Weak Brill-Noether for vector bundles on the projective plane

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0. Introduction

We work over an algebraically closed field k. Throughout the paper a sheaf will be a torsion-free coherent sheaf of rank at least 2 on $\mathbb{P}_2$. For a sheaf E on $\mathbb{P}_2$, we let $\chi(E)$ be the Euler-Poincaré characteristic, and $P_E(n) := \chi(E(n))$ the Hilbert polynomial. Given a polynomial $P \in \mathbb{Q}[n]$, we let $M_{\mathbb{P}_2}(P)$ (or just $M(P)$) be the moduli space of S-equivalence classes of Gieseker-semistable sheaves with Hilbert polynomial P. The Euler-Poincaré characteristic, the rank and the slope $\mu(E)$ of a sheaf E are determined by its Hilbert polynomial. Therefore we will put $\chi(P) := P(0)$, $rk(P) := rk(E)$ and $\mu(P) := \mu(E) = deg(c_1(E)/rk(E))$ for any sheaf E with Hilbert polynomial P.

Let E be a sheaf with $h^2(E) = 0$. Then clearly $h^0(E) \geq \chi^+(E) := max(\chi(E), 0)$. We will call E special if $h^0(E) > \chi^+(E)$. If E is a semistable sheaf with $\mu(E) > -3$ then $h^2(E) = 0$. We let $Sp(P) \subset M(P)$ be the closed subset of special S-equivalence classes of sheaves, where a class is called special if at least one of its representatives is special. In fact, if we assume $\mu(P) > -3$, then $Sp(P)$ has a natural scheme structure (see lemma 1.3).

Theorem 1. *Assume $\mu(P) > -3$. Then*

(1) *If $\chi(P) \neq 0$, then $Sp(P)$ has codimension at least 2.*
(2) *If $\chi(P) = 0$, then $Sp(P)$ is an irreducible reduced divisor or empty.*

Theorem 1 is used in [LP] to give a new and simpler proof of the structure of the Picard group of the moduli space $M(P)$.

We will indeed prove a more general result dealing with moduli stacks of sheaves on $\mathbb{P}_2$, rather then with moduli spaces of semistable sheaves.

A sheaf E is called prioritary if $\mathrm{Ext}^2(E, E(-1)) = 0$ (see [H-L]), and it is called of rigid splitting type (rst) if the restriction to a general line l is rigid, i.e. if it splits as

$E_{|l} = \mathcal{O}_{\mathbb{P}_1}(\alpha)^{\oplus a} \oplus \mathcal{O}_{\mathbb{P}_1}(\alpha+1)^{\oplus b}$. It is easy to see that semistable sheaves and sheaves of rigid splitting type are prioritary (cfr. e.g. [H-L], prop. 1.2). Let $\mathcal{P}rior(P)$ be the moduli stack of prioritary sheaves with Hilbert polynomial P. Let $\mathcal{R}st(P) \subset \mathcal{P}rior(P)$ be the moduli stack of rst sheaves and let $\mathcal{U}(P) \subset \mathcal{R}st(P)$ be the open substack of sheaves E with $h^2(E) = 0$. Let $\mathcal{S}p(P) \subset \mathcal{U}(P)$ be the closed substack of special sheaves (see lemma 1.3).

Theorem 2.

(1) *If $\chi(P) \neq 0$, then $\mathcal{S}p(P)$ has codimension at least 2 in $\mathcal{U}(P)$.*
(2) *If $\chi(P) = 0$, then $\mathcal{S}p(P)$ is an irreducible reduced divisor or empty.*

We will prove theorem 2 by descending induction on P using a degeneration argument. A similar induction was used in [H-L] to prove the irreducibility of $\mathcal{P}rior(P)$, thereby giving a new proof of the irreducibility of $M(P)$. Theorem 1 will be proved from theorem 2 essentially by showing that in the moduli space of stable sheaves the locus of non-rst sheaves has codimension at least 2.

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1. Background material

A versal family of sheaves on $\mathbb{P}_2$ will be a flat family of sheaves $\mathcal{E}$ with smooth base S such that the Kodaira-Spencer map $T_{S,s} \longrightarrow \mathrm{Ext}^1(\mathcal{E}_s, \mathcal{E}_s)$ is surjective at every point $s \in S$. If a family of prioritary locally free sheaves is versal, then its restriction to any line l is also versal. To see this it is enough to check that the map $H^1(E \otimes E^*) \longrightarrow H^1((E \otimes E^*)|_l)$ is surjective and this follows from the prioritary condition $H^2(E \otimes E^*(-1)) = 0$.

Proposition 1.1.

(1) *Let E be a rst sheaf. If $h^0(E(-1)) > 0$, then $h^2(E(-1)) = 0$ (hence $h^2(E) = 0$).*
(2) *If E is rst, then there exists an n such that $h^0(E(n)) = h^2((n)) = 0$.*
(3) *The Hilbert polynomial P_E of a prioritary sheaf is nonpositive (i.e. there exists an n with $P_E(n) \leq 0$).*

Proof. This follows essentially from the proof of proposition 1.3 of [H-L]. There (1) is proved under the assumption that E is generic prioritary, however the proof uses only that E is locally free and rst. If E is not locally free, one can use the exact sequence

$$0 \longrightarrow E \longrightarrow E^{**} \longrightarrow Q \longrightarrow 0,$$

where Q is a sheaf with 0-dimensional support, hence has no higher cohomology. (2) follows directly from (1) and (3) follows from (2) plus the fact that a generic prioritary sheaf is rst ([H-L] prop. 1.2). □

We will use algebraic stacks in the sense of Artin [A], which is more general than that of Deligne-Mumford [D-M]; our main reference will be [L-MB]. There exists a moduli stack $\mathcal{C}oh(P)$ of coherent sheaves on $\mathbb{P}_2$ with Hilbert polynomial P, and there is a universal sheaf over $\mathcal{C}oh(P) \times \mathbb{P}_2$. The prioritary sheaves form a smooth open substack $\mathcal{P}rior(P)$ of $\mathcal{C}oh(P)$. For a smooth scheme S, a smooth morphism $S \longrightarrow \mathcal{P}rior(P)$ is the same as a versal family of prioritary sheaves on $\mathbb{P}_2$ with base S. Local properties of $\mathcal{P}rior(P)$ (in the smooth topology), like codimension of substacks, can therefore be checked on versal families.

Notation 1.2. We denote by $\mathcal{M}(P)$ the moduli stack of semistable sheaves. We will use the following notation: For any locally closed substack $\mathcal{A}(P)$ of $\mathcal{P}rior(P)$ we denote $\mathcal{A}^s(P)$ the open substack of stable sheaves in $\mathcal{A}(P)$, $\mathcal{A}_0(P)$ the open substack of locally free sheaves, $\mathcal{A}_{\leq 1}(P)$ the open substack of sheaves with at most one simple singularity and no other singularities, $\mathcal{A}_1(P) := \mathcal{A}_{\leq 1} \setminus \mathcal{A}_0$ with the reduced structure, $\mathcal{A}_{\geq 1}(P)$ the closed substack of sheaves having at least one simple singularity (with the reduced structure), $\mathcal{A}_{>1}(P) := \mathcal{A}_{\geq 1}(P) \setminus \mathcal{A}_1(P)$, we also denote $\mathcal{A}_{1,x}(P)$, $\mathcal{A}_{\geq 1,x}(P)$, $\mathcal{A}_{>1,x}(P)$ the substacks of $\mathcal{A}_1(P)$, $\mathcal{A}_{\geq 1}(P)$, $\mathcal{A}_{>1}(P)$ of sheaves with a simple singularity at x.

For any locally closed subscheme $A(P)$ of $M(P)$ we define $A^s(P)$, $A_0(P)$, $A_{\leq 1}(P)$, $A_1(P)$, $A_{\geq 1}(P)$, $A_{>1}(P)$, $A_{1,x}(P)$, $A_{\geq 1,x}(P)$, $A_{>1,x}(P)$ corrispondingly.

By theorème 3.1 and corollaire 3.2 of [H-L] the moduli stack $\mathcal{P}rior(P)$ is irreducible, and the generic element is locally free. In particular if $\mathcal{P}rior(P)$ contains the sheaf $F_0 = \bigoplus_{j=1}^r \mathcal{O}(b_j)$, then it is reduced to a point, as its tangent space at F_0 has dimension zero.

Lemma 1.3.

(1) *$\mathcal{S}p(P)$ has a natural structure of a closed substack of $\mathcal{U}(P)$. For $\chi(P) = 0$ it is either $\mathcal{U}(P)$ or a Cartier divisor or empty.*

(2) *If $\mu(P) > -3$ then $Sp(P)$ has a natural structure of a closed substack of $M(P)$.*

(3) *The codimension of $\mathcal{S}p^s(P) \subset \mathcal{M}^s(P)$ is the same as that of $Sp^s(P) \subset M^s(P)$.*

Proof. (1) Giving a substack structure to $\mathcal{S}p(P) \subset \mathcal{U}(P)$ is equivalent to giving, for each family of rst sheaves E with $h^2(E) = 0$, a subscheme structure to the locus of special sheaves, which is compatible with pullback. In fact it would be enough to do this only for versal families. We use the construction of ([H] 7.6). Given a family $\mathcal{E}$ over $B \times \mathbb{P}_2$, B a scheme, we can embed $\mathcal{E}$ into a locally free sheaf $\mathcal{F}$, flat over B, with no higher direct images, so that the fibre homomorphisms $\mathcal{E}|_{\{b\}\times\mathbb{P}_2} \longrightarrow \mathcal{F}|_{\{b\}\times\mathbb{P}_2}$ are also injections. The cokernel $\mathcal{G}$ of $\mathcal{E} \longrightarrow \mathcal{F}$ is flat over B and has no higher direct images. Thus $\pi_*\mathcal{F}$ and $\pi_*\mathcal{G}$ are locally free, where $\pi : B \times \mathbb{P}_2 \longrightarrow B$ is the projection. The subscheme structure of the locus of special sheaves is given by the vanishing of the maximal minors of the induced map $\pi_*\mathcal{F} \longrightarrow \pi_*\mathcal{G}$. This subscheme structure is independent of the choice of $\mathcal{F}$ (by the theory of Fitting ideals) and compatible with base change. In case $\chi(\mathcal{E}|_{\{b\}\times\mathbb{P}_2}) = 0$, the sheaves $\pi_*\mathcal{F}$ and $\pi_*\mathcal{G}$ have the same rank, hence $\mathcal{S}p(P)$ is either equal to $\mathcal{U}(P)$, or a Cartier divisor or empty.

(2) We have a natural morphism $c : \mathcal{M}(P) \longrightarrow M(P)$: for a scheme T we have the functor $\mathcal{M}(P)(T) \longrightarrow \mathrm{Hom}(T, M(P))$, associating to a family of semistable sheaves on $\mathbb{P}_2$ parametrized by T the induced morphism $T \longrightarrow M(P)$. By the same arguments as above as in (1), the sublocus of $\mathcal{M}(P)$ of sheaves E in $\mathcal{M}(P)$ with $H^0(E) > \chi^+(E)$ carries a natural structure as a closed substack $\mathcal{S}$ of $\mathcal{M}(P)$. Let $Sp(P)$ be the scheme-theoretic image of $\mathcal{S}$, i.e. the smallest subscheme of $M(P)$, through which $c|_{\mathcal{S}}$ factors.

(3) The fibres of $c|_{\mathcal{M}^s(P)} : \mathcal{M}^s(P) \longrightarrow M^s(P)$ have all the same dimension (i.e. -1) and $\mathcal{S}p^s(P)$ is the preimage of $Sp^s(P)$. Therefore (3) follows. □

It is easy to see that $Sp(P)$ is irreducible and reduced if $\mathcal{S}p(P)$ is (this follows from the corresponding statement for the scheme theoretic image of schemes).

Let E be a semistable sheaf with $\mu(E) > -3$ and $\bigoplus_i E_i$ the graded object associated to the Jordan-Hölder filtration. Then obviously $h^0(E) \leq \sum h^0(E_i)$, hence if E is special, then $\bigoplus E_i$ also is, and there exists an i_0 with E_{i_0} special.

We will show that, when $\chi(P) = 0$, $Sp(P) \subset M(P)$ is irreducible and that $Sp(P) \cap (M(P) \setminus M^s(P))$ has codimension at least 2 in $M(P)$. As $M(P)$ is normal $Sp(P)$ has an induced structure of a Cartier divisor.

Remark 1.4. Let $\mathcal{F}$ be a universal sheaf over $\mathcal{R}st_0(P+1) \times \mathbb{P}_2$. Then $\mathcal{R}st_1(P)$ is canonically isomorphic to the projectivization $\mathbb{P}(\mathcal{F})$ (for the definition of the projectivisation of a coherent sheaf over a stack see [L-MB] 7.2.3.). In fact the universal sheaf $\mathcal{E}$ on $T = \mathbb{P}(\mathcal{F}) \times \mathbb{P}_2$ can be defined by the exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow p^*(\mathcal{F}) \longrightarrow p_1^*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))|_{\tilde{\Delta}} \longrightarrow 0,$$

where p_1, p_2 are the projections of $\mathbb{P}(\mathcal{F}) \times \mathbb{P}_2$ to $\mathbb{P}(\mathcal{F})$ and $\mathbb{P}_2$, $p : \mathbb{P}(\mathcal{F}) \times \mathbb{P}_2 \longrightarrow \mathcal{R}st_0(P+1) \times \mathbb{P}_2$ is the composition of p_1 with the projection $\pi : \mathbb{P}(\mathcal{F}) \longrightarrow \mathcal{R}st_0(P+1) \times \mathbb{P}_2$, and $\widetilde{\Delta}$ is the inverse image of the diagonal via the mapping $\mathbb{P}(\mathcal{F}) \times \mathbb{P}_2 \longrightarrow \mathbb{P}_2 \times \mathbb{P}_2$ which is the product of p_2 and the composition of p with the projection $\mathcal{R}st_0(P+1) \times \mathbb{P}_2 \longrightarrow \mathbb{P}_2$. Fibrewise, this exact sequence is just

$$0 \longrightarrow E \longrightarrow E^{**} \longrightarrow k_x \longrightarrow 0,$$

where E is a sheaf with a simple singularity at x and k_x the skyscraper sheaf with fibre k at x.

By [H-L] théorème 3.1 and corollaire 3.2 $\mathcal{P}rior(P)$ is irreducible and the generic point of $\mathcal{P}rior(P)$ is locally free. Therefore $\mathcal{P}rior_{\geq 1}(P)$ has at least codimension 1 in $\mathcal{P}rior(P)$. By the same arguments as in the previous paragraph $\mathcal{R}st_{\geq 1}(P)$ is a $\mathbb{P}_{r(P)-1}$-bundle over the open substack of $\mathcal{R}st(P+1) \times \mathbb{P}_2$ of (F, x) such that F is locally free at x. In particular it is irreducible and $\mathcal{R}st_{>1}(P)$ has codimension at least 1. Repeating this argument shows that the generic element in $\mathcal{R}st_{>1}(P)$ has exactly 2 simple singularities.

By restriction we have an isomorphism $\mathcal{U}_1(P) \simeq \mathbb{P}(\mathcal{F}|_{\mathcal{U}_0(P+1)\times\mathbb{P}_2})$ as $h^2(E^{**}) = 0$ if and only if $h^2(E) = 0$. We also denote by $\pi : \mathcal{U}_1(P) \longrightarrow U_0(P+1) \times \mathbb{P}_2$ the restiction of the projection.

Every sheaf E has a graded resolution of length at most 1

$$0 \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}(a_i) \xrightarrow{M} \bigoplus_{j=1}^{n+r} \mathcal{O}(b_j) \longrightarrow E \longrightarrow 0 \qquad (*)$$

where $a_i \geq a_{i+1}$, $b_j \geq b_{j+1}$, $r > 0$. The resolution is minimal if the entries of M contain no nonzero constants: this implies $a_i < b_{i+1}$ (see prop. 2.3 of [H-L]). By leaving out the corresponding summands we can always change a resolution to a minimal resolution.

Definition 1.5. Given $a_1 \geq \ldots \geq a_n$, $b_1 \geq \ldots \geq b_{n+r}$ we call $\mathcal{R}st((a_i),(b_j))$ (resp. $\mathcal{U}((a_i),(b_j))$) the locally closed substack of $\mathcal{R}st(P)$ (resp. $\mathcal{U}(P)$) of sheaves having a graded resolution of type $(*)$. (Here $P(n) = \sum_j \binom{n+b_j+1}{2} - \sum_i \binom{n+a_i+1}{2}$.)

Note that $h^0(E)$ is the same for all $E \in \mathcal{R}st((a_i),(b_j))$, therefore $\mathcal{U}((a_i),(b_j))$ is either contained in or disjoint from $\mathcal{S}p(P)$.

Lemma 1.6. *Assume $(a_i)_{i=1}^{n}$, $(b_j)_{j=1}^{n+r}$ fulfill $a_i \leq b_{i+1}$ for all i and that $n > 0$. If $\mathcal{R}st((a_i),(b_j))$ is nonempty, then it intersects $\mathcal{R}st_{\geq 1}(P)$.*

Proof. This is shown in [H-L] proposition 2.1 and proposition 2.6. There the assumption is made that $\mathcal{R}st((a_i)(b_j))$ contains a generic prioritay sheaf E, such that $(*)$ is a minimal resolution. In the proof it is however only used that E is rst and $a_i \leq b_{i+1}$ for all i. $\square$

Corollary 1.7. *Every component of $\mathcal{S}p(P)$ intersects $U_{\geq 1}(P)$.*

Proof. Let $\mathcal{U}((a_i),(b_j)) \neq \emptyset$ be contained in $\mathcal{S}p(P)$ with $a_i \leq b_{i+1}$ for $i = 1 \dots n$. If $n = 0$, then $\mathcal{U}((a_i),(b_j)) = \mathcal{P}rior(P) = \{\bigoplus_j \mathcal{O}(b_j)\}$ hence $\mathcal{S}p(P) = \emptyset$. If $n > 0$ then $\mathcal{R}st((a_i),(b_j))$ intersects $\mathcal{R}st_{\geq 1}(P)$ by lemma 1.6. Furthermore $\mathcal{U}((a_i),(b_j)) = \mathcal{R}st((a_i),(b_j))$; in fact for some, and hence for all $E \in \mathcal{R}st((a_i),(b_j))$ we have $h^0(E) > 0$ (by speciality) hence $h^2(E) = 0$ by proposition 1.1. $\square$

Definition 1.8. Let $\mathcal{A}$ be an algebraic stack over k and K and extension of k. A point $x_1 \in \mathcal{A}(K)$ is called a generalization of $x \in \mathcal{A}(k)$, if x lies in the closure of the image of the morphism $Spec(K) \longrightarrow \mathcal{A}$ given by x_1. We call x_1 a generic point of $\mathcal{A}$ if this morphism is dominant and separable (see also [H-L]).

Proposition 1.9. *$\mathcal{R}st(P) \setminus \mathcal{R}st_0(P)$ is irreducible and a generic point lies in $\mathcal{R}st_1(P)$.*

For the proof we will need an elementary result from linear algebra.

Lemma 1.10. *Let V, W be vector spaces of dimension n and $n + r$ respectively, $V_1, \dots, V_n$ (resp. $W_1, \dots, W_n$) be flags of linear subspaces such that $dim(V_i) = i$ (respectively $dim(W_i) \geq i + 1$) for all i. Let $FL \subset Hom(V, W)$ be the affine scheme of flag maps (i. e. maps φ such that $\varphi(V_i) \subset W_i$) having rank at most $n-1$. For $\varphi \in FL$, let $l(\varphi) := \min\{i | dim\, \varphi(V_i) < i\}$.*

Then every $\phi \in FL$ has a generalization ψ such that $rk\, \psi = n-1$ and $W_{l(\psi)} \cap \psi(V) = \psi(V_l)$.

Proof. We use descending induction on $l := l(\varphi)$, the case $l = n - 1$ being trivial. Let φ be such that either $\mathrm{rk}(\varphi) \leq n - 2$ or $W_l \cap \varphi(V) \neq \varphi(V_l)$.

By assumption there exists $v_l \in V_l \cap \ker \varphi$, $v_l \notin V_{l-1}$; complete v_l to a flag basis $v_1, \dots, v_n$ of V (a basis such that $v_1, \dots, v_i$ span V_i). If $\mathrm{rk}\, \varphi \leq n - 2$, choose $w \in W_l \setminus \varphi(V_l)$; otherwise choose $w \in W_l \cap \varphi(V)$, $w \notin \varphi(V_l)$. Put

$$\psi_t(v_i) = \begin{cases} tw & \text{if } i = l \\ \varphi(v_i) & \text{otherwise.} \end{cases}$$

Then ψ_t is a generalization of φ in FL and $l(\psi_t) > l(\varphi)$. $\square$

Proof of Proposition 1.9. The second assertion follows from the first as $\mathcal{R}st_1(P)$ is irreducible and dense in $\mathcal{R}st_{\geq 1}(P)$ by remark 1.4. So it is enough to prove that every non-locally free sheaf in $\mathcal{R}st(P)$ can be generalized to a sheaf having one simple singularity (and possibly others). Let $E \in \mathcal{R}st(P) \setminus \mathcal{R}st_0(P)$. Choose $x \in \mathbb{P}_2$ such that E is not locally free at x, and let

$$0 \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}(-a_i) \xrightarrow{f} \bigoplus_{j=1}^{n+r} \mathcal{O}(-b_j) \longrightarrow E \longrightarrow 0$$

be a minimal resolution (again $a_i \geq a_{i+1}$, $b_j \geq b_{j+1}$). Let H be the space of matrices with polynomial entries

$$H = \{h = (h_{i,j}) \mid h_{i,j} \in H^0(\mathcal{O}_{\mathbb{P}_2}(a_i - b_j)),\ h_{i,j} = 0 \text{ if } a_i \leq b_j,\ rk(h_{i,j}(x)) \leq n-1\}.$$

Let $H_0 \subset H$ be the open subset of maps h, for which the induced map $\bigoplus \mathcal{O}(-a_i) \xrightarrow{h} \bigoplus \mathcal{O}(-b_j)$ is injective and the cokernel E_h is torsion free. The E_h form a flat family over H_0; as $f \in H_0$ and E is rst, for every generalisation h of f we have that E_h is also rst.

Fix affine coordinates (y, z) in $\mathbb{P}_2$ such that x is the origin; to each section of $\mathcal{O}_{\mathbb{P}_2}(r)$ we can associate an element of $\mathcal{O}_x$, hence we can consider the quotients $H' = H/m_x$ and $H" = H/m_x^2$, where $m_x \subset \mathcal{O}_x$ is the maximal ideal.

Let now $V_i \subset k^n$ (resp. $W_i \in k^{n+r}$) be the subspace where at most the first i (resp. the first j, with $b_j > a_i \geq b_{j-1}$) coordinates are nonzero. Then H' is naturally isomorphic to FL, and the natural projections $H \longrightarrow H'$, $H \longrightarrow H"$ and $H" \longrightarrow H$ are (trivial) vector bundles. Let f', $f"$ be the images of f in H', $H"$.

By lemma 1.10 f' has a generalization ψ with rank $\psi = n-1$, $W_l \cap \psi(V) = \psi(V_l)$, where $l = l(\psi)$. Let $h"$ be the generic point of the fibre over ψ of $H" \to H'$. We will prove that the maximal minors of $h"$ generate m_x/m_x^2 by constructing a g in the fibre with the same property. As $\dim \psi(V_l) = l - 1 = \dim(W_l \cap \psi(V))$, we can choose w_1, w_2 in W_l such that their images in $W_l/(W_l \cap \psi(V))$ are linearly independent. Let $v_1, \ldots, v_l$ be a flag basis of V such that $v_l \in \ker \psi$. Define linear maps ψ_1, $\psi_2 : V \to W$ by

$$\psi_j(v_i) = \begin{cases} w_j & \text{if } i = l \\ 0 & \text{otherwise.} \end{cases}$$

Then if $g \in H"$ is defined by

$$g = \psi + y\psi_1 + z\psi_2$$

in suitable bases for V and W we have

$$g = \begin{pmatrix} I_{n-1} & 0 \\ 0 & y \\ 0 & z \\ 0 & 0 \end{pmatrix}$$

hence its maximal minors generate m_x/m_x^2.

Let now $h \in H$ be the generic element of the fibre of $H \to H"$ over $h"$. As ψ has rank $n-1$ and the maximal minors of $h"$ generate m_x/m_x^2, E_h has a simple singularity at x; on the other hand h is a generalization of f as required. □

2. Proof of Theorem 2

We want to prove theorem 2 by descending induction on P; more precisely we will prove that if theorem 2 holds for $P+1$, then it holds for P. To start the induction, note that, given P, for large enough m the polynomial $P+m$ is positive. In this case $\mathcal{U}(P+m)$ is empty by proposition 1.1 and the theorem is trivial. So we assume the theorem for $P+1$ and have to prove it for P. By corollary 1.7 each irreducible component of $\mathcal{S}p(P)$ intersects $\mathcal{U}_{\geq 1}(P)$. Therefore it is enough to prove the result for $\mathcal{S}p_{\geq 1}(P) \subset \mathcal{U}_{\geq 1}(P)$.

Lemma 2.1.

(1) $\mathcal{S}p(P) \subset \mathcal{U}(P)$ *has codimension* ≥ 1.

(2) $\mathcal{S}p_{>1}(P) \subset \mathcal{U}_{>1}(P)$ *has codimension* ≥ 1.

Proof. (1) We prove the result by descending induction on P, the beginning of the induction being again trivial. So we again have to show the result for $\mathcal{U}_{\geq 1}(P)$. By remark 1.4 we see that $\mathcal{U}_{>1}(P)$ has at least codimension 1 in $\mathcal{U}_{\geq 1}(P)$ and thus it is enough to show that $\mathcal{S}p_1(P) \subset \mathcal{U}_1(P)$ has codimension ≥ 1.

Case $\chi \geq 0$. By induction $\mathcal{S}p_0(P) \subset \mathcal{U}_0(P)$ has codimension ≥ 1. Let $\pi : \mathcal{U}_1(P) \to \mathcal{U}_0(P+1) \times \mathbb{P}_2$ be as in remark 1.4. If $F \in \mathcal{U}_0(P+1)$ is nonspecial, and $E \in \pi^{-1}(F,x)$, then by the exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow k_x \longrightarrow 0 \qquad (**)$$

E is special if and only if the induced map $H^0(F) \to k_x$ is the zero map.

As each $F \in \mathcal{U}_0(P+1)$ has at least one section, the locus

$$N_0 := \{(F,x) \in \mathcal{U}_0(P+1) \times \mathbb{P}_2 \mid \text{all sections of } F \text{ vanish at } x\}$$

has at least codimension 1. Now for $(F,x) \notin N_0$ and F nonspecial the codimension of $\mathcal{S}p_1(P)$ in $\pi^{-1}(F,x)$ is at least 1, so (1) follows.

Case $\chi < 0$. In this case $\mathcal{S}p_1(P) \subset \pi^{-1}(\mathcal{S}p_0(P+1) \times \mathbb{P}_2)$, and $\mathcal{S}p_0(P+1)$ has codimension ≥ 1 in $\mathcal{U}_0(P+1)$ by induction.

(2) By remark 1.4 the generic element of $\mathcal{U}_{>1}(P)$ has exactly two simple singularities, therefore it is enough to prove the result for $\mathcal{U}_{2,x}(P)$, the stack of sheaves having exactly two simple singularities, one of which at x. Now again $\mathcal{U}_{2,x}(P)$ is a $\mathbb{P}_{r-1}$-bundle over $\mathcal{U}_{1,x}(P+1) \times \{\mathbb{P}_2 \setminus x\}$ and by the proof of (1) we know that $\mathcal{S}p_{1,x}(P+1)$ has codimension at least 1 in $\mathcal{U}_{1,x}(P+1)$. Arguing as in the proof of (1) shows the result.

□

By part (2) of lemma 2.1 it is enough to prove theorem 2 for $\mathcal{S}p_1(P) \subset \mathcal{U}_1(P)$.

Case (1): $\chi(P) > 0$.

For any $(F,x) \in \mathcal{U}_0(P+1)$ let $ev_{F,x} : H^0(F, \mathbb{P}_2) \longrightarrow F(x)$ be the evaluation map at x. For $i = 0, 1$ let

$$N_i := \left\{ (F,x) \in \mathcal{U}_0(P+1) \times \mathbb{P}_2 \;\middle|\; rk(ev_{F,x}) \leq i \right\}$$

with the reduced structure as a closed substack.

Lemma 2.2. *N_i has codimension at least $2-i$ for $i = 0, 1$.*

Proof. As $\mathcal{S}p(P+1)$ has codimension at least 2 in $\mathcal{U}_0(P+1)$ by induction, it is enough to prove the result for $\big(N_i \setminus \mathcal{S}p(P+1)\big) \times \mathbb{P}_2$.
Case $i = 0$: No fibre of the projection $N_0 \longrightarrow \mathcal{U}_0(P+1)$ has dimension 2, and if the the fibre over F has dimension 1, then $h^0(F) = h^0(F(-1))$, so $F(-1)$ is special; in fact $h^2(F(-1))$ is zero because $h^0(F) \neq 0$, applying proposition 1.1. So the locus of nonspecial F such that the fibre of N_0 over F has codimension 1 belongs to the inverse image of $\mathcal{S}p(Q+1)$ via the isomorphism $\mathcal{U}_0(P+1) \to \mathcal{U}_0(Q+1)$, given by $F \mapsto F(-1)$, where Q is the polynomial defined by $Q(n) = P(n-1)$. So the result follows by lemma 2.1.
Case $i = 1$: Let $F \in \mathcal{U}_0(P+1)$ be a nonspecial sheaf such that the subsheaf spanned by global sections is a subsheaf of rank 1, i.e. of the form $\mathcal{O}(l) \otimes \mathcal{I}_Z$ for $\mathcal{I}_Z$ the ideal sheaf of a zero dimensional subscheme of $\mathbb{P}_2$. By taking double duals and using that F is locally free we see that $\mathcal{I}_Z = \mathcal{O}_{\mathbb{P}_2}$. The inclusion $\mathcal{O}_{\mathbb{P}_2}(l) \longrightarrow F$ induces an isomorphism of global

sections. By $h^0(F) \geq 2$ we see that $l \geq 1$. As the isomorphism $H^0(\mathcal{O}_{\mathbb{P}_2}(l)) \longrightarrow H^0(F)$ is induced by the inclusion above, we also have isomorphisms $H^0(\mathcal{O}_{\mathbb{P}_2}(k)) \longrightarrow H^0(F(k-l))$ for all $k \leq l$. By lemma 2.1 we can assume that $F(k-l)$ is nonspecial for $k = -1, \ldots, l$. So we have $\chi(F(k-l)) = \chi(\mathcal{O}_{\mathbb{P}_2}(k))$ for $k = 0, \ldots, l$, in particular $\chi(F(-1-l)) \leq 0$, $\chi(F(-l)) = 1$, $\chi(F(1-l)) = 3$. The Riemann-Roch theorem implies that the second difference function of P_F is the rank of F and so F has to have rank 1. So we get a contradiction. $\square$

Let again $\pi : \mathcal{U}_1(P) \longrightarrow \mathcal{U}_0(P+1) \times \mathbb{P}_2$ be the projection. By induction $\mathcal{S}p(P+1)$ has codimension at least 2 in $\mathcal{U}_0(P+1)$, so it is enough to prove the result for $\pi^{-1}((\mathcal{U}_0(P+1) \setminus \mathcal{S}p(P+1)) \times \mathbb{P}_2)$. For $E \in \pi^{-1}(F, x)$ with F nonspecial we have that $E \in \mathcal{S}p(P)$ if and only if the map $H^0(F) \to k_x$ induced by the exact sequence (**) is the zero map.

So for $(F, x) \in \mathcal{U}_0(P+1) \times \mathbb{P}_2$, F nonspecial, the codimension of $\mathcal{S}p_1(P)$ in $\pi^{-1}(F, x)$ is 0 or 1 if and only if $(F, x) \in N_0$ (resp. $N_1 \setminus N_0$); hence by lemma 2.1 the proof is complete.

Case (2): $\chi(P) = 0$.

Either $\mathcal{S}p(P)$ is equal to $\mathcal{U}(P)$, or it is a Cartier divisor (by lemma 1.3). Let D_i be irreducible Cartier divisors on an algebraic stack, and assume there is a substack Z such that $D_i \cap Z$ is a nonempty divisor for all i; then one can see that $D = \sum D_i$ is a reduced and irreducible divisor if $D \cap Z$ is. This is easy if the stack is a scheme, and the general case follows. We will use this to prove case (2).

Let $(F, x) \in \mathcal{U}_0(P+1) \times \mathbb{P}_2$ such that F is nonspecial and its nonzero sections do not vanish at x. The fiber of $\pi : \mathcal{U}_1(P) \to \mathcal{U}_0(P+1) \times \mathbb{P}_2$ over (F, x) is naturally isomorphic to $\mathbb{P}(F(x))$; we have an exact sequence on $T := \mathbb{P}(F(x)) \times \mathbb{P}_2$

$$0 \to \mathcal{E} \to F(x) \otimes \mathcal{O}_T \to p^*(\mathcal{O}_{\mathbb{P}(F(x))}(1))_{|\mathbb{P}(F(x)) \times \{x\}} \to 0, \qquad (***)$$

where p is the projection to $\mathbb{P}(F(x))$. As $F(x) \otimes \mathcal{O}_T$ has no higher cohomology on the fibres, by lemma 1.3 the (scheme-theoretic) intersection of $\mathcal{S}p_1(P)$ with $\mathbb{P}(F(x))$ is the determinant locus of the mapping

$$p_*(F(x) \otimes \mathcal{O}_T) \to p_* \left(p^*(\mathcal{O}_{\mathbb{P}(F(x))}(1))_{|\mathbb{P}(F(x)) \times \{x\}}\right)$$

induced by the exact sequence (***). This mapping is isomorphic to $\mathcal{O}_{\mathbb{P}(F(x))} \to \mathcal{O}_{\mathbb{P}(F(x))}(1)$, where the isomorphism $p_*(F(x) \otimes \mathcal{O}_T) \to \mathcal{O}_{\mathbb{P}(F(x))}$ is given by choosing a (nonzero) section of F, and it is nonzero because the section does not vanish at x. So $\mathcal{S}p_1(P) \cap \pi^{-1}(F, x)$ is a reduced, irreducible divisor.

It is now enough to prove that every component of $\mathcal{S}p_1(P)$ intersects $\pi^{-1}(F, x)$ to complete the proof; we will in fact prove that every component surjects on $\mathcal{U}_0(P+1) \times \mathbb{P}_2$.

Note that if F is nonspecial in $\mathcal{U}_0(P+1)$ then a nonzero section must vanish in codimension at least 2; otherwise we would have a section of $F(-1)$, hence $h^0(F) \geq 3$. So the locus

$$\{(F, x) \in \mathcal{U}_0(P+1) \times \mathbb{P}_2 | \text{ either } F \in \mathcal{S}p(P+1) \text{ or } h^0(F \otimes \mathcal{I}_{\{x\}}) = 1\}$$

has codimension at least two, and so the image of a component of $\mathcal{S}p_1(P)$ cannot be contained in it.

As out of this locus the intersection of the component with the fibre has codimension one, and as each component of $\mathcal{S}p(P)$ is a divisor, the required surjectivity is proven.

Case (3): $\chi(P) < 0$.

From the injection $H^0(E) \to H^0(E^{**})$ it follows that $\mathcal{S}p_1(P) \subset \pi^{-1}(\mathcal{S}p_0(P+1) \times \mathbb{P}_2)$. So if $\chi(P) < -1$ then $\mathcal{S}p_1(P)$ has codimension at least 2 by induction.

Assume therefore that $\chi(P) = -1$. Let $\mathcal{V}sp(P+1) \subset \mathcal{S}p(P+1)$ be the locus of sheaves F such that $h^0(F) \geq 2$. We note that $\mathcal{V}sp(P+1)$ has codimension at least 2 in $\mathcal{U}(P+1)$. To see this, arguing as before it is enough to prove that $\mathcal{V}sp_1(P+1)$ has codimension ≥ 2 in $U_1(P+1)$; but $\mathcal{V}sp_1(P+1)$ is contained in the inverse image of $\mathcal{S}p_0(P+2) \times \mathbb{P}_2$, so by induction we are done.

For $F \in \mathcal{S}p(P+1) \setminus \mathcal{V}sp(P+1)$ the nonzero sections of F vanish in codimension at least 2. Therefore the locus

$$\{(F, x) \in \mathcal{S}p_0(P+1) \times \mathbb{P}_2 \mid \text{either } F \in \mathcal{V}sp(P+1) \text{ or } h^0(F \otimes \mathcal{I}_{\{x\}}) = 1\}$$

has codimension at least two, and as the fibers of $\mathcal{S}p_1(P)$ over its complement in $\mathcal{S}p_0(P+1) \times \mathbb{P}_2$ (which is a divisor in $\mathcal{U}_0(P+1) \times \mathbb{P}_2$ by induction) have codimension one, the result is proven. □

3. Proof of Theorem 1 from Theorem 2

By lemma 1.3 theorem 2 implies that theorem 1 holds for $Rst^s(P)$. We now want to prove theorem 1 for $M^s(P)$ by showing that $M^s(P) \setminus Rst^s(P)$ has codimension at least 2 in $M^s(P)$. Finally we will show that theorem 1 for $M^s(P)$ for all P implies theorem 1 for $M(P)$.

Lemma 3.1. *$M^s(P) \setminus Rst^s(P)$ has codimension at least 2 in $M^s(P)$.*

Proof. We first prove that $M_0^s(P) \setminus Rst_0^s(P)$ has codimension at least 2 in $M_0^s(P)$. By the theorem of Grauert-Mülich [G-M] a stable bundle of rank 2 is rst, so we can assume the rank is at least 3. It is enough to show that in versal families of stable bundles on $\mathbb{P}_2$ non-rst bundles occur in codimension at least 2. We assume that there exists a versal family $\mathcal{E}$ of stable bundles on $\mathbb{P}_2$ over a scheme S, such that non-rst bundles occur over an irreducible divisor D. Let $\mathbb{P}_2^*$ be the space of lines in $\mathbb{P}_2$ and $\Gamma \subset \mathbb{P}_2 \times \mathbb{P}_2^*$ the incidence variety with projection p_2 to $\mathbb{P}_2^*$. Let $\widetilde{\mathcal{E}}$ by the pullback of $\mathcal{E}$ to $\Gamma \times S$. For every vector bundle F on $\mathbb{P}_1$ we have a locally closed subset $Z_F := \{(l,t) \in \mathbb{P}_2^* \times S \mid \mathcal{E}_t|_l \simeq F\}$ of $\mathbb{P}_2^* \times S$. Z_F is smooth of codimension $dim(\mathrm{Ext}^1_{\mathbb{P}_1}(F,F))$ or empty: in fact let $U \subset \mathbb{P}_2^*$ be an open subset over which there exists a trivialization $f : \mathbb{P}_1 \times U \longrightarrow \Gamma \cap p_2^{-1}(U)$. Then by the remarks before proposition 1.1 $f^*(\widetilde{\mathcal{E}})$ is a versal family of bundles on $\mathbb{P}_1$ and thus $Z_F \cap U$ is smooth of codimension $dim(\mathrm{Ext}^1_{\mathbb{P}_1}(F,F))$ or empty.

As $\mathcal{E}$ is not rst over D, there is a nonempty Z_{F_1} with $dim(\mathrm{Ext}^1_{\mathbb{P}_1}(F_1,F_1)) = 1$. However the only possibility for such an F_1 is $F_1 = \mathcal{O}(a-1) \oplus \mathcal{O}(a)^{\oplus r-2} \oplus \mathcal{O}(a+1)$. So in particular if we put $F_0 := \mathcal{O}(a)^{\oplus r}$ then Z_{F_0} is open and dense in $\mathbb{P}_2^* \times S$. By [B] the only degeneration G of F_1 with $dim(\mathrm{Ext}^1_{\mathbb{P}_1}(G,G)) \leq 3$ is $G = \mathcal{O}(a-2) \oplus \mathcal{O}(a+2)$. However this can only occur if the rank is 2, which we have excluded in our assumption. Hence after possibly replacing S by an open subset with nonempty intersection with D, we can assume that Z_{F_1} is closed. Hence $(\mathbb{P}_2^* \times S) \setminus Z_{F_0}$ is smooth along $\mathbb{P}_2^* \times D$, hence $\mathbb{P}_2^* \times D$ is a connected component. Let W be the union of the other components of $(\mathbb{P}_2^* \times S) \setminus Z_{F_0}$. The image $\pi(W)$ under the projection π to S is closed in S and disjoint from D. For all $t \in S \setminus (D \cup \pi(W))$ the restriction of $\mathcal{E}_t$ to every line is F_0. Thus $\mathcal{E}_t \simeq \mathcal{O}(a)^r$, which is however not stable.

It is now enough to show that the complement of $M_1^s(P) \cap Rst(P)$ has codimension at least 1 in $M_1^s(P)$. If E is a stable (or simple) sheaf with exactly one simple singularity, then $F = E^{**}$ is prioritary. To prove this, take any line l containing the singular point of E; then by the exact sequence

$$0 \longrightarrow \mathrm{Hom}(F, F(-1)) \longrightarrow \mathrm{Hom}(F,F) \longrightarrow \mathrm{Hom}(F, F_{|l})$$

every nonzero element of $\mathrm{Hom}(F, F(-1))$ induces a nonzero element of $\mathrm{Hom}(E,E)$ vanishing along l.

Therefore $\mathcal{M}_1^s(P)$ is an open substack of a $\mathbb{P}_{r-1}$-bundle over $\mathcal{P}rior_0(P+1) \times \mathbb{P}_2$ and $\mathcal{R}st_1^s(P)$ is the preimage of $\mathcal{R}st_0(P+1) \times \mathbb{P}_2$. As the complement of $\mathcal{R}st_0(P+1)$

in $\mathcal{P}rior_0(P+1)$ has codimension at least 1 (by [H-L], proposition 1.2), we get that the complement of $\mathcal{R}st_1^s(P)$ in $\mathcal{M}_1^s(P)$ has codimension at least 1; the corresponding statement for the moduli scheme follows from lemma 1.3. □

Proof of Theorem 1. By lemma 3.1 theorem 2 implies theorem 1 for $M^s(P)$. $M(P) \setminus M^s(P)$ is a finite union of locally closed subsets $M(\{P_i\}_{i\in I})$, where $\{P_i\}_{i\in I}$ is a finite set of polynomials with $\sum_i P_i = P$ and $P_i/r(P_i) = P/r(P)$. $M(\{P_i\}_{i\in I})$ is the image of the quasifinite morphism

$$\prod_i M^s(P_i) \longrightarrow M(P); \quad (E_i)_{i\in I} \mapsto [\oplus_i E_i].$$

Now by the remarks after lemma 1.3 if $[\oplus_i E_i]$ is special, then $\oplus_i E_i$ is special, and so one of the E_i has to be special. By the condition $P_i/r(P_i) = P/r(P)$ and lemma 3.1 we see that $Sp(P) \cap M(\{P_i\}_{i\in I})$ has codimension at least 2 in $M(\{P_i\}_{i\in I})$ for $\chi(P) \neq 0$ and codimension at least 1 for $\chi(P) = 0$.

This shows the theorem for $\chi(P) \neq 0$. If $M^s(P) \neq \emptyset$, then $M(P) \setminus M^s(P)$ has at least codimension 1 so by the previous paragraph and preview noelemma 1.3 the result follows also in the case $\chi(P) = 0$. Finally if $M^s(P) = \emptyset$, then $M(P)$ is reduced to a point ([Dr-L] prop. 4.5 and thm. 4.10) and by the previous paragraph $Sp(P) = \emptyset$. □

References

[A] M. Artin, *Versal deformations and algebraic stacks*, Invent. Math. **27** (1974), 165-189.

[B] E. Brieskorn, *Über holomorphe $\mathbb{P}_n$-Bündel über $\mathbb{P}_1$*, Math. Ann. **157** (1965), 343-357.

[B-H1] J. Brun and A. Hirschowitz, *Droites de saut des fibrés stables de rang élevé sur $\mathbb{P}_2$*, Math. Zeit. **181** (1982), 171-178.

[B-H2] J. Brun and A. Hirschowitz, *Variété des droites sauteuses du fibré instanton général*, Comp. Math. **53** (1984), 325-336.

[B-H3] J. Brun and A. Hirschowitz, *Restrictions planes du fibré instanton général*, J. reine angew. Math. **399** (1989), 27-37.

[B-H4] J. Brun and A. Hirschowitz, *Le problème de Brill Noether pour les idéaux de $\mathbb{P}_2$*, Ann. Sc. ENS Paris **20** (1987), 171-200.

[D-M] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. I.H.E.S. **36** (1969), 75-110.

[Dr] J. M. Drezet, *Groupe de Picard des variétés de modules de faisceaux semi-stables sur* $\mathbb{P}_2(\mathbb{C})$, Ann. Inst. Fourier **38** (1988), 105-168.

[Dr-L] J. M. Drezet and J. Le Potier, *Fibrés stables et fibrés exceptionels sur* $\mathbb{P}_2$, Ann. scient. Éc. Norm. Sup. **18** (1985), 193-244.

[G-M] H. Grauert and G. Mülich, *Vektorbündel vom Rang 2 über dem n-dimensionalen komplex projektiven Raum*, Manuscripta Math. **16** (1975), 75-100.

[H] A. Hirschowitz, *Rank Techiques and Jump stratifications*, Vector bundles on Algebraic varieties, Proc. Bombay 1984, Oxford U. Press, 1987, pp. 159-205.

[H-L] A. Hirschowitz and Y. Laszlo, *Fibrés géneriques sur le plan projectif*, Math. Ann. **297** (1993), 85-102.

[L-MB] G. Laumon and L. Moret-Bailly, *Champs algébriques*, preprint (1992).

[LP] J. Le Potier, *Espaces de modules de faisceaux semi-stables sur le plan projectif*, preprint (1993).