Moduli Space of Abelian Varieties with Level Structure over Function Fields

by

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§ 1. Introduction and main results

Nadel [N] lately proved a very interesting theorem on the bound of level structures on principally polarized Abelian varieties which are defined over 1–dimensional complex function fields of genus ≤ 1 and non–constant. The purpose of this paper is to deal with the moduli problem of those Abelian varieties defined over an arbitrary function field of dimension ≥ 1, and to extend Nadel's theorem. We know that the moduli space of families of g–dimensional principally polarized Abelian varieties over a smooth complex projective manifold R, having degenerations at most over a given hypersurface of R, forms a quasi–projective scheme of finite type over $\mathbb{C}$ ([F]) (boundedness), and is a finite union of quotients of symmetric bounded domains ([No], cf. also [MN]). Here we put no condition on the degeneration locus, but take into account level structures to obtain the boundedness of the moduli space.

Let k be the rational function field of R. Let $\mathrm{A}(g, n, R)$ denote the moduli space of all g–dimensional principally polarized Abelian varieties A with level n–structure over R which are non–constant (cf. Definition (3.1)). Here the term "level n–structure" is used in the same sense as in [N]; i.e., it is a $2g$–tuple $(x_1, \ldots, x_{2g})$ of k–rational points $\in A(k)$ which generates the subgroup of all n–torsion points of $A(\bar{k})$, where $\bar{k}$ is the algebraic closure of k. Moreover, let $\mathrm{A}_{\mathrm{deg}}(g, n, R)$ be the subspace of those elements $A \longrightarrow R$ of $\mathrm{A}(g, n, R)$ which have degenerations. Set

$$\gamma(R) = \inf\left\{ \int_R c_1(K_R) \wedge \Omega^{n-1} \right\} \geq -\infty ,$$

where $c_1(K_R)$ denotes the first Chern class of the canonical bundle K_R over R and Ω runs over all Hodge metric forms on R. We denote by $k(g)$ the smallest k such that the Siegel modular cusp–forms of weight k have no common zero.

Main Theorem 1. *Assume that* $g \geq 5$.

i) *If* $n > gk(g)/2$, *then* $\mathrm{A}(g, n, R)$ *is a quasi–projective scheme of finite type over* $\mathbb{C}$.

ii) *If* $\gamma(R) \leq 0$ *and* $n > gk(g)/2$, *then* $\mathrm{A}(g, n, R) = \phi$; *if* $\gamma(R) > 0$ *and* $n > (1 + \gamma(R))gk(g)/2$, *then* $\mathrm{A}_{\mathrm{deg}}(g, n, R) = \phi$.

Remark 1. The case of $1 \leq g \leq 4$ can be reduced to the case of $g = 5$ (see [N, p. 176]).

Remark 2. If $\gamma(R) \leq 0$, then $\mathrm{A}(g, n, R) = \mathrm{A}_{\mathrm{deg}}(g, n, R)$ (see Proposition (4.3)).

Let $\mathrm{A}'(g, n, R)$ denote the subspace of $\mathrm{A}(g, n, R)$ consisting of those $A \in \mathrm{A}(g, n, R)$ such that the polarization divisor Θ_t of A_t $(t \in R)$ is non–singular for some $t \in R$. Let

$$\mathrm{A}'_{\mathrm{deg}}(g, n, R) = \mathrm{A}'(g, n, R) \cap \mathrm{A}_{\mathrm{deg}}(g, n, R) .$$

For $A'(g, n, R)$ and $A'_{\text{deg}}(g, n, R)$ we prove the following better bounds on n than those in the Main Theorem 1.

Theorem 2. *Assume that* $g \geq 5$.

i) *If* $n > 3g(g+3)/(g+1)$, *then* $A'(g, n, R)$ *is a quasi–projective scheme of finite type over* $\mathbb{C}$.

ii) *If* $\gamma(R) \leq 0$ *and* $n > 3g(g+3)/(g+1)$, *then* $A'(g, n, R) = \phi$; *if* $\gamma(R) > 0$ *and* $n > (1 + \gamma(R))3g(g+3)/(g+1)$, *then* $A'_{\text{deg}}(g, n, R) = \phi$.

In the proof we will use similar but more precise current inequalities than those in [N]. Nadel reduced the non–existence of high level structure to that of certain transcendental holomorphic mapping from $\mathbb{C}$ and used the Nevanlinna calculus. Instead, we will carry out a sort of Nevanlinna claculus just on R , and then deduce estimates on certain Chern numbers, which yield our assertions.

Unfortunately, it is not yet proved if Nadel's theorem holds in its form. We will discuss his result at the end of § 4 (Remark 2).

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§ 2. Lemmas on currents

Let M be an m–dimensional paracompact complex manifold and φ a locally integrable function on M. We denote by $\partial\bar{\partial}[\varphi]$ the $\partial\bar{\partial}$–derivative of φ in the sense of currents (cf., e.g., [L]).

(2.1) Lemma. *Let φ be a holomorphic function on M and $a > 0$ a C^{∞}–function on M. Then we have the following:*

i) (Poincaré–Lelong) $\frac{i}{2\pi}\partial\bar{\partial}[\log|\varphi|^2] = (\varphi)$, *where* (φ) *denotes the divisor determined by* φ.

ii) $\frac{i}{2\pi}\partial\bar{\partial}[\log(\log|\varphi|^2 a)^2] = \frac{i}{2\pi}\partial\bar{\partial}\log(\log|\varphi|^2 a)^2$.

For a proof, see [GrK].

Let D be a complex hyperface of M with only normal crossings. Let $p_0 \in D$ and take a holomorphic local coordinate neighborhood $U(z_1, \dots, z_m)$ around p_0 such that U is biholomorphic to the unit polydisc $\Delta^m = \{(z_1, \dots, z_m) : |z_j| < 1\}$, $p_0 = (0, \dots, 0)$ and

$$D \cap U = \{z_1 \cdots z_k = 0\} \quad (1 \le k \le m) .$$

Let ω_0 be the following form on $\Delta^m - D$:

$$(2.2) \qquad \omega_0 = \sum_{j=1}^{k} \frac{i}{2\pi} \frac{dz_j \wedge d\bar{z}_j}{|z_j|^2(\log|z_j|^2)^2} + \sum_{j=k+1}^{m} \frac{i}{2\pi} dz_j \wedge d\bar{z}_j .$$

A real (1,1) – form η is said to have at most Poincaré growth near D if $|\eta| = O(\omega_0)$ around all points of D. Note that the coefficients of such η are locally integrable around any point of D.

(2.3) Lemma. *Let M and D be as above. Let $\varphi : M - D \longrightarrow \mathbb{R}$ be a C^∞ – function which is bounded from above around every point of D. Let ω_1 and ω_2 be C^∞ real (1,1) – form an $M - D$ such that $\omega_1 \geq 0$ and ω_2 has at most Poincaré growth near D. Suppose that*

$$\frac{i}{2\pi} \partial\bar{\partial} \varphi \geq \omega_1 + \omega_2$$

on $M - D$. Then the coefficients of ω_1 are locally integrable on M and

$$\frac{i}{2\pi} \partial\bar{\partial} [\varphi] \geq \omega_1 + \omega_2$$

on M as currents.

Proof. While this is implicitly proved in Propositions 1.2 and 1.3 of [N], we here give a simplified proof for the completeness. We may assume that M is biholomorphic to Δ^m and $D = \{z_1 \cdots z_k = 0\}$. Let ω_0 be the form defined by (2.2). By the assumption, there is a constant $c > 0$ such that $c\omega_0 + \omega_2 \geq 0$. Put

$$\psi = \frac{1}{2}\log \frac{1}{\prod_{j=1}^{k} (\log |z_j|^2)^2} + \sum_{j=k+1}^{m} |z_j|^2 .$$

Then ψ is bounded from above around any point of D and

$$\frac{i}{2\pi}\, \partial\bar{\partial}\, \psi = \omega_0 .$$

We have

$$\frac{i}{2\pi}\, \partial\bar{\partial}(\varphi + c\psi) \geq \omega_1 + (c\omega_0 + \omega_2) \geq 0 ,$$

so that $\varphi + c\psi$ is plurisubharmonic on $M - D$. Since $\varphi + c\psi$ is bounded from above around any point of D, $\varphi + c\psi$ is uniquely extended to a plurisubharmonic function on M ([GR]). Thus

$$\eta = \frac{i}{2\pi}\, \partial\bar{\partial}[\varphi + c\psi]$$

is a positive (1,1) – current on M. Let

$$\eta = \eta_{\mathrm{reg}} + \eta_{\mathrm{sing}}$$

be the Lebesgue decomposition of η into the regular part η_{reg} and the singular part η_{sing}, which are both positive (1,1) – currents. Since $\eta_{\mathrm{reg}} \geq \omega_1 + \omega_2 + c\omega_0$,

$$\frac{i}{2\pi}\, \partial\bar{\partial}[\varphi + c\psi] \geq \omega_1 + \omega_2 + c\omega_0 .$$

By Lemma (2.1), ii), $\frac{i}{2\pi}\partial\bar{\partial}[\psi] = \omega_0$, and hence

$$\frac{i}{2\pi}[\varphi] \geq \omega_1 + \omega_2 .$$

Q.E.D.

§ 3. Inequalities for Chern numbers

Let $\mathbf{H}_g$ be the Siegel upper–half space of degree g and $\Gamma(n) \subset Sp(2g, \mathbb{Z})$ the Siegel modular group of level $n \geq 3$. Then $\Gamma(n)$ is torsion free. Let $\overline{\Gamma(n)\backslash\mathbf{H}_g}^*$ be the Satake compactification of the quotient variety $\Gamma(n)\backslash\mathbf{H}_g$ and put

$$D^* = \overline{\Gamma(n)\backslash\mathbf{H}_g}^* - \Gamma(n)\backslash\mathbf{H}_g .$$

Let R be a complex projective manifold. Let $A \longrightarrow R$ be a principally polarized Abelain variety over R with level n–structure . Then it naturally induces a meromorphic mapping

$$f_A : t \in R \longrightarrow [A_t] \in \overline{\Gamma(n)\backslash\mathbf{H}_g}^* .$$

Since $\Gamma(n)\backslash\mathbf{H}_g$ is complete hyperbolic and hyperbolically imbedded into $\overline{\Gamma(n)\backslash\mathbf{H}_g}^*$, $f_A^{-1}(D^*)$ is a hypersurface of R and f_A is holomorphic in $R - \operatorname{Sing} f_A^{-1}(D^*)$, where $\operatorname{Sing} f_A^{-1}(D^*)$ denotes the singular locus of $f_A^{-1}(D^*)$ (cf. [K]).

(3.1) Definition. *We say that $A \longrightarrow R$ is non–constant if*

$$\operatorname{rank} f_A = \max\{\operatorname{rank} df_{At} : t \in R - f_A^{-1}(D^*)\} = \dim R .$$

Let $\mathbf{A}(g, n, R)$ be the moduli space of all non–constant principally polarized Abelian varieties over R with level n–structure , and $\mathbf{A}_{\deg}(g, n, R)$ the subspace of all $A \longrightarrow R$ of $\mathbf{A}(g, n, R)$ which have degenerations; i.e., $f_A(R) \cap D^* \neq \phi$.

Let ω be the Bergman metric form on $\Gamma(n)\backslash\mathbf{H}_g$ normalized as

$$\frac{i}{2}\,\partial\bar\partial \log \omega^m = \omega ,$$

$$m = \dim \mathbf{H}_g = \frac{g(g+1)}{2} .$$

Here we use the notation, $\partial\bar\partial \log \omega^m$ in the following sense: If $\omega^m = a(z) \cdot idz_1 \wedge d\bar z_1 \wedge \ldots \wedge idz_m \wedge d\bar z_m$ with holomorphic local coordinates $(z_1, \ldots , z_m)$ on M , then

$$\partial\bar\partial \log \omega^m = \partial\bar\partial \log a(z) .$$

It follows that

(3.2) $\qquad$ holomorphic sectional curvature of $\omega \le -\frac{1}{m}$.

Let S be a locally closed $s-$dimensional submanifold of $\mathbf{H}_g$ with imbedding $\iota : S \longrightarrow \mathbf{H}_g$.

(3.3) Lemma. $\frac{i}{2}\,\partial\bar\partial \log \iota^* \omega^s \ge \frac{1}{m}\,\iota^* \omega$.

Proof. We endow S with the induced metric h by $\overset{*}{\iota}\omega$. Let K be the curvature form of h. Let $X \in T_pM$ be a holomorphic tangent vector at $p \in M$ and $\{e_1, \dots, e_s\}$ be an orthonormal basis of T_pM. Since $-\partial\bar{\partial}\log \overset{*}{\iota}\omega^s$ is the Ricci curvature of h, we see by [GK]

$$(3.4) \qquad (\partial\bar{\partial}\log \overset{*}{\iota}\omega^s)(X, \bar{X}) = -\sum_{j=1}^{s} K(e_j, \bar{e}_j, X, \bar{X}).$$

We may assume that $X = ce_1$ with $c = \|X\|$, and note that K is non–positive. Therefore we obtain from (3.2) and (3.4)

$$(\partial\bar{\partial}\log \overset{*}{\iota}\omega^s)(X, \bar{X}) \geq -|c|^2 K(e_1, \bar{e}_1, e_1, \bar{e}_1) \geq |c|^2 \frac{1}{m} = \frac{1}{m}\|X\|^2 .$$

Q.E.D.

We fix a Hodge metric form Ω on R. For a (1,1)–current T on R, we set

$$\nu(T) = \nu(T; \Omega) = \int_R T \wedge \Omega^{r-1},$$

$$r = \dim R .$$

Let $\overline{\Gamma(n)\backslash \mathbf{H}_g}$ be the torioidal compactification of $\Gamma(n)\backslash \mathbf{H}_g$ such that $\overline{\Gamma(n)\backslash \mathbf{H}_g}$ is smooth projective and $D = \overline{\Gamma(n)\backslash \mathbf{H}_g} - \Gamma(n)\backslash \mathbf{H}_g$ is a hypersurface with only normal crossings. Then we have a holomorphic mapping from $\overline{\Gamma(n)\backslash \mathbf{H}_g}$ onto $\overline{\Gamma(n)\backslash \mathbf{H}_g}^*$ which is the identity on $\Gamma(n)\backslash \mathbf{H}_g$. Let $f_A : R \longrightarrow \overline{\Gamma(n)\backslash \mathbf{H}_g}^*$ be the meromorphic mapping induced from $A \in \mathbf{A}(g, n, R)$. Then f_A defines a meromorphic mapping from R into

$\overline{\Gamma(n)\backslash \mathbf{H}_g}$, which is again denoted by $f_A : R \longrightarrow \overline{\Gamma(n)\backslash \mathbf{H}_g}$. We denote by $I(f_A)$ the indeterminancy locus of $f_A : R \longrightarrow \overline{\Gamma(n)\backslash \mathbf{H}_g}$. Then codim $I(f_A) \geq 2$. Let $[D] \longrightarrow \overline{\Gamma(n)\backslash \mathbf{H}_g}$ be the line bundle determined by D. Let $\sigma \in H^0(\overline{\Gamma(n)\backslash \mathbf{H}_g}, [D])$ be a section of $[D]$ such that $(\sigma) = D$. Take a hermitian metric $|\ \ |$ in $[D]$ so that $|\sigma| < 1$, and denote by $c_1(D)$ its first Chern form.

(3.5) Lemma. $\nu(\overset{*}{f}_A\omega/\pi) \leq m\{\nu(\overset{*}{f}_A c_1(D)) + \nu(c_1(K_R))\}$.

Proof. Put

$$S = f_A^{-1}(D) \cup I(f_A) \cup \{x \in R - I(f_A)\ ;\ \text{rank}\ df_{Ax} < r\}.$$

Then S is a thin analytic subset of R. Let $\alpha : \tilde{R} \longrightarrow R$ be a blowing up with center contained in S such that $\tilde{S} = \alpha^{-1}(S)$ is a hypersurface with only normal crossings and f_A is lifted to a holomorphic mapping $\tilde{f}_A : \tilde{R} \longrightarrow \overline{\Gamma(n)\backslash \mathbf{H}_g}$. We identify $\tilde{R} - \tilde{S}$ with $R - S$. Let Φ be a volume form on R. It follows from Lemmas (2.1) and (3.3) that

$$\text{(3.6)} \qquad \frac{i}{2\pi}\,\partial\bar{\partial}\log\left[\frac{\overset{*}{f}_A\omega^r}{\Phi}\,|\sigma \circ f_A|^2\right] \geq \frac{1}{m\pi}\,\overset{*}{f}_A\omega - \frac{i}{2\pi}\,\partial\bar{\partial}\log\Phi - \overset{*}{f}_A c_1(D)$$

$$= \frac{1}{m\pi}\,\overset{*}{f}_A\omega - c_1(K_R) - \overset{*}{f}_A c_1(D)$$

on $\tilde{R} - \tilde{S}$. Note that ω has at most Poincaré growth near D. Hence the function

$$\frac{(\overset{*}{f}_A\omega)^r}{\Phi}\ |\sigma\circ f_A|^2$$

is bounded from above, and $\overset{*}{f}_A\omega \geq 0$. By Lemma (2.3) and (3.6) we have

$$\frac{i}{2\pi}\ \partial\bar\partial\left[\log\left[\frac{(\tilde{\overset{*}{f}}_A\omega)^r}{\Phi}\ |\ \sigma\circ\tilde f_A|^2\right]\right] \geq \frac{1}{m\pi}\tilde{\overset{*}{f}}_A\omega - \overset{*}{\alpha}c_1(K_R) - \tilde{\overset{*}{f}}_A c_1(D)$$

as currents on $\tilde R$. We deduce that

$$\frac{1}{m}\nu(\overset{*}{f}\omega/\pi) - \nu(c_1(K_R)) - \nu(\overset{*}{f}_A c_1(D))$$

$$\leq \int_{\tilde R}\frac{i}{2\pi}\partial\bar\partial\left[\log\left[\frac{(\overset{*}{f}\omega)^r}{\Phi}\ |\ \sigma\circ\tilde f|^2\right]\right]\wedge\overset{*}{\alpha}\Omega^{r-1} = 0\ .$$

Q.E.D.

We denote by K_n the canonical line bundle over $\overline{\Gamma(n)\backslash\mathbb{H}_g}$.

(3.7) Lemma. $\nu(\overset{*}{f}_A c_1(K_n)) + \nu(\overset{*}{f}_A c_1(D)) \leq \nu(\overset{*}{f}_A\omega/\pi)$.

Proof. By $[M_2$, Proposition 3.4], there is a positive constant C and N such that

$$\frac{1}{|\sigma|^2(\log|\sigma|^2)^{2N}}\cdot\frac{\Phi}{\omega^m} \leq C \tag{3.8}$$

(cf. [N, p. 168]) . We have

$$(3.9) \qquad \frac{i}{2\pi}\,\partial\bar\partial \log\left[\frac{1}{|\sigma\circ f_A|^2(\log|\sigma\circ f_A|^2)^{2N}}\cdot\frac{\Phi\circ f_A}{\omega^m\circ f_A}\right]$$

$$= \overset{*}{f}_A c_1(K_n) - \frac{1}{\pi}\overset{*}{f}_A\omega + \overset{*}{f}_A c_1(D) - N\frac{i}{2\pi}\,\partial\bar\partial\log(\log|\sigma\circ f_A|^2)^2$$

on $\tilde{R}-\tilde{S}$. By Lemma (2.1), ii), Lemma (2.3), (3.8) and (3.9), we see that

$$\frac{i}{2\pi}\,\partial\bar\partial\left[\log\left[\frac{1}{|\sigma\circ\tilde{f}_A|^2(\log|\sigma\circ\tilde{f}_A|^2)^{2N}}\cdot\frac{\Phi\circ\tilde{f}_A}{\omega^m\circ\tilde{f}_A}\right]\right]$$

$$\geq \tilde{f}_A^{*} c_1(K_n) - \frac{1}{\pi}\tilde{f}_A^{*}\omega + \tilde{f}_A^{*} c_1(D) - N\frac{i}{2\pi}\,\partial\bar\partial\,[\log(\log|\sigma\circ\tilde{f}_A|^2)^2]$$

as currents on $\tilde{R}$. Therefore we deduce

$$0 \geq \int_{\tilde{R}}\tilde{f}_A^{*}c_1(K_n)\wedge\overset{*}{\alpha}\,\Omega^{r-1} - \frac{1}{\pi}\int_{\tilde{R}}\tilde{f}_A^{*}\omega\wedge\overset{*}{\alpha}\,\Omega^{r-1}$$

$$+\int_{\tilde{R}}\tilde{f}_A^{*}c_1(D)\wedge\overset{*}{\alpha}\,\Omega^{r-1}$$

$$= \nu(\overset{*}{f}_A c_1(K_n)) + \nu(\overset{*}{f}_A c_1(D)) - \nu(\overset{*}{f}_A\omega/\pi)\,.$$

Q.E.D.

§ 4. Proofs of main results

(a) We use the same notation as in § 3. Moreover, in this section we fix a Hodge metric form Ω on R as follows. If $\gamma(R) = -\infty$, we take Ω so that $\nu(c_1(K_R);\Omega) < 0$; if $\gamma(R) > -\infty$, we take Ω so that $\nu(c_1(K_R);\Omega) = \gamma(R)$. In any case we put

$$(4.1) \qquad \gamma(R) = \nu(c_1(K_R);\Omega)$$

in this section. We also assume that

$$(4.2) \qquad g \geq 5, \quad n \geq 3.$$

First we show the following:

(4.3) Proposition. *If* $\gamma(R) \leq 0$, *then* $\mathbf{A}(g, n, R) = \mathbf{A}_{\deg}(g, n, R)$.

Proof. Let $A \in \mathbf{A}(g, n, R)$ be an arbitrary element and $f_A : R \longrightarrow \overline{\Gamma(n)\backslash \mathbf{H}_g}$ the meromorphic mapping induced by A (see § 3). We show that $f_A(R) \cap D \neq \phi$. Assume that $f_A(R) \cap D = \phi$; i.e., $f_A(R) \subset \Gamma(n)\backslash \mathbf{H}_g$. Then $f_A : R \longrightarrow \overline{\Gamma(n)\backslash \mathbf{H}_g}$ is holomorphic. Put

$$S = \{x \in R ; \operatorname{rank} df_{Ax} < r\}.$$

Then S is a thin analytic subset of R. It follows from Lemma (3.3) that

$$\frac{i}{2}\,\partial\bar{\partial}\log f_A^*\omega^\tau \geq \frac{1}{m}\,f_A^*\omega$$

on $R-S$. Let Φ be a volume form on R and put $c_1(K_R) = (i/2\pi)\partial\bar{\partial}\log\Phi$. Then

$$\frac{i}{2\pi}\,\partial\bar{\partial}\log\frac{f_A^*\omega^\tau}{\Phi} \geq \frac{1}{m\pi}\,f_A^*\omega - c_1(K_R)$$

on $R-S$. Since $f_A^*\omega^\tau/\Phi$ is bounded, Lemma (2.3) implies

$$\frac{i}{2\pi}\,\partial\bar{\partial}\log\left[\frac{f_A^*\omega^\tau}{\Phi}\right] \geq \frac{1}{m\pi}\,f_A^*\omega - c_1(K_R)$$

as currents on R. Hence

$$\frac{1}{m\pi}\,\nu\,(f_A^*\omega) - \gamma(R) \leq 0\;,$$

so that $\gamma(R) > 0$.

Q.E.D.

(b) **Proof of Main Theorem 1.** Let $A \in \mathbf{A}(g, n, R)$ be an arbitrary element and $f_A : R \longrightarrow \overline{\Gamma(n)\backslash\mathbf{H}_g}$ be the induced meromorphic mapping. By definition of $k(g)$ (§ 1), there is a Siegel modular cusp form τ such that

$$\overline{\{\tau=0\}} \not\supset f_A(R)\;,$$

where $\overline{\{\tau=0\}}$ denotes the closure of $\{\tau=0\}$ in $\overline{\Gamma(n)\backslash\mathbf{H}_g}$. Let $(w_{11}, w_{12}, \dots, w_{gg}) = (w_1, w_2, \dots, w_m)$ be the standard coordinate system on $\mathbf{H}_g$.

Then

$$\eta = \tau^{g+1}(dw \wedge \dots \wedge dw_m)^{k(g)}$$

defines a $k(g)$ – pluricanonical meromorphic form on the non–singular part of $\overline{\mathrm{Sp}(2g, \mathbb{Z})\backslash \mathbf{H}_g^*}$. Here we note the following result due to Tai [T].

(4.4) If $g \geq 5$, every holomorphic pluricanonical form on the non–singular part of $\overline{\mathrm{Sp}(2g, \mathbb{Z})\backslash \mathbf{H}_g^*}$ extends holomorphically on the whole nonsingular model of $\overline{\mathrm{Sp}(2g, \mathbb{Z})\backslash \mathbf{H}_g^*}$.

By [N, p. 174] we see that

(4.5) the projection $\pi_n : \overline{\Gamma(n)\backslash \mathbf{H}_g} \longrightarrow \overline{\mathrm{Sp}(2g, \mathbb{Z})\backslash \mathbf{H}_g}$ is ramified to order at least n over D.

Since τ is a cusp form, we infer from (4.4) and (4.5) that $\pi_n^* \eta$ defines a holomorphic section

$$\xi \in \mathrm{H}^0(\overline{\Gamma(n)\backslash \mathbf{H}_g}, k(g)(K_n + D) - n(g+1)D)$$

such that $\{\xi = 0\} \not\supset f_A(R)$, where the assumption $n > gk(g)/2$ is used. Therefore $\nu(f_A^*(\xi)) \geq 0$, so that

$$k(g)\{\nu(\overset{*}{f_A}c_1(K_n)) + \nu(\overset{*}{f_A}c_1(D))\} - n(g+1)\nu(\overset{*}{f_A}c_1(D)) \geq 0 \ . \tag{4.6}$$

Combining (4.6) with Lemma (3.7), we get

$$k(g)\nu(\overset{*}{f_A}\omega/\pi) - n(g+1)\nu(\overset{*}{f_A}c_1(D)) \geq 0 \ . \tag{4.7}$$

This and Lemma (3.5) imply

$$k(g)\nu(\overset{*}{f_A}\omega/\pi) - n(g+1)\frac{1}{m}\nu(\overset{*}{f_A}\omega/\pi) + n(g+1)\gamma(R) \geq 0 \ .$$

Therefore

$$\left[\frac{2}{g}n - k(g)\right]\nu(\overset{*}{f_A}\omega/\pi) \leq n(g+1)\gamma(R) \ . \tag{4.8}$$

Since $n > gk(g)/2$, $\nu(\overset{*}{f_A}\omega)$ is uniformly bounded, so that $A(g, n, R)$ is quasiprojective. This shows i).

If $\gamma(R) \leq 0$ and $n > gk(g)/2$, then (4.8) implies that $A(g, n, R) = \phi$. Now suppose that $\gamma(R) > 0$ and $A \in A_{\deg}(g, n, R)$. It follows from (4.7) and Lemma (3.5) that

$$m\nu(\overset{*}{f_A}c_1(D)) + m\gamma(R) - n\frac{g+1}{k(g)}\nu(\overset{*}{f_A}c_1(D)) \geq 0 \ ,$$

so that

$$\left[\frac{n}{k(g)} - \frac{g}{2}\right]\nu(\overset{*}{f_A}c_1(D)) \leq \frac{g}{2}\gamma(R) \ .$$

Since $\nu(f_A^* c_1(D)) \geq 1$, we get

$$n \leq \frac{g}{2} k(g)(1 + \gamma(R)) .$$

Thus ii) is proved.

Q.E.D.

(c) **Proof of Theorem 2.** Here we use a result of Mumford $[\mathrm{M}_3]$. Let $N_0 \subset \mathrm{Sp}(2g, \mathbb{Z})\backslash \mathbb{H}_g$ be the set of principally polarized Abelian varieties (A, Θ) of which theta divisors Θ are singular. Let $\overline{\mathrm{Sp}(2g, \mathbb{Z})\backslash \mathbb{H}_g}^{(1)}$ be the partial compactification of rank 1 degenerations of $\mathrm{Sp}(2g, \mathbb{Z})\backslash \mathbb{H}_g$ (see $[\mathrm{M}_3, \S\, 1]$). Put

$$D^{(1)} = \overline{\mathrm{Sp}(2g, \mathbb{Z})\backslash \mathbb{H}_g}^{(1)} - \mathrm{Sp}(2g, \mathbb{Z})\backslash \mathbb{H}_g .$$

and let $\overline{\mathrm{Sp}(2g, \mathbb{Z})\backslash \mathbb{H}_g}^{(1),0}$ denote the smooth part of $\overline{\mathrm{Sp}(2g, \mathbb{Z})\backslash \mathbb{H}_g}^{(1)}$. Then

$$K_{\overline{\mathrm{Sp}(2g, \mathbb{Z})\backslash \mathbb{H}_g}^{(1),0}} = (g+1)\lambda - D^{(1)} , \tag{4.9}$$

where λ is the line bundle associated to Siegel modular forms ($[\mathrm{M}_3$, Proposition 1.7]). Let $\overline{N}_0$ denote the closure of N_0 in $\overline{\mathrm{Sp}(2g, \mathbb{Z})\backslash \mathbb{H}_g}^{(1),0}$. The divisor class $[\overline{N}_0]$ of $\overline{N}_0$ is given by

$$[\overline{N}_0] = \frac{(g+3)}{2} g!\, \lambda - \frac{(g+1)!}{12} [D^{(1)}] \tag{4.10}$$

([M$_3$, Theorem (2.10)]). Let $A \in \mathbf{A}'(g, n, R)$ and $f_A : R \longrightarrow \overline{\Gamma(n)\backslash \mathbb{H}_g}$ be the induced meromorphic mapping. Then $f_A(R)$ is not contained in the closure of $\pi_n^{-1}(N_0)$ in $\overline{\Gamma(n)\backslash \mathbb{H}_g}$. We infer from (4.10), (4.9), (4.5) and (4.4) that

$$\frac{(g+3)\,g!}{2(g+1)}\,\{\nu(f_A^* K_n) + \nu(f_A^* c_1(D))\} - \frac{(g+1)!}{12}\, n\nu(f_A^* c_1(D)) \geq 0\,,$$

and hence

$$6(g+3)\{\nu(f_A^* K_n) + \nu(f_A^* c_1(D))\} - (g+1)^2 n\nu(f_A^* c_1(D)) \geq 0\,. \tag{4.11}$$

Instead of (4.6), we use (4.11) and apply the same arguments as in (b) to obtain our assertions.

Q.E.D.

Remark 1. For the assertion i)'s of the Main Theorem 1 and Theorem 2, it is sufficient to assume in Definition (3.1) that $\operatorname{rank} f_A > 0$.

Remark 2. If $\dim R = 1$, then $\gamma(R) = -e(R)$, where $e(R)$ is the Euler number of R. Hence, if the genus of $R \leq 1$, then $\gamma(R) \leq 0$; in this case, Nadel [N] proved in fact that there is a proper algebraic subset E of $\overline{\mathrm{Sp}(2g, \mathbb{Z})\backslash \mathbb{H}_g}$ such that if $n \geq \max\{28, g(g+1)/2\}$, then $f_A(R) \subset \pi_n^{-1}(E)$ for all $A \in \mathbf{A}(g, n, R)$.

Remark 3. We can give a variation of Theorem 2 by making use of the theta–null–divisor ϑ_{null}. That is, using the standard notation of theta functions, we let

$$F(w) = \prod \vartheta \begin{bmatrix} a \\ t \end{bmatrix}(0, w)\,,\ w \in \mathbb{H}_g$$

be the product of all even–characteristic theta functions $\vartheta\begin{bmatrix} a \\ t \end{bmatrix}(0, w)$ (see [M_3, p. 370]). Then the divisor $\vartheta_{\mathrm{null}}$ defined by F satisfies

$$[\vartheta_{\mathrm{null}}] = 2^{g-2}(2^g + 1)\lambda - 2^{2g-5}[D] \tag{4.12}$$

on $\overline{\mathrm{Sp}(2g, \mathbb{Z})\backslash \mathrm{H}_g^{(1),0}}$. We restrict ourselves to consider only those $A \in \mathrm{A}(g, n, R)$ such that $\pi_n \circ \tilde{f}_A(R)$ is not contained in the closure of the support of $\vartheta_{\mathrm{null}}$. Then we obtain the bound $4g(1 + 2^{-g})$ (resp. $4g(1 + 2^{-g})(1 + \gamma(R))$) on n for the similar assertion to Theorem 2, i) (resp. ii)).

Remark 4 (T. Shioda). We consider the analogue of the Main Theorem 1 over a number field K. Let A be a principally polarized Abelian variety defined over K. Let $\hat{A}$ be the dual of A and A_n (resp. $\hat{A}_n$) the n–torsion subgroup of A (resp. $\hat{A}$). Let μ_n be the group of n–th roots of 1. Then we have Weil's $\bar{e}_n$–pairing

$$\bar{e}_n : A_n \times \hat{A}_n \longrightarrow \mu_n$$

which is surjective (cf. [M_1]). By the assumption, $\hat{A} = A$. Hence, if $A_n \subset A(K)$, then $\mu_n \subset K$. This implies that if $A(g, n, K) \neq \phi$, then n is bounded by the number of all roots of 1 in K. Hence, the estimate on n is as follows:

$$\varphi(n) \leq [K ; \mathbb{Q}] ,$$

where $[K ; \mathbb{Q}]$ is the extension degree of K over $\mathbb{Q}$ and $\varphi(n)$ is Euler's function

$$\varphi(n) = n \cdot \prod_{p \mid n} \left(1 - \frac{1}{p}\right).$$

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