On a Chisini Conjecture

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Abstract

Chisini's conjecture asserts that for a cuspidal curve $B \subset \mathbb{P}^2$ a generic morphism f of a smooth projective surface onto $\mathbb{P}^2$ of degree ≥ 5, branched along B, is unique up to isomorphism. We prove that if $\deg f$ is greater than the value of some function depending on the degree, genus, and number of cusps of B, then the Chisini conjecture holds for B. This inequality holds for many different generic morphisms. In particular, it holds for a generic morphism given by a linear subsystem of the mth canonical class for almost all surfaces with ample canonical class.

Introduction

Let $B \subset \mathbb{P}^2$ be an irreducible plane curve over $\mathbb{C}$ with ordinary cusps and nodes, as the only singularities. Denote by $2d$ the degree of B, and let g be the genus of its desingularization, $c = \#\{\text{cusps of}\, B\}$, and $n = \#\{\text{nodes of}\, B\}$. We shall call B *the discriminant curve* of a generic morphism if there exists a finite morphism $f : S \to \mathbb{P}^2$, $\deg f \geq 3$, satisfying the following conditions:

(i) S is a non-singular irreducible projective surface;

(ii) f is unramified over $\mathbb{P}^2 \setminus B$;

(iii) $f^*(B) = 2R + C$, where R is irreducible and non-singular, and C is reduced;

(iv) $f_{|R} : R \to B$ coincides with the normalization of B.

We shall call such f *a generic morphism.*

Note that if $S \subset \mathbb{P}^r$, f is the restriction to S of a generic projection of $\mathbb{P}^r$ onto $\mathbb{P}^2$, and B is the branch curve of f, then (S, f) is a generic morphism and B is its discriminant curve.

Two generic morphisms (S_1, f_1), (S_2, f_2) with the same discriminant curve B are said to be equivalent if there exists an isomorphism $\varphi : S_1 \to S_2$ such that $f_1 = f_2 \circ \varphi$. In the sequel, "f is unique" means "f is unique up to equivalence".

The following assertion is known as Chisini's Conjecture.

Conjecture 1 *Let B be the discriminant curve of a generic morphism $f : S \to \mathbb{P}^2$ of degree* $\deg f \geq 5$. *Then, for B, the generic morphism f is unique.*

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If $B \subset \mathbb{P}^2$ is the dual curve of a smooth cubic, then B is the discriminant curve of four generic non-equivalent morphisms ([6], [5]). Three of them have degree four, and the last one has degree three. This is the only known (up to now) example of a discriminant curve, for which there exist several non-equivalent generic morphisms.

In general case, as it follows from [5], the number of non-equivalent generic morphisms with a given discriminant curve B is less than or equal to 2^{2g+c-1}.

B. Moishezon proved the Chisini Conjecture for the discriminant curves of generic projections of smooth hypersurfaces in $\mathbb{P}^3$. His proof is based on the presentation of the fundamental group of the complement in $\mathbb{P}^2$ of the discriminant curve of projection, obtained by him in [13]. A short review of other results relating to the Chisini Conjecture, and of some attempts to prove it can be found in [5].

The main result of this paper is

Theorem 1 *Let B be the discriminant curve of a generic morphism $f : S \to \mathbb{P}^2$ of $\deg f = N$. If*

$$N > \frac{4(3d+g-1)}{2(3d+g-1)-c}. \tag{1}$$

Then, for B, the generic morphism f is unique and thus, the Chisini Conjecture holds for B.

Theorem 1 shows that if the degree of a generic morphism with given discriminant curve B is sufficiently large, then this generic morphism is unique for B. Almost all generic morphisms interesting from algebraic geometric point of view satisfy this condition. More precisely, let $E = f^*(\mathbb{P}^1)$ be the preimage of a generic line $\mathbb{P}^1 \subset \mathbb{P}^2$. In order to obtain the following theorems, which are consequences of the main result, we check inequality (1) for morphisms given by three-dimensional subsystems of different linear systems $|E|$ on surfaces of different types.

Theorem 2 *Let S be a surface of general type with ample canonical bundle K_S, $f : S \to \mathbb{P}^2$ a generic morphism such that $E \equiv mK_S$, $m \in \mathbb{N}$ ($\equiv$ means numerical equivalence). Then, for the discriminant curve B of f, the generic morphism f is unique except, possibly, for:*

1) $m = 1,\ K_S^2 = 5,\ p_a = 1,\ \deg B = 20,\ g = 51,\ c = 108,\ n = 12;$

2) $m = 1,\ K_S^2 = 5,\ p_a = 2,\ \deg B = 20,\ g = 51,\ c = 96,\ n = 24;$

3) $m = 1,\ K_S^2 = 6,\ p_a = 1,\ \deg B = 24,\ g = 61,\ c = 132,\ n = 60,$

where $p_a = \chi(\mathcal{O}_S)$ is the arithmetic genus of S.

In all the exceptional cases, if for B there exist non-equivalent generic morphisms, then these morphisms have degree ≤ 6.

Theorem 3 *Let S be a Del Pezzo surface and $f : S \to \mathbb{P}^2$ be a generic morphism such that $E \in |-mK_S|$, $m \in \mathbb{N}$. Then, for the discriminant curve B of f, the generic morphism f is unique.*

Theorem 4 *Let $f : S \to \mathbb{P}^2$ be any generic morphism of $S = \mathbb{P}^1 \times \mathbb{P}^1$. Then, for the discriminant curve B of f, the generic morphism f is unique.*

Theorem 5 *Let S be a K3 surface and $f : S \to \mathbb{P}^2$ any generic morphism. Then, for the discriminant curve B of f, the generic morphism f is unique.*

Theorem 6 *Let S be an Enriques surface and $f : S \to \mathbb{P}^2$ any generic morphism. Then, for the discriminant curve B of f, the generic morphism f is unique except, possibly, for $\deg f = 4$. In the exceptional case, $\deg B = 12$, $g = 19$, $c = 36$, $n = 0$ and if such a morphism exists, then:*
1) for B, there exist at least two non-equivalent generic morphisms,
2) any generic morphism f' with such a discriminant curve B has $\deg f' \leq 4$.
In particular, Chisini's Conjecture holds for the discriminant curves of the generic morphisms of Enriques surfaces.

Theorem 7 *Let S be an abelian surface and $f : S \to \mathbb{P}^2$ any generic morphism. Then, for the discriminant curve B of f, the generic morphism f is unique except, possibly, for $\deg f = 6$. In the exceptional case, $\deg B = 18$, $g = 28$, $c = 72$, $n = 36$ and if, for B, there exists a generic morphism f' which is not equivalent to f, then $\deg f' \leq 6$.*

Theorem 8 *Let $S \subset \mathbb{P}^N$ be a complete intersection and $f : S \to \mathbb{P}^2$ the restriction of a generic projection. Then, for the discriminant curve B of f, the generic morphism f is unique.*

Theorem 9 *Let S be a projective non-singular surface and L an ample divisor on S, $f : S \to \mathbb{P}^2$ a generic morphism given by a three-dimensional subsystem $\{E\} \subset |mL|$, $m \in \mathbb{Q}$, and B its discriminant curve. Then there exists a constant m_0 (depending on L^2, (K_S, L), K_S^2, p_a) such that, for B, the generic morphism f is unique if $m \geq m_0$. In particular, if $L = K_S$, then one can take $m_0 = 2$.*

Theorem 10 *The Chisini Conjecture holds for the dual curve B of a nodal plane curve except, possibly, for:*
1) $\deg B = 30,\ g = 10,\ c = 72,\ n = 324$;
2) $\deg B = 20,\ g = 6,\ c = 45,\ n = 120$;
3) $\deg B = 18,\ g = 5,\ c = 39,\ n = 92$;
4) $\deg B = 16,\ g = 4,\ c = 33,\ n = 68$.
In all the exceptional cases, if, for B, there exist non-equivalent generic morphisms, then these morphisms have degree ≤ 6.

Theorem 11 *The Chisini Conjecture holds for a curve B of genus $g \leq 3$.*

Theorem 12 *The Chisini Conjecture holds for B satisfying the inequality*

$$d > 3(g - 1).$$

Unfortunately, in general case we have no a satisfactory description, purely in terms of algebraic geometry, of the set of discriminant curves with given degree, genus, and the number of cusps. But it is possible to give such a description in some particular cases. For example, in [23], Zariski showed that a sextic with 6 ordinary cusps is the discriminant curve of some generic morphism if and only if these 6 cusps lie on a conic, and he proved that there exist sextics with 6 cusps which do not lie on a conic. We also give, in terms of algebraic geometry, a description of discriminant curves of morphisms given by three-dimensional subsystems of the mth canonical class for $m \geq 21$ (see section 4.2 below). On the other hand, the set of the discriminant curves can be completely described in terms of the fundamental group of their complement in $\mathbb{P}^2$ (see Proposition 1 below).

In section 1, we recall some well-known facts on generic morphisms and their discriminant curves. Section 2 is devoted to the proof of Theorem 1. In section 3, we check inequality (1) in different cases in order to prove Theorems 2 - 12. Section 4 contains a more detailed investigation of the case when a generic morphism is given by a linear subsystem of the mth canonical class. In section 5, we briefly discuss the question about the number of irreducible components of moduli space of discriminant curves with given degree, genus, and the number of cusps, and we apply Theorem 2 to find new examples of Zariski's pairs.

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1 Auxiliary results.

1.1. Let B be a discriminant curve of a generic morphism $f : S \to \mathbb{P}^2$, $\deg f = N$, $E = f^*(\mathbb{P}^1)$. We have $(E^2)_S = N$.

Lemma 1 $\deg B = 2d$ *is even and, consequently,* $d \in \mathbb{N}$.

Proof. By Hurwitz's formula,

$$2g(E) - 2 = -2N + \deg B.$$

Hence $\deg B$ is even.

Since $g(E) \geq 0$, we have the following inequality

$$\deg f \leq d + 1. \tag{2}$$

Now we recall some inequalities contracting the genus, degree, and the number of cusps of B and following from Plücker's formulas and Nori's result [15]. For the dual curve B^* of B, put $\delta = \deg B^*$, $\gamma = \#\{\text{cusps of } B^*\}$ and $\nu = \#\{\text{nodes of } B^*\}$. From Plücker's formulas

$$\begin{array}{rcl} \delta & = & 2d(2d-1) - 2n - 3c\,; \\ 2d & = & \delta(\delta-1) - 2\nu - 3\gamma\,; \\ 2g & = & (2d-1)(2d-2) - 2n - 2c\,; \\ 2g & = & (\delta-1)(\delta-2) - 2\nu - 2\gamma \end{array}$$

it follows that

$$\begin{array}{rcl} \delta & = & 4d - c + 2g - 2\,; \\ \gamma & = & 6d - 2c + 6g - 6. \end{array}$$

Since $\delta \geq 0$ and $\gamma \geq 0$, we have

Lemma 2

$$\begin{aligned} c &\leq 4d + 2g - 2\,; \\ c &\leq 3d + 3g - 3\,. \end{aligned}$$

Corollary 1 $c < 2(3d + g - 1)$.

Lemma 3 $3d + g - 1 \leq 2c$.

Proof. If B is a discriminant curve of a generic morphism, then $\pi_1(\mathbb{P}^2 \setminus B)$ is not abelian. Hence by [15], $B^2 - 6c \leq 2n$. For a plane cuspidal curve B of $\deg B = 2d$ we have $g + c + n = (2d-1)(d-1)$. Hence $B^2 - 2n = 4d^2 - 2n = 2c + 6d + 2g - 2 \leq 6c$.

Lemma 4

$$\begin{aligned} R^2 &= 2d^2 - c - n = && (3) \\ &= 3d + g - 1 && (4) \end{aligned}$$

Proof. We have $K_S = -3E + R$. Hence (4) follows from adjunction formula:

$$\frac{(K_S + R, R)}{2} = \frac{(-3E + 2R, R)}{2} = -3d + R^2 = g - 1.$$

We obtain (3) if instead of g we substitute $g = (2d-1)(d-1) - c - n$.

Lemma 5

$$N \leq \frac{4d^2}{3d + g - 1}, \tag{5}$$

and (5) is the equality if and only if either $E \equiv mK_S$ for some $m \in \mathbb{Q}^*$, or $K_S \equiv 0$.

Proof. By Hodge's Index Theorem,

$$\begin{vmatrix} E^2 & (E,R) \\ (E,R) & R^2 \end{vmatrix} = \begin{vmatrix} N & 2d \\ 2d & 3d + g - 1 \end{vmatrix} \leq 0,$$

i.e.

$$N \leq \frac{4d^2}{3d + g - 1}\,,$$

and we have the equality if and only if E and R are linearly dependent in $NS(S) \otimes \mathbb{Q}$, where $NS(S)$ is the Neron-Severi group of S. In the last case, since $K_S = -3E + R$, either $E \equiv mK_S$ for some m, or $K_S \equiv 0$.

Lemma 6

$$\begin{aligned} K_S^2 &= 9N + 2d^2 - 12d - c - n = \\ &= 9N - 9d + g - 1. \end{aligned}$$

Proof. $K_S^2 = (-3E + R, -3E + R) = 9E^2 - 6(E,R) + R^2 = 9N - 12d + R^2$.

Lemma 7 *The topological Euler characteristic*

$$\begin{aligned} e(S) &= 3N + 4d^2 - 6d - 3c - 2n = \\ &= 3N + 2(g-1) - c. \end{aligned}$$

Proof. Using a generic pencil of lines in $\mathbb{P}^2$ and its preimage $\{E_t\}$ in S, we get a formula for $e(S)$:

$$\begin{aligned} e(S) + N &= \delta + 2(2 - 2g(E)) = 2d(2d-1) - 3c - 2n - 2(K_S + E, E) = \\ &= 2d(2d-1) - 3c - 2n - 2(-2E + R, E) = 4N + 4d^2 - 6d - 3c - 2n. \end{aligned}$$

From Noether's formula $K_S + e(S) = 12p_a$ it follows

Lemma 8 *The Euler characteristic of $\mathcal{O}_S$*

$$\begin{aligned} p_a = 1 - q + p_g &= N + \frac{d(d-3)}{2} - \frac{c}{3} - \frac{n}{4} = \\ &= N + \frac{3g - 3 - 9d - c}{12}. \end{aligned}$$

Corollary 2 *([14]).*

$$\begin{aligned} c &\equiv 0 \ (mod\, 3)\,; \\ n &\equiv 0 \ (mod\, 4)\,. \end{aligned}$$

Lemma 9 *The divisor R is ample on S.*

Proof. It suffices to show that $(R, \Gamma) > 0$ for any irreducible curve Γ. If $(R, \Gamma) \leq 0$, then by Hodge's Index Theorem, $\Gamma^2 < 0$ and $(R, \Gamma) = 0$, since by Lemma 4, $R^2 > 0$ and R is irreducible. If $\Gamma^2 < 0$ and Γ is irreducible, then $(K_S, \Gamma) \geq -1$, i.e. $-3(E, \Gamma) + (R, \Gamma) \geq -1$, which contradicts $(R, \Gamma) = 0$.

1.2. Let us fix $p \in \mathbb{P}^2 \setminus B$ and denote by $\pi_1 = \pi_1(\mathbb{P}^2 \setminus B,\, p)$ the fundamental group of the complement of B. Choose any point $x \in B \backslash Sing\, B$ and consider a line $\Pi = \mathbb{P}^1 \subset \mathbb{P}^2$ intersecting B transversely at x. Let $\gamma \subset \Pi$ be a circle of small radius with center at x. If we choose an orientation on $\mathbb{P}^2$, then it defines an orientation on γ. Let Γ be a loop consisting of a path L in $\mathbb{P}^2 \setminus B$ joining the point p with a point $q \in \gamma$, the circuit in positive direction along γ beginning and ending at q, and a return to p along the path L in the opposite direction. Such loops Γ (and the corresponding elements in π_1) will be called *geometric generators*. It is well-known that π_1 is generated by geometric generators, and any two geometric generators are conjugated in π_1, since B is irreducible.

For each singular point s_i of B we choose a small neighborhood $U_i \subset \mathbb{P}^2$ such that $B \cap U_i$ is defined (in local coordinates in U_i) by equation $y^2 = x^3$ if s_i is a cusp, and $y^2 = x^2$ if s_i is a node. Let p_i be a point in $U_i \setminus B$. It is well-known that if s_i is a cusp, then $\pi_1(U_i \setminus B, p_i)$ is isomorphic to the braid group Br_3 of 3-string braids and generated by two geometric generators (say a and b) satisfying the following relation

$$aba = bab.$$

If s_i is a node, then $\pi_1(U_i \setminus B, p_i)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ generated by two commuting geometric generators.

Let us choose smooth paths γ_i in $\mathbb{P}^2 \setminus B$ joining p_i and p. This choice defines homomorphisms $\psi_i : \pi_1(U_i \setminus B, p_i) \to \pi_1$. Denote the image $\psi_i(\pi_1(U_i \setminus B, p_i))$ by G_i if s_i is a cusp, and Γ_i if s_i is a node.

A generic morphism of degree N determines a homomorphism $\varphi : \pi_1 \to \mathfrak{S}_N$, where $\mathfrak{S}_N$ is the symmetric group. This homomorphism φ is determined uniquely up to inner automorphism of $\mathfrak{S}_N$.

Proposition 1 *The set of the non-equivalent generic morphisms of degree N possessing the same discriminant curve B is in one to one correspondence with the set of the epimorphisms $\varphi : \pi_1(\mathbb{P}^2 \setminus B) \to \mathfrak{S}_N$ (up to inner automorphisms of $\mathfrak{S}_N$) satisfying the following conditions:*

(i) for a geometric generator γ the image $\varphi(\gamma)$ is a transposition in $\mathfrak{S}_N$;

(ii) for each cusp s_i the image $\varphi(G_i)$ is isomorphic to $\mathfrak{S}_3$ generated by two transpositions;

(iii) for each node s_i the image $\varphi(\Gamma_i)$ is isomorphic to $\mathfrak{S}_2 \times \mathfrak{S}_2$ generated by two commuting transpositions.

Proof. It is well-known that each homomorphism $\varphi : \pi_1 \to \mathfrak{S}_N$ defines a finite morphism $f : S \to \mathbb{P}^2$ of degree N, unramified over the complement of B, and such that S is a normal surface, and vice versa.

Condition (i) is equivalent to one that $f^*(B) = 2R + C$, where R is irreducible and C is reduced (cf. [10] and [11]).

Conditions (ii) and (iii) are equivalent to one that S and R are non-singular at the points in $f^{-1}(s_i)$ (cf. [5]).

Since S is irreducible and $\deg f = N$, $\varphi(\pi_1)$ must act transitively on the set $\overline{N} = \{1, ..., N\}$, and since π_1 is generated by the geometric generators, $\varphi(\pi_1)$ is generated by some subset of transpositions. It is easy to check that subgroup of $\mathfrak{S}_N$, generated by some subset of transpositions acting on $\overline{N}$ transitively, must coincide with $\mathfrak{S}_N$. Therefore φ must be an epimorphism.

Remark 1 *By Proposition 1, if we allow $N = 2$, then a two-sheeted covering $f : S \to \mathbb{P}^2$, branched along a non-singular curve $B \subset \mathbb{P}^2$, can be also considered as a generic morphism.*

Remark 2 *If $B \subset \mathbb{P}^2$ is the dual curve of a smooth cubic, then by Proposition 1, using the presentation of $\pi_1(\mathbb{P}^2 \setminus B)$ obtained in [22], it is easy to show that B is the discriminant curve of exactly four generic non-equivalent morphisms.*

1.3. Let $s_j \in B$ be a cusp and (x, y) local coordinates around s_j such that $B \cap U_j$ is given by $y^2 = x^3$. Choose a neighborhood $V_j \subset S$ of $p_j \in R$, $f(p_j) = s_j$, such that $f = f_{|V_j} : V_j \to U_j$ is a three-sheeted covering ramified along $R \cap V_j$. It is well known that $f : V_j \to U_j = U$ is unique up to equivalence and, in particular, f is equivalent to the standard covering $\overline{f} : \overline{V} \to U$ given by the normalized equation of third degree:

$$\begin{cases} \overline{V} = \{(w, x, y) \ \mid \ (x, y) \in U, \ w^3 - 3xw + 2y = 0 \ \}\,; \\ \overline{f}(w, x, y) = (x, y)\,. \end{cases}$$

$\overline{V}$ is non-singular and (x, w) are local coordinates in $\overline{V}$. The ramification divisor $\overline{R} = \{(x, w) \in \overline{V} \mid x - w^2 = 0\}$ of $\overline{f}$ is smooth and $\overline{f}^{-1}(B) = 2\overline{R} + \overline{C}$, where $\overline{C} = \{(x, w) \in \overline{V} \mid 4x - w^2 = 0\}$. Note that $\overline{R}$ is tangent to $\overline{C}$ at the origin $o = (0, 0)$, and the intersection multiplicity of $\overline{R}$ and $\overline{C}$ at o is equal to 2.

It was mentioned above that $\pi_1(U \setminus B) \simeq \mathrm{Br}_3 =< a, b \mid aba = bab >$. Then $\overline{f} : \overline{V} \to U$ corresponds to the homomorphism $\overline{\varphi} : \pi_1(U \setminus B) \to \mathfrak{S}_3$ given by $\overline{\varphi}(a) = (1, 2)$ and $\overline{\varphi}(b) = (2, 3)$.

Put $\overline{W} = \{(w_1, w_2, w_3) \mid w_1 + w_2 + w_3 = 0\}$. We have morphisms $\widetilde{f} : \overline{W} \to U$ and $g : \overline{W} \to \overline{V}$ given by

$$
\begin{aligned}
x &= -\frac{1}{3}(w_1w_2 + w_1w_3 + w_2w_3), \\
y &= -\frac{1}{2}w_1w_2w_3, \\
w &= w_1
\end{aligned}
$$

such that $\widetilde{f} = \overline{f} \circ g$, $\deg \widetilde{f} = 6$ and $\deg g = 2$. The morphism g is a two-sheeted covering branched along $\overline{C}$, $g^*(\overline{C}) = 2\widetilde{C}_2$, where $\widetilde{C}_2$ is given in coordinates (w_1, w_2) by $w_1 + 2w_2 = 0$; and $g^*(\overline{R}) = \widetilde{R} + \widetilde{C}_1$, where $\widetilde{R}$ and $\widetilde{C}_1$ are given by $w_1 = w_2$ and $2w_1 + w_2 = 0$ respectively.

Note that $\widetilde{f}$ corresponds to the homomorphism $\widetilde{\varphi} : \pi_1(U \setminus B) \to \mathfrak{S}_6 = \mathfrak{S}(\mathfrak{S}_3)$ defined by $\overline{\varphi}$.

2 Proof of Theorem 1.

2.1. Assume that there exist two non-equivalent generic morphisms (S_1, f_1) and (S_2, f_2) with the same discriminant curve B, $\deg f_1 = N_1$ and $\deg f_2 = N_2$. Put $f_1^*(B) = 2R_1 + C_1$ and $f_2^*(B) = 2R_2 + C_2$. Let us consider

$$S_1 \times_{\mathbb{P}^2} S_2 = \{ (x, y) \in S_1 \times S_2 \mid f_1(x) = f_2(y) \}$$

and let $X = \widetilde{S_1 \times_{\mathbb{P}^2} S_2}$ be the normalization of $S_1 \times_{\mathbb{P}^2} S_2$. Denote by $g_1 : X \to S_1$, $g_2 : X \to S_2$, and $f_{1,2} : X \to \mathbb{P}^2$ the corresponding natural morphisms. We have $\deg g_1 = N_2$, $\deg g_2 = N_1$, and $\deg f_{1,2} = N_1N_2$.

Proposition 2 *If (S_1, f_1) and (S_2, f_2) are non-equivalent, then X is irreducible.*

Proof. The morphism $f_{1,2}$ corresponds to the homomorphism

$$\varphi_{1,2} = \varphi_1 \times \varphi_2 : \pi_1 \to \mathfrak{S}_{N_1} \times \mathfrak{S}_{N_2} \subset \mathfrak{S}_{N_1N_2},$$

where $\varphi_1 : \pi_1 \to \mathfrak{S}_{N_1}$ (resp. φ_2) is an epimorphism corresponding to f_1 (resp. f_2). Put $G = \varphi_{1,2}(\pi_1)$. The group G, as a subgroup of $\mathfrak{S}_{N_1} \times \mathfrak{S}_{N_2}$, acts on $\overline{N_1} \times \overline{N_2}$. Without loss of generality we can assume that for some geometric generator γ the image $\varphi_{1,2}(\gamma) = ((1,\ 2), (1,\ 2))$ is a product of transpositions $(1,\ 2) \in \mathfrak{S}_{N_i}$. Let $p_i : G \to \mathfrak{S}_{N_i}$ be the restriction to G of the projection $pr_i : \mathfrak{S}_{N_1} \times \mathfrak{S}_{N_2} \to \mathfrak{S}_{N_i}$.

Lemma 10 *Let G be a subgroup of $\mathfrak{S}_{N_1} \times \mathfrak{S}_{N_2}$, $N_i > 2$, such that $p_i : G \to \mathfrak{S}_{N_i}$ is an epimorphism for $i = 1, 2$, and such that $((1, 2), (1, 2)) \in G$. Let $St_{(1,1)} \subset G$ be the stabilizer of $(1, 1) \in \overline{N_1} \times \overline{N_2}$. Then the index of $St_{(1,1)}$ in G*

$$(G : St_{(1,1)}) = N_1 N_2$$

except for the case when $N_1 = N_2 = N$ and $G = \Delta \subset \mathfrak{S}_N \times \mathfrak{S}_N$ (up to inner automorphism of one of factors), where Δ is the diagonal subgroup.

Proof. The inclusion $\overline{N_i - 1} \simeq \{2, \dots, N_i\} \subset \{1, 2, \dots, N_i\}$ defines the embedding $\mathfrak{S}_{N_i - 1} \subset \mathfrak{S}_{N_i}$. Then

$$St_{(1,1)} = G \cap (\mathfrak{S}_{N_1 - 1} \times \mathfrak{S}_{N_2 - 1}).$$

Put

$$H_1 \times \{e_2\} = G \cap (\mathfrak{S}_{N_1} \times \{e_2\}) \qquad \text{and} \qquad \{e_1\} \times H_2 = G \cap (\{e_1\} \times \mathfrak{S}_{N_2}),$$

where e_i is the unit element of $\mathfrak{S}_{N_i}$. We note that $\ker p_2 = H_1 \times \{e_2\}$ and $\ker p_1 = \{e_1\} \times H_2$. Therefore $H_1 \times \{e_2\}$ and $\{e_1\} \times H_2$ are normal subgroups of G. Since $p_i : G \to \mathfrak{S}_{N_i}$ is an epimorphism for $i = 1, 2$, H_i is a normal subgroup of $\mathfrak{S}_{N_i}$.

It is well known that if H is a normal subgroup of $\mathfrak{S}_N$, then either $H = \mathfrak{S}_N$, or $H = \mathfrak{A}_N$ is the alternating group, or $H = \{e\}$, and if $N = 4$, then there exist one more possibility: H is the Klein four group

$$K_4 = \{e,\ (1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3)\}.$$

Consider all possible cases.

Case I : $H_1 = \mathfrak{S}_{N_1}$. Since p_2 is an epimorphism, $G = \mathfrak{S}_{N_1} \times \mathfrak{S}_{N_2}$. Therefore $\mid G \mid= N_1!N_2!$ and $\mid St_{(1,1)} \mid= (N_1 - 1)!(N_2 - 1)!$. Hence $(G : St_{(1,1)}) = N_1 N_2$.

Case II : $H_1 = \mathfrak{A}_{N_1}$. Since p_2 is an epimorphism and $\ker p_2 = H_1 \times \{e\}$,

$$\mid G \mid= \frac{N_1!N_2!}{2}.$$

Similarly, if we consider p_1, then we obtain $|\ker p_1| = N_2!/2$, hence $H_2 = \mathfrak{A}_{N_2}$. Therefore $(\sigma_1, \sigma_2) \in G$ if and only if σ_1 and σ_2 have the same sign. Hence

$$\mid St_{(1,1)} \mid= \frac{(N_1 - 1)!(N_2 - 1)!}{2}$$

and $(G : St_{(1,1)}) = N_1 N_2$.

Case III : $H_1 = \{e_1\}$. Therefore p_2 is an isomorphism and there exist two possibilities: either $H_2 = \{e_2\}$ and p_1 is also an isomorphism or $H_2 \neq \{e_2\}$. If p_1 and p_2 are isomorphisms, then $N_1 = N_2 = N$ and $G = \Delta \subset \mathfrak{S}_N \times \mathfrak{S}_N$ up to automorphism of one of factors, and since $((1, 2), (1, 2)) \in G$, this automorphism must be inner. If $H_2 \neq \{e_2\}$, then $p_1 \circ p_2^{-1} : \mathfrak{S}_{N_2} \to \mathfrak{S}_{N_1}$ is an epimorphism (not isomorphism). Since $N_i > 2$, $\mathfrak{S}_{N_2}$ must coincide with $\mathfrak{S}_4$, $\mathfrak{S}_{N_1} = \mathfrak{S}_3$ and $H_2 = K_4$. The rest of the proof of this case will be left to the reader.

Case IV : $N_1 = 4$ and $H_1 = K_4$. The case $H_2 = \mathfrak{S}_{N_2}$ is impossible. In fact, if we consider p_1, then we obtain $\mid G \mid = 4!N_2!$. On the other hand, if to consider p_2, then $\mid G \mid = 4N_2!$, a contradiction.

The case $H_2 = \mathfrak{A}_{N_2}$ is also impossible. In fact, if we consider p_1, then $\mid G \mid = 4!N_2!/2$. On the other hand, if to consider p_2, then $\mid G \mid = 4N_2!$, a contradiction.

The case $H_2 = \{e_2\}$ coincides (up to indexing) with one in Case III.

The case $N_2 = 4$ and $H_2 = K_4$ will be left to the reader.

To complete the proof of Proposition 2 we note that $\deg f_{1,2} = N_1N_2$ and there exists an irreducible component $X_{(1,1)}$ of X such that $\deg f_{|X_{(1,1)}} = (G : St_{(1,1)})$. Hence, by Lemma 9, $\deg f_{|X_{(1,1)}} = N_1N_2$ and, consequently, X is irreducible always except the case when $N_1 = N_2 = N$ and $G \simeq \Delta \subset \mathfrak{S}_N \times \mathfrak{S}_N$. But the exceptional case corresponds to one when (S_1, f_1) and (S_2, f_2) are equivalent.

Proposition 3 *X is non-singular.*

Proof. We need to check the smoothness of X only at $z \in f_{1,2}^{-1}(B)$ such that $p_1 = g_1(z) \in R_1 \subset S_1$ and $p_2 = g_2(z) \in R_2 \subset S_2$. Put $f_{1,2}(z) = s$ and choose a small neighborhood $V_1 \subset S_1$ of p_1 (resp. $V_2 \subset S_2$ of p_2) and a small neighborhood $U \subset \mathbb{P}^2$ of s such that $f_i(V_i) = U$ and, in the chosen neighborhoods, there exist local holomorphic coordinates for which equations defining f_i have the simplest form.

Let $s \in B$ be a non-singular point of B or a node. Then $f_i : V_i \to U$ is given by

$$\begin{array}{rcl} u_{i,1}^2 & = & v_1\,; \\ u_{i,2} & = & v_2\,, \end{array}$$

where $v_1 = 0$ is an equation of $B \cap U$ (or one of the branches of B if s is a node). Therefore $V_1 \times_U V_2$ in $V_1 \times V_2$ is given by equations

$$\begin{array}{rcl} u_{1,1}^2 & = & u_{2,1}^2\,; \\ u_{1,2} & = & u_{2,2}\,, \end{array}$$

or, equivalently,

$$\begin{array}{rcl} u_{1,1} & = & \pm u_{2,1}\,; \\ u_{1,2} & = & u_{2,2}\,. \end{array}$$

Hence $V_1 \times_U V_2$ consists of two irreducible non-singular components one of which corresponds to the sign $+$ and the other does to $-$. Therefore the normalization $\widetilde{V_1 \times_U V_2}$ of $V_1 \times_U V_2$ is the disjoint union of two non-singular surfaces.

Let $s = s_j \in B$ be a cusp and (x, y) local coordinates around s chosen in section 1.3. Let $V_1 \subset S_1$ (resp. for S_2) be a neighborhood of $p_1 = p_{1,j}$ such that $f_1 = f_{1|V_1} : V_1 \to U$ is a three-sheeted covering ramified along $R_1 \cap V_1$. Put $Y = V_1 \times_U V_2$ and let $\widetilde{Y}$ be the normalization of Y. Denote by $g_i : \widetilde{Y} \to V_i$ and $f_{1,2} : \widetilde{Y} \to U$ the corresponding natural morphisms. Since (V_1, f_1) and (V_2, f_2) are equivalent, $f_{1,2}$ corresponds to the homomorphism

$$\varphi_{1,2} = \varphi_1 \times \varphi_2 : \pi_1(U \setminus B) \to \Delta \subset \mathfrak{S}_3 \times \mathfrak{S}_3 \subset \mathfrak{S}_9,$$

thus, $\varphi_{1,2}(\pi_1(U \setminus B))$ acts on $\overline{9} \simeq \overline{3} \times \overline{3}$. It is easy to check that there are two orbits of the action of $\varphi_{1,2}(\pi_1(U \setminus B))$: one of them is the orbit of $(1,1)$, and the other one is the orbit of $(1,2)$. Therefore $\widetilde{Y}$ is the disjoint union of $\widetilde{Y}_{(1,1)}$ and $\widetilde{Y}_{(1,2)}$. It is easily seen (cf. Lemma 1.6 in [5]) that $(\widetilde{Y}_{(1,1)}, f_{1,2})$ is isomorphic to $(\overline{V}, \overline{f})$ (in notation of 1.3) and $g_i : \widetilde{Y}_{(1,1)} \to V_i$ is an isomorphism for $i = 1,\ 2$; $(\widetilde{Y}_{(1,2)}, f_{1,2})$ is isomorphic to $(\overline{W}, \widetilde{f})$, and each $(\widetilde{Y}_{(1,2)}, g_i)$ is isomorphic to $(\overline{W}, g)$. Hence X is non-singular.

Remark 3 *If (S_1, f_1) and (S_2, f_2) with the same discriminant curve are equivalent, then*

1) X is non-singular;

2) X is the disjoint union of two irreducible components: $X = X_{(1,1)} \bigsqcup X_{(1,2)}$, such that $g_{i|X_{(1,1)}} : X_{(1,1)} \to S_i$ is an isomorphism, $i = 1,\ 2$, and $\deg g_{i|X_{(1,2)}} = N-1$, where $N = N_1 = N_2$.

2.2. Let $\widetilde{R} \subset X$ be a curve $g_1^{-1}(R_1) \cap g_2^{-1}(R_2)$, $\widetilde{C} = g_1^{-1}(C_1) \cap g_2^{-1}(C_2)$, $\widetilde{C}_1 = g_1^{-1}(R_1) \cap g_2^{-1}(C_2)$, and $\widetilde{C}_2 = g_1^{-1}(C_1) \cap g_2^{-1}(R_2)$.

Proposition 4

$$\begin{array}{rcl} \widetilde{R}^2 & = & 2(3d+g-1)-c\,, \\ \widetilde{C}_1^2 & = & (N_2-2)(3d+g-1)-c\,, \\ \widetilde{C}_2^2 & = & (N_1-2)(3d+g-1)-c\,, \\ (\widetilde{R}, \widetilde{C}_i) & = & c \qquad \textit{for } i = 1,\ 2. \end{array}$$

Proof. It follows from the local considerations in the proof of Proposition 3 that $\deg g_{1|\widetilde{R}} = 2$ and $g_{1|\widetilde{R}}$ is étale.

One can check that $\widetilde{R}$ and $\widetilde{C}_i$ are intersected only at points over the cusps of B. Consider one of cusps of B, say $s = s_j$, and $p_i = p_{i,j} \in R_i \cap f^{-1}(s)$. In notation of the proof of Proposition 3, let U be a neighborhood of s. It is easily seen that one of the branches of $\widetilde{R} \cap \widetilde{Y}$ lies in $\widetilde{Y}_{(1,1)}$ and the other one does in $\widetilde{Y}_{(1,2)}$. Since $(\widetilde{Y}_{(1,1)}, f_{1,2})$ is isomorphic to (V_i, f_i), the intersection $\widetilde{Y}_{(1,1)} \cap \widetilde{C}_i = \emptyset$ for $i = 1,\ 2$. Consider $\widetilde{Y}_{(1,2)} \cap \widetilde{R}$ and $\widetilde{Y}_{(1,2)} \cap \widetilde{C}_i$. Since each $(\widetilde{Y}_{(1,2)}, g_i)$ is equivalent to $(\overline{W}, g)$, where $(\overline{W}, g)$ was defined in section 1.3, we can identify $(\widetilde{Y}_{(1,2)}, g_1)$ with $(\overline{W}, g)$. Then $\widetilde{Y}_{(1,2)} \cap \widetilde{R}$, $\widetilde{Y}_{(1,2)} \cap \widetilde{C}_1$, and $\widetilde{Y}_{(1,2)} \cap \widetilde{C}_2$ can be identified, respectively, with $\widetilde{R}$, $\widetilde{C}_1$, and $\widetilde{C}_2 \subset \overline{W}$. Since in the neighborhood $\overline{W}$ the intersection multiplicity of $\widetilde{R}$ and $\widetilde{C}_i$ is equal to 1, hence $(\widetilde{R}, \widetilde{C}_i) = c$.

To calculate $\widetilde{R}^2$, consider again the local case. Let $g : \overline{W} \to V$, $g(0) = 0$, be a two-sheeted covering given in local analytic coordinates by

$$\begin{array}{rcl} w_1^2 & = & v_1\,; \\ w_2 & = & v_2\,, \end{array}$$

Denote by $C \subset V$ a curve given by $v_1 = 0$ and R given by $v_2^2 = v_1$. Then C is the branch curve, $g^*(C) = 2\widetilde{C}_2$, and $g^*(R) = \widetilde{R} + \widetilde{C}_1$, where $\widetilde{C}_2$ is given by $w_1 = 0$, and $\widetilde{R}$ and $\widetilde{C}_1$ are given by $w_2 = \pm w_1$. Let $\sigma : \widetilde{V} \to V$ be the composition of two σ-processes with centers at points and such that $\sigma^{-1}(R + C) = R + C + L_1 + L_2$ is a divisor with normal crossings, where L_1 is the exceptional divisor of the first σ-process, L_2 is the exceptional divisor of the second σ-process, and, for simplicity of notation, we again denote by R and C, respectively, the strict preimages of $R \subset V$ and $C \subset V$. Since we performed two σ-processes with centers at points lying in R, R^2 is decreased by 2 (if R is considered as a complete curve). We can perform two σ-processes $\widetilde{\sigma} : \widetilde{W} \to \overline{W}$ (the first one with center at the origin and the second one with center at the intersection point of the strict preimage of the curve $\{w_2 = 0\}$ and the exceptional divisor of the first σ-process). It is easy to check that we again obtain a morphism $\overline{g} : \widetilde{W} \to \widetilde{V}$. Since we performed only one σ-process with center at a point lying in $\widetilde{R}$, $\widetilde{R}^2$ is decreased by 1. Besides, $\overline{g}_{|\widetilde{R}} : \widetilde{R} \to R$ is an isomorphism (locally), and $\overline{g}$ is non-ramified at each point lying in $\widetilde{R}$.

The considerations described above allow us to calculate $\widetilde{R}^2$. Indeed, performing at each point $p_{1,j} \in R_1$ two σ-processes as above, R_1^2 is decreased by $2c$. Performing at each point of $g_1^{-1}(p_{1,j}) \cap \widetilde{R}$ two σ-processes either as above or if a neighborhood of the point in consideration is isomorphic to $\widetilde{Y}_{(1,1)}$ we perform σ-processes as in V_1 as, in view of the fact that $g_1 : \widetilde{Y}_{(1,1)} \to V_1$ is an isomorphism. After performing all these σ-processes, $\widetilde{R}^2$ is decreased by $3c$, and we can find a neighborhood $\widetilde{V}_1$ of the strict preimage of R_1 and a neighborhood $\widetilde{W}$ of the strict preimage of $\widetilde{R}$ such that the restriction $\widetilde{g}_{1|\widetilde{W}} : \widetilde{W} \to \widetilde{V}$ of the obtained morphism $\widetilde{g}_1$ to $\widetilde{W}$ is a non-ramified two-sheeted covering. Hence

$$\widetilde{R}^2 - 3c = 2(R_1^2 - 2c).$$

Thus, by Lemma 4, $\widetilde{R}^2 = 2(3d + g - 1) - c$.

Since $\deg g_1 = N_2$,

$$N_2 R_1^2 = (\widetilde{R} + \widetilde{C}_1, \widetilde{R} + \widetilde{C}_1) = \widetilde{R}^2 + 2(\widetilde{R}, \widetilde{C}_1) + \widetilde{C}_1^2.$$

Hence

$$\widetilde{C}_1^2 = (N_2 - 2)(3d + g - 1) - c.$$

Proposition 4 is proved.

2.3. To complete the proof of Theorem 1, we apply Hodge's Index Theorem. Since by Corollary 1, $\widetilde{R}^2 > 0$,

$$\begin{vmatrix} \widetilde{R}^2 & (\widetilde{R}, \widetilde{C}_i) \\ (\widetilde{C}_i, \widetilde{R}) & \widetilde{C}_i^2 \end{vmatrix} = \begin{vmatrix} 2(3d + g - 1) - c & c \\ c & (N_j - 2)(3d + g - 1) - c \end{vmatrix} \leq 0,$$

i.e.

$$2(N_j - 2)(3d + g - 1)^2 - N_j(3d + g - 1)c \leq 0\,.$$

Hence

$$N_j[2(3d + g - 1) - c] \leq 4(3d + g - 1)\,.$$

Thus, if there exist two non-equivalent generic morphisms (S_1, f_1) and (S_2, f_2), then their degrees

$$N_j \leq \frac{4(3d + g - 1)}{2(3d + g - 1) - c}.$$

3 Uniqueness of generic morphisms for certain types of discriminant curves.

Let us write inequality (1) in the form

$$N[2(3d+g-1)-c]-4(3d+g-1)>0 \tag{6}$$

3.1. *Proof of Theorem* 2. Put $k=K_S^2$, then $E^2=m^2k$ and hence

$$N=\deg f=m^2k. \tag{7}$$

We have $K_S=f^*(K_{\mathbb{P}^2})+R=-3E+R$, hence $R\equiv(3m+1)K_S$. Therefore $\deg B=(E,R)=m(3m+1)k$, i.e.

$$d=\frac{m(3m+1)k}{2}. \tag{8}$$

By adjunction formula, $2(g-1)=(K_S+R,R)$. Therefore

$$g-1=\frac{(3m+2)(3m+1)k}{2}. \tag{9}$$

From (8) and (9) it follows that

$$3d+g-1=(3m+1)^2k. \tag{10}$$

By Lemma 8,

$$c=(12m^2+9m+3)k-12p_a. \tag{11}$$

Let us substitute (7), (10), and (11) into (6). We have

$$\begin{array}{rcl} m^2k[(6m^2+3m-1)k+12p_a]-4(3m+1)^2k & = & \\ m^2k[(6m^2+3m-1)k+12p_a-4(3+\frac{1}{m})^2] & > & 0. \end{array}$$

If $m\geq 2$, then

$$[(6m^2+3m-1)k+12p_a-4(3+\frac{1}{m})^2]\geq 29k+12p_a-49>0,$$

and, since $k\geq 1$ and $p_a\geq 1$, the last inequality does not hold only for $k=1$ and $p_a=1$. In the exceptional case if $m\geq 3$, then

$$[(6m^2+3m-1)k+12p_a-4(3+\frac{1}{m})^2]>62+12-45>0.$$

The case $m=2$, $k=p_a=1$ is impossible. Indeed, in this case by (8) - (11), we have $\deg B=14$, $g=29$, and $c=57$, and the number of nodes n must be non-negative. But by adjunction formula,

$$n=\frac{1}{2}(\deg B-1)(\deg B-2)-g-c=13\cdot 6-29-57<0,$$

a contradiction. Therefore inequality (1) holds always for $m \geq 2$.

If $m = 1$, then

$$[(6m^2 + 3m - 1)k + 12p_a - 4(3 + \frac{1}{m})^2] = 8k + 12p_a - 64 > 0,$$

and, consequently, inequality (1) is equivalent to

$$2k + 3p_a > 16.$$

Since $k = E^2 = N \geq 3$, the last inequality does not hold only for

1) $k = 3, \ p_a \leq 3$;
2) $k = 4, \ p_a \leq 2$;
3) $k = 5, \ p_a \leq 2$;
4) $k = 6, \ p_a = 1$.

The cases 1) and 2) are impossible. Indeed, by (8) - (11), $\deg B = 4k$, $g = 10k + 1$, and $c = 12(2k - p_a)$, and the number of nodes n must be non-negative. But by adjunction formula,

$$n = \frac{1}{2}(\deg B - 1)(\deg B - 2) - g - c = (4k-1)(2k-1) - 10k - 1 - 24k + 12p_a = 4(2k^2 - 10k + 3p_a) < 0$$

for $k = 3, \ p_a \leq 3$ and $k = 4, \ p_a \leq 2$.

The assertion on the degree of generic morphisms in the exceptional cases follows from Lemma 5.

3.2. *Proof of Theorem* 3. Put $k = K_S^2$, then $E^2 = m^2k$ and hence

$$N = \deg f = m^2k. \tag{12}$$

We have $K_S = f^*(K_{\mathbb{P}^2}) + R = -3E + R$, hence $R \in |(-3m + 1)K_S|$. Therefore $\deg B = (E, R) = m(3m - 1)k$, i.e.

$$d = \frac{m(3m-1)k}{2}. \tag{13}$$

By adjunction formula, $2(g - 1) = (K_S + R, R)$. Therefore

$$g - 1 = \frac{(3m-2)(3m-1)k}{2}. \tag{14}$$

It follows from (13) and (14) that

$$3d + g - 1 = (3m - 1)^2k. \tag{15}$$

By Lemma 8,

$$c = (12m^2 - 9m + 3)k - 12. \tag{16}$$

Let us substitute (12), (15), and (16) into (6). We have

$$\begin{array}{lcl} m^2k[2(3m-1)^2k - (12m^2 - 9m + 3)k + 12] - 4(3m-1)^2k & = & \\ m^2k[(6m^2 - 3m - 1)k + 12 - 4(3 - \frac{1}{m})^2] & = & \\ m^2k[(6(m - \frac{1}{4})^2 - \frac{11}{8})k + 12 - 4(3 - \frac{1}{m})^2] & > & 0. \end{array}$$

If $m \geq 3$, then inequality (1) holds, since

$$[(6m^2 - 3m - 1)k + 12 - 4(3 - \frac{1}{m})^2] > 26k + 12 - 36 > 0.$$

If $m = 2$, then inequality (1) holds, since

$$[(6m^2 - 3m - 1)k + 12 - 4(3 - \frac{1}{m})^2] = 17k + 12 - 25 > 0.$$

If $m = 1$, then inequality (1) also holds, since in this case $k \geq 3$, hence

$$[(6m^2 - 3m - 1)k + 12 - 4(3 - \frac{1}{m})^2] = 2k + 12 - 16 > 0.$$

3.3. *Proof of Theorem* 4. Let $f^{-1}(\mathbb{P}^1) = E \in |aL_1 + bL_2|$, where L_1 and L_2 are the natural generators of $Pic\, S$. Without loss of generality, we can assume that $a \geq b > 0$. Then

$$N = \deg f = 2ab. \tag{17}$$

We have $K_S = -2L_1 - 2L_2$ and $K_S = f^*(K_{\mathbb{P}^2}) + R = -3E + R$, hence $R \in |(3a-2)L_1 + (3b-2)L_2|$. Therefore $\deg B = (E, R) = a(3b-2) + b(3a-2)$, i.e.

$$d = 3ab - a - b. \tag{18}$$

By adjunction formula, $2(g-1) = (K_S + R, R)$. Therefore

$$g - 1 = 9(ab - a - b) + 8. \tag{19}$$

It follows from (18) and (19) that

$$3d + g - 1 = 18ab - 12a - 12b + 8. \tag{20}$$

By Lemma 7,

$$c = 24ab - 18a - 18b + 12. \tag{21}$$

Let us substitute (17), (20), and (21) into (6). We have

$$\begin{aligned} 2ab[12ab - 6a - 6b + 4] - 4(18ab - 12a - 12b + 8) &= \\ 4ab[3a(b-1) + 3b(a-1) + \frac{12}{b} + \frac{12}{a} - 16 - \frac{8}{ab}] &> 0\,. \end{aligned}$$

If $a \geq b \geq 2$, then inequality (1) holds, since

$$3a(b-1) + 3b(a-1) + \frac{12}{b} + \frac{12}{a} - 16 - \frac{8}{ab} \geq 3a + 6(a-1) + \frac{8}{a} + \frac{12}{b} - 16 = 9a + \frac{8}{a} + \frac{12}{b} - 22 > 0.$$

If $a > b = 1$, then inequality (1) also holds, since

$$3a(b-1) + 3b(a-1) + \frac{12}{b} + \frac{12}{a} - 16 - \frac{8}{ab} = 3(a-1) + 12 + \frac{4}{a} - 16 = 3a + \frac{4}{a} - 7 > 0.$$

If $a = b = 1$, then f is a two-sheeted covering of $\mathbb{P}^2$ branched along smooth conic.
3.4. *Proof of Theorems* 5-7. Let

$$\deg f = N = E^2 = 2k \tag{22}$$

(E^2 is even, since $2K_S$ is trivial). We have $K_S = f^*(K_{\mathbb{P}^2}) + R = -3E + R$, hence $R \equiv 3E$. Therefore $\deg B = 6k$, i.e.

$$d = 3k. \tag{23}$$

By adjunction formula, $2(g-1) = R^2$. Therefore

$$g - 1 = 9k. \tag{24}$$

It follows from (23) and (24) that

$$3d + g - 1 = 18k. \tag{25}$$

By Lemma 8,

$$c = 24k - 12p_a. \tag{26}$$

Let us substitute (22), (25), and (26) into (6). We have

$$\begin{array}{lcl} 2k[12k + 12p_a] - 72k & = & \\ 24k[k + p_a - 3] & > & 0\,. \end{array}$$

and $k \geq 2$, since $N > 2$. If S is a K3 surface, then inequality (1) holds, since $p_a = 2$.

If S is an Enriques surface, then $p_a = 1$ and inequality (1) also holds except, possibly, for $k = 2$. For abelian varieties ($p_a = 0$) inequality (1) holds except, possibly, for $k = 2$ and $k = 3$.

For abelian surfaces the case $k = 2$ is impossible, since such a curve B does not exist. Indeed, in this case it follows from (23), (24), and (26) that $\deg B = 12$, $g = 19$, $c = 48$. Then

$$n = \frac{1}{2}(\deg B - 1)(\deg B - 2) - g - c = 55 - 19 - 48 < 0,$$

a contradiction.

For Enriques surface in the exceptional case we have $\deg f = 4$. Consider the epimorphism $\varphi : \pi_1 = \pi_1(\mathbb{P}^2 \setminus B) \to \mathfrak{S}_4$ corresponding to f. It follows from (23), (24), and (26) that $\deg B = 12$, $g = 19$, and $c = 36$, hence $n = 0$. Therefore, by Proposition 1, the epimorphism $\varphi' : \pi_1 \to \mathfrak{S}_3$, which is the composition of φ and the natural epimorphism $\mathfrak{S}_4 \to \mathfrak{S}_3 = \mathfrak{S}_4 / K_4$, where K_4 is the Klein four group, gives rise to a generic morphism $f' : S' \to \mathbb{P}^2$ of degree 3 with the same discriminant curve B.

The assertion on the degree of generic morphisms in the exceptional cases follows from Lemma 5.
3.5. *Proof of Theorem* 8. Let $S = X(m_1, ..., m_k) \subset \mathbb{P}^{k+2}$ be a complete intersection of multidegree $(m_1, ..., m_k)$, $m_i > 1$. Then for a generic projection onto $\mathbb{P}^2$

$$\deg f = N = \prod_{i=1}^{k} m_i\,. \tag{27}$$

By adjunction formula, $K_S = (m_1 + ... + m_k - k - 3)E$. Since $K_S = f^*(K_{\mathbb{P}^2}) + R = -3E + R$, we have $R = (m_1 + ... + m_k - k)E$. Therefore $\deg B = (m_1 + ... + m_k - k)E^2$, i.e.

$$d = \frac{1}{2}(\sum_{i=1}^{k}(m_i - 1))\prod_{i=1}^{k} m_i\,. \tag{28}$$

By adjunction formula, $2(g-1) = R^2 + (R, K_S)$. Therefore

$$g - 1 = \frac{1}{2}(\sum_{i=1}^{k}(m_i - 1))(2\sum_{i=1}^{k}(m_i - 1) - 3)\prod_{i=1}^{k} m_i. \tag{29}$$

It follows from (28) and (29) that

$$3d + g - 1 = (\sum_{i=1}^{k}(m_i - 1))^2\prod_{i=1}^{k} m_i. \tag{30}$$

By Lemma 7,

$$c = 3N + 2(g-1) - e(S) = 3\prod_{i=1}^{k} m_i + (\sum_{i=1}^{k}(m_i - 1))(2\sum_{i=1}^{k}(m_i - 1) - 3)\prod_{i=1}^{k} m_i - e(S). \tag{31}$$

Let us substitute (27), (30), and (31) into (6). We have

$$\prod_{i=1}^{k} m_i[3\prod_{i=1}^{k} m_i(\sum_{i=1}^{k}(m_i - 1)) + e(S) - 4(\sum_{i=1}^{k}(m_i - 1))^2] > 0\,.$$

If $k \geq 2$, then

$$[3\prod_{i=1}^{k} m_i(\sum_{i=1}^{k}(m_i - 1)) + e(S) - 4(\sum_{i=1}^{k}(m_i - 1))^2] > (\sum_{i=1}^{k}(m_i - 1))[3\prod_{i=1}^{k} m_i - 4(\sum_{i=1}^{k}(m_i - 1))] > 0$$

and, obviously, the last inequality is true.

If $k = 1$, then $e(S) = m_1^3 - 4m_1^2 + 6m_1$, hence

$$\begin{array}{lcl} 3\prod_{i=1}^{k} m_i(\sum_{i=1}^{k}(m_i - 1)) + e(S) - 4(\sum_{i=1}^{k}(m_i - 1))^2 & = & \\ 3m_1(m_1 - 1) + m_1^3 - 4m_1^2 + 6m_1 - 4(m_1 - 1)^2 & = & \\ m_1^3 - 5m_1^2 + 11m_1 - 4 & = & \\ (m_1 - 2)^3 + (m_1 - 2)^2 + 3(m_1 - 2) + 6 & \geq & 6 > 0 \end{array}$$

for $m \geq 2$, i.e. inequality (1) also holds in the case $k = 1$.

3.6. *Proof of Theorem* 9. Put $k = K_S^2$, $L^2 = a$, and $(K_S, L) = b$. Note that $a > 0$. If $E = mL$, then

$$N = \deg f = m^2 a. \tag{32}$$

We have $K_S = f^*(K_{\mathbb{P}^2}) + R = -3E + R$, i.e. $R = K_S + 3mL$. Therefore

$$\begin{array}{rcl} R^2 & = & 9m^2a + 6mb + k, \\ (K_S, R) & = & 3mb + k. \end{array}$$

By adjunction formula,

$$2(g-1) = (K_S + R, R) = 9m^2 + 9mb + 2k \tag{33}$$

and by Lemma 4,

$$3d + g - 1 = 9m^2a + 6mb + k. \tag{34}$$

By (32), (33), and Lemma 7,

$$c = 12m^2a + 9mb + 2k - e(S). \tag{35}$$

Let us substitute (32), (34), and (35) into (6). We have

$$\begin{array}{rcl} m^2a[6m^2a + 3mb + e(S)] - 4(9m^2a + 6mb + k) & = & \\ m^2a[6m^2a + 3mb + e(S) - 36a - \dfrac{24b}{ma} - \dfrac{4k}{m^2a}] & > & 0, \end{array}$$

Now it is clear that there exist a constant m_0 such that the last inequality holds for $m \geq m_0$.

The proof in the case $L = K_S$ coincides with one of Theorem 2.

3.7. *Proof of Theorem* 10. Let $B^* \subset \mathbb{P}^{2*}$ be a nodal curve of genus g, $\deg B^* = \delta$, and B the dual curve of B^*. Then (cf. [5]) B is the discriminant curve of some generic morphism f of degree δ. In fact, let S be the normalization of

$$X = \{\, ((x_1, x_2, x_3), (y_1, y_2, y_3)) \in \mathbb{P}^2 \times \mathbb{P}^{2*} \ \mid\ \sum_i x_i y_i = 0\,,\ (y_1, y_2, y_3)) \in B^* \,\}$$

and f be induced by the projection $pr_1 : P^2 \times \mathbb{P}^{2*} \to P^2$. It is clear that $(x_1, x_2, x_3) \notin B$ iff $\sum_i x_i y_i = 0$ is not tangent to B^*, i.e. $f^{-1}((x_1, x_2, x_3))$ has exactly δ points and

$$N = \deg f = \delta, \tag{36}$$

hence B is the branch curve of f and it is clear that f is a generic morphism.

It follows from Plücker's formulas that

$$d = \delta + (g-1) \tag{37}$$

and

$$c = 3\delta + 6(g-1). \tag{38}$$

Let us substitute (36) - (38) into inequality (1) :

$$\delta > \frac{4(3\delta + 4(g-1))}{3\delta + 2(g-1)} = 8 - \frac{12\delta}{3\delta + 2(g-1)}.$$

Thus, inequality (1) holds for $\delta \geq 8$.

Consider the cases $\delta \leq 7$.

The case $\delta = 7$. Inequality (1) does not hold iff

$$\frac{12\delta}{3\delta + 2(g-1)} \leq 1 \Longrightarrow 9\delta \leq 2(g-1) \Longrightarrow g \geq 33.$$

On the other hand,

$$g \leq \frac{(\delta-1)(\delta-2)}{2} = \frac{6 \cdot 5}{2} < 33.$$

Thus, in the case $\delta = 7$ inequality (1) is true.

The case $\delta = 6$. Inequality (1) does not hold iff

$$\frac{12\delta}{3\delta + 2(g-1)} \leq 2 \Longrightarrow 3\delta \leq 2(g-1) \Longrightarrow g \geq 10.$$

On the other hand,

$$g \leq \frac{(\delta-1)(\delta-2)}{2} = \frac{5 \cdot 4}{2} = 10.$$

Thus, in the case $\delta = 6$ inequality (1) is not true iff $\nu = 0$, i.e., possibly, there exist two non-equivalent generic morphisms only when B possesses the following invariants: $\deg B = 30$, $g = 10$, $c = 72$, $n = 324$. For such B, if there exist another (non-equivalent to f) generic morphism f_1 of $\deg f_1 = N_1$, then

$$N_1 \leq \frac{4(3d + (g-1))}{6d + 2(g-1) - c} = \frac{4(3 \cdot 15 + (10-1))}{6 \cdot 15 + 2(10-1) - 72} = 6.$$

The case $\delta = 5$. Inequality (1) does not hold iff

$$\frac{12\delta}{3\delta + 2(g-1)} \leq 3 \Longrightarrow \delta \leq 2(g-1) \Longrightarrow g \geq 4.$$

On the other hand,

$$g \leq \frac{(\delta-1)(\delta-2)}{2} = \frac{5 \cdot 4}{2} = 6.$$

Thus, in the case $\delta = 5$ inequality (1) is not true iff either $\nu = 0$, or $\nu = 1$, or $\nu = 2$, i.e., possibly, there exist two non-equivalent generic morphisms only in the following three cases:

0) $\deg B = 20$, $g = 6$, $c = 45$

1) $\deg B = 18$, $g = 5$, $c = 39$.

2) $\deg B = 16$, $g = 4$, $c = 33$.

In all cases 0) - 2) if there exist two non-equivalent generic morphisms f and f_1 with the same discriminant curve B, then the computation, similar to one described above, gives rise to $\deg f_1 = N_1 \leq 5$.

The case $\delta = 4$. We have $g \leq 3$. If $g = 3$, then $\deg B = 12$ and $c = 24$. It is easy to check that if there exist two non-equivalent generic morphisms f and f_1 with such discriminant curve

B, then $\deg f_1 = N_1 \leq 5$. Let us show that such a curve can not be the discriminant curve of a generic morphism $f_1 : S_1 \to \mathbb{P}^2$ of degree 5. In fact, in this case by Lemmas 6 - 8, $K^2_{S_1} = -7$, $e(S_1) = -5$, and $p_a = -1$. Hence S_1 is a ruled surface over a curve C of genus $g = 2$. Let $\overline{S}_1$ be a relatively minimal model of S_1. Then $e(\overline{S}_1) = -4$, hence $e(S_1) \geq -4$, a contradiction.

The case $g \leq 2$ will be considered in the proof of Theorem 11.

3.8. *Proof of Theorem* 11. In the next subsection we shall prove Theorem 12. By that theorem, the Chisini Conjecture holds for a curve B of $g \leq 1$, and if in the case $g = 2$ there exist two non-equivalent generic morphisms f_1 and f_2, then $d \leq 3$.

Consider the case $g = 2$ and $d = 3$. Then the inequality opposite to (1) takes the following form

$$N_i \leq \frac{40}{20 - c}.$$

Therefore, if $N_i \geq 5$, then

$$5 \leq \frac{2c}{20 - c} \Longleftrightarrow 12 \leq c \leq 19.$$

On the other hand,

$$c = (2d - 1)(d - 1) - g - n = 8 - n \leq 8,$$

a contradiction.

A curve B with invariants $g = 2$ and $\deg B = 4$ (i.e. $d = 2$) can not be a discriminant curve. Indeed, in this case either $c = 1$, or $n = 1$, which contradicts Corollary 2.

Consider the case $g = 3$. By Theorem 12, if there exist two non-equivalent generic morphisms f_1 and f_2 such that $N_1 \geq 5$, then $d \leq 6$. By Lemmas 2 and 3,

$$\frac{3}{2}d + 1 \leq c \leq 3d + 6, \tag{39}$$

and the inequality opposite to (1) takes the following form

$$5 \leq \frac{12d + 8}{6d + 4 - c} \Longleftrightarrow \frac{18d + 12}{5} \leq c \leq 6d + 3. \tag{40}$$

If $d = 6$, then by (39), $10 \leq c \leq 24$. On the other hand, by (40), $c \geq 24$ and, consequently, $c = 24$ and $n = 28$. Thus, we obtain the case considered already in the proof of Theorem 10.

If $d = 5$, then by (39), $9 \leq c \leq 21$. On the other hand, by (40), $c \geq 21$ and, consequently, $c = 21$ and $n = 12$. By Plücker's formula, $\delta = 10 \cdot 9 - 2 \cdot 12 - 3 \cdot 21 = 3$, contrary to $g = 3$.

If $d = 4$, then by (39), $7 \leq c \leq 18$. On the other hand, by (40), $c \geq 17$ and, consequently, by Corollary 2, $c = 18$ and $n = 0$. By Plücker's formula, $\delta = 8 \cdot 7 - 3 \cdot 21 \leq 0$, a contradiction.

If $d = 3$, then by (39), $6 \leq c \leq 15$. On the other hand, by (40), $c \geq 14$ and, consequently, by Corollary 2, $c = 15$, which contradicts the inequality $g = (2d - 1)(d - 1) - c - n \geq 0$.

If $d = 2$, i.e. $\deg B = 4$. Hence B is non-singular. Thus, B can not be a discriminant curve.

3.9. *Proof of Theorem* 12. Consider again the inequality opposite to (1)

$$N_i \leq \frac{4(3d + (g - 1))}{6d + 2(g - 1) - c}.$$

Thus, if there exist two non-equivalent generic morphisms such that one of them has degree $N_i \geq 5$, then

$$\begin{array}{rll} \dfrac{4(3d+(g-1))}{6d+2(g-1)-c} & \geq 5 & \Longrightarrow \\ 5c & \geq 18d+6(g-1) & \Longrightarrow \text{ (by Lemma 2)} \\ 15d+15(g-1) & \geq 18d+6(g-1) & \Longleftrightarrow \\ 3(g-1) & \geq d. & \end{array}$$

4 Canonical discriminant curves.

4.1. A curve B is said to be a(n m-)canonical discriminant curve if B is the discriminant curve of a generic morphism $f : S \to \mathbb{P}^2$ given by a linear subsystem $\{E\} \subseteq |mK_S|$, $m \in \mathbb{N}$.

Let $B \subset \mathbb{P}^2$ be a curve of even degree $2d$ with ordinary cusps and nodes as the only singularities and $\widetilde{\nu} : R \to B$ the normalization. Put $\mathfrak{e} = \widetilde{\nu}^{-1}(\mathbb{P}^1 \cap B)$, $\mathfrak{c} = 2\sum' \widetilde{\nu}^{-1}(s_i)$, and $\mathfrak{n} = \sum'' \widetilde{\nu}^{-1}(s_i)$, where we denote by $\sum'$ (resp. by $\sum''$) summation over all cusps (resp. nodes) $s_i \in B$.

Proposition 5 *Let B and R be as above. If B is a canonical discriminant curve, then*

(i) $\dfrac{2d}{g-3d-1} := m \in \mathbb{N}$,

(ii) $\dfrac{(g-3d-1)^2}{3d+g-1} := k \in \mathbb{N}$,

(iii) $\dfrac{4d^2}{3d+g-1} := N \in \mathbb{N}$,

(iv) $N + \dfrac{3g-3-9d-c}{12} := p_a \in \mathbb{N}$,

(v) There exists a divisor $\mathfrak{k} \in Pic\, R$ such that

$$K_R = (3m+2)\mathfrak{k} \qquad and \qquad \mathfrak{e} = m\mathfrak{k}.$$

Besides,

$$\dim H^0(R, \mathcal{O}_R(r\mathfrak{k})) = \frac{r(r-1)}{2}k + p_a$$

for $r = 2, \dots, 3m$.

(vi) $\mathfrak{c} + \mathfrak{n} = [(2d-6)m-2]\mathfrak{k}$,

Proof. By (8) and (9),

$$\left\{ \begin{array}{rcl} 2g-2 & = & (9m^2+9m+2)k \\ 6d & = & (9m^2+3m)k \end{array} \right. \Longrightarrow$$

$$\begin{array}{rcll} g-1-3d & = & (3m+1)k & \Longrightarrow \\ m & = & \dfrac{2d}{g-3d-1}. & \end{array}$$

Now (ii) follows from (8), (iii) follows from (7), and (iv) does from (11)

The element $\mathfrak{k} \in Pic\, R$ is the restriction of K_S to R. Since K_S is ample,

$$\dim H^1(S, \mathcal{O}_S(-rK_S)) = 0$$

for $r > 0$. Therefore from the exact sequence

$$0 \longrightarrow \mathcal{O}_S((r-3m-1)K_S) \longrightarrow \mathcal{O}_S(rK_S) \longrightarrow \mathcal{O}_R(rK_S) \longrightarrow 0$$

it follows that $H^0(S, \mathcal{O}_S(rK_S))$ is isomorphic to $H^0(R, \mathcal{O}_R(rK_S))$ for $1 \leq r \leq 3m$. By Riemann-Roch formula, we have

$$\dim H^0(S, \mathcal{O}_S(rK_S)) = \frac{r(r-1)}{2}K_S^2 + p_a$$

for $r > 1$.

To prove (vi), let us blow up all singular points of B. Denote the composition of these $c+n$ σ-processes by $\sigma : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ and let $L_i = \sigma^{-1}(s_i)$ be the exceptional curve over $s_i \in Sing\, B$. Then $\sigma^*(B) = R + 2\sum L_i$, where the strict preimage R of B is a non-singular curve, since all singularities of B are ordinary cusps and nodes. We have

$$\begin{array}{rcl} K_{\widetilde{\mathbb{P}^2}} & = & -3\sigma^*(\mathbb{P}^1) + \sum L_i\,, \\ R & = & \deg B\sigma^*(\mathbb{P}^1) - 2\sum L_i\,. \end{array}$$

Hence by adjunction formula,

$$\mathcal{O}_R(K_R) = \mathcal{O}_R(K_{\widetilde{\mathbb{P}^2}} + R) = \mathcal{O}_R((\deg B - 3)\sigma^*(\mathbb{P}^1) - \sum L_i).$$

But $\mathcal{O}_R(\mathfrak{c} + \mathfrak{n}) = \mathcal{O}_R(\sum L_i)$. Therefore

$$\mathcal{O}_R(\mathfrak{c} + \mathfrak{n}) = \mathcal{O}_R(((2d-6)m-2)\mathfrak{k}),$$

since $\mathcal{O}_R(\sigma^*(\mathbb{P}^1)) = \mathcal{O}_R(\mathfrak{e}) = \mathcal{O}_R(m\mathfrak{k})$ and $\mathcal{O}_R(K_R) = \mathcal{O}_R((3m+2)\mathfrak{k})$.

Proposition 6 *Let B satisfy conditions (i) - (vi) of Proposition 5. If B is a discriminant curve of a generic morphism $f : S \to \mathbb{P}^2$ of $\deg f = N$, then S is a surface of general type with ample canonical bundle and f is given by a three-dimensional linear subsystem of $|mK_S + \alpha|$, where $\alpha \in Pic\, S$ is an element of order 2. Moreover, if m is even, then $\alpha = 0$.*

Proof. Let f be given by linear subsystem of $|E|$. We have

$$E^2 = N = \frac{4d^2}{3d+g-1},$$

$R^2 = 3d + g - 1$ and $(E, R) = 2d$. Then

$$\begin{vmatrix} E^2 & (E,R) \\ (R,E) & R^2 \end{vmatrix} = \begin{vmatrix} \dfrac{4d^2}{3d+g-1} & 2d \\ 2d & 3d+g-1 \end{vmatrix} = 0.$$

Therefore by Hodge's Index Theorem,

$$R \equiv \frac{3d+g-1}{2d}E \equiv \frac{3m+1}{m}E.$$

Since $K_S = -3E + R$, we have

$$\begin{array}{rcl} E & \equiv & mK_S, \\ R & \equiv & (3m+1)K_S, \end{array}$$

and if $E = mK_S + \alpha$, where $\alpha \equiv 0$, and $\mathcal{O}_R(K_S) = \mathcal{O}_R(\mathfrak{k} + \beta)$, where $\deg \beta = 0$, then

$$\mathcal{O}_R(E) = \mathcal{O}_R(m\beta + \alpha + m\mathfrak{k}) = \mathcal{O}_R(m\mathfrak{k})$$

and

$$\mathcal{O}_R(R) = \mathcal{O}_R(K_S + 3E) = \mathcal{O}_R((3m+1)\beta + 3\alpha + (3m+1)\mathfrak{k}).$$

Hence $\mathcal{O}_R(K_R) = \mathcal{O}_R((3m+2)\beta + 3\alpha + (3m+2)\mathfrak{k})$. Therefore

$$\begin{array}{rcl} \mathcal{O}_R(m\beta) & = & \mathcal{O}_R(-\alpha), \\ \mathcal{O}_R((3m+2)\beta) & = & \mathcal{O}_R(-3\alpha). \end{array}$$

Hence $2\beta = 0$ and $\mathcal{O}_R(2\alpha) = \mathcal{O}_R$, and if m is even, then $\mathcal{O}_R(\alpha) = \mathcal{O}_R$.

Proposition 6 will be proved once we prove

Lemma 11 *(cf. Appendix to Chapter V in [23]) Let S be a smooth projective surface and $i : R \hookrightarrow S$ a smooth irreducible curve. If $\mathcal{O}_S(R)$ is ample, then $i^* : Pic^0 S \to Pic\, R$ is injective, where $Pic^0 S \subset Pic\, S$ is the subgroup of numerically equivalent to zero classes of divisors.*

Proof. In the commutative diagram

$$\begin{tikzcd}
 & & 0 = H^1(S, \mathcal{O}_S(-R)) \arrow[d] & Pic^0 S \arrow[d, hook] & & \\
0 \arrow[r] & H^1(S, \mathbb{Z}) \arrow[r] \arrow[d] & H^1(S, \mathcal{O}_S) \arrow[r] \arrow[d, "j^*"] & H^1(S, \mathcal{O}_S^*) \arrow[r] \arrow[d, "i^*"] & H^2(S, \mathbb{Z}) \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & H^1(R, \mathbb{Z}) \arrow[r] & H^1(R, \mathcal{O}_R) \arrow[r] & H^1(R, \mathcal{O}_R^*) \arrow[r] & H^2(R, \mathbb{Z}) \arrow[r] & {}
\end{tikzcd}$$

with exact rows, the morphism j^* is embedding. Therefore if for $\alpha \in Pic^0 S$ the image $i^*(\alpha) = 0$, then $r\alpha = 0$ for some $r \in \mathbb{N}$, since $Tors\, H^2(S, \mathbb{Z})$ is a finite abelian group and $H^1(S, \mathbb{Z}) \to H^1(R, \mathbb{Z})$ is embedding. An element $\alpha \in Pic\, S$ of finite order r defines a non-ramified abelian covering $\varphi : S_r \to S$ of $\deg \varphi = r$. If $\alpha \neq 0$ and $i^*(\alpha) = 0$, then $\varphi^{-1}(R)$ is the disjoint union of r irreducible curves, which contradicts the ampleness of R.

4.2. In notation of section 4.1, consider the natural homomorphism

$$\mu : \mathrm{Sim}^2 H^0(R, \mathcal{O}_R(\mathfrak{e})) \longrightarrow H^0(R, \mathcal{O}_R(2\mathfrak{e})).$$

The kernel $\ker \mu$ generates the homogeneous ideal I in the homogeneous coordinate ring $\mathfrak{R} = \oplus Sym^r H^0(R, \mathcal{O}_R(\mathfrak{e}))$ of the projective space $\mathbb{P} = \mathbb{P}H^0(R, \mathcal{O}_R(\mathfrak{e}))$. Put $S_I = Proj\, \mathfrak{R}/I$. The normalization $\widetilde{\nu} : R \to B$ determines a 3-dimensional subspace

$$L = \widetilde{\nu}^*(H^0(B, \mathcal{O}_B(\mathbb{P}^1 \cap B))) \subseteq H^0(R, \mathcal{O}_R(\mathfrak{e})).$$

The subspace L defines a projection $pr : \mathbb{P} \to \mathbb{P}^2$ with base locus $\mathbb{P}L \subset \mathbb{P}$. Let $f_I : S_I \to \mathbb{P}^2$ be the restriction of pr to S_I.

Proposition 7 *Let B satisfy conditions (i) - (vi) of Proposition 5. Assume that m is even and ≥ 21. Then the curve B is an m-canonical discriminant curve if and only if S_I is a non-singular surface of* $\deg S_I = N$ *and f_I is a generic morphism with discriminant curve B.*

Proof. Let A be a very ample divisor on S and D a numerically effective one. By [7], if the embedding of S into $\mathbb{P}$ is given by $|K_S + 4A + D|$, then the homogeneous ideal $I(S)$ is generated by quadrics.

By [3], $A = 5K_S$ is very ample (we assume that K_S is ample). Therefore, if S is embedded by $|mK_S|$, $m \geq 21$, then $I(S)$ is generated by quadrics. On the other hand, $R \in |(3m+1)K_S|$ and $R \subset S \subset \mathbb{P}$ is embedded by $|m\mathfrak{k}|$. As it was mentioned in the proof of Proposition 5, the restriction map

$$H^0(S, \mathcal{O}_S(rmK_S)) \longrightarrow H^0(R, \mathcal{O}_R(rmK_S)) = H^0(R, \mathcal{O}_R(r\mathfrak{e}))$$

is an isomorphism for $r = 1, 2$. Hence the set of quadrics containing S coincides with the one containing R, and Proposition 7 follows from Propositions 5 and 6.

Remark 4 *Proposition 7 holds also for odd $m \geq 21$ if in view of Proposition 6 we slightly change the definition of m-canonical discriminant curves. Moreover, Proposition 7 is a particular case of more general assertion.*

Proposition 7′ *Let B satisfy conditions (i) - (iv) of Proposition 5. Assume that $m \geq 21$. Then the curve B is the discriminant curve of morphism given by three-dimensional subsystem of $|L|$, where L is numerically equivalent to the mth canonical class, if and only if S_I is a non-singular surface of* $\deg S_I = N$ *and f_I is a generic morphism with discriminant curve B.*

5 On Zariski's pairs

5.1. The set of plane curves of degree $2d$ is naturally parameterized by the points in $\mathbb{P}^{d(2d+3)}$. The subset of plane irreducible curves of degree $2d$ and genus g with c ordinary cusps and some nodes, as the only singularities, corresponds to a quasi-projective subvariety $\mathcal{M}(2d, g, c) \subset P^{d(2d+3)}$ ([21]). One can show that if two non-singular points of the same irreducible component of $\mathcal{M}_{red}(2d, g, c)$ correspond to curves B_1 and B_2, then the pairs $(\mathbb{P}^2, B_1)$ and $(\mathbb{P}^2, B_2)$ are diffeomorphic. In particular, in this case the fundamental groups $\pi_1(\mathbb{P}^2 \setminus B_1)$ and $\pi_1(\mathbb{P}^2 \setminus B_1)$ are isomorphic.

The following Proposition is a simple consequence of Proposition 1 and local considerations in 1.2 and 1.3.

Proposition 8 *Let* $(\mathbb{P}^2, B_1)$ *and* $(\mathbb{P}^2, B_2)$ *be two diffeomorphic (resp. homeomorphic) pairs. If* B_1 *is the discriminant curve of a generic morphism* (S_1, f_1)*, then* B_2 *is also the discriminant curve of some generic morphism* (S_2, f_2)*. Moreover, if* (S_1, f_1) *is unique, then the same is true for* (S_2, f_2) *and* S_1 *and* S_2 *are diffeomorphic (resp. homeomorphic).*

Conversely, for $S \subset \mathbb{P}^r$ a projection $f : S \to \mathbb{P}^2$ is defined by a point in Grassmannian $\mathrm{Gr}_{r+1,r-2}$ (the base locus of the projection). It is well-known that the set of generic projections is in one to one correspondence with some Zariski's open subset U_S of $\mathrm{Gr}_{r+1,r-2}$. A continuous variation of a point in U_S gives rise to a continuous family of generic projections of S whose branch curves belong to the same continuous family of plane cuspidal curves. Therefore discriminant curves of two generic projections of $S \subset \mathbb{P}^r$ belong to the same irreducible component of $\mathcal{M}(2d, g, c)$.

Moreover, if two surfaces S_1 and S_2 of general type with the same $K_S^2 = k$ and $p_a = p$ are embedded by the mth canonical class into the same projective space $\mathbb{P}^r$ and belong to the same irreducible component of coarse moduli space $\mathcal{M}_S(k, p)$ of surfaces with given invariants ([8]), then there exist generic projections f_1 of S_1 and f_2 of S_2 belonging to the same continuous family of generic projections. Therefore, discriminant curves of two generic projections of S_1 and S_2, belonging to the same irreducible component of a moduli space $\mathcal{M}_S(k, p)$, belong to the same irreducible component of $\mathcal{M}(2d, g, c)$ (cf. [21]). By Theorem 2 and Propositions 5 and 6, for a surface of general type with ample canonical class the triple of integers (m, k, p) is uniquely determined by the invariants (d, g, c) of mth canonical discriminant curve, and vice versa. Hence by Proposition 1 and Theorem 2,

$$i(2d(k,p), g(k,p), c(k,p)) \geq i(k,p),$$

where $i(2d, g, c)$ (resp. $i(k, p)$) is the number of irreducible components of $\mathcal{M}(2d, g, c)$ (resp. $\mathcal{M}_S(k, p)$).

In [4], F. Catanese showed that for each positive integer h there exist integers k, p such that $\mathcal{M}_S(k, p)$ has at least h irreducible components. Hence for each positive integer h there exist integers d, g, c such that $\mathcal{M}(2d, g, c)$ has at least h irreducible components. Note that the lower bound estimates for $i(k, p)$, obtained in [4] and [12], hold also for $i(2d, g, c)$, where d, g, and c are the invariants of the corresponding mth canonical discriminant curves.

5.2. Recently, there were published several articles (see, for example, [1], [16], [19], [20]) devoted to so called Zariski's pairs. By Artal-Bartolo's definition, two plane curves $C_1, C_2 \subset \mathbb{P}^2$ are called a Zariski pair if they have the same degree and homeomorphic tubular neighborhoods in $\mathbb{P}^2$, but the pairs $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ are not homeomorphic.

The first example of such pairs was obtained by O. Zariski and it is just two curves of degree 6 with 6 cusps mentioned in the Introduction.

In view of Theorem 2, applying Proposition 8, it is easy to prove the existence of a lot of Zariski pairs. To this end, we can find pairs of non-homeomorphic minimal surfaces of general type with ample canonical class and the same K_S^2 and p_a, and consider the corresponding mth canonical discriminant curves. For example,

Proposition 9 *For each integer $m \geq 5$, there is at least one Zariski's triple (not only a pair) of plane cuspidal curves $B_{m,i}$, $i = 3, 4, 5$, of degree $m(3m+1)$ and genus $g = \frac{1}{2}(3m+1)(3m+2)+1$ with $c = 3(4m^2 + 3m - 3)$ cusps.*

Proof. There exist ([9], [18], [2]) five non-homeomorphic surfaces S_i ($i = 1, \dots, 5$) of general type with $p_g = 0$, $p_a = 1$, and $K^2_{S_i} = 1$. They are distinguished by $\mathrm{Tors} H_1(S_i, \mathbb{Z}) = \mathbb{Z}/i\mathbb{Z}$. At least for $i \geq 3$ there exists a surface S_i with ample canonical class possessing such invariants. Let $\phi_{m,i} : S_i \hookrightarrow \mathbb{P}^{\frac{1}{2}m(m-1)}$ be the mth canonical embedding, $m \geq 5$ ([3]), and $f_{m,i} : S_i \to \mathbb{P}^2$ a generic projection with discriminant curve $B_{m,i}$. Applying the computation in the proof of Theorem 2, the discriminant curve $B_{m,i}$ has

$$\deg B_{m,i} = m(3m+1), \quad g = \frac{1}{2}(3m+1)(3m+2)+1, \quad c = 3(4m^2+3m-3).$$

By Proposition 8, since S_i and S_j are not homeomorphic for $i \neq j$, the pairs $(\mathbb{P}^2, B_{m,i})$ and $(\mathbb{P}^2, B_{m,j})$ also are not homeomorphic.

The easiest way to construct two non-homeomorphic surfaces S_1 and S_2 of general type, for which general morphisms, given by linear subsystems of the mth canonical class, give rise to Zariski's pair B_1 and B_2, is to construct two surfaces with different irregularities q_1 and q_2, where $q_i = \dim H^1(S_i, \mathcal{O}_{S_i})$, and the same K^2_S and p_a. For example, let A be an abelian surface and $C \subset A$ a non-singular curve such that the class of C in $Pic\, A$ is divisible by 2. Consider the two-sheeted covering $\varphi : S \to A$ branched along C. It is easy to check that S is a surface of general type with ample canonical bundle and irregularity $q \geq 2$. If $C^2 = 8p$, then the simplest computation gives rise to $K^2_S = 4p$ and $p_a = p$. On the other hand, in [17], U. Persson proved that for any positive integers x, y satisfying

$$2x - 6 \leq y \leq 8(x - cx^{2/3}), \tag{41}$$

where $c = \frac{9}{\sqrt[3]{12}}$, there exists a simply connected minimal surface of general type with $K^2 = y$ and $p_a = x$. It is easily seen that $x = p$ and $y = 4p$ satisfy (41) if $p \geq 486$.

5.3. Question. *Let $S_1, S_2 \subset \mathbb{P}^r$ be two diffeomorphic (resp. homeomorphic) surfaces of general type embedded by the mth canonical class. Are pairs $(\mathbb{P}^2, B_1)$ and $(\mathbb{P}^2, B_2)$ diffeomorphic (resp. homeomorphic), where B_i is the discriminant curve of a generic projection of S_i onto $\mathbb{P}^2$?*

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