# DIVISOR ARRANGEMENTS AND ALGEBRAIC SURFACES

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Abstract. We give a definition of arrangement of divisors on a surface that generalize the notion of arrangement of lines on the plane. To such an arrangement, we associate surfaces and we compute their Chern numbers. Using the arrangement of the 30 elliptic curves of the Fano surface of the Fermat cubic threefold, we obtain a surface with Chern ratio $2\frac{26}{27}$.

Key words: Kummer coverings, Arrangements of divisors, Fano surface of a cubic threefold.
MSC class: 14J29; 14E20.

0.1. **Introduction.** In the spirit of the construction of surfaces by arrangement of lines on the plane done by of F. Hirzebruch, we give a definition of divisor arrangement on a smooth complex projective surface $S$. This is a given set $\Lambda$ of curves $L_1,..,L_t$ on $S$ and certain divisors $H_1,..,H_k$ linear combination of these curves. This definition leads to the construction of surfaces $S_n = S_n(\Lambda)$ for any integer $n > 1$ prime to a certain integer $m(\Lambda) > 0$ associated to the arrangement.
When $n$ varies, the ratio of the Chern numbers of $S_n$ are arbitrarily close to the ratio of the logarithmic Chern numbers of the divisor $L = \sum L_i$. This is the same fact for the surfaces recently constructed by cyclic covering (G. Urzua [13]).
This construction is done to obtain new examples of surfaces of general type with high Chern ratios. Recall that the upper bound for the ratios of the Chern numbers of a surface is 3 and we know only 3 arrangements of lines on the plane that reach this bound [3]. More generally, it is a difficult talk to construct surfaces with Chern ratio close to 3. We obtain:

**Theorem 1.** *The Fano surface $S$ of the Fermat cubic threefold possesses* 30 *elliptic curves that form an arrangement. The ratio of the Chern numbers of the associated surface $S_2$ is equal to $2\frac{26}{27}$.*

As a by product, we remark in paragraph 0.6 that our construction of surfaces enables us to recover the following two apparently very different constructions A) and B):
A) The surfaces $\mathcal{H}_n$ constructed by F. Hirzebruch by an arrangement of lines of the plane.

B) Some surfaces $S_{\times C}C'$ (considered by Sommese [12]) where $f: S \to C$ is a fibration and $C' \to C$ is a particular covering.

0.2. **Definition of divisor arrangement.** Let $S$ be a smooth complex projective surface. We denote by $Div(S)$ the group of divisor on $S$.
Let $L_1, .., L_t$ be $t \geq 2$ smooth curves on $S$. If $H = \sum n_i L_i$ is a divisor, we note $v_i(H) = n_i$ the valuation at $L_i$ of $H$.

**Definition 1.** An **arrangement** $\Lambda = \Lambda(L_i, H_j)$ of the surface $S$ is given by the smooth curves $L_1, .., L_t$ $(t \geq 3)$ and effective divisors $H_0, .., H_k$ $(k \geq 1)$ such that:
a) A singular point of $L = \sum_{i=1}^{i=t} L_i$ is the tranverse intersection point of exactly two curves $L_i, L_j$. We denote by $I$ denote the set of singular points of $L$.
b) The divisors $H_0, .., H_k$ are linearly equivalent and

$$H_i = v_1(H_i)L_1 + .. + v_t(H_i)L_t.$$

c) The divisors $H_0, .., H_k$ are $\mathbb{Q}$ -linearly independents in $Div(S) \otimes \mathbb{Q}$.
d) For each curve $L_i$, there exists a divisor $H_a$ such that $v_i(H_a) = 1$.
e) If $L_i$ and $L_j$ cut each other in a point, then there exist 2 divisors $H_a$ and $H_b$ such that:

$$v_i(H_a) = 1,\ v_j(H_a) = 0 \text{ and } v_j(H_b) = 1$$

or such that:

$$v_i(H_b) = 0,\ v_j(H_b) = 1 \text{ and } v_i(H_a) = 1.$$

f) The linear system generated by $H_0, .., H_k$ is without base points.
We call $L_1, .., L_t$ (resp. $H_1, .., H_k$) the curves (resp. the divisors) of the arrangement $\Lambda = \Lambda(L_i, H_j)$.

*Remark* 1. Let $L_1, .., L_t$ be curves and $H_1, .., H_k$ be divisors which verify hypothesis b),..,f), such that the intersection of any two curves $L_i, L_j$ is transverse but for which there exist points in which 3 or more curves $L_i$ meet. Let $\pi : S' \to S$ be the blow-up of $S$ at these points, $L'_i$ the strict transform of $L_i$ and $L'_{t+1}, .., L'_{t+r}$ the exceptionnal divisors of $\pi$, then $\Lambda = \Lambda(L'_i, \pi^* H_j)$ is an arrangement of $S'$.

0.3. **Construction of surfaces by an arrangement of curves.** Let us consider a surface $S$ and an arrangement $\Lambda = \Lambda(L_i, H_j)$. Let $\mathcal{L}$ be the invertible sheaf $\mathcal{L} = \mathcal{O}_S(H_1)$. Let us denote by $s_0, ..., s_k$ the sections of $\mathcal{L}$ such that $(s_i) = H_k$ (where $(u)$ is the divisor of a section $u$). These sections define a morphism:

$$\begin{array}{rcl} g : S & \to & \mathbb{P}^k \\ s & \to & (s_0 : ... : s_k). \end{array}$$

Let $n > 1$ be an integer. Let $S'_n$ be:

$$S'_n = S_{\times \mathbb{P}^k} \mathbb{P}^k$$

where the morphism $\mathbb{P}^k \to \mathbb{P}^k$ is the morphism

$$(x_0 : ... : x_k) \to (x_0^n : ... : x_k^n).$$

We define below (definition 2) an integer $m(\Lambda) \in \mathbb{N}^*$, named the number of the arrangement, such that if $n$ is prime to $m(\Lambda)$, then $S'_n$ is irreducible. In this case, we denote by $S_n$ the desingularization of $S'_n$ and

$$f : S_n \to S$$

the natural morphism. This is the surface that we will study.

*Remark* 2. The idea of this study was found in [5] where M.N. Ishida consider $S = \mathbb{P}^2$ and $L_1 = H_1, .., L_6 = H_6$ the lines of the complete quadrilateral. But our construction differs because it allows to use arrangement of curves $L_i$ that are not linearly equivalent (see Theorem 3).

We have:

**Theorem 2.** *Suppose that $n$ is prime to $m(\Lambda)$. The morphism $f : S_n \to S$ has degree $n^k$. It is ramified with order $n$ above $L = \sum_{i=1}^{i=t} L_i$ and with order $n^2$ above the set $I$ of singular points of $L = \sum L_i$. The Euler number of $S_n$ is:*

$$e(S_n) = n^{k-2}(n^2 e(S \setminus L) + ne(L \setminus I) + e(I)).$$

*Let $K_S$ be a canonical divisor of $S$ and $K_n$ a canonical divisor of $S_n$, then*

$$(K_n)^2 = n^{k-2}(nK_S + (n-1)L)^2.$$

*If $e(S) \neq e(L)$, then the limit of $\frac{K_n^2}{e(S_n)}$ (where $n$ varies among the integers prime to $m(\Lambda) > 0$) is equal to the ratio*

$$\frac{(K_S + L)^2}{e(S \setminus L)}$$

*of the logarithmic Chern numbers of $L$.*

Let us prove Theorem 2.
As a set, the surface $S'_n$ is :

$$\{(s, (a_0 : .. : a_k))/a_i^n = s_i\}.$$

Let $F = \mathbb{C}(S)$ be the function field of $S$. For $1 \leq i \leq k$, denote by $f_i \in \mathbb{C}(S)$ the rational function such that $f_i = \frac{s_i}{s_0}$. The associated divisor of $f_i$ is $H_i - H_0$. We want to compute the degree of the smallest field $F'$ that contains the $n^{th}$ roots of $f_1, .., f_k$. To this aim, we use the following Proposition:

**Proposition 1.** *([6], Appendix, Thm 10.3). Let $F$ be a field that contains the $n^{th}$ roots of unity. Let $B \subset F^*$ be a finitely generated group and let $F'$ be the smallest subfield (in an algebraic closure of $F$) that contains the $n^{th}$ root of each element of $B$. The degree of $F'$ over $F$ is the order of the group $B/B \cap (F^*)^n$.*

We apply this Proposition to the group $B \subset F^*$ generated by $f_1, .., f_k$. We have a surjective morphism

$$\begin{array}{rcl} \phi : \mathbb{Z}^k & \to & B/B \cap (F^*)^n \\ (a_1, .., a_k) & \to & f_1^{a_1} ... f_k^{a_k}. \end{array}$$

The kernel of $\phi$ contains $n\mathbb{Z}^k$ and by Proposition 1 the field $F'$ has degree $n^k$ if and only if $Ker(\phi) = n\mathbb{Z}^k$.
For $g \in \mathbb{C}(S)$ let $(g)$ be the associated divisor. Write $(f_j) = \sum_{i=1}^{i=t} m_{ij} L_i$ and let $M$ be the $t \times k$ matrix $M = (m_{ij})_{1 \leq i \leq t; 1 \leq j \leq k}$. Let $m(\Lambda) \in \mathbb{N}$ be the integer such that the ideal generated by all the size $k$ minors of $M$ equals $m(\Lambda)\mathbb{Z}$.

**Definition 2.** We call $m(\Lambda)$ the number of the arrangement $\Lambda$.

We denote by $\widetilde{M}$ the matrix with coefficients in $\mathbb{Z}/n\mathbb{Z}$ obtained by reduction modulo $n$ of the coefficients of $M$. The Kernel of $\phi$ is equal to of $n\mathbb{Z}^k$ if and only if the equation:

$$\widetilde{M} A = 0$$

has no non trivial solution $A =^t (a_1, ..., a_k)$ in $(\mathbb{Z}/n\mathbb{Z})^k$. This is so if and only if $m(\Lambda)$ and $n$ are coprime. (Note that the hypothesis c) of definition 1 is necessary, otherwise, we have $m(\Lambda) = 0$).

Suppose now that $m(\Lambda)$ and $n$ are coprime. Then $S'_n$ is an irreducible surface and the function field of $S_n$ and $S'_n$ is:

$$\mathbb{C}(S)(f_1^{\frac{1}{n}}, .., f_k^{\frac{1}{n}}).$$

As this Kummer extension of $\mathbb{C}(S)$ has degree $n^k$, the morphism $f$ has degree $n^k$. Let $s$ be a point of $S$ and let $t$ be a point of $S_n$ above $s$. If $s$ is not a point of $L$ then the local ring of $S_n$ at $t$ is isomorphic to:

$$\mathcal{O}_{S,s}(f_1^{\frac{1}{n}}, .., f_k^{\frac{1}{n}}).$$

The morphism $f$ is not ramified above $s$. There exists an affine open $U_s$ of $S$ and an open $V_s$ of $S_n$ such that $V_s$ is isomorphic to

$$Spec(\mathcal{O}_S(U_s)(f_1^{\frac{1}{n}}, .., f_k^{\frac{1}{n}})).$$

Let us suppose now that $s$ is a point of $I$. Allowing that the indicies may be permuted, we can suppose that $s$ is an intersection point of $L_1$ and $L_2$ and that it is not an element of $H_0$ (the system which contains $H_0, .., H_k$ is free from base points). By the condition e) of the definition, it can be also assumed that the divisors $H_1$ and $H_2$ verify:

$$v_1(H_1) = 1,\ v_2(H_1) = 0 \text{ and } v_2(H_2) = 1.$$

The regular element $f_2$ is written : $f_2 = f_1^{v_1(H_2)} g_2$ and $(f_1, g_2)$ is a parameter system at $s$ because $L_1$ and $L_2$ meet tranversally. Moreover $f_1$ (resp. $g_2$)

is a local equation of $L_1$ (resp. $L_2$) in $s$. For $i \in \{3, .., k\}$, there exists an invertible element $g_i \in \mathcal{O}_{S,s}$ such that:

$$f_i = f_1^{v_1(H_i)} g_2^{v_2(H_i)} g_i.$$

The local ring of $S_p$ at $t$ is isomorphic to the integral closure of

$$\mathcal{O}_{S,s}[f_1^{\frac{1}{n}}, f_2^{\frac{1}{n}}, .., f_k^{\frac{1}{n}}]$$

and this ring is:

$$\mathcal{O}_{S,s}(g_3^{\frac{1}{n}} ....., g_k^{\frac{1}{n}})[f_1^{\frac{1}{n}}, g_2^{\frac{1}{n}}].$$

Hence $f$ has ramification index $n^2$ at $t$. There exists an affine open $U_s$ of $S$ and an open $V_s$ of $S_n$ such that $V_s$ is isomorphic to

$$Spec(\mathcal{O}_S(U_s)(g_3^{\frac{1}{n}} ....., g_k^{\frac{1}{n}})[f_1^{\frac{1}{n}}, g_2^{\frac{1}{n}}]).$$

Let $s$ be a point of $L$ outside $I$. Allowing that the indicies may be permuted, we can suppose that $s$ is a point of $L_1$ and that the divisors $H_1$ and $H_0$ verify:

$$v_1(H_1) = 1 \text{ and } v_1(H_k) = 0.$$

Let be $i \in \{2, .., k\}$, there exist $h_i \in \mathcal{O}_{S,s}$ invertible such that:

$$f_i = f_1^{v_1(H_i)} h_i.$$

The local ring of $S_n$ at $t$ is isomorphic to:

$$\mathcal{O}_{S,s}(h_2^{\frac{1}{n}} ....., h_k^{\frac{1}{n}})[f_1^{\frac{1}{n}}]$$

and $f$ is ramified with order $n$ in $t$. There exists an affine open $U_s$ of $S$ and an open $V_s$ of $S_n$ such that $V_s$ is isomorphic to

$$Spec(\mathcal{O}_S(U_s)(h_2^{\frac{1}{n}}, ..., h_k^{\frac{1}{n}})[f_1^{\frac{1}{n}}]).$$

Now we calculate the Chern numbers of $S_n$. The morphism $f$ is ramified with order $n$ over $L$, hence $K_n$ is numerically equivalent to:

$$f^*(K_S + \frac{n-1}{n} L).$$

The morphism $f$ is an étale covering of $f^{-1}(S \setminus L)$ of degree $n^k$. This is a covering of $f^{-1}(L \setminus I)$ of degree $n^{k-1}$ and above each point of $I$, there are $n^{k-2}$ points. The Euler number of $S_n$ is equal to

$$n^k e(S \setminus L) + n^{k-1} e(L \setminus I) + n^{k-2} e(I).$$

That ends the proof of Theorem 2.

0.4. **Arrangement of the 30 elliptic curves on the Fano surface of the Fermat cubic threefold.** Let $S$ be the Fano surface that parametrizes the lines of the Fermat cubic threefold

$$F = \{x_1^3 + ... + x_5^3 = 0\} \hookrightarrow \mathbb{P}^4.$$

If $s$ is a point of $S$, we denote by $L_s$ the line on $F$ that corresponds to the point $s$. Let $\mu_3$ be the third roots of unity. For $1 \leq i < j \leq 5$, $\beta \in \mu_3$, the hyperplane $\{x_i + \beta x_j = 0\}$ cuts out a cone on $F$ denoted by $\mathcal{C}_{ij}^\beta$. The curve that parametrizes the lines on $\mathcal{C}_{ij}^\beta$ is an elliptic curve $E_{ij}^\beta$ that is naturally embedded in the Fano surface $S$. These 30 cones are the only one contained in $F$ and the 30 elliptic curves of $S$ verify:

$$E_{ij}^\beta E_{st}^\gamma = \begin{cases} 1 & \text{if } \{i,j\} \cap \{s,t\} = \emptyset \\ -3 & \text{if } E_{ij}^\beta = E_{st}^\gamma \\ 0 & \text{otherwise.} \end{cases}$$

and a canonical divisor $K_S$ on $S$ verifies $K_S^2 = 45$ (for these facts see [9]). For $1 \leq u < v \leq 5$, let $B_{uv}$ be $B_{uv} = B_{vu} = \sum_{\mu_3} E_{uv}^\beta$.

**Theorem 3.** *The* 10 *divisors:*

$$K_{ij} = 2B_{ij} + B_{rs} + B_{rt} + B_{st}$$

*(*$\{i,j,r,s,t\} = \{1,..,5\}$*) are canonical divisors of* $S$.
*The* 6 *divisors* $K_{12}, K_{14}, K_{23}, K_{25}, K_{35}, K_{45}$ *and the* 30 *elliptic curves form an arrangement* $\Lambda$ *of* $S$*. The number of the arrangement divides* 3*. Let* $n$ *be an integer prime to* 3*. The Euler caracteristic of* $S_n$ *is :*

$$e(S_n) = (162n^2 - 270n + 135)n^3$$

*and the first Chern number is:*

$$(K_n)^2 = 45(3n-2)^2 n^3.$$

*We have* $(K_2)^2/e(S_2) = 2\frac{26}{27}$ *and* $\lim_{n \notin 3\mathbb{Z}} \frac{(K_n)^2}{e(S_n)} = \frac{5}{2}$.

*Proof.* By the tangent bundle Theorem 12.37 of [2], we can identify the forms $x_1, .., x_5 \in H^o(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)$ to a basis of the space of global sections of the cotangent sheaf $\Omega_S$. Let be $\omega_1$, $\omega_2$ elements of $H^o(S, \Omega_S)$ such that $\omega_1 \wedge \omega_2 \neq 0$. The identification between the two spaces is such that the subjacent set to the canonical divisor associated to $\omega_1 \wedge \omega_2$ parametrizes the points $s$ on $S$ such that the line $L_s \hookrightarrow F$ cuts the space $\{\omega_1 = \omega_2 = 0\} \hookrightarrow \mathbb{P}^4$. For $1 \leq i < j \leq 5$, the intersection of $F$ and the plane $x_i = x_j = 0$ is a smooth elliptic curve $E$. Let $r < s < t$ be integers such that $\{i,j,r,s,t\} = \{1,2,3,4,5\}$. The curve $E$ contains the 9 edges of the cones

$$\mathcal{C}_{rs}^\beta, \mathcal{C}_{rt}^\beta, \mathcal{C}_{st}^\beta, \beta \in \mu_3$$

and $E$ is the base curve of the cones $\mathcal{C}_{ij}^\beta = F \cap \{x_i + \beta x_j = 0\}$, $\beta \in \mu_3$.
Let $p$ be a point of $F$. The scheme $S_p$ that parametrizes the lines on $F$ going through $p$ is an intersection of a cubic and a quadric in a plane (see [8]). If

there is a finite number of lines through $p$, this scheme has degree 6.
A line $L$ on $F$ is called double if there exists a plane $X$ such that

$$XF = 2L + L'$$

where $L'$ is the residual line. A double line going through a point $p$ contribute for a degree at least 2 to the scheme $S_p$.
Suppose that $p$ is a point of $E \hookrightarrow F$ hat is not the vertex of a cone. Then there is three lines through $p$ that come from the 3 cones $F \cap \{x_i + \beta x_j = 0\}$, $\beta \in \mu_3$. As each line of a cone is double, these 3 lines are the only one that goes through $p$.
That implies that each line $L$ on $F$ that goes through a point of $E = F \cap \{x_i = x_j = 0\}$ corresponds to a point $s$ of one of the following 12 elliptic curves:

$$E_{ij}^{\beta}, E_{rs}^{\beta}, E_{rt}^{\beta}, E_{st}^{\beta}, \beta \in \mu_3$$

contained in the Fano surface $S$. Hence the subjacent set to the canonical divisor $K_{ij}$ associated to the form $x_i \wedge x_j$ is the union of these 12 curves. The group of symmetries that preserves the plane $x_i = x_j = 0$ and the cubic $F$ acts on the Fano surface and preserves the canonical divisor $K_{ij}$. That implies that there exist some integers $a, b$ such that

$$K_{ij} = aB_{ij} + b(B_{rs} + B_{rt} + B_{st}).$$

Let $K_S$ be a canonical divisor on $S$. As $K_S^2 = K_S K_{ij} = 45$ and $K_S E_{uv}^{\beta} = 3$, we have :

$$9a + 27b = 45.$$

Since $a$ and $b$ are positive integers, the unique solution is $a = 2$ and $b = 1$.
The 6 divisors $K_{12}, K_{14}, K_{23}, K_{25}, K_{35}, K_{45}$ and the 30 elliptic curves verify properties a),..,d) and f) of definition 1.
For the property e), we consider the following tables:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 12, 34 | 12, 35 | 12, 45 | 13, 24 | 13, 25 | 13, 45 | 14, 23 | 14, 25 |
| 35, 25 | 45, 14 | 35, 23 | 25, 35 | 45, 14 | 25, 23 | 25, 45 | 23, 14 |

and

| | | | | | | |
|---|---|---|---|---|---|---|
| 14, 35 | 15, 23 | 15, 24 | 15, 34 | 23, 45 | 24, 35 | 25, 34 |
| 23, 12 | 23, 14 | 23, 35 | 23, 12 | 14, 12 | 35, 12 | 14, 12. |

On the first line of these tables there are the indices $ij, st$ such that the curve $E_{ij}^{\beta}$ cuts the curve $E_{st}^{\gamma}$ ($\gamma, \beta \in \mu_3$). The second line gives the indices $uv, xy$ such that the divisors $H_a = K_{uv}$, $H_b = K_{xy}$ and the curves $L' = E_{ij}^{\beta}$ and $L'' = E_{st}^{\gamma}$ verify the properties e) of the definition 1. By example, we look at the $8^{\text{th}}$ column of the first table. In that case the divisor $K_{23}$ contains $E_{14}^{\beta}$ with multiplicity 1 and does not contain $E_{25}^{\gamma}$, moreover $K_{14}$ contains $E_{25}^{\gamma}$ with multiplicity 1.
Thus the 6 divisors $K_{12}, K_{14}, K_{23}, K_{25}, K_{35}, K_{45}$ and the 30 elliptic curves form an arrangement. We easily check that the number of this arrangement divides 3.

The Euler characteristic of $S$ is $e(S) = 27$ [2]. Each of the 30 elliptic curves on $S$ cuts 9 elliptic curves and $L = \sum B_{ij}$ contains 135 singular points, hence

$$e(L \setminus I) = 30(0 - 9) = -270$$

and

$$e(L) = e(L \setminus I) + e(I) = -135$$

where $I$ is the set of singular points of $L$. The calculation of $e(S_n)$ ensues. In order to simplify the computation of $(K_n)^2$, we can use the fact (proved in [9]) that $L$ is a bicanonical divisor of $S$. □

0.5. **Arrangements of hyperplane sections.** Let $S$ be a surface. For each integer $t \geq 3$, let $L_1, .., L_t$ be smooth hyperplane sections of an embedding of $S$ such that for all $i \neq j$, the intersection of $L_i$ and $L_j$ is transverse and such that by a point of $S$ goes at most 2 hyperplane sections. For $i \in \{1, ..., t\}$, take $H_i = L_j$. The curves $L_i$ and the divisors $H_i$ form an arrangement $\Lambda_t$ of $S$. The number of this arrangement is 1.
For $n \in \mathbb{N}^*$, let $S_n^t$ be the surface associated to this arrangement. The numbers $K_S L$, $L^2$ and $e(S \setminus L)$, $e(L \setminus I)$ and $e(I)$ are easily calculated and we have $\lim_n \frac{c_1(S_n^n)^2}{e(S_n^n)} = 2$.

**Corollary 1.** *Let $S$ be a smooth projective surface. There are smooth surfaces $S'$ with a morphism $S' \to S$ such that the Chern ratios of $S'$ are arbitrarilly close to* 2.

This fact was known for fibred surfaces [1]. We can ask what is the upper bound $a$ of the ratio $c_1(S')/e(S')$ when $S'$ varies among all surfaces of general type which have a morphism $S' \to S$. For the plane, the response is $a = 3$.

0.6. **Arrangements of lines on the plane and arrangements of smooth fibers of a fibration.** Take $L_1, .., L_k$, $k$ lines on the plane and let be $\pi : S \to \mathbb{P}^2$ the blow-up of points where 3 or more lines meet. The lines $L_i$ and the divisors $H_i = L_i$ verifiy the properties b),..,e) of the definition 1. By remark 1, we obtain an arrangement of $S$. The number of this arrangement is 1 and the surfaces $S_n$ are isomorphic to the surfaces $\mathcal{H}_n$ constructed by F. Hirzebruch.

Now, let us consider $S$ a surface with a fibration $f : S \to C$. Let $\delta_1, .., \delta_k$ be effective divisors of the same linear system on $C$ such that the divisors $H_i = f^*\delta_i$ are smooth. Let $L_1, ..., L_t$ be the irreducibles componants of the $H_j$. The divisors $H_i$ and the fibers $L_i$ form an arrangement $\Lambda(L_i, H_j)$ of $S$, with number $m(\Lambda) = 1$.
Let $C_n \to C$ be the ramified cover of $C$ of degree $n$ above the points subjacent to the divisor $\sum \delta_i$. The surface $S_n$ is the fibred product

$$S \times_C C_n.$$

We obtain the same surfaces as Sommese did in [12].

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