathrm{e}^{|x|}}{2}\frac{|x|}{1+|x|} \tag{B.19}$$

from the lower bound for W in (B.11), and $q\alpha_q = \alpha_q + 1$, that:

$$\overline{\mathcal{P}}_N[B_\delta(\mu)] \geq \frac{e^{N(\alpha_q \ln N + \tau_\epsilon)}}{2^{\beta N(N-1)}} \int_{\Omega_\delta} \exp\Big\{ \pi\beta(\omega_1+\omega_2) N^{\alpha_q} \sum_{a<b}^{N} |\lambda_a - \lambda_b| - N^{\alpha_q+1} c_q(1+\epsilon) \sum_{a=1}^{N} |\lambda_a|^q \Big\} \prod_{a<b}^{N} \Big\{ g_N(\lambda_a - \lambda_b) \Big\}^{\beta} \cdot \mathrm{d}^N\boldsymbol{\lambda} \,. \tag{B.20}$$

where we have set

$$g_N(\lambda) = \frac{\pi\omega_1 N^{\alpha_q}\,|\lambda|}{1+\pi\omega_1 N^{\alpha_q}\,|\lambda|} \cdot \frac{\pi\omega_2 N^{\alpha_q}\,|\lambda|}{1+\pi\omega_2 N^{\alpha_q}\,|\lambda|} \tag{B.21}$$

Now, we would like to replace λ_a by $x_a^{N,M}$. Since the configurations $\boldsymbol{\lambda} \in \Omega_\delta$ satisfy $|x_a^{N,M} - \lambda_a| < \delta/2$, we have:

$$\sum_{a<b} |\lambda_a - \lambda_b| \;\;\geq\;\; -N(N-1)\frac{\delta}{2} \;+\; \sum_{a<b} (x_b^{N,M} - x_a^{N,M}) \tag{B.22}$$

Since $q > 1$, we also deduce from the mean value theorem:

$$|\lambda_a|^q \leq \big|x_a^{N,M}\big|^q + \frac{q\delta}{2}(|x_a^{N,M}| + \delta/2)^{q-1} \tag{B.23}$$

and thus

$$-(1+\epsilon)|\lambda_a|^q \;=\; -(1+\epsilon)\,\big|x_a^{N,M}\big|^q \;+\; \frac{\delta}{c_q}\,h_{\epsilon,\delta}(x_a^{N,M}) \qquad h_{\epsilon,\delta}(x) \;=\; \frac{qc_q}{2}(1+\epsilon)\cdot(\,|x| + \delta/2)^{q-1} \tag{B.24}$$

These inequalities yield the lower bound:

$$\overline{\mathcal{P}}_N[B_\delta(\mu)] \geq \exp\Big\{ C\,N^2 - N^{2+\alpha_q}\Big(\delta\Big\{ C' + \int h_{\epsilon,\delta}(\xi)\mathrm{d}L_N^{(\boldsymbol{x}^{N,M})}(\xi) \Big\} + \mathcal{E}_{(\mathrm{ply})}[L_N^{(\boldsymbol{x}^{N,M})}] + \epsilon c_q \int |\xi|^q\; \mathrm{d}L_N^{(\boldsymbol{x}_{N,M})}(\xi) \Big) \Big\} \cdot G_{N,\delta} \tag{B.25}$$

for some irrelevant, N and δ independent, constants $C, C' > 0$. Furthermore, the factor $G_{N,\delta}$ reads

$$G_{N,\delta} \;\;=\;\; \int_{\Omega_\delta^{\mathrm{ord}}} \prod_{a>b}^{N} \{g_N(\lambda_b - \lambda_a)\}^{\beta} \cdot \mathrm{d}^N\boldsymbol{\lambda} \tag{B.26}$$

in which $\Omega_\delta^{\rm ord} = \Omega_\delta \cap \{\boldsymbol{\lambda} \in \mathbb{R}^N \;:\; \lambda_1 < \cdots < \lambda_N\}$.

To find a lower bound for $G_{N,\delta}$, we can restrict further to configurations such that $u_a = \lambda_a - x_a^{N,M}$ increases with $a \in [\![1, N]\!]$, and satisfies $|u_1| < \delta/(2N)$ and $|u_{a+1} - u_a| \leq \delta/2N$ for any $a \in [\![1, N-1]\!]$. Using that $\xi \mapsto g_N(\xi)$ is increasing on $\mathbb{R}_+$, we have:

$$G_{N,\delta} \geq \int_{[-\delta/2,\delta/2]^N} \prod_{a=1}^{N-1} \{g_N(u_{a+1} - u_a)\}^{\beta(N-a)} \cdot \mathrm{d}^N u \;\geq \int_{[0,\delta/2N]^N} \prod_{a=2}^{N} \{g_N(v_a)\}^{\beta(N-a+1)} \cdot \mathrm{d}^N \boldsymbol{v} \tag{B.27}$$

Now, using an arithmetic-geometric upper bound for the denominator in $g_N(v)$, we can write:

$$\forall v \in [0, \delta/2N], \qquad g_N(v) \geq \frac{N^{\alpha_q+1}\pi\sqrt{\omega_1\omega_2}}{2\delta} \cdot |v|^2 \geq \widetilde{C} N^{\alpha_q+1} \cdot |v|^2 \tag{B.28}$$

for some irrelevant $C' > 0$ independent of δ provided that $\delta < 1$. So, we arrive to:

$$G_{N,\delta} \geq \frac{\delta}{2N} \cdot (\widetilde{C}\, N^{\alpha_q-1})^{\beta N(N-1)/2} \cdot \prod_{a=2}^{N-1} \frac{(\delta/2N)^{2\beta(N-a+1)}}{2\beta(N-a+1)+1} \geq e^{\widetilde{C}'\, N^2 \ln N} \tag{B.29}$$

for some $\widetilde{C}' > 0$ independent of δ. Hence, ultimately

$$\overline{\mathcal{P}}_N[B_\delta(\mu)] \geq e^{\widetilde{C}'' N^2 \ln N} \exp\left\{-N^{2+\alpha_q}\left[\delta\Big(C' + \int h_{\epsilon,\delta}(\xi)\, \mathrm{d}L_N^{(\boldsymbol{x}^{N,M})}(\xi)\Big) + \mathcal{E}_{(\rm ply)}[L_N^{(\boldsymbol{x}^{N,M})}] + \epsilon c_q \int |\xi|^q\, \mathrm{d}L_N^{(\boldsymbol{x}^{N,M})}(\xi)\right]\right\} \tag{B.30}$$

To establish the desired result (B.14), we only need to focus on the last exponential. If ϕ is a C^1 function of p real variables, we denote:

$$\phi^{[M]}(\xi) = \min\left[\phi(\xi)\,;\, ||\phi||_{L^\infty([-M\,;M]^p)}\right] \tag{B.31}$$

which has the advantage of being bounded and Lipschitz. Since μ_M is supported on $[-M\,;M]$, so must be the classical positions $x_a^{N,M}$, and we can apply the truncation to all the functions against which $L_N^{(\boldsymbol{x}^{N,M})}$ is integrated. In particular, we make appear the truncated functional:

$$\mathcal{E}_{(\rm ply)}^{[M]}[\mu] \;=\; \int E^{[M]}(\xi,\eta)\, \mathrm{d}\mu(\xi)\mathrm{d}\mu(\eta)\;. \tag{B.32}$$

The advantage is that now, all functions to be integrated are Lipschitz bounded. Since, $D_V(\mu_M, L_N^{(\boldsymbol{x}^{N,M})}) \to 0$ when $N \to \infty$, we get:

$$\liminf_{N\to\infty} \frac{\ln \overline{\mathcal{P}}_N[B_\delta(\mu)]}{N^{2+\alpha_q}} \;\geq\; -\delta\Big(C + \int h_{\epsilon,\delta}(\xi)\, \mathrm{d}\mu_M(\xi)\Big) - \mathcal{E}_{(\rm ply)}^{[M]}[\mu_M] - \epsilon c_q \int \{\max(|\xi|, M)\}^q\, \mathrm{d}\mu_M(\xi)\;. \tag{B.33}$$

The right-hand is an affine function of ϵ, and at this stage, we can send ϵ to 0:

$$\liminf_{N\to\infty} N^{-(2+\alpha_q)}\, \ln \overline{\mathcal{P}}_N[B_\delta(\mu_M)] \geq -\delta\Big(C + \int h_{0,\delta}(\xi)\, \mathrm{d}\mu_M(\xi)\Big) - \mathcal{E}_{(\rm ply)}[\mu_M]\;. \tag{B.34}$$

Now, for any fixed δ, there exists M_δ such that, for any $M \geq M_\delta$, $D_V(\mu, \mu_M) \leq \delta$, and consequently:

$$\liminf_{N\to\infty} N^{-(2+\alpha_q)} \ln \overline{\mathcal{P}}_N[B_{2\delta}(\mu)] \geq -\delta\Big(C + \int h_{0,\delta}(\xi)\,\mathrm{d}\mu_M(\xi)\Big) - \mathcal{E}_{(\mathrm{ply})}[\mu_M]\,. \tag{B.35}$$

We could replace $\mathcal{E}^{[M]}_{(\mathrm{ply})}$ by $\mathcal{E}_{(\mathrm{ply})}$ here because μ_M is supported on $[-M\,;M]$. Now, we can consider sending $M \to \infty$. Since we have the bound:

$$\forall \xi, \eta \in \mathbb{R}, \qquad E(\xi,\eta) \leq C'\,(1 + |\xi|^q + |\eta|^q\,), \qquad h_{0,\delta} \leq C'\,(1 + |\xi|^q\,) \tag{B.36}$$

and we assumed that $\int |\xi|^q\, \mathrm{d}\mu(\xi) < +\infty$, we get by dominated convergence:

$$\liminf_{N\to\infty} N^{-(2+\alpha_q)} \ln \overline{\mathcal{P}}_N[B_{2\delta}(\mu)] \geq -\delta\Big(C + \int h_{0,\delta}(\xi)\,\mathrm{d}\mu(\xi)\Big) - \mathcal{E}_{(\mathrm{ply})}[\mu]\,. \tag{B.37}$$

Last but not least, sending $\delta \to 0$, the first term disappears and we find:

$$\liminf_{\delta\to 0}\liminf_{N\to\infty} N^{-(2+\alpha_q)} \ln \overline{\mathcal{P}}_N[B_{2\delta}(\mu)] \geq -\mathcal{E}_{(\mathrm{ply})}[\mu]\,. \tag{B.38}$$

■

B.3 Upper bound

In this paragraph, we complete our estimate by an upper bound on the probability of small Vasershtein balls:

Lemma B.3

$$\limsup_{\delta\to 0}\limsup_{N\to\infty} N^{-(2+\alpha_q)} \ln \overline{\mathcal{P}}_N[B_\delta(\mu)] \leq -\mathcal{E}_{(\mathrm{ply})}[\mu] \tag{B.39}$$

Proof — Let $\mu \in \mathcal{M}^1(\mathbb{R})$. In order to establish an upper bound, we use that $|\sinh(x)| \leq \mathrm{e}^{|x|}$ and the upper bound in (B.11) for the potential W. This makes appear again the function $\mathcal{E}_{(\mathrm{ply})}$ of (B.13):

$$\overline{\mathcal{P}}_N[B_\delta(\mu)] \leq e^{N(\alpha_q \ln N + \tau_\epsilon)} \int_{L_N^{(\lambda)} \in B_\delta(\mu)} \exp\Big\{-N^{2+\alpha_q}\Big(-2c_q\epsilon\int|\xi|^q\,\mathrm{d}L_N^{(\lambda)} + \mathcal{E}_{(\mathrm{ply})}[L_N^{(\lambda)}]\Big)\Big\} \prod_{a=1}^{N} e^{-N^{\alpha q+1}\,c_q\epsilon|\lambda_a|^q}\cdot\mathrm{d}^N\boldsymbol{\lambda} \tag{B.40}$$

where we have put aside one exponential decaying with rate ϵ to ensure later convergence of the integral. If $M > 0$, let us define the truncated functional:

$$\mathcal{E}^{\{M,\epsilon\}}_{(\mathrm{ply})}[\mu] = \int E^{\{M,\epsilon\}}(\xi,\eta)\,\mathrm{d}\mu(\xi)\mathrm{d}\mu(\eta), \qquad E^{\{M,\epsilon\}} = \min\big[M\,;\,E(\xi,\eta) - c_q\epsilon\,(\,|\xi|^q + |\eta|^q\,)\big]\,. \tag{B.41}$$

Since $E^{\{M,\epsilon\}}$ is a Lipschitz function bounded by M, with Lipschitz constant bounded by $O(M^{1-1/q})$, we deduce the following bounds when the event $L_N^{(\lambda)} \in B_\delta(\mu)$ is realised:

$$\big|\mathcal{E}^{\{M,\epsilon\}}_{(\mathrm{ply})}[L_N^{(\lambda)}] - \mathcal{E}^{\{M,\epsilon\}}_{(\mathrm{ply})}[\mu]\big| \leq C\,\delta\,M\,, \tag{B.42}$$

for some constant $C > 0$ independent of N, δ and ϵ. Therefore:

$$\overline{\mathcal{P}}_N[B_\delta(\mu)] \leq \exp\left\{C' N \ln N + N^{\alpha_q+2}\left(CM\cdot\delta - \mathcal{E}^{\{M,\epsilon\}}_{(\mathrm{ply})}[\mu]\right)\right\}\cdot\left(\int_{\mathbb{R}} \mathrm{e}^{-c_q\epsilon\,|\lambda|^q}\,\mathrm{d}\lambda\right)^N \tag{B.43}$$

It follows that:

$$\limsup_{N\to\infty} N^{-(2+\alpha_q)}\,\ln\widetilde{\mathcal{P}}_N[B_\delta(\mu)] \leq CM\cdot\delta - \mathcal{E}^{\{M,\epsilon\}}_{(\mathrm{ply})}[\mu] \tag{B.44}$$

We observe that $-E^{\{M,\epsilon\}}$ is an increasing function of ϵ. We can now let $\epsilon\to 0$ by applying the monotone convergence theorem:

$$\limsup_{N\to\infty} N^{-(2+\alpha_q)}\,\ln\widetilde{\mathcal{P}}_N[B_\delta(\mu)] \leq C\,M\cdot\delta - \mathcal{E}^{\{M,0\}}_{(\mathrm{ply})}[\mu]\ . \tag{B.45}$$

Then, sending $\delta\to 0$ erases the first term, and finally letting $M\to\infty$ using again monotone convergence:

$$\limsup_{\delta\to 0}\limsup_{N\to\infty} N^{-(2+\alpha_q)}\,\ln\widetilde{\mathcal{P}}_N[B_\delta(\mu)] \leq -\mathcal{E}_{(\mathrm{ply})}[\mu]\ , \tag{B.46}$$

Notice that monotone convergence proves this last inequality even in the case where $\mathcal{E}_{(\mathrm{ply})}[\mu] = +\infty$. ■

B.4 Partition function and equilibrium measure

By applying the reasoning described in [37], to the lower bounds (Lemma B.2) and upper bounds (Lemma B.3), along with the property of exponential tightness (Lemma B.1), we deduce that $\mathcal{E}_{(\mathrm{ply})}$ is a good rate function for large deviations, *i.e.*

$$\begin{array}{ll}\text{for any open set } \Omega\subseteq\mathcal{M}^1(\mathbb{R}), & \liminf\limits_{N\to+\infty} N^{-(2+\alpha_q)}\,\ln\overline{\mathcal{P}}_N[\Omega] \;\geq\; -\inf\limits_{\mu\in\Omega}\mathcal{E}_{(\mathrm{ply})}[\mu]\ ,\\ \text{for any closed set } F\subseteq\mathcal{M}^1(\mathbb{R}) & \limsup\limits_{N\to+\infty} N^{-(2+\alpha_q)}\,\ln\overline{\mathcal{P}}_N[F] \;\leq\; -\inf\limits_{\mu\in F}\mathcal{E}_{(\mathrm{ply})}[\mu]\ .\end{array} \tag{B.47}$$

These two estimates, taken for $\Omega = F = \mathcal{M}^1(\mathbb{R})$, lead to

$$\lim_{N\to\infty} N^{-(2+\alpha_q)}\,\ln\mathfrak{z}_N = -\inf_{\mu\in\mathcal{M}^1(\mathbb{R})}\mathcal{E}_{(\mathrm{ply})}[\mu]\ . \tag{B.48}$$

The proof of the statements relative to the existence, uniqueness and characterisation of the minimiser of $\mathcal{E}_{(\mathrm{ply})}$ is identical to those for the usual logarithmic energy [77] – and even simpler since there is no log singularity here. The minimiser is denoted $\mu^{(\mathrm{ply})}_{\mathrm{eq}}$ and it is characterised by the existence of a constant $C^{(\mathrm{ply})}_{\mathrm{eq}}$ such that:

$$c_q\,|\xi|^q - \pi\beta(\omega_1+\omega_2)\int|\xi-\eta|\,\mathrm{d}\mu^{(\mathrm{ply})}_{\mathrm{eq}}(\eta) \;=\; C^{(\mathrm{ply})}_{\mathrm{eq}} \quad \text{for } \xi\,,\quad \mu^{(\mathrm{ply})}_{\mathrm{eq}} \text{ everywhere} \tag{B.49}$$

$$c_q\,|\xi|^q - \pi\beta(\omega_1+\omega_2)\int|\xi-\eta|\,\mathrm{d}\mu^{(\mathrm{ply})}_{\mathrm{eq}}(\eta) \;\geq\; C^{(\mathrm{ply})}_{\mathrm{eq}} \quad \text{for any } \xi\in\mathbb{R} \tag{B.50}$$

The construction of the solution of this regular integral equation is left as an exercise to the reader. We only give the final result in the announcement of Theorem 2.3. Actually, the fact that (2.4) is a solution can be checked directly by integration by parts, and we can conclude by uniqueness.

■

C Properties of the N-dependent equilibrium measure

We give here elements for the proof of Theorem 2.4, which establishes the main properties of the minimiser of:

$$\mathcal{E}_N[\mu] \;=\; \frac{1}{2}\int\left(V(\xi)\;+\;V(\eta)\;-\;\frac{\beta}{N^{\alpha}}\ln\left\{\prod_{p=1}^{2}\sinh\left[\pi N^{\alpha}\omega_p(\xi-\eta)\right]\right\}\right)\mathrm{d}\mu(\xi)\mathrm{d}\mu(\eta)\,. \tag{C.1}$$

among probability measure μ on $\mathbb{R}$, with N considered as a fixed parameter. As for any probability measures μ, ν and $\alpha \in [0,1]$,

$$\mathcal{E}_N[\alpha\mu+(1-\alpha)\nu]-\alpha\mathcal{E}_N[\mu]-(1-\alpha)\mathcal{E}_N[\nu]=-\alpha(1-\alpha)\mathfrak{D}^2[\mu-\nu,\mu-\nu]\,,$$

$\mathcal{E}_N$ is strictly convex, and the standard arguments of potential theory [69, 77] show that it admits a unique minimiser, denoted $\mu_{\mathrm{eq}}^{(N)}$. More precisely, one can prove that $\mu_{\mathrm{eq}}^{(N)}$ has a continuous density $\rho_{\mathrm{eq}}^{(N)}$ (as soon as V is C^2) and is supported on a compact of $\mathbb{R}$ (since the potential here is confining for any given value of N) a priori depending on N, see *e.g.* [17, Lemma 2.4]. What we really need to justify in our case is that:

(0) the support of $\mu_{\mathrm{eq}}^{(N)}$ is contained in a compact independent of N ;

(i) $\mu_{\mathrm{eq}}^{(N)}$ is supported on a segment ;

(ii) $\rho_{\mathrm{eq}}^{(N)}$ does not vanish on the interior of this segment and vanishes like a square root at the edges.

As a preliminary, we recall that the characterisation of the equilibrium measure is obtained by writing that $\mathcal{E}_N[\mu_{\mathrm{eq}}^{(N)}+\epsilon\nu] \geq \mathcal{E}_N[\mu_{\mathrm{eq}}^{(N)}]$ for all $\epsilon>0$, all measures ν with zero mass and such that $\mu_{\mathrm{eq}}^{(N)}+\epsilon\nu$ is non-negative. The resulting condition can be formulated in terms of the effective potential introduced in (3.3):

$$V_{N;\mathrm{eff}}(\xi) \;=\; U_{N;\mathrm{eff}}(\xi)-\inf_{\mathbb{R}} U_{N;\mathrm{eff}}, \qquad U_N(\xi)=V(\xi)\;-\;2\fint s_N(\xi-\eta)\,\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(\eta) \tag{C.2}$$

with the two-point interaction kernel:

$$s_N(\xi) \;=\; \frac{\beta}{2N^{\alpha}}\ln\Big[\sinh\left(\pi\omega_1 N^{\alpha}\xi\right)\,\sinh\left(\pi\omega_2 N^{\alpha}\xi\right)\Big]\,. \tag{C.3}$$

The equilibrium measure is characterised by the condition:

$$V_{N;\mathrm{eff}}(\xi)\geq 0\,, \qquad \text{with equality } \mu_{\mathrm{eq}}^{(N)} \text{ almost everywhere} \tag{C.4}$$

Proof — of (0). Let $m_N>0$ such that the support of $\mu_{\mathrm{eq}}^{(N)}$ is contained in $[-m_N, m_N]$. For $|\xi|>2m_N$, we have an easy lower bound:

$$\Big|\int s_N(\xi-\eta)\,\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(\eta)\Big| \geq \frac{\beta}{2N^{\alpha}}\ln\Big[\sinh\left(\pi\omega_1 N^{\alpha}m_N\right)\,\sinh\left(\pi\omega_2 N^{\alpha}m_N\right)\Big]\geq\frac{\beta\pi(\omega_1+\omega_2)}{2}\,m_N+\mathrm{O}(1) \tag{C.5}$$

where the remainder is bounded uniformly when $N\to\infty$ and $m_N\to\infty$. By the growth assumption on the potential, there exists constant $C, C'>\epsilon>0$ such that

$$V(\xi)\geq C|\xi|^{1+\epsilon}+C' \tag{C.6}$$

Therefore, we can choose $m:=2m_N$ large enough and independent of N such that $V_{N;\mathrm{eff}}(\xi)>0$ for any $|\xi|>m$. This guarantees that the support of $\mu_{\mathrm{eq}}^{(N)}$ is included in the compact $[-m\,;m]$ for any N.

Proof — of (*i*).

We observe that $-s_N$ is strictly convex:

$$s_N''(\xi) \;=\; -\frac{\beta N^a}{2}\sum_{p=1}^{2}\frac{(\pi\omega_p)^2}{(\sinh\pi\omega_p\xi)^2} \;<\; 0\;. \tag{C.7}$$

Since V is assumed strictly convex and $\mu_{\mathrm{eq}}^{(N)}$ is a positive measure, it implies that $V_{N;\mathrm{eff}}$ is strictly convex. Therefore, the locus where it reaches its minimum must be a segment. So, there exists $a_N < b_N$ such that $[a_N\,;b_N]$ is the support of $\mu_{\mathrm{eq}}^{(N)}$. This strict convexity also ensures that

$$V_{N;\mathrm{eff}}'(\xi) \;>\; 0 \qquad \text{for any} \qquad \xi > b_N\,, \quad V_{N;\mathrm{eff}}'(\xi) \;<\; 0 \qquad \text{for any} \qquad \xi < a_N\;. \tag{C.8}$$

Proof — of (*ii*).

This piece of information is enough so as to build the representation:

$$\rho_{\mathrm{eq}}^{(N)}(\xi) = \mathcal{W}_N[V']\cdot\mathbf{1}_{[a_N\,;b_N]}(\xi) \tag{C.9}$$

for the equilibrium measure. Indeed, we constructed $\mathcal{W}_N[H]$ in Section 5.4 so that it provides the unique solution to:

$$\forall\xi\in]a_N\,;b_N[,\qquad \fint_{a_N}^{b_N} s_N[N^a(\xi-\eta)]\,\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(\eta) \;=\; V'(\xi) \tag{C.10}$$

which extends continuously on $[a_N\,;b_N]$, and this was only possible when $\mathcal{X}_N[V']=0$ in terms of the linear form introduced in Definition 3.18. Since the equilibrium measure exists, this imposes the constraint:

$$\mathcal{X}_N[V'] = 0\;. \tag{C.11}$$

Besides, since the total mass of (C.9) must be 1, we must also have:

$$\int_{a_N}^{b_N}\mathcal{W}_N[V'] = 1\;. \tag{C.12}$$

At this stage, we can use Corollary 7.10, which shows that (C.11)-(C.12) determine uniquely the large-N asymptotic expansion of a_N and b_N, in particular there exists $a<b$ such that $(a_N,b_N)\to(a,b)$ with rate of convergence $N^{-\alpha}$. Besides, the leading behaviour at $N\to\infty$ of $\mathcal{W}_N$ is described by Proposition 6.4 and 6.6. It follows from the reasonings outlined in the proof of Proposition 8.2 that

$$\rho_{\mathrm{eq}}^{(N)}(\xi) \;=\; \mathcal{W}_N[V'](\xi) = \begin{cases} \frac{V''(\xi)}{2\pi\beta(\omega_1+\omega_2)} + \mathrm{O}(N^{-\alpha}) & \xi\in\Big[a_N+\dfrac{(\ln N)^2}{N^\alpha}\,;b_N-\dfrac{(\ln N)^2}{N^\alpha}\Big] \\ V''(b_N)\,\mathfrak{a}_0(N^\alpha(b_N-\xi)) \;+\; \mathrm{O}\Big(\dfrac{(\ln N)^3}{N^\alpha}\sqrt{N^\alpha(b_N-\xi)}\Big) & \xi\in[b_N-(\ln N)^2\cdot N^{-\alpha}\,;b_N] \\ V''(a_N)\,\mathfrak{a}_0(N^\alpha(\xi-a_N)) \;+\; \mathrm{O}\Big(\dfrac{(\ln N)^3}{N^\alpha}\sqrt{N^\alpha(\xi-a_N)}\Big) & \xi\in[a_N\,;a_N+(\ln N)^2\cdot N^{-\alpha}] \end{cases} \tag{C.13}$$

Therefore, for N large enough, $\rho_{\mathrm{eq}}^{(N)}(\xi)>0$ on $[a_N\,;b_N]$. The vanishing like a square root at the edges then follows from he properties of the $\mathfrak{a}$'s established in Lemma 6.10. In fact, one even has

$$\lim_{\xi \to b_N^-} \frac{\rho_{\mathrm{eq}}^{(N)}(\xi)}{\sqrt{b_N - \xi}} \;=\; N^{\alpha/2}\Big(V''(b_N) \cdot \lim_{x \to 0} x^{-1/2}\mathfrak{a}_0(x) \;+\; \mathrm{O}(N^{-\alpha})\Big\} \;=\; \frac{N^{\alpha/2}V''(b)}{\pi\beta\sqrt{\pi(\omega_1+\omega_2)}} + \mathrm{O}(N^{-\alpha/2}) \,. \tag{C.14}$$

This concludes the proof. ■

D The Gaussian potential

In this appendix we focus on the case of a Gaussian potential and establish two results. On the one hand, we establish in Lemma D.1 that, for N large enough, there exists a unique sequence of Gaussian potential $V_{G;N} = g_N\lambda^2 + t_N\lambda$ such that their associated equilibrium measure has support $\sigma_{\mathrm{eq}}^{(N)} = [a_N\,;b_N]$. On the other hand we show, in Proposition D.2, that the partition function associated with any Gaussian potential can be explicitly evaluated, and thus is amenable to a direct asymptotic analysis when $N \to \infty$.

Lemma D.1 *There exists a unique sequence of Gaussian potentials*

$$V_{G;N} \;=\; g_N\lambda^2 + t_N\lambda \tag{D.1}$$

such that their associated equilibrium measure has support $\sigma_{\mathrm{eq}}^{(N)} = [a_N\,;b_N]$. The coefficients g_N, t_N take the form

$$g_N \;=\; \pi\beta(\omega_1+\omega_2)\Big\{b_N - a_N \;+\; N^{-\alpha}\sum_{p=1}^{2}\frac{1}{\pi\omega_p}\ln\Big(\frac{\omega_1\omega_2}{\omega_p(\omega_1+\omega_2)}\Big)\Big\}^{-1} \;+\; \mathrm{O}(N^{-\infty}) \tag{D.2}$$

and

$$t_N \;=\; -(a_N+b_N)g_N \;+\; \mathrm{O}(N^{-\infty}) \,. \tag{D.3}$$

Proof — Let $V_G(\lambda) = g\lambda^2 + t\lambda$ be any Gaussian potential. Since it strictly convex, all previous results apply. Suppose that V_G gives rise to an equilibrium measure supported on $\sigma_{\mathrm{eq}}^{(N)} = [a_N\,;b_N]$. This means that the potential V_G has to satisfy the system of two equations that are linear in V_G':

$$\int_{a_N}^{b_N}\mathcal{W}_N[V_G'](\xi)\,\mathrm{d}\xi \;=\; 1 \qquad \text{and} \qquad \int_{\mathbb{R}+\mathrm{i}\epsilon'}\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\chi_{11}(\mu)\int_{a_N}^{b_N}V_G'(\eta)\mathrm{e}^{iN^{\alpha}\mu(\eta-b_N)}\,\mathrm{d}\eta \;=\; 0 \,. \tag{D.4}$$

It follows from the multi-linearity in (g,t) of V_G and from the evaluation of single integrals carried out in Lemma 7.2 and Proposition 7.6 that there exist two linear forms L_1, L_2 of (g,t) whose norm is a $\mathrm{O}(N^{-\infty})$ and such that

$$1 \;=\; \frac{g}{\pi\beta(\omega_1+\omega_2)}\Big\{(b_N - a_N) + \frac{1}{N^{\alpha}}\cdot\sum_{p=1}^{2}\frac{1}{\omega_p\pi}\ln\Big(\frac{\omega_1\omega_2}{\omega_p(\omega_1+\omega_2)}\Big)\Big\} \;+\; L_1(g,t) \tag{D.5}$$

where we have used that

$$\int_0^{+\infty}\mathfrak{b}_0(x)\,\mathrm{d}x \;=\; \frac{1}{2\pi\beta(\omega_1+\omega_2)}\cdot\sum_{p=1}^{2}\frac{1}{\omega_p\pi}\ln\Big(\frac{\omega_1\omega_2}{\omega_p(\omega_1+\omega_2)}\Big) \tag{D.6}$$

a formula that can be established with the help of (6.72) and (6.92). One also obtains that

$$0 \;=\; \frac{2}{N^{\alpha}\sqrt{\omega_1+\omega_2}}\Big(g(b_N+a_N)+t\Big) \;+\; L_2(g,t)\;. \tag{D.7}$$

In virtue of the unique solvability of perturbations of linear solvable systems, the existence and uniqueness of the potential $V_{G;N}$ follows. ■

Proposition D.2 *The partition function $Z_N[V_G]$ at $\beta=1$ associated with the Gaussian potential $V_G(\lambda)=g\lambda^2+t\lambda$ can be explicitly computed as*

$$Z_N[V_G]_{|\beta=1} \;=\; \frac{N!}{2^{N(N-1)}}\left(\frac{\pi}{gN^{1+\alpha}}\right)^{N/2}\exp\left\{\frac{N^{2+\alpha}t^2}{4g}+\frac{\pi^2(\omega_1+\omega_2)^2}{12g}N^{\alpha}(N^2-1)\right\}\prod_{j=1}^{N}(1-\mathrm{e}^{-\frac{2jN^{\alpha}}{gN}\pi^2\omega_1\omega_2})^{N-j}\;. \tag{D.8}$$

Proof — We can get rid of the linear term in the potential by a translation of the integration variables. Then

$$Z_N[V_G]_{|\beta=1} \;=\; \exp\left\{\frac{N^{2+\alpha}t^2}{4g}\right\}\cdot Z_N[\widetilde{V}_G]_{|\beta=1} \quad \text{where} \qquad \widetilde{V}_G(\lambda)=g\lambda^2\;. \tag{D.9}$$

Further, the products over hyperbolic sinh's can be recast as two Van-der-Monde determinants

$$\prod_{a<b}^{N}\Big\{\sinh\left[\pi\omega_1N^{\alpha}(\lambda_a-\lambda_b)\right]\sinh\left[\pi\omega_2N^{\alpha}(\lambda_a-\lambda_b)\right]\Big\} \;=\; \prod_{a=1}^{N}\left\{\frac{\mathrm{e}^{-\pi(\omega_1+\omega_2)N^{\alpha}(N-1)\lambda_a}}{2^{N-1}}\right\}\cdot\prod_{p=1}^{2}\mathrm{det}_N\left[\mathrm{e}^{2\pi\omega_pN^{\alpha}\lambda_j(k-1)}\right]\;. \tag{D.10}$$

Inserting this formula into the multiple integral representation for $Z_N[V_G]_{|\beta=1}$ and using the symmetry of the integrand, one can replace one of the determinants by $N!$ times the product of its diagonal elements. Then, the integrals separate and one gets:

$$Z_N[\widetilde{V}_G]_{|\beta=1} \;=\; \frac{N!}{2^{N(N-1)}}\cdot\mathrm{det}_N\left[\int_{\mathbb{R}}\mathrm{e}^{-\pi(\omega_1+\omega_2)N^{\alpha}(N-1)\lambda}\mathrm{e}^{2\pi N^{\alpha}[\omega_1(k-1)+\omega_2(j-1)]\lambda}\cdot\mathrm{e}^{-gN^{1+\alpha}\lambda^2}\mathrm{d}\lambda\right]\;. \tag{D.11}$$

The integral defining the $(k,j)^{\text{th}}$ entry of the determinant is Gaussian and can thus be computed. This yields, upon factorising the trivial terms arising in the determinant,

$$Z_N[\widetilde{V}_G]_{|\beta=1} \;=\; \Big(\frac{\pi}{gN^{1+\alpha}}\Big)^{\frac{N}{2}}\frac{N!}{2^{N(N-1)}}\cdot\prod_{j=1}^{N}\mathrm{e}^{\frac{\pi^2}{4g}N^{\alpha-1}(\omega_1^2+\omega_2^2)(2j-1-N)^2}\cdot D_N\;, \tag{D.12}$$

where

$$D_N \;=\; \mathrm{det}_N\left[\exp\Big\{\frac{\pi^2}{2g}N^{\alpha-1}\omega_1\omega_2(2k-N-1)(2j-1-N)\Big\}\right]\;. \tag{D.13}$$

The last determinant can be reduced to a Van-der-Monde. Indeed, we have:

$$\begin{aligned} D_N \;&=\; \exp\Big\{\frac{\pi^2}{2g}\omega_1\omega_2(N-1)^2N^{\alpha}\Big\}\cdot\prod_{j=1}^{N}\Big\{\mathrm{e}^{-2\frac{\pi^2}{g}\omega_1\omega_2N^{\alpha-1}(N-1)(j-1)}\Big\}\cdot\mathrm{det}_N\left[\exp\Big\{2\frac{\pi^2}{g}N^{\alpha-1}\omega_1\omega_2(k-1)(j-1)\Big\}\right] \\ &=\; \exp\left\{-\frac{\pi^2}{2g}\omega_1\omega_2(N-1)^2N^{\alpha}\right\}\cdot\prod_{k>j}^{N}\left(\mathrm{e}^{2\frac{\pi^2}{g}N^{\alpha-1}\omega_1\omega_2(k-1)} \;-\; \mathrm{e}^{2\frac{\pi^2}{g}N^{\alpha-1}\omega_1\omega_2(j-1)}\right)\;. \end{aligned} \tag{D.14}$$

In order to present the last product into a convergent form, we factor out the largest exponential of each term. The product of these contributions is computable as

$$\prod_{k>l}^{N}\Big(\mathrm{e}^{2\frac{\pi^2}{g}N^{\alpha-1}\omega_1\omega_2(k-1)}\Big) \;=\; \prod_{k=1}^{N-1}\mathrm{e}^{2\frac{\pi^2}{g}N^{\alpha-1}\omega_1\omega_2 k^2} \;=\; \exp\Big\{\frac{\pi^2}{3g}\omega_1\omega_2 N^{\alpha}(N-1)(2N-1)\Big\}\,, \tag{D.15}$$

where we took advantage of

$$\sum_{p=1}^{N} p^2 \;=\; \frac{N(N+1)(2N+1)}{6}\,. \tag{D.16}$$

Putting all of the terms together leads to the claim. ■

The large-N asymptotic behaviour of the partition function at $\beta = 1$ and associated to a Gaussian potential can be extracted from (D.8).

Proposition D.3 *Assume* $0 < \alpha < 1$. *We have the asymptotic expansion:*

$$\begin{aligned}\ln Z_N[V_G]\big|_{\beta=1} \;=\;& N^{2+\alpha}\cdot\Big[\frac{t^2}{4g} + \frac{\pi^2(\omega_1+\omega_2)^2}{12g}\Big] \;-\; N^2\cdot\ln 2 \;-\; N^{2-\alpha}\cdot\frac{g}{12\omega_1\omega_2}\\ &+\; N^{2-2\alpha}\cdot\frac{g^2\,\zeta(3)}{(2\pi^2\omega_1\omega_2)^2} \;+\; (1-\alpha)\,N\ln N \;+\; N\cdot\ln\Big(\frac{2/e}{\sqrt{\omega_1\omega_2}}\Big)\\ &\qquad -\; N^{\alpha}\cdot\frac{\pi^2(\omega_1+\omega_2)^2}{12g} \;+\; \ln N\cdot\frac{\alpha+5}{12} \;+\; \frac{1}{12}\ln\Big(\frac{128\pi^8\omega_1\omega_2}{g}\Big) \;+\; \zeta'(-1) \;+\; \mathrm{o}(1)\,.\end{aligned} \tag{D.17}$$

Proof — The sole problematic terms demanding some further analysis is the last product in (D.8). The latter can be recast as :

$$\prod_{\ell=1}^{N}(1-\mathrm{e}^{-\tau_N\ell})^{N-\ell} \;=\; \left[\frac{M_0(\mathrm{e}^{-N\tau_N};\mathrm{e}^{-\tau_N})}{M_0(1;\mathrm{e}^{-\tau_N})}\right]^N\cdot\frac{M_1(1;\mathrm{e}^{-\tau_N})}{M_1(\mathrm{e}^{-N\tau_N};\mathrm{e}^{-\tau_N})} \qquad \text{where} \quad \tau_N \;=\; \frac{2N^{\alpha}}{gN}\pi^2\omega_1\omega_2 \tag{D.18}$$

and $M_r(a,q)$ corresponds to the infinite products $M_r(a;q) = \prod_{\ell=1}^{\infty}(1-aq^{\ell})^{-\ell^r}$.

We will exploit the fact that asymptotics of $M_r(a;e^{-\tau})$ when $\tau \to 0^+$ up to $o(1)$ can be read-off from the singularities of the Mellin transform of its logarithm

$$\mathfrak{M}_r(a;s) = \int_0^{\infty}\ln M_r(a;e^{-t})\,t^{s-1}\mathrm{d}t \qquad \text{where} \qquad \ln M_r(a;q) \;\equiv\; -\sum_{\ell=1}^{+\infty}\ell^r\ln(1-aq^{\ell})\,. \tag{D.19}$$

The above Mellin transform is well-defined for $\mathrm{Re}(s) > r+1$ and can be easily computed. For any $|a| \leq 1$, we have:

$$\mathfrak{M}_r(a;s) \;=\; \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ell^r\,a^m}{m}\int_0^{\infty}t^{s-1}\,e^{-t\ell m}\,\mathrm{d}t = \Gamma(s)\,\zeta(s-r)\,\mathrm{Li}_{s+1}(a)\,. \tag{D.20}$$

Above, ζ refers to the Riemann zeta function whereas $\mathrm{Li}_s(z)$ is the polylogarithm which is defined by its series expansion in a variable z inside the unit disk:

$$\mathrm{Li}_s(z) = \sum_{k\geq 1}\frac{z^k}{k^s} \tag{D.21}$$

Note that, when $\mathrm{Re}\, s > 1$, the series also converges uniformly up to the boundary of the unit disk. We remind that the first two polylogarithms can be expressed in terms of elementary functions:

$$\mathrm{Li}_0(z) \;=\; \frac{z}{1-z} \qquad \text{and} \qquad \mathrm{Li}_1(z) \;=\; -\ln(1-z)\,. \tag{D.22}$$

In both cases $|a| < 1$ or $a = 1$, $\mathfrak{M}_r(a;s)$ admits a meromorphic extension from $\mathrm{Re}\, s > 1$ to $\mathbb{C}$. When $|a| < 1$ this is readily seen at the level of the series expansion of the polylogarithm whereas when $a = 1$, this follows from $\mathrm{Li}_{s+1}(1) = \zeta(s+1)$. Furthermore, this meromorphic continuation is such that $\mathfrak{M}_r(a; x+\mathrm{i}y) = \mathrm{O}(e^{-c|y|})$, $c > 0$, when $y \to \pm\infty$. This estimate is uniform for a in compact subsets of the open unit disk and for x belonging to compact subsets of $\mathbb{R}$. The same type of bounds also holds for $a = 1$, namely $\mathfrak{M}_r(1; x+\mathrm{i}y) = \mathrm{O}(e^{-c|y|})$, $c > 0$, when $y \to \pm\infty$ for x belonging to a compact subset of $\mathbb{R}$. This is a consequence of three facts:

- $\Gamma(x+\mathrm{i}y)$ decays exponentially fast when $|y| \to +\infty$ and x is bounded, as follows from the Stirling formula ;
- $|\zeta(x+\mathrm{i}y)| \leq C|x+\mathrm{i}y|^c$ for some $c > 0$ valid provided that x is bounded [41] ;
- $\mathrm{Li}_{x+\mathrm{i}y}(a)$ is uniformly bounded for x in compact subsets of $\mathbb{R}$ and a in compact subsets of the open unit disk, as is readily inferred from the series representation (D.21).

Thanks to the inversion formula for the Mellin transform

$$\ln M_r(a, \mathrm{e}^{-\tau}) \;=\; \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \tau^{-s}\, \mathfrak{M}_r(a;s)\, \frac{\mathrm{d}s}{2\mathrm{i}\pi} \qquad \text{with} \quad c > r+1\,, \tag{D.23}$$

we can compute the $\tau \to 0$ asymptotic expansion of $\ln M_r(a, \mathrm{e}^{-\tau})$ – this principle is the basis of the transfer theorems of [47]. To do so, we deform the contour of integration to the region $\mathrm{Re}\, s < 0$. The residues at the poles of $\mathfrak{M}_r(a;s)$ are picked up in the process. There are two cases to distinguish since the polylogarithm factor in (D.20) is entire if $|a| < 1$, while for $a = 1$ one has $\mathrm{Li}_{s+1}(1) = \zeta(s+1)$ what generates an additional pole at $s = 0$. We remind that:

$$\Gamma(s) \underset{s\to 0}{=} \frac{1}{s} - \gamma_E + \mathrm{O}(s)\,, \qquad \zeta(s) \;=\; \frac{1}{s-1} + \gamma_E + \mathrm{O}(s) \tag{D.24}$$

where γ_E is the Euler constant. For $a < 1$, $\mathfrak{M}_r(a;s)$ has simple poles at $s = 1+r$ and $s = 0$:

$$\mathfrak{M}_r(a;s) \;=\; \frac{\mathrm{Li}_{2+r}(a)\, r!}{s-(1+r)} + \mathrm{O}(1)\,, \qquad \mathfrak{M}_r(a;s) \;=\; \frac{-\zeta(-r)\ln(1-a)}{s} + \mathrm{O}(1)\,. \tag{D.25}$$

Notice that here $r \in \{0,1\}$ and the Riemann zeta function has the special values $\zeta(0) = -1/2$ and $\zeta(-1) = -1/12$. Therefore,

$$\ln M_r(a; e^{-\tau}) \;=\; \frac{r!\,\mathrm{Li}_{2+r}(a)}{\tau^{1+r}} - \zeta(-r)\ln(1-a) + \mathrm{o}(1)\,, \qquad \tau \to 0^+ \tag{D.26}$$

and the remainder is uniform for a uniformly away from the boundary of the unit disk. For $a = 1$, $\mathfrak{M}_r(a;s)$ has the same simple pole at $s = 1+r$ with residue $r!\,\zeta(2+r)$, but now a double pole at $s = 0$:

$$\mathfrak{M}_r(1;s) \;=\; \frac{\zeta(-r)}{s^2} + \frac{\zeta'(-r)}{s} + \mathrm{O}(1)\,, \qquad \mathfrak{M}_r(1;s) \;=\; \frac{r!\,\zeta(2+r)}{s-(1+r)} + \mathrm{O}(1) \tag{D.27}$$

and we remind the special value $\zeta'(0) = -\ln(2\pi)/2$. In this case, we thus have:

$$M_r(1; e^{-\tau}) \;=\; \frac{r!\,\zeta(2+r)}{t^2} - \zeta(-r)\ln t + \zeta'(-r) + \mathrm{o}(1), \qquad \tau \to 0^+\,. \tag{D.28}$$

Collecting all the terms from (D.26)-(D.28), we obtain the asymptotics of the product (D.18) that are uniform in a belonging to compact subsets of the unit disk:

$$\ln\bigg[\prod_{\ell=1}^{N-1}(1-e^{-\tau_N})^{N-\ell}\bigg] = \frac{\zeta(3)-\mathrm{Li}_2(\mathrm{e}^{-N\tau_N})}{\tau_N^2} + \frac{N}{\tau_N}\Big(\mathrm{Li}_1(\mathrm{e}^{-N\tau_N})-\frac{\pi^2}{6}\Big) + \Big(\frac{N}{2}-\frac{1}{12}\Big)\ln\Big(\frac{1-\mathrm{e}^{-N\tau_N}}{\tau_N}\Big) + \frac{N\ln(2\pi)}{2} + \zeta'(-1) + \mathrm{o}(1)\,. \qquad \text{(D.29)}$$

Here, we have used the special value $\zeta(2) = \pi^2/6$. It remains to insert in (D.29) the value of the parameter of interest $\tau_N = N^{\alpha-1}\,2\pi^2\omega_1\omega_2/g$, and return to the original formula. The announced result for the Gaussian partition function (D.17) follows, upon using the Stirling approximation $N! \sim \sqrt{2\pi}N^{N+1/2}e^{-N}$ for the factorial prefactor.

We remark that for $\alpha \geq 1$, $\tau_N \geq 0$ is not anymore going to 0 when $N \to \infty$, therefore the asymptotic regime will be different.

■

E Summary of symbols

Empirical and equilibrium measures

$\mathcal{E}_{(\mathrm{ply})}[\mu]$	(B.13)	energy functional for the baby integral of § 2.1
$E(\xi,\eta)$	(B.13)	its kernel function
$\mu_{\mathrm{eq}}^{(\mathrm{ply})}$	§ B.4	minimiser of $\mathcal{E}_{(\mathrm{ply})}$
$\mathcal{E}_N[\mu]$	(2.33)	N-dependent energy functional
$\mathcal{E}_\infty[\mu]$	(2.27)	same one at $N=\infty$
$\mathfrak{D}[\mu,\nu]$	Def. 3.2	pseudo-distance between probability measures induced by $\mathcal{E}_N$
$\mu_{\mathrm{eq}}^{(N)}$	(2.35)-(2.36)	N-dependent equilibrium measure (maximiser of $\mathcal{E}_N$)
$\rho_{\mathrm{eq}}^{(N)}$	Thm. 2.4	density of $\mu_{\mathrm{eq}}^{(N)}$
$[a_N,b_N]$	Thm. 2.4	support of $\mu_{\mathrm{eq}}^{(N)}$
x_a^N	Def. 3.4	classical positions for $\mu_{\mathrm{eq}}^{(N)}$
$V_{N;\mathrm{eff}}$	(3.3)	effective potential
$L_N^{(\lambda)}$	(2.48)	empirical measure
$\tilde{\boldsymbol{\lambda}}$	Def. 3.6	deformation of $\boldsymbol{\lambda}$ enforcing a minimal spacing
$L_{N;u}^{(\lambda)}$	Def. 3.6	convolution of $L_N^{(bs\lambda)}$ with uniform law of small support
$\mathcal{L}_N^{(\lambda)}$	Def. 3.1	centred empirical measure with respect to $\mu_{\mathrm{eq}}^{(N)}$
$\mathbb{M}_{N;\kappa}^{(n)}$	Def. 3.8	probability measure including exponential regularization of n variables

Partition functions

$Z_N[V]$	(1.10)	partition function of the sinh model with potential V
$V_{G;N}$	Lemme D.1	Gaussian potential leading to support $[a_N,b_N]$

Pairwise interactions

$s_N(\xi)$	(3.2)	pairwise interaction kernel
$S(\xi)$	(2.42)	derivative of $\beta \ln \left|\sinh(\pi\omega_1\xi)\sinh(\pi\omega_2\xi)\right|$ *viz.* $\frac{1}{2}\partial_\xi s_N(N^{-\alpha}\xi)$
$S_{\mathrm{reg}}(\xi)$	(8.6)	S minus its pole at 0
$\mathcal{S}_N$	(2.42)	integral operator with kernel $S(N^\alpha(\xi_1 - \xi_2))$
$\mathcal{S}_{N;\gamma}$	(4.1)	same one with extended support
$\mathscr{S}_{N;\gamma}$	(4.3)	same one in rescaled and centered variables

Operators

$\mathcal{K}_\kappa$	Def. 3.8	multiplication by a decreasing exponential
$\Xi^{(p)}$	Def. 3.12	operator inserting a copy of ξ_1 at position p
$\mathcal{U}_N$	(3.54)	master operator
$\mathcal{D}_N$	(3.57)	hyperbolic analog of the non-commutative derivative
$\mathcal{V}_N$	Prop 8.1	building block of $\mathcal{U}_N^{-1}$
$\mathcal{W}_N$	(2.44)	inverse of $\mathcal{S}_N$
$\mathcal{X}_N$	Def. 3.18	linear form related to $\mathscr{I}_{11}$
$\widetilde{\mathcal{X}}_N$	Def. 3.18	projection to the hyperplane $\mathfrak{X}_s([a_N\,; b_N]) = \mathrm{Ker}\,\mathcal{X}_N$
$\widetilde{\mathcal{U}}_N^{-1}, \widetilde{\mathcal{W}}_N$	(3.101)	operators composed to the right with $\widetilde{\mathcal{X}}_N$
$\mathscr{W}_N$	(5.58)	operator $\mathcal{W}_N$ in rescaled and centered variables
$\widetilde{\mathscr{W}}_{\vartheta;z_0}$	(5.15)	a pseudo-inverse of $\mathscr{S}_{N;\gamma}$.
$\mathscr{I}_{11}, \mathscr{I}_{12}$	Prop. 5.4	functionals appearing in the inversion of $\mathscr{S}_{N;\gamma}$
$\mathscr{J}_{1a}(\lambda)$	(5.34)	related functionals
$w_{k;a}^{(1/2)}, w_{k;a}^{(1)}$	(5.36)	functionals appearing in the large λ expansion of the latter
$H^\wedge$	Def. 6.2	reflection of the function H (exchanging left and right boundary)
$\mathcal{G}_N$	(9.2)	2-variable operator related to $\mathcal{W}_N$
$\mathcal{T}_{\mathrm{even}}, \mathcal{T}_{\mathrm{odd}}$	(9.23)	some even/odd averaging operator

Decomposition of operators for asymptotic analysis

$\mathcal{W}^{(\infty)}, \delta\mathcal{W}$	(6.1)	leading and subleading terms in $\mathcal{W}_N$ when $N \to \infty$
$\mathcal{W}_R, \mathcal{W}_L$	(6.4)	contribution of the right/left boundary to $\mathcal{W}_N$
$\mathcal{W}_{R;k}$	Prop. 6.6	terms contributing to the latter up to $O(N^{-k\alpha})\ldots$
$\Delta_{[k]}\mathcal{W}_R$	Prop. 6.6	$\ldots$ and the remainder
$\mathcal{W}_{R;k}^{(\mathrm{as})}, \Delta_{[k]}\mathcal{W}_{R;k}^{(\mathrm{as})}$	Lemma 6.11	putting aside exponentially small terms in $\mathcal{W}_{R;k}$
$\mathcal{W}_{\mathrm{bk}}$	(6.4)	contribution of the bulk to $\mathcal{W}_N$
$\mathcal{W}_{\mathrm{bk};k}$	Prop. 6.6	the terms contributing to the latter up to $O(N^{-k\alpha})\ldots$
$\Delta_{[k]}\mathcal{W}_{\mathrm{bk}}$	Prop. 6.6	$\ldots$ and the remainder
$\mathcal{W}_{\mathrm{bk};k}^{(\mathrm{as})}, \Delta_{[k]}\mathcal{W}_{\mathrm{bk};k}^{(\mathrm{as})}$	Lemma 6.11	putting aside exponentially small terms in the bulk operator
$\mathcal{W}_{\mathrm{exp}}$	(6.4)	exponentially small contribution
$(\Delta_{[k]}\mathcal{W}_N)_R$	(9.8)	local right boundary remainder

Similar notations are used throughout Section 9 for the decompositions of $\mathcal{G}$ and the various $\mathfrak{I}$.

Riemann-Hilbert problems

$R(\lambda)$	(4.19)	reflection coefficient
κ_N	(4.18)	coefficient of $1/\lambda$ term
$R_{\uparrow/\downarrow}(\lambda)$	(4.25)-(4.26)	Wiener-Hopf factors of $R(\lambda)$
$\upsilon(\lambda)$	(4.24)	related, piecewise holomorphic function
Φ	Lemma 4.1	$2d$-vector in correspondence with solutions of $\mathscr{S}_{N;\gamma}[f] = g$.
$\chi(\lambda)$	Prop. 4.3	2×2 matrix solution of the homogeneous Riemann–Hilbert problem with jump G_χ
$\chi^{(\mathrm{as})}_{\uparrow/\downarrow}(\lambda)$	(7.7)	leading part of $\chi(\lambda)$ when $N \to \infty$
$\chi^{(\mathrm{exp})}_{\uparrow/\downarrow}(\lambda)$	(7.8)	exponentially small part of $\chi(\lambda)$
χ_k	(4.65)	matrix coefficients in the large λ expansion of $\chi(\lambda)$
G_χ	(4.8)	jump matrix of the Riemann–Hilbert problem of Φ and χ
$\Psi(\lambda)$	(4.55) and Fig. 1	2×2 matrix related to $\chi(\lambda)$
$\Pi(\lambda)$	(4.49) and Fig. 2	related 2×2 matrix
$\Delta\Pi(\lambda)$	(7.8)	difference between $\Pi(\lambda)$ minus identity
G_Ψ	(4.39)-(4.40)	jump matrix of the auxiliary Riemann–Hilbert problem
$\varkappa_\epsilon$	(4.42)	rate of exponential decay of $G_\Psi - I_2$
$\mathcal{R}_{\uparrow/\downarrow}(\lambda)$	(4.31)	some factors of the jump matrix
$\mathcal{R}^{(\infty)}_{\uparrow/\downarrow}$	(4.32)	their non-oscillatory parts
$M_{\uparrow/\downarrow}(\lambda)$	(4.33)	some factors of the jump matrix
$P_R(\lambda), P_{L;\uparrow/\downarrow}(\lambda)$	(4.34)	some factors in the auxiliary Riemann–Hilbert problem
θ_R	(4.55)	a constant involved in the auxiliary Riemann–Hilbert problem
$\Upsilon(\lambda)$	(5.5)-(5.13)	polynomial remainder in the inhomogeneous Riemann–Hilbert problem
$\boldsymbol{H}(\lambda)$	(4.8)	$2d$-vector on the right-hand side of the inhomogeneous Riemann–Hilbert problem
$\widehat{\boldsymbol{H}}(\lambda)$	(5.14)	related quantity

Auxiliary functions, contours, and constants

$J_k(\lambda)$	(5.41)	model integral appearing in the asymptotics of $\mathcal{J}_{1a}(\lambda)$
x_R, x_L	Def. 6.1	reduced variables centered at the right and left boundary
$\Gamma_{\uparrow/\downarrow}$	Figure 4	contours in the upper/lower half-plane
$\mathscr{C}^{(\pm)}_{\mathrm{reg}}$	Def. 6.3 and Figure 4	contours between $\Gamma_{\uparrow/\downarrow}$ and $\mathbb{R}$
$J(x)$	Def. 6.3	related to the Fourier transform of $1/R(\lambda)$
$\varrho_0(x)$	(6.34)-(6.35)	proportional to a primitive of $J(x)$
$\varrho_\ell(x)$	Def. 6.5	related to higher primitives
$\varpi_\ell(x)$	Def. 6.5	integrals of $x^\ell J(x)$ from x to ∞
u_ℓ	Def. 6.5	coefficients in the Taylor expansion of $1/R(\lambda)$ at $\lambda = 0$
$\mathfrak{u}_\ell(x)$	Def. 6.9, (8.48)	related to the ℓ^{th} order truncation of the Taylor series of $1/R$
$\mathfrak{a}_\ell(x),\mathfrak{b}_\ell(x)$	Def. 6.9	combinations of the above, involved in asymptotics of $\mathcal{W}_N$
$\daleth_p$	Def. 7.1	negative moments of $1/R_\downarrow$
$\daleth_{s,\ell}$	Def. 7.5	s^{th} order moment of $\mathfrak{b}_\ell$
$\gimel_\ell$	Def. 9.4	ℓ^{th} order moments related to J and S
$\mathcal{P}_\ell, \mathcal{Q}_\ell$	(9.73)	some universal multivariable polynomial
$\mathfrak{g}_{R;\ell}, \mathfrak{g}_{\mathrm{bk};\ell}$	(9.77)-(9.80)	a specialisation of the latter involving the functions above

Answer for the partition function

$\mathfrak{I}_{\mathrm{s}}[H,G]$	Def. 7.4	bilinear pairing induced by $\mathcal{W}_N$
$\mathfrak{I}^{(1)}_{\mathrm{s};\beta}[H,G]$	(3.115)	related expression appearing only for $\beta \neq 1$
$\mathfrak{I}^{(1)}_{\mathrm{s};\beta}[H,G]$	(3.116)	related expression appearing only for $\beta \neq 1$
$\mathfrak{I}_{\mathrm{d}}[H,G]$	(9.1)	related expression
$\mathfrak{I}_{\mathrm{d};\beta}[H,G]$	(3.118)	related expression appearing only for $\beta \neq 1$
$\Omega[V,V_0]$	(2.22)	a functional appearing in the interpolation
$\mathfrak{c}(x)$	Def. 9.9	a function involving the $\mathfrak{a}$'s and $\mathfrak{b}$'s appearing in expansion of $\mathfrak{I}_{\mathrm{d}}$
$\aleph_0$	Def. 9.9	a constant involving integrals of J, S and $R_{\uparrow/\downarrow}$, appears in expansion of $\mathfrak{I}_{\mathrm{d}}$

Norms

$\mathcal{N}^{(\ell)}_N[\phi]$	Def. 3.14	weighted norms involving W^{∞}_k norms for $k \in [\![0 ; \ell]\!]$
$\mathfrak{n}_\ell[V]$	Def. 3.15	some estimates for the magnitude of potential

Miscellaneous

$q(z)$	(8.8)	squareroot
$q_R(z)$	(8.23)	squareroot at the right boundary

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