MOTIVES AND SIEGEL MODULAR FORMS (APPENDIX TO BÖCHERER AND HEIM: CRITICAL VALUES FOR L-FUNCTIONS ON $GSp_2 \times Gl_2$)

By Hiroyuki Yoshida

Introduction

For a Siegel modular form $f \in S_k(Sp(n,\mathbf{Z}))$, we normalize the Petersson norm so that

$$\langle f, f\rangle = \mathrm{vol}(Sp(n,\mathbf{Z})\backslash\mathfrak{H}_n)^{-1}\int_{Sp(n,\mathbf{Z})\backslash\mathfrak{H}_n} |f(z)|^2(\det y)^{k-n-1}dxdy.$$

Let $f \in S_k(Sp(2,\mathbf{Z}))$ and $g \in S_l(SL(2,\mathbf{Z}))$ be non-zero Hecke eigenforms. Let $L(s, f\otimes g)$ be the tensor product L-function, the product taken for the spinor L-function of f and Hecke's L-function of g. We assume the existence of the motive $M(f)$ attached to f (cf. §4) and Deligne's conjecture. We also assume that the Fourier coefficients of f are contained in a totally real algebraic number field. (This condition can be fulfilled by multiplying f by a non-zero constant.)

Theorem A. *If $l > 2k-2$, then for $n \in \mathbf{Z}$, $2k-2 \leqq n \leqq l-1$, we have*

$$\left(\frac{L(n, f\otimes g)}{\pi^{4n-4k+8}\langle g, g\rangle^2}\right)^\sigma = \frac{L(n, f^\sigma\otimes g^\sigma)}{\pi^{4n-4k+8}\langle g^\sigma, g^\sigma\rangle^2} \qquad \text{for} \quad \sigma \in Aut(\mathbf{C}).$$

Theorem B. *If $l < 2k-2$, then for $n \in \mathbf{Z}$ such that $k \leqq n \leqq k+l-3$ if $k \geqq l$, $l \leqq n \leqq 2k-3$ if $k \leqq l$, we have*

$$\left(\frac{L(n, f\otimes g)}{\pi^{4n-2k-l+8}\langle f, f\rangle\langle g, g\rangle}\right)^\sigma = \frac{L(n, f^\sigma\otimes g^\sigma)}{\pi^{4n-2k-l+8}\langle f^\sigma, f^\sigma\rangle\langle g^\sigma, g^\sigma\rangle} \qquad \text{for} \quad \sigma \in Aut(\mathbf{C}).$$

Let us explain the contents of this paper. In §1, we determine polynomials on $M(n,n)$ which transform according to characters under the left action of a parabolic subgroup of $GL(n)$ and also under the right action of $GL(n^+)\times GL(n^-)$, $n^+ + n^- = n$. Our method is elementary and we can work over an arbitrary field. In §2, we define "hidden period invariants" of a motive using our results in §1. We shall show that Deligne's period invariants $c^\pm$ of motives, which are obtained by various algebraic operations of given motives, can be expressed using them. We illustrate the efficacy of this method by taking a motive of rank 4 as an example in §3. In §4, we deduce Theorems A and B applying the results in §3 to the motive $M(f)$.

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§1. On certain relatively invariant polynomials

In this section, we work over an arbitrary base field F. By $M(m,n)$, we denote the set of all $m \times n$ matrices and regard $M(m,n)$ as an algebraic variety defined over F. Throughout this section, P_n denotes the minimal parabolic subgroup of $GL(n)$ consisting of all lower triangular matrices.

Let P be the lower parabolic subgroup of $GL(n)$ corresponding to the partition $n = s_1 + s_2 + \cdots + s_m$. For $(a_1, a_2, \cdots, a_m) \in \mathbf{Z}^m$, we define an algebraic character λ_1 of P by

$$\lambda_1\left(\begin{pmatrix} p_{11} & 0 & \dots & 0 \\ * & p_{22} & \dots & 0 \\ * & * & \ddots & \vdots \\ * & * & * & p_{mm} \end{pmatrix}\right) = \det(p_{11})^{a_1}\det(p_{22})^{a_2}\cdots\det(p_{mm})^{a_m}, \tag{1.1}$$

where $p_{ii} \in GL(s_i)$. We decompose n as $n = n^+ + n^-$ with non-negative integers n^+ and n^- and embed $GL(n^+) \times GL(n^-)$ as diagonal blocks into $GL(n)$. For k^+, $k^- \in \mathbf{Z}$, we define an algebraic character λ_2 of $GL(n^+) \times GL(n^-)$ by

$$\lambda_2\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = (\det a)^{k^+}(\det b)^{k^-}, \qquad a \in GL(n^+),\ b \in GL(n^-). \tag{1.2}$$

We consider a polynomial function $f(x)$ on $M(n,n)$ rational over F which satisfies

$$f(px\gamma) = \lambda_1(p)\lambda_2(\gamma)f(x) \qquad \text{for all } \ p \in P,\ \gamma \in GL(n^+) \times GL(n^-). \tag{1.3}$$

We call f to be *of the type* (λ_1, λ_2) or *of the type* $\{(a_1, a_2, \cdots, a_m); (k^+, k^-)\}$. All of such an f generate a graded algebra over F. Our main results in this section assert that every subspace of a given grade of this algebra is at most one dimensional and that this algebra is isomorphic to the polynomial ring with explicitly given generators. We refer the reader to Shimura [Sh2], Theorem 2A and Howe [Ho] among vast literatures on the subjects related to our results.

The restriction of λ_1 to P_n , which we denote by λ_1^*, is given by

$$\lambda_1^*\left(\begin{pmatrix} t_{11} & 0 & \dots & 0 \\ * & t_{22} & \dots & 0 \\ * & * & \ddots & \vdots \\ * & * & * & t_{nn} \end{pmatrix}\right) = t_{11}^{w_1} t_{22}^{w_2} \cdots t_{nn}^{w_m}, \tag{1.4}$$

$$w_j = a_i \qquad \text{if} \quad s_1 + s_2 + \cdots + s_{i-1} < j \leqq s_1 + s_2 + \cdots + s_i. \tag{1.5}$$

We call (λ_1, λ_2) *admissible* or $T = \{(a_1, a_2, \cdots, a_m), (k^+, k^-)\}$ *an admissible data* if the following conditions (1.6) $\sim$ (1.10) hold.

$$w_1 \geqq w_2 \geqq \cdots \geqq w_{n^+} \geqq k^+ \geqq 0, \tag{1.6}$$

$$k^- \geqq w_{n^++1} \geqq w_{n^++2} \geqq \cdots \geqq w_n \geqq 0. \tag{1.7}$$

$$\begin{aligned} &w_{n^-+1} = w_{n^-+2} = \cdots = w_{n^+} = k^+ \qquad \text{if} \quad n^+ > n^-, \\ &w_{n^++1} = w_{n^++2} = \cdots = w_{n^-} = k^- \qquad \text{if} \quad n^+ < n^-. \end{aligned} \tag{1.8}$$

$$w_i + w_{n+1-i} = k^+ + k^-, \qquad 1 \leqq i \leqq \min(n^+, n^-). \tag{1.9}$$

$$\begin{aligned} &w_1 \geqq w_2 \geqq \cdots \geqq w_{\min(n^+,n^-)} \geqq \max(k^+, k^-) \\ &\quad \geqq \min(k^+, k^-) \geqq w_{\max(n^+,n^-)+1} \geqq \cdots \geqq w_n. \end{aligned} \tag{1.10}$$

Theorem 1. *Let the notation be the same as above. There exists a non-zero polynomial function $f(x)$ on $M(n,n)$ of the type (λ_1, λ_2) if and only if (λ_1, λ_2) is admissible. If this is the case, such an f is unique up to the constant multiple and is given by (1.17) up to the constant multiple.*

For the proof of Theorem 1, we first show the next lemma.

Lemma 2. *Let m and n be natural numbers. Let χ_1 and χ_2 be the algebraic characters of P_m and P_n defined by*

$$\chi_1(\alpha) = \alpha_{11}^{a_1}\alpha_{22}^{a_2}\cdots\alpha_{mm}^{a_m}, \qquad \chi_2(\beta) = \beta_{11}^{b_1}\beta_{22}^{b_2}\cdots\beta_{nn}^{b_m},$$

$$\alpha = \begin{pmatrix} \alpha_{11} & 0 & \dots & 0 \\ * & \alpha_{22} & \dots & 0 \\ * & * & \ddots & \vdots \\ * & * & * & \alpha_{mm} \end{pmatrix}, \qquad \beta = \begin{pmatrix} \beta_{11} & 0 & \dots & 0 \\ * & \beta_{22} & \dots & 0 \\ * & * & \ddots & \vdots \\ * & * & * & \beta_{nn} \end{pmatrix}$$

where a_i, $b_j \in \mathbf{Z}$. Put $l = \min(m,n)$. For $x \in M(m,n)$ and $i \leqq l$, let $\det_i(x)$ denote the determinant of the upper right $i \times i$-submatrix of x. Then there exists a non-zero polynomial function $g(x)$ on $M(m,n)$ which satisfies

$$g(\alpha x \beta) = \chi_1(\alpha)\chi_2(\beta)g(x) \qquad \text{for all } \alpha \in P_m,\ \beta \in P_n$$

if and only if the following conditions (1.11) and (1.12) hold.

$$\begin{aligned} &a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0, \qquad a_{n+1} = a_{n+2} = \cdots = a_m = 0, \\ &b_i = a_{n+1-i}, \quad 1 \leqq i \leqq n \qquad \text{if} \quad m \geqq n. \end{aligned} \tag{1.11}$$

(1.12)
$$\begin{aligned} &a_1 \geqq a_2 \geqq \cdots \geqq a_m \geqq 0, \\ &b_i = a_{n+1-i}, \quad n-m+1 \leqq i \leqq n, \qquad b_1 = b_2 = \cdots = b_{n-m} = 0 \qquad \text{if} \quad m \leqq n. \end{aligned}$$

If these conditions hold, $g(x)$ is equal to

$$\prod_{i=1}^{l-1} \det_i(x)^{a_i - a_{i+1}} \det_l(x)^{a_l}$$

up to the constant multiple.

Proof. Let $g(x)$ be a non-zero polynomial function as in our lemma. We note that every a_i is non-negative since $\chi_1(\alpha) = g(\alpha x)/g(x)$, $\alpha \in P_m$ is a polynomial of α. Similarly every b_j is non-negative. We also note that

$$\det_i(\alpha x) = \alpha_{11}\alpha_{22}\cdots\alpha_{ii}\det_i(x), \tag{1.13}$$

$$\det_i(x\beta) = \beta_{nn}\beta_{n-1\,n-1}\cdots\beta_{n+1-i\,n+1-i}\det_i(x) \tag{1.14}$$

for $\alpha \in P_m$, $\beta \in P_n$ as in our lemma.

First let us consider the case $m = n$. Put

$$w = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \in GL(n).$$

Then $P_n w P_n$ is Zariski dense in $M(n,n)$. Hence $g(w) \neq 0$. Put

$$Z = \{x \in M(n,n) \mid \det_i(x) \neq 0 \quad \text{for} \quad 1 \leqq i \leqq n\}.$$

Let $x \in Z$. We can write $xw = pu$ with $p = (p_{ij}) \in P_n$, $u \in GL(n)$ is an upper unipotent matrix. We have

$$p_{ii} = (-1)^{i-1} \det_i(x)/\det_{i-1}(x), \qquad 1 \leqq i \leqq n,$$

with the convention $\det_0(x) = 1$. Since $x = pw \cdot wuw$, we get

$$g(x) = (-1)^\delta \prod_{i=1}^{n-1} \det_i(x)^{a_i - a_{i+1}} (\det x)^{a_n} g(w), \qquad \delta = \sum_{i=1}^{n} (i-1)a_i.$$

Since Z is Zariski dense in $M(n,n)$, this rational function of x must be a polynomial of x. Since $\det_1(x)$, $\det_2(x)$, $\cdots$, $\det_n(x)$ are mutually prime irreducible polynomials, we see that $a_i \geqq a_{i+1}$ for $1 \leqq i \leqq n-1$ and that $g(x)$ is equal to $\prod_{i=1}^{n-1} \det_i(x)^{a_i - a_{i+1}} \det(x)^{a_n}$ up to the constant multiple. The relation between a_i and b_j follows from (1.13) and (1.14).

Next assume $m > n$. Write $x \in M(m,n)$ as $x = \begin{pmatrix} y \\ z \end{pmatrix}$, $y \in M(n,n)$, $z \in M(m-n,n)$ and put $h(y) = g(\begin{pmatrix} y \\ 0 \end{pmatrix})$, $y \in M(n,n)$. By

$$\text{(1.15)} \qquad \begin{pmatrix} 1_n & 0 \\ u & 1_{m-n} \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y \\ uy + z \end{pmatrix}, \qquad u \in M(m-n,n),$$

we see easily that $h(y) \neq 0$. (The subset $\det y \neq 0$ is Zariski dense.) Since $h(y)$ has the transformation property of the type in our lemma, we have

$$h(y) = c \prod_{i=1}^{n-1} \det_i(y)^{a_i - a_{i+1}} (\det y)^{a_n}$$

with a constant c. Then we see that $g(x)$ is equal to $c \prod_{i=1}^{n-1} \det_i(x)^{a_i - a_{i+1}} (\det_n(x))^{a_n}$ by (1.15). Remaining assertions for the case $m > n$ follow immediately. The case $m < n$ can be treated similarly. This completes the proof.

Proof of Theorem 1. We first prove the theorem for the case $P = P_n$ and for polynomial functions of the type (λ_1, λ_2). Thus λ_1 is λ_1^* which is given by (1.4). Suppose that a non-zero polynomial function of the type (λ_1, λ_2) exists. By the same reason as in Lemma 2, we have $w_i \geqq 0$, $1 \leqq i \leqq n$, k^+, $k^- \geqq 0$. If $n^- = 0$,

$f(x)$ is a constant multiple of $(\det x)^{k^+}$. Since the conditions (1.6) $\sim$ (1.10) are equivalent to $w_1 = w_2 = \cdots = w_n = k^+$, we see that the assertion holds. Similarly the case $n^- = 0$ is clear.

We assume $n^+ \geqq 1$ and $n^- \geqq 1$. . Define a polynomial function $g(x)$ on $M(n^+, n^-)$ by

$$g(x) = f\left(\begin{pmatrix} 1_{n^+} & x \\ 0 & 1_{n^-} \end{pmatrix}\right), \qquad x \in M(n^+, n^-).$$

We see that

$$P_n \cdot \left\{ \begin{pmatrix} 1_{n^+} & x \\ 0 & 1_{n^-} \end{pmatrix} \mid x \in M(n^+, n^-) \right\} \cdot (GL(n^+) \times GL(n^-))$$

is Zariski dense in $M(n, n)$. Hence $g \neq 0$ and f is uniquely determined by g. For $\alpha \in P_{n^+}$ and $\beta \in P_{n^-}$, the relation

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta^{-1} \end{pmatrix} \begin{pmatrix} 1_{n^+} & x \\ 0 & 1_{n^-} \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1_{n^+} & \alpha x \beta \\ 0 & 1_{n^-} \end{pmatrix}$$

yields

$$g(\alpha x \beta) = \lambda_1\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta^{-1} \end{pmatrix}\right) \lambda_2\left(\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \beta \end{pmatrix}\right) g(x).$$

By Lemma 2, we obtain the conditions (1.6) $\sim$ (1.9). Lemma 2 also implies that f is unique up to the constant multiple. To see that the condition (1.10) holds, put

$$f^*(x) = f(x\omega), \qquad \omega = \begin{pmatrix} 0 & 0 & 1_{n^-} \\ 0 & 1_t & 0 \\ 1_{n^-} & 0 & 0 \end{pmatrix} \in GL(n), \quad t = n^+ - n^-$$

assuming $n^+ \geqq n^-$. Then f^* satisfies

$$f^*(px\gamma) = \lambda_1(p)\lambda_2^*(\gamma) f(x) \qquad \text{for all } \ p \in P_n,\ \gamma \in GL(n^-) \times GL(n^+),$$

$$\lambda_2^*\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = (\det a)^{k^-} (\det b)^{k^+}, \qquad a \in GL(n^-),\ b \in GL(n^+).$$

By the conditions (1.6) and (1.7) satisfied by f^*, we get (1.10). The case $n^+ \leqq n^-$ can be handled similarly.

Let us show the existence of $f \neq 0$ of the type (λ_1, λ_2) when (λ_1, λ_2) is admissible. Put $k = k^+ + k^-$, $N = n^+k^+ + n^-k^-$. We first note that

$$(1.16) \qquad w_1 + w_2 + \cdots + w_n = N$$

which follows from (1.8) and (1.9). Let K be an algebraically closed field containing F. For row vectors $x_i^\pm \in K^{n^\pm}$ $(1 \leqq i \leqq n)$, $y_j^\pm \in K^{k^\pm}$ $(1 \leqq j \leqq k)$, we define a square matrix Z of size N as follows. Divide the rows of Z according as the partition (1.16). In the i-th block, we put

$$\begin{pmatrix} x_i^+ \otimes y_1^+ & x_i^- \otimes y_1^- \\ x_i^+ \otimes y_2^+ & x_i^- \otimes y_2^- \\ \vdots & \vdots \\ x_i^+ \otimes y_{w_i}^+ & x_i^- \otimes y_{w_i}^- \end{pmatrix},$$

regarding $x_i^\pm \otimes y_j^\pm \in K^{d^\pm k^\pm}$. Put

$$x = \begin{pmatrix} x_1^+ & x_1^- \\ x_2^+ & x_2^- \\ \vdots & \vdots \\ x_n^+ & x_n^- \end{pmatrix}, \qquad y = \begin{pmatrix} y_1^+ & y_1^- \\ y_2^+ & y_2^- \\ \vdots & \vdots \\ y_k^+ & y_k^- \end{pmatrix}, \qquad h(x,y) = \det Z.$$

We regard x (resp. y) as a variable on $M(n,n)$ (resp. $M(k,k)$). When y is fixed, we see easily that the polynomial $h(x,y)$ of x is of the type $\{(w_1, w_2, \cdots, w_n); (k^+, k^-)\}$. By the uniqueness assertion already established, we see that

$$h(x,y) = f(x)g(y) \tag{1.17}$$

with polynomials $f(x)$ and $g(y)$ with coefficients in F. Our task is to show $h \neq 0$. We note the following "duality" between f and g. Put

$$w_i' = |\{j \mid 1 \leqq j \leqq n, \quad w_j \geqq i\}|$$

for $1 \leqq i \leqq k$. Then it can be verified that $U = \{(w_1', w_2', \cdots, w_k'); (n^+, n^-)\}$ is an admissible data and that $g(y)$ is of the type U.

Now we proceed to prove $h \neq 0$ by induction on N. The case $n^+n^- = 0$ or $k^+k^- = 0$ being easy, we assume n^+, n^-, k^+, $k^- \geqq 1$. We first consider the case $w_n \geqq 1$. For $x \in GL(n,K)$, $y \in M(k,k,K)$, suppose that

$$\sum_{i=1}^{n} \sum_{j=1}^{w_i} c_{ij}(x_i^\pm \otimes y_j^\pm) = 0 \tag{1.18}$$

with $c_{ij} \in K$. We choose $y_1^\pm \in K^{k^\pm}$ as non-zero vectors and take $y_2^\pm, \cdots, y_k^\pm$ from complements of the spaces generated by $y_1^\pm$. Then (1.18) is equivalent to

$$\sum_{i=1}^{n} c_{i1} x_i^\pm = 0,$$

$$\sum_{i=1}^{n} \sum_{j=2}^{w_i} c_{ij}(x_i^\pm \otimes y_j^\pm) = 0. \tag{1.19}$$

We get $c_{i1} = 0$, $1 \leqq i \leqq n$ since $\det x \neq 0$. The data $\{(w_1 - 1, w_2 - 1, \cdots, w_n - 1); (k^+ - 1, k^- - 1)\}$ is admissible. Hence, by the induction hypothesis, we can choose $x_i^\pm$ $(1 \leqq i \leqq n)$, $y_j^\pm$ $(2 \leqq j \leqq k)$ so that (1.19) implies all $c_{ij} = 0$. Next we consider the case $w_n = 0$. By (1.9), we have $w_1 = k$. Therefore $w_k' \geqq 1$ and we are reduced to the previous case by the duality.

Now the assertion for the general P clearly follows from the special case $P = P_n$ in view of the definition of $h(x,y)$. This completes the proof.

Theorem 3. *Let the notation be the same as in Theorem 1. If there exists p^+ (resp. p^-) such that $1 \leqq p^+ \leqq m$ (resp. $1 \leqq p^- \leqq m$) such that $s_1 + s_2 + \cdots + s_{p^+} = n^+$ (resp. $s_1 + s_2 + \cdots + s_{p^-} = n^-$), let $f^+(x)$ (resp. $f^-(x)$) denote the determinant of the upper left $n^+ \times n^+$-submatrix (resp. the upper right $n^- \times n^-$-submatrix) of x. Let A denote the set of integers a such that $1 \leqq a < \min(n^+, n^-)$,*

$$s_1 + s_2 + \cdots + s_p = a, \qquad s_1 + s_2 + \cdots + s_q = d - a$$

for some $1 \leqq p < q \leqq m$, and that $T_a = \{(\overbrace{2, \cdots, 2}^{p}, \overbrace{1, \cdots, 1}^{q-p}, \overbrace{0, \cdots, 0}^{m-q}); (1,1)\}$ is an admissible data. For $a \in A$, let $f_a(x)$ be a non-zero polynomial of the type T_a. Then every polynomial of the admissible type $\{(a_1, a_2, \cdots, a_m); (k^+, k^-)\}$ can be expressed uniquely as a monomial of $\det(x)$, *$f^+(x)$, $f^-(x)$ and $f_a(x)$, $a \in A$.*

Proof. For two admissible data

$$T = \{(a_1, a_2, \cdots, a_m); (k^+, k^-)\}, \qquad U = \{(b_1, b_2, \cdots, b_m); (l^+, l^-)\},$$

we define their sum by $T + U = \{(a_1 + b_1, a_2 + b_2, \cdots, a_m + b_m); (k^+ + l^+, k^- + l^-)\}$. Then $T + U$ is admissible. By the existence and the uniqueness proved in Theorem 1, it suffices to show that any admissible data T can be written uniquely as a sum of admissible data of $\det(x)$, $f^+(x)$, $f^-(x)$ and $f_a(x)$, $a \in A$. The task is somewhat tedious but can be done without difficulty using induction on $\sum_{i=1}^m a_i$. This completes the proof.

§2. On the hidden periods of a motive

Let E be an algebraic number fields of finite degree. Let J_E denote the set of all isomorphisms of E into $\mathbf{C}$. We put $R = E \otimes_{\mathbf{Q}} \mathbf{C} \cong \mathbf{C}^{J_E}$. Let M be a motive over $\mathbf{Q}$ with coefficients in E. Then M has (at least) three realizations Betti, de Rham and λ-adic.

First let $H_B(M)$ denote the Betti realization of M; $H_B(M)$ is a finite dimensional vector space over E, whose dimension $d = d(M)$ is called *the rank of M*. The complex conjugation F_∞ acts E-linearly on $H_B(M)$. We can decompose

$$H_B(M) = H_B^+(M) \oplus H_B^-(M) \tag{2.1}$$

where $H_B^\pm(M)$ denotes the eigenspace of $H_B(M)$ with the eigenvalue ± 1. We denote the dimension of $H_B^\pm(M)$ over E by $d^\pm = d^\pm(M)$. Furthermore $H_B(M)$ has the Hodge decomposition:

$$H_B(M) \otimes_{\mathbf{Q}} \mathbf{C} = \oplus_{p,q \in \mathbf{Z}} H^{pq}(M), \tag{2.2}$$

where $H^{pq}(M)$ is a free $E \otimes_{\mathbf{Q}} \mathbf{C}$-module. If $H^{pq}(M) = \{0\}$ whenever $p + q \neq w$, M is called *of pure weight w*, which we shall assume hereafter.

We skip the description of the λ-adic realization of M. We simply note that the L-function $L(M, s)$ of M taking values in R is defined using it and the gamma factor of the functional equation can be described by the Hodge decomposition (2.2) (see [D]).

Let $H_{DR}(M)$ be the de Rham realization of M which is a d-dimensional vector space over E. Let $\{F^p\}$ be the Hodge filtration on $H_{DR}(M)$; $\{F^p\}$ is a decreasing filtration by vector spaces over E. When it is necessary to emphasize the dependence on M, we write $F^p(M)$ for F^p. Let

$$I : H_B(M) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(M) \otimes_{\mathbf{Q}} \mathbf{C}$$

be the comparison isomorphism. We have

$$(2.3) \qquad I(\oplus_{p' \geqq p} H^{p'q}(M)) = F^p(M) \otimes_{\mathbf{Q}} \mathbf{C}.$$

Now we are going to define a period matrix of M. Let $\{v_1^+, v_2^+, \cdots, v_{d^+}^+\}$ (resp. $\{v_1^-, v_2^-, \cdots, v_{d^-}^-\}$) be a basis of $H_B^+(M)$ (resp. $H_B^-(M)$) over E. We write the Hodge filtration as

$$H_{DR}(M) = F^{i_i} \underset{\neq}{\supset} F^{i_2} \underset{\neq}{\supset} \cdots \underset{\neq}{\supset} F^{i_m} \underset{\neq}{\supset} F^{i_{m+1}} = \{0\}$$

so that there are no different filtrations between successive members. The choice of numbers i_μ may not be unique for F^{i_μ}. Put $t_\mu = \dim_E F^{i_\mu}$, $1 \leqq \mu \leqq m$. We take a basis $\{w_1, w_2, \cdots, w_d\}$ of $H_{DR}(M)$ over E so that $\{w_{d-t_\mu+1}, \cdots, w_d\}$ is a basis of F^{i_μ} for $1 \leqq \mu \leqq m$. Writing

$$(2.4) \qquad I(v_j^\pm) = \sum_{i=1}^{d} x_{ij}^\pm w_i, \qquad x_{ij}^\pm \in R, \quad 1 \leqq j \leqq d^\pm,$$

we obtain a matrix $X^\pm = (x_{ij}^\pm) \in M(d, d^\pm, R)$. We put

$$s_i = t_i - t_{i+1}, \quad 1 \leqq i \leqq m-1, \qquad s_m = t_m.$$

Let P_M be the lower parabolic subgroup of $GL(d)$ which corresponds to the partition $d = s_1 + s_2 + \cdots + s_m$. Then the coset of $X^\pm$ in

$$P_M(E) \backslash M(d, d^\pm, R) / GL(d^\pm, E)$$

does not depend on the choices of basis. We put $X = (X^+ \, X^-) \in M(d, d, R)$ and call it *a period matrix of* M. The coset of X in

$$P_M(E) \backslash M(d, d^\pm, R) / (GL(d^+, E) \times GL(d^-, E))$$

is well defined. Here $GL(d^+, E) \times GL(d^-, E)$ is embedded in $GL(d)$ as diagonal blocks.

We are going to apply our results in §1 taking the base field F as $\mathbf{Q}$, P as P_M and $n = d$, $n^\pm = d^\pm$. Let $f(x)$ be a non-zero polynomial of the type $\{(a_1, a_2, \cdots, a_m); (k^+, k^-)\}$. Let $X \in M(d, d, R)$ be a period matrix of M. Then $f(X)$ is uniquely determined up to the multiplication by elements of $E^\times$. We call $f(X)$ *a period invariant of* M *of the type* $\{(a_1, a_2, \cdots, a_m); (k^+, k^-)\}$. Hereafter we understand the equality between period invariants mod $E^\times$.

Let $f(x) = \det(x)$, $x \in M(d, d)$. Then $f(x)$ is of the type $\{(1, 1, \cdots, 1); (1, 1)\}$ and $f(X)$ is Deligne's period $\delta(M)$. Assume that $t_1 - t_{p^+ + 1} = d^+$ for some p^+. Let $f^+(x)$ be the determinant of the upper left $d^+ \times d^+$-submatrix of $x \in M(d, d)$.

Then $f^+(x)$ is of the type $\{(\overbrace{1,1,\cdots,1}^{p^+},0,\cdots,0);(1,0)\}$ and $f^+(X)$ is Deligne's period $c^+(M)$. Similarly if $t_1 - t_{p^-+1} = d^-$ for some p^-, let $f^-(x)$ be the determinant of the upper right $d^- \times d^-$-submatrix of x. Then $f^-(x)$ is of the type $\{(\overbrace{1,1,\cdots,1}^{p^-},0,\cdots,0);(0,1)\}$ and $f^-(X)$ is Deligne's period $c^-(M)$. Either one of the above conditions is equivalent to that $F^\pm(M)$ can be defined (cf. [D], §1, [Y], §2). We have $F^\mp(M) = F^{i_{p^\pm+1}}(M)$; $F^\pm(M)$ can be defined if M has a critical value.

Deligne's conjecture states that $L(M,0)/c^+(M) \in E$ if 0 is critical for M. Other period invariants are hidden for the relation with $L(M,0)$ but will appear when we make various algebraic operations on M (tensor products with other motives, exterior powers, etc.). Deligne showed $c^\pm(M) \neq 0$. The non-vanishing of other period invariants will be proved at the end of this section.[1]

We write X in the form

$$X = \begin{pmatrix} \mathfrak{X}_1^+ & \mathfrak{X}_1^- \\ \mathfrak{X}_2^+ & \mathfrak{X}_2^- \\ \vdots & \vdots \\ \mathfrak{X}_d^+ & \mathfrak{X}_d^- \end{pmatrix}, \qquad \mathfrak{X}_i^+ \in R^{d^+}, \quad \mathfrak{X}_i^- \in R^{d^-}.$$

If $t_1 - t_\mu \leqq i < t_1 - t_{\mu+1}$, we call $\mathfrak{X}_i^\pm$ *of weight* i_μ and write $w(\mathfrak{X}_i^+) = w(\mathfrak{X}_i^-) = i_\mu$.

Let us consider the exterior square $\wedge^2 M$ and the symmetric square $\mathrm{Sym}^2 M$ of the motive M.

Proposition 4. *Assume that $F^\pm(\wedge^2 M)$ can be defined. Let $F^\pm(\wedge^2 M) = F^{q^\pm}(\wedge^2 M)$. Then $c^+(\wedge^2 M) = f^+(X)$ with a polynomial $f^+(x)$ of the type $\{(a_1,a_2,\cdots,a_m);(k^+,k^-)\}$, where, taking $\mathfrak{X}_i^+$ so that $w(\mathfrak{X}_i^+) = i_\mu$,*

$$a_\mu = |\{l \mid 1 \leqq l \leqq d,\ w(\mathfrak{X}_l^+) \neq i_\mu,\ i_\mu + w(\mathfrak{X}_l^+) < q^-\}| + \begin{cases} t_\mu - t_{\mu+1} - 1 & \text{if} \quad 2i_\mu < q^-, \\ 0 \quad \text{if} \quad 2i_\mu \geqq q^-, \end{cases} \tag{2.5}$$

$$k^+ = d^+(M) - 1, \qquad k^- = d^-(M) - 1. \tag{2.6}$$

We have $c^-(\wedge^2 M) = f^-(X)$ with a polynomial $f^-(x)$. The type of $f^-(x)$ is given by

$$a_\mu = |\{l \mid 1 \leqq l \leqq d,\ w(\mathfrak{X}_l^+) \neq i_\mu,\ i_\mu + w(\mathfrak{X}_l^+) < q^+\}| + \begin{cases} t_\mu - t_{\mu+1} - 1 & \text{if} \quad 2i_\mu < q^+, \\ 0 \quad \text{if} \quad 2i_\mu \geqq q^+, \end{cases} \tag{2.7}$$

$$k^+ = d^-(M), \qquad k^- = d^+(M). \tag{2.8}$$

[1]It is not difficult to prove that if $f(X)$ is non-zero and if $f(X) \bmod E^\times$ is well defined for a polynomial $f(x)$, f must be of the above form.

Proof. We have

$$H_B^+(\wedge^2 M) = (H_B^+(M) \wedge H_B^+(M)) \oplus (H_B^-(M) \wedge H_B^-(M)),$$
$$H_B^-(\wedge^2 M) = H_B^+(M) \wedge H_B^-(M),$$
$$I(v_j^\pm \wedge v_k^\pm) = \sum_{1 \leqq i < l \leqq d} (x_{ij}^\pm x_{lk}^\pm - x_{lj}^\pm x_{ik}^\pm)(w_i \wedge w_l).$$

Let $W = (W^+ \ W^-)$ be a period matrix of $\wedge^2 M$. The formula above shows that W^+ can be given by arranging $(\mathfrak{X}_i^+ \wedge \mathfrak{X}_l^+,\ \mathfrak{X}_i^- \wedge \mathfrak{X}_l^-)$, $1 \leqq i < l \leqq d$ according to the increasing order of $w(\mathfrak{X}_i^+) + w(\mathfrak{X}_l^+)$, regarding $\mathfrak{X}_i^\pm \wedge \mathfrak{X}_l^\pm \in R^{d^\pm(d^\pm - 1)/2}$. The assertions for $c^+(\wedge^2 M)$ follow easily from this observation. The formula for $c^-(\wedge^2 M)$ can be shown similarly. This completes the proof.

Proposition 5. *Assume that $F^\pm(Sym^2 M)$ can be defined and let $F^\pm(Sym^2 M) = F^{q^\pm}(Sym^2 M)$. Then $c^\pm(Sym^2 M) = g^\pm(X)$ with a polynomial $g^\pm$. The type of g^+ is given by*

$$a_\mu = |\{l \mid 1 \leqq l \leqq d,\ w(\mathfrak{X}_l^+) \neq i_\mu,\ i_\mu + w(\mathfrak{X}_l^+) < q^-\}| + \begin{cases} t_\mu - t_{\mu+1} + 1 & \text{if} \quad 2i_\mu < q^-, \\ 0 \quad \text{if} \quad 2i_\mu \geqq q^-, \end{cases} \tag{2.9}$$

$$k^+ = d^+(M) + 1, \qquad k^- = d^-(M) + 1. \tag{2.10}$$

The type of g^- is given by

$$a_\mu = |\{l \mid 1 \leqq l \leqq d,\ w(\mathfrak{X}_l^+) \neq i_\mu,\ i_\mu + w(\mathfrak{X}_l^+) < q^+\}| + \begin{cases} t_\mu - t_{\mu+1} + 1 & \text{if} \quad 2i_\mu < q^+, \\ 0 \quad \text{if} \quad 2i_\mu \geqq q^+, \end{cases} \tag{2.11}$$

$$k^+ = d^-(M), \qquad k^- = d^+(M). \tag{2.12}$$

The proof is similar to that of Proposition 4, so we omit it.

Let N be a motive over $\mathbf{Q}$ with coefficients in E. Let us consider periods of $M \otimes N$.

Lemma 6. *We have*

$$F^p(M \otimes N) = \sum_{k+l=p} F^k(M) \otimes F^l(N).$$

Proof. By (2.3), we get

$$F^p(M \otimes N) \otimes_{\mathbf{Q}} \mathbf{C} = \sum_{k+l=p} F^k(M) \otimes F^l(N) \otimes_{\mathbf{Q}} \mathbf{C}$$

and the assertion follows from this fact.

Let

$$H_{DR}(N) = F^{j_1}(N) \supsetneqq F^{j_2}(N) \supsetneqq \cdots \supsetneqq F^{j_v}(N) \supsetneqq F^{i_{v+1}}(N) = \{0\},$$
$$H_{DR}(M \otimes N) = F^{k_1}(M \otimes N) \supsetneqq \cdots \supsetneqq F^{k_z}(M \otimes N) \supsetneqq F^{i_{z+1}} M \otimes N) = \{0\}$$

be the Hodge filtrations. The next lemma follows easily from Lemma 6.

Lemma 7. *Choose $v_1^{(r)}, \cdots, v_{a_r}^{(r)} \in F^{i_r}(M)$ so that they make a basis of $F^{i_r}(M)/F^{i_{r+1}}(M)$ when taken modulo $F^{i_{r+1}}(M)$. Choose $w_1^{(s)}, \cdots, w_{b_s}^{(s)} \in F^{j_s}(N)$ so that they make a basis of $F^{j_s}(N)/F^{j_{s+1}}(N)$ when taken modulo $F^{i_{s+1}}(N)$. Then*

$$\{v_k^{(r)} \otimes w_l^{(s)} \mod F^{k_{t+1}}(M \otimes N) \mid i_r + j_s = k_t,\ 1 \leqq k \leqq a_r,\ 1 \leqq l \leqq b_s\}$$

makes a basis of $F^{k_t}(M \otimes N)/F^{k_{t+1}}(M \otimes N)$.

Proposition 8. *Let $X = (X^+\ X^-) \in M(d(M), d(M), R)$ and $Y = (Y^+\ Y^-) \in M(d(N), d(N), R)$ be period matrices of M and N respectively. Write $Y^\pm$ as*

$$Y^\pm = \begin{pmatrix} \mathfrak{Y}_1^\pm \\ \mathfrak{Y}_2^\pm \\ \vdots \\ \mathfrak{Y}_{d(N)}^\pm \end{pmatrix}.$$

Assume that $F^\pm(M \otimes N)$ can be defined and let $F^\pm(M \otimes N) = F^{q^\pm}(M \otimes N)$. Then we have

$$c^\pm(M \otimes N) = f^\pm(X) g^\pm(Y)$$

where $f^\pm$ (resp. $g^\pm$) is a polynomial on $M(d(M), d(M))$ (resp. $M(d(N), d(N))$) with coefficients in $\mathbf{Z}$. The type of $f^\pm$ is given by

$$a_\mu = |\{l \mid 1 \leqq l \leqq d(N),\ i_\mu + w(\mathfrak{Y}_l^+) < q^\mp\}|, \tag{2.13}$$

$$k^+ = d^\pm(N), \qquad k^- = d^\pm(N). \tag{2.14}$$

Proof. We have

$$H_B^+(M \otimes N) = (H_B^+(M) \otimes H_B^+(N)) \oplus (H_B^-(M) \otimes H_B^-(N)).$$

By Lemma 7, we see that $c^+(M \otimes N)$ is given as the determinant of the square matrix of size $d^+(M)d^+(N) + d^-(M)d^-(N)$:

$$(\mathfrak{X}_i^+ \otimes \mathfrak{Y}_l^+,\ \mathfrak{X}_i^- \otimes \mathfrak{Y}_l^- \mid w(\mathfrak{X}_i^+) + w(\mathfrak{Y}_l^+) < q^-).$$

Therefore $c^+(M \otimes N) = h^+(X, Y)$ with a polynomial $h^+(x, y)$ with coefficients in $\mathbf{Z}$ on $M(d(M), d(M)) \oplus M(d(N), d(N))$. When y is fixed, we see that the type of $h^+(x, y)$ as a polynomial of x is given by (2.13) and (2.14). By the uniqueness assertion of Theorem 1, we see that $h^+(x, y) = f^+(x)g^+(y)$ with polynomials $f^+(x)$, $g^+(y)$ with coefficients in $\mathbf{Z}$. Thus we established our assertion for $c^+(M \otimes N)$. The assertions for $c^-(M \otimes N)$ can be shown similarly. This completes the proof.

Corollary 1. *If $d^+(M) = d^-(M)$, $d^+(N) = d^-(N)$ and $q^+ = q^-$, then we have $c^+(M \otimes N) = c^-(M \otimes N)$.*

Corollary 2. *Let M be a motive over $\mathbf{Q}$ with coefficients in E. Then the period invariant of the type $\{(a_1, a_2, \cdots, a_m), (k^+, k^-)\}$ is non-zero.*

Proof. Let X be a period matrix of M. We use the notation of Theorem 3. It suffices to show that $f_a(X) \neq 0$ for any $a \in A$. Let N be a motive over $\mathbf{Q}$ with coefficients in E of pure weight $w' \geqq 1$ such that

$$H_B(N) \otimes \mathbf{C} = H^{0,w'}(N) \oplus H^{w',0}(N).$$

Examples of such an N can be obtained as the motive attached to an elliptic modular form (cf. §4). We choose the numbering of the Hodge filtration of $H_{DR}(M)$ so that i_μ is the maximum number. Then we see that

$$i_c + i_{m+1-c} = w, \quad 1 \leqq c \leqq m, \qquad s_\mu = s_{m+1-\mu}, \quad 1 \leqq \mu \leqq m. \tag{2.15}$$

Let $a \in A$ and take $w' = w - 1 - 2i_p$. The conditions that the type of $f^+(x)$ coincides with that of $f_a(x)$ are

$$q^- - i_{p+1} \leqq w' < q^- - i_p, \qquad i_q < q^- \leqq i_{q+1}. \tag{2.16}$$

By (2.15), we obtain $p + q = m$, $w' \geqq 1$. Since $q^- = (w + w' + 1)/2$ (cf. [Y], §2.2) and $i_p + i_{q+1} = w$, we see that (2.16) holds. This completes the proof.

§3. Motives of rank 4

Let M be a motive over $\mathbf{Q}$ with coefficients in E. We assume that M is of rank 4 and of pure weight $w \geqq 3$. We assume further that

$$H_B(M) \otimes \mathbf{C} = H^{0,w}(M) \oplus H^{w,0}(M) \oplus H^{c,w-c}(M) \oplus H^{w-c,c}(M) \tag{3.1}$$

is the Hodge decomposition with

$$0 < 2c < w. \tag{3.2}$$

Then the Hodge filtration is

$$H_{DR}(M) = F^0 \supsetneqq F^c \supsetneqq F^{w-c} \supsetneqq F^w \supsetneqq \{0\}$$

and $d^+(M) = d^-(M) = 2$. Since F_∞ acts on $H^{w,w}(\wedge^2 M)$ as -1, we have $F^+(\wedge^2 M) = F^{w+1}(\wedge^2 M)$, $F^-(\wedge^2 M) = F^w(\wedge^2 M)$. By Proposition 4, we have

$$c^\pm(\wedge^2 M) = f^\pm_{\wedge^2}(X)$$

with a polynomial $f^\pm_{\wedge^2}(x)$ on $M(4,4)$. The type of $f^+_{\wedge^2}(x)$ is $\{(2,1,1,0);(1,1)\}$ and the type of $f^-_{\wedge^2}(x)$ is $\{(3,2,2,1);(2,2)\}$. Therefore we obtain

$$c^-(\wedge^2 M) = c^+(\wedge^2 M)\delta(M). \tag{3.3}$$

Since F_∞ acts on $H^{w,w}(\mathrm{Sym}^2 M)$ as 1, we have $F^+(\mathrm{Sym}^2 M) = F^w(\mathrm{Sym}^2 M)$, $F^-(\mathrm{Sym}^2 M)$
$= F^{w+1}(\mathrm{Sym}^2 M)$. By Proposition 5, we have

$$c^\pm(\mathrm{Sym}^2(M)) = f^\pm_{\mathrm{Sym}^2}(X),$$

where the type of $f^+_{\mathrm{Sym}^2}(x)$ is $\{(5,4,2,1);(3,3)\}$ and the type of $f^-_{\mathrm{Sym}^2}(x)$ is $\{(4,3,1,0);(2,2)\}$. Hence

$$c^+(\mathrm{Sym}^2(M)) = c^+(M)c^-(M)c^+(\wedge^2 M)\delta(M), \tag{3.4}$$

$$c^-(\mathrm{Sym}^2(M)) = c^+(M)c^-(M)c^+(\wedge^2 M). \tag{3.5}$$

Next let N be a motive over $\mathbf{Q}$ with coefficients in E. We assume that N is of rank 2 and of pure weight $w' \geqq 1$. We also assume

$$H_B(N) \otimes \mathbf{C} = H^{0,w'}(N) \oplus H^{w',0}(N) \tag{3.6}$$

is the Hodge decomposition. The motive $M \otimes N$ is of pure weight $w + w'$. We assume that $F^\pm(M \otimes N)$ can be defined and are going to calculate $c^\pm(M \otimes N)$. If $w = w'$ or $2c = w - w'$, then $F^\pm(M \otimes N)$ cannot be defined. We have

$$w \neq w' \qquad \text{and} \qquad 2c \neq w - w'. \tag{3.7}$$

Let Y be a period matrix of N. We have

$$c^\pm(M \otimes N) = f^\pm(X)g^\pm(Y)$$

with a polynomial $f^\pm$ (resp. $g^\pm$) on $M(4,4)$ (resp. $M(2,2)$). By (3.7), we see that $F^+(M \otimes N) = F^-(M \otimes N)$. By Corollary 1 to Proposition 8, we have

$$c^+(M \otimes N) = c^-(M \otimes N). \tag{3.8}$$

Put $F^+(M \otimes N) = F^{q^+}(M \otimes N)$. We must distinguish three cases.

Case A. The case where $w' > w$.

We can take $q^+ = w'$. By Proposition 8, the type of $f^+(x)$ is $\{(1,1,1,1);(1,1)\}$ and the type of $g^+(x)$ is $\{(4,0);(2,2)\}$. Hence

$$c^\pm(M \otimes N) = \delta(M)c^+(N)^2 c^-(N)^2. \tag{3.9}$$

Case B. The case where $w' + 2c > w > w'$.

We can take $q^+ = \min(w, w' + c)$. By Proposition 8, the type of $f^+(x)$ is $\{(2,1,1,0);$
$(1,1)\}$ and the type of $g^+(x)$ is $\{(3,1);(2,2)\}$. Hence

$$c^\pm(M \otimes N) = c^+(\wedge^2 M)c^+(N)c^-(N)\delta(N). \tag{3.10}$$

Case C. The case where $w > w' + 2c$.

We can take $q^+ = w - c$. By Proposition 8, the type of $f^+(x)$ is $\{(2,2,0,0);(1,1)\}$ and the type of $g^+(x)$ is $\{(2,2);(2,2)\}$. Hence

$$c^\pm(M \otimes N) = c^+(M)c^-(M)\delta(N)^2. \tag{3.11}$$

§4. Siegel modular forms of genus 2

Let $f \in S_k(Sp(2,\mathbf{Z}))$ be a non-zero Hecke eigenform. Let E be an algebraic number field which contains eigenvalues of f of all Hecke operators. Let $L(s,f)$ denote the spinor L-function attached to f. The functional equation proved by Andrianov [A] is

$$(4.1)\quad R(s,f) = (-1)^k R(2k-2-s,f), \qquad R(s,f) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)L(s,f).$$

Let $M(f)$ be the (hypothetical) motive over $\mathbf{Q}$ with coefficients in E such that $L(M(f),s)= (L(s,f^\sigma))_{\sigma\in J_E}$. Put $M = M(f)$. The rank of M must be 4 and from the functional equation (4.1), we infer that M is of pure weight $2k-3$ and that

$$(4.2)\quad H_B(M)\otimes_{\mathbf{Q}}\mathbf{C} = H^{0,2k-3}(M)\oplus H^{2k-3,0}(M)\oplus H^{k-2,k-1}(M)\oplus H^{k-1,k-2}(M).$$

Thus M is a special case $w = 2k-3$, $c = k-2$ of the motive considered in §3.

Let $g \in S_l(SL(2,\mathbf{Z}))$ be a normalized Hecke eigenform. We take E as an algebraic number field containing eigenvalues of f and g under all Hecke operators. Let $N = N(g)$ be the motive over $\mathbf{Q}$ with coefficients in E such that $L(N(g),s) = (L(s,g^\sigma))_{\sigma\in J_E}$ (see Scholl [Sc]). Then N is of pure weight $l-1$ and

$$(4.3)\qquad H_B(N)\otimes_{\mathbf{Q}}\mathbf{C} = H^{0,l-1}(N)\oplus H^{l-1,0}(N).$$

Since $l \geqq 2$, Case C does not occur and we obtain

$$(4.4)\qquad c^\pm(M\otimes N) = \delta(M)c^+(N)^2c^-(N)^2 \qquad \text{if} \quad l > 2k-2,$$

$$(4.5)\qquad c^\pm(M\otimes N) = c^+(\wedge^2 M)c^+(N)c^-(N)\delta(N) \qquad \text{if} \quad l < 2k-2$$

from (3.9) and (3.10). We note that $M\otimes N$ has a critical value if and only if $l \neq 2k-2$. It is well known that (cf. [Sh1], [Y], (4.11))

$$(4.6)\qquad c^+(N)c^-(N) = i^{1-l}\pi(\langle g^\sigma, g^\sigma\rangle)_{\sigma\in J_E}.$$

Since $\wedge^2 N \cong T(1-l)$, $\wedge^4 M \cong T(6-4k)$, $T(m)$ being a Tate motive, we have

$$(4.7)\qquad \delta(N) = 1\otimes(2\pi\sqrt{-1})^{1-l}, \qquad \delta(M) = 1\otimes(2\pi\sqrt{-1})^{6-4k}.$$

To write down $c^\pm(M\otimes N)$ in terms of data of modular forms, a crucial step is to identify $c^+(\wedge^2 M)$ with the normalized Petterson norm of f. To this end, we proceed as follows.

Let $L_{\rm st}(s,f)$ be the standard L-function attached to f. Comparing Euler factors, we see easily that

$$(4.8)\qquad L(\wedge^2 M, s) = \zeta(s-2k+3)(L_{\rm st}(s-2k+3,f^\sigma))_{\sigma\in J_E}.$$

(The functional equation of $L_{\rm st}(s,f)$ is given with respect to $s \longrightarrow 1-s$ in Andrianov-Kalinin [AK] and we can check that the gamma factor given there matches with that predicted by the theory of motives. See also Shimura [Sh3].) Using (3.3), (4.7) and [D], 5.1.8, we obtain

$$(4.9)\qquad \begin{gathered} L(\wedge^2 M, n)/(1\otimes 2\pi\sqrt{-1})^{4n-4k+6}c^+(\wedge^2 M) \in E \\ \text{if } n \text{ is odd and } 2k-1 \leqq n \leqq 3k-5 \end{gathered}$$

by Deligne's conjecture. Several authors (Harris [Ha], Sturm [St], Böcherer [Bo], Mizumoto [M]) obtained results on the critical values of the standard L-functions. We find [Bo], Satz 14 is most convenient for our present purpose. Assuming that the Fourier coefficients of f are contained in a totally real algebraic number field, it states that[2]

$$(4.10) \qquad \left(\frac{L_{\rm st}(n,f)}{\pi^{3n+2k}\langle f,f\rangle}\right)^{\sigma} = \frac{L_{\rm st}(n,f^{\sigma})}{\pi^{3n+2k}\langle f^{\sigma},f^{\sigma}\rangle}$$

for $\sigma \in \mathrm{Aut}(\mathbf{C})$ if $n \in \mathbf{Z}$ satisfies $n \equiv 0 \mod 2,\ 2 < n \leqq k-2$. Comparing (4.8), (4.9) with (4.10), we obtain the identification:

$$(4.11) \qquad c^{+}(\wedge^2 M) = \pi^{6-2k}(\langle f^{\sigma},f^{\sigma}\rangle)_{\sigma\in J_E}.$$

Summing up, we obtain

$$(4.12) \qquad c^{\pm}(M(f)\otimes M(g)) = (2\pi\sqrt{-1})^{8-4k}(\langle g^{\sigma},g^{\sigma}\rangle)^2_{\sigma\in J_E} \qquad \text{if} \quad l > 2k-2,$$

$$(4.13) \qquad c^{\pm}(M(f)\otimes M(g)) = (2\pi\sqrt{-1})^{8-2k-l}(\langle f^{\sigma},f^{\sigma}\rangle\langle g^{\sigma},g^{\sigma}\rangle)_{\sigma\in J_E} \qquad \text{if} \quad l < 2k-2.$$

Theorems A and B given in the introduction now follow immediately from (4.12), (4.13) and from Deligne's conjecture. For the convenience in the future, we note the formulas[3]

$$(4.14) \qquad c^{+}(\mathrm{Sym}^2(M(f))) = (2\pi\sqrt{-1})^{12-6k}c^{+}(M(f))c^{-}(M(f))(\langle f^{\sigma},f^{\sigma}\rangle)_{\sigma\in J_E},$$

$$(4.15) \qquad c^{-}(\mathrm{Sym}^2(M(f))) = (2\pi\sqrt{-1})^{6-2k}c^{+}(M(f))c^{-}(M(f))(\langle f^{\sigma},f^{\sigma}\rangle)_{\sigma\in J_E}$$

which follow from (3.4), (3.5), (4.7) and (4.11).

Remark. Let V be a 4-dimensional vector space over $\mathbf{Q}$ endowed with a non-degenerate alternating form. We let $GSp(2)$ act on V in standard way. The representation of $GSp(2)$ on $\wedge^2 V$ splits into a direct sum of a one dimensional representation and a 5-dimensional irreducible representation ρ such that $\rho|Sp(2)$ is an orthogonal representation (cf. Bourbaki [Bou], Chap. VIII, §13, Exercices, 3)). Since the image of the λ-adic representation is contained in $GSp(2)$, Tate's conjecture implies that $\wedge^2 M$ splits into a direct sum of motives according as this decomposition:

$$\wedge^2 M = T(3-2k)\oplus\rho(M),$$

where $\rho(M)$ is a motive over $\mathbf{Q}$ of rank 5 with coefficients in E.

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Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan
email address: yoshida@kusm.kyoto-u.ac.jp